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Variational Principles in Cosmology.

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Summary. — The difficulties with the application of Lagrangian or Hamiltonian formulations to spatially homogeneous cosmology are examined from a new point of view and a simple explanation is given for the necessary modifications of those formulations. A rather natural restriction on the shift vector field freedom minimizes the extent of the required modifications, leaving the dynamical Einstein equations in the form of a Lagrangian/Hamiltonian system driven by a nonpotential force. The symmetry group of this classical mechanical system is a representation of the diffeomorphism group corresponding to the restricted class of shift vector fields.

1. – Introduction.

Spatially homogeneous cosmology has fascinated relativists ever since TAUB introduced the topic nearly three decades ago (1), using group theory dating back to the work of Bianchi in the early part of the century (2). Over this period of time, the subject has provided a testing ground for ideas on singularities, global structure, Hamiltonian dynamics, particle creation effects, isotropization of cosmological models and gravitational degrees of freedom, to name a few.

Spatially homogeneous space-times are dynamic space-times (in the sense of having a nontrivial time development of Cauchy data) in which a natural

(1) A. H. TAUB: *Ann. Math.*, **53**, 472 (1951).

(2) L. BIANCHI: *Lezioni sulla teoria dei gruppi finiti di trasformazioni* (1918), p. 550; *Sugli spazi a tre dimensioni che ammettono un gruppo continuo di movimenti*, in *Collected Works of Bianchi*, Vol. **9**, p. 17.

slicing exists, thus making them ideal candidates for the application of three-plus-one techniques which involve the splitting of space-time into space and time ([3]). Because the hypersurfaces of the natural slicing are the orbits of a 3-parameter isometry group, the spatial aspects of the three-plus-one splitting are essentially all explicitly known for these space-times, leaving ordinary and not partial differential equations to govern their evolutionary aspects. This is one of the main attractions of these models, namely that one must deal only with ordinary differential equations which can, in fact, be treated by using the familiar methods of classical mechanics. However, these equations are still rather complicated and one needs to exploit the symmetry present in the problem, not only at the space-time level which produced ordinary rather than partial differential equations, but in the system of ordinary differential equations itself, which of course reflects the underlying symmetry.

A discussion of spatially homogeneous cosmology recognizing the various roles played by Lie groups has been developed in a series of papers by the author and it would be inappropriate to repeat much of that here ([4-6]). Instead, sufficient background will be introduced to provide a setting in which the reader's attention can be focused on the variational approach to the Einstein field equations for this class of space-times. This topic has been dealt with in either a somewhat confused or rather complicated manner in the literature ([7-9]), but, when looked at from the proper perspective, the unusual features which characterize it can be simply understood.

First it should be remarked that, although the details of Lie group theory may not be that familiar to all those interested in cosmology, the basic ideas involved in understanding spatially homogeneous space-times from the perspective of that theory are simple and lead to valuable finite-dimensional examples of many features of three-plus-one relativity that have been discussed over the past decade, among these being linearization stability, the initial-value problem, decompositions of symmetric tensors, lapse and shift freedom, gravitational degrees of freedom and the interplay of kinematics and dynamics. There have also arisen very interesting differences from the usual context of spatially compact or asymptotically flat space-times where formal manipula-

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([3]) C. W. MISNER, K. S. THORNE and J. A. WHEELER: *Gravitation* (San Francisco, Cal., 1973).

([4]) R. T. JANTZEN: *Commun. Math. Phys.*, **64**, 211 (1979); *Spatially homogeneous cosmology: background and dynamics*, in *Relativistic Cosmology and Bianchi Universes*, edited by R. RUFFINI, to appear.

([5]) R. T. JANTZEN: *Conformal geometry and spatially homogeneous cosmology*, to appear in *Ann. Ist. H. Poincaré*.

([6]) R. T. JANTZEN: *Variation of parameters in cosmology*, to appear in *Ann. Phys.*

([7]) M. P. RYAN jr.: *J. Math. Phys.*, **15**, 812 (1974).

([8]) G. E. SNEDDON: *J. Phys. A*, **9**, 229 (1976).

([9]) A. H. TAUB and M. A. H. MACCALLUM: *Commun. Math. Phys.*, **25**, 173 (1972).

tions can be made rigorous. The well-known trouble with the careless application of variational principles is merely one example. Less known are related features of the spatially homogeneous case which lead to the breakdown of formal and intuitive notions gained by the study of spatially compact or asymptotically flat space-times (5). Ideas about kinematics and dynamics and their interplay with various decompositions of symmetric tensors must all be carefully re-examined as in the case of variational principles, essentially because the function spaces involved are so fundamentally different from the above-mentioned classes of space-times, the difference being largely due to boundary conditions. Of course in the compact spatially homogeneous case called Bianchi type IX, there are no surprises and everything goes through as expected. This case has been nicely treated from a Hamiltonian viewpoint by RYAN (10) and from a non-Hamiltonian viewpoint by many others, among whom are LIFSHITZ, KHALATNIKOV and BELINSKII (11). For the remaining Bianchi types, the natural slices are noncompact when assumed to be simply connected as is done here. It is the lack of compactness coupled with the transitive symmetry group acting on these slices which leads to the difficulties referred to above.

2. – Preliminary details.

The manifold M of a spatially homogeneous (SH) space-time $(M, {}^4\mathrm{g})$ is here taken to be the product manifold $R \times G$ of the real line R and a 3-dimensional simply connected Lie group G. Let $g(\tilde{g})$ be the Lie algebra of left (right) invariant vector fields on G, $g^*(\tilde{g}^*)$ the corresponding dual space of left (right) invariant 1-forms on G, and $\mathscr{M}(g)$ the space of left invariant Riemannian metrics on G. The largest subgroup of the group of diffeomorphisms $\mathscr{D}(G)$ of G which acts naturally on the space of either left or right invariant tensor fields (or densities) on G is the semi-direct product group of translations and automorphisms of G:

$$\mathscr{D}(g) = L(G) \times \mathrm{Aut}\,(G) = R(G) \times \mathrm{Aut}(G) \subset \mathscr{D}(G)\,, \tag{2.1}$$

$L(G)$ and $R(G)$ stand for the groups of left and right translations, respectively. The generating Lie algebra $\mathfrak{X}(g)$ for the action of $\mathscr{D}(g)$ on G is

$$\mathfrak{X}(g) = \tilde{g} \oplus \mathrm{aut}\,(G) = g \oplus \mathrm{aut}\,(G) \subset \mathfrak{X}(G)\,. \tag{2.2}$$

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(10) M. P. RYAN jr. and L. C. SHEPLEY: *Homogeneous Relativistic Cosmologies* (Princeton, N. J., 1975) (see bibliography).

(11) V. A. BELINSKII, I. M. KHALATNIKOV and E. M. LIFSHITZ: *Adv. Phys.*, **19**, 525 (1970); E. M. LIFSHITZ, I. M. KHALATNIKOV and I. M. LIFSHITZ: *Sov. Phys. JETP*, **30**, 173 (1971).

Here aut (G) is the generating Lie algebra for the action of Aut (G) on G. The action of $\mathscr{D}(g)$ on g by dragging along is a homomorphism $\mathscr{A}:\mathscr{D}(g) \to \text{Aut}\,(g) \subset GL(g)$ (with kernel $L(G)$) of $\mathscr{D}(g)$ onto the group of automorphisms of the Lie algebra g, a subgroup of the general linear group of g. Similarly, the action of $\mathfrak{X}(g)$ on g by Lie bracketing on the left is a Lie algebra homomorphism $\text{ad}:\mathfrak{X}(g) \to \text{aut}\,(g) = \text{der}\,(g) \subset gl(g)$ (with kernel $\tilde{g}$) of $\mathfrak{X}(g)$ onto the Lie algebra of Aut (g), namely the derivations of g. These two statements necessarily hold only if G is simply connected.

It is convenient to introduce a basis $e = \{e_a\}$ of g and its dual basis $\{\omega^a\}$ of g^*. The components of the structure constant tensor of g with respect to e are

$$C^a{}_{bc} = \varepsilon_{bcd}\, n^{ad} + a_f\, \delta^{fa}_{bc} = \omega^a([e_b, e_c])\,. \tag{2.3}$$

The basis e generates matrix representations of $gl(g)$ and $GL(g)$ and of Lie algebras and Lie subgroups, respectively, of each, as well as of homomorphisms into these spaces; these will be indicated by the subscript e. For example, one has $\text{aut}_e(g) \subset gl(3, R)$, $\text{Aut}_e(g) \subset GL(3, R)$, $\text{ad}_e:\mathfrak{X}(g) \to \text{aut}_e(g)$ and $\mathscr{A}_e:\mathscr{D}(g) \to \text{Aut}_e(g)$. The natural basis of $gl(3, R)$ will be denoted by $\{\boldsymbol{e}^b{}_a\}$, so that one has, for instance, if $X \in \mathfrak{X}(g)$,

$$\text{ad}_e(X) = \text{ad}_e(X)^a{}_b\, \boldsymbol{e}^b{}_a = \omega^a(\pounds_X e_b)\, \boldsymbol{e}^b{}_a\,. \tag{2.4}$$

Similarly, $\mathscr{M}(g)$ is mapped onto $\mathscr{M} \subset GL(3, R)$, the 6-dimensional submanifold of matrices of components of inner products on R^3:

$$\mathsf{g} = g_{ab}\,\omega^a \otimes \omega^b \in \mathscr{M}(g) \mapsto \boldsymbol{g} = g_{ab}\,\boldsymbol{e}^b{}_a \in \mathscr{M}\,. \tag{2.5}$$

The action of $\mathscr{D}(g)$ on $\mathscr{M}(g)$ by dragging along maps homomorphically (with kernel $L(G)$) onto the natural left action of $\text{Aut}_e(g)$ on $\mathscr{M}$:

$$\boldsymbol{g} \mapsto \bar{\boldsymbol{g}} = \boldsymbol{S}^{-1T}\boldsymbol{g}\boldsymbol{S}^{-1}\,, \qquad \boldsymbol{S} \in \text{Aut}_e(g)\,. \tag{2.6}$$

This action reflects the freedom remaining in the choice of basis e of g once the structure constant tensor components $C^a{}_{bc}$ are fixed. For a given group G there is usually a preferred set of components which will be called canonical components. An explicit set is given in ([4]) for each isomorphism class (Bianchi type) of 3-dimensional Lie groups and this set will be understood here to be the canonical set. A basis in which the structure constant tensor components are canonical will be called a canonical basis. The group $\text{Aut}_e(G)$ is the subgroup of $GL(3, R)$ which leaves $C^a{}_{bc}$ invariant under the natural tensor action of that group:

$$C^a{}_{bc} \mapsto S^a{}_d\, C^d{}_{fg}\, S^{-1f}{}_b\, S^{-1g}{}_c = C^a{}_{bc}\,, \qquad \boldsymbol{S} \in \text{Aut}_e(g)\,, \tag{2.7}$$

and so depends only on the values of these components. One can, therefore, speak of the canonical matrix automorphism group $\mathrm{Aut}_e(g)$ as the representation of $\mathrm{Aut}(g)$ in any canonical basis.

Let $T\mathcal{M}$ and $T^*\mathcal{M}$ be the tangent and cotangent bundles of $\mathcal{M}$. By interpreting $\{g_{ab}\}$ as « co-ordinates » on $\mathcal{M}$, they may be lifted in a standard way to « co-ordinates » $\{g_{ab}, \dot{g}_{ab}\}$ on $T\mathcal{M}$ and « canonical co-ordinates » $\{g_{ab}, \pi^{ab}\}$ on $T^*\mathcal{M}$ (4). The action of $\mathrm{Aut}_e(g)$ on $\mathcal{M}$ lifts to an action on $T\mathcal{M}$ and a canonical action (in the sense of canonical transformations) on $T^*\mathcal{M}$:

$$(2.8)\qquad \begin{cases} \dot{\boldsymbol{g}} \mapsto \boldsymbol{S}^{-1T}\dot{\boldsymbol{g}}\boldsymbol{S}^{-1}, & \boldsymbol{S} \in \mathrm{Aut}_e(g), \\ \boldsymbol{\pi} \mapsto \boldsymbol{S}\boldsymbol{\pi}\boldsymbol{S}^T, & \boldsymbol{\pi} = \pi^{ab}\boldsymbol{e}^b{}_a. \end{cases}$$

The momentum function on $T^*\mathcal{M}$ which generates the canonical action on $T^*\mathcal{M}$ of the 1-parameter subgroup of $\mathrm{Aut}_e(g)$ generated by $\boldsymbol{A} = A^a{}_b\boldsymbol{e}^b{}_a \in \mathrm{aut}_e(g)$ is given by (4)

$$(2.9)\qquad P(\boldsymbol{A}) = -2\pi^{ab}A^\#{}_{ab}, \qquad A^\#{}_{ab} = g_{c(a}A^c{}_{b)}.$$

If $\mathbf{N}_t \subset \mathfrak{X}(g)$ is a time-dependent vector field with flow $h_t \subset \mathcal{D}(g)$, its action on g by dragging along produces a time-dependent automorphism $\boldsymbol{S}_t^{-1} = \mathcal{A}_e(h_t) \subset \subset \mathrm{Aut}_e(g)$, which acts as a time-dependent diffeomorphism of $\mathcal{M}$ via (2.6). The lift of this diffeomorphism to $T^*\mathcal{M}$ is a time-dependent canonical transformation generated by the time-dependent momentum function:

$$(2.10)\qquad P(\mathrm{ad}_e(\mathbf{N}_t)) = -2\pi^{ab}\,\mathrm{ad}_e(\mathbf{N}_t)^\#{}_{ab}.$$

Now consider the following class of metrics on M:

$$(2.11)\qquad {}^4\mathrm{g} = -N_t{}^2\,\mathrm{d}t \otimes \mathrm{d}t + g_{ab}(t)\,\omega^a \otimes \omega^b.$$

Here the lapse N_t is a function on R and the matrix $\boldsymbol{g}_t = g_{ab}(t)\,\boldsymbol{e}^b{}_a$ is a parametrized curve in $\mathcal{M}$ so that the induced metric ${}^3\mathrm{g}_t = g_{ab}(t)\,\omega^a \otimes \omega^b$ is a time-dependent left invariant metric on G. $(M, {}^4\mathrm{g})$ is then a spatially homogeneous space-time. The components of the left invariant extrinsic curvature tensor and canonical momentum density are given by

$$(2.12)\qquad 2N_tK_{t\,ab} = -\dot{g}_{t\,ab}, \qquad \pi_t{}^{ab} = -g_t{}^{\frac{1}{2}}\,(K_t{}^{ab} - K_t{}^c{}_c\,g_t{}^{ab}),$$

where the simpler subscript notation for time dependence is used throughout.

However, it is possible to introduce a nonzero shift vector field $\mathbf{N} = N_t{}^a\,e_a = \bar{N}^a_t\,\bar{e}_a$ which is a parametrized curve in $\mathfrak{X}(g)$ so that the metric assumes the following form when expressed in terms of the new spatial frame $\bar{e}_t$ which is obtained by dragging along the frame e by the flow h_t of $\mathbf{N}_t$ (4):

$$(2.13)\qquad \begin{cases} {}^4\mathrm{g} = -N_t^2\,\mathrm{d}t\otimes\mathrm{d}t + \bar{g}_{ab}(t)(\bar{\omega}^a + \bar{N}^a_t\,\mathrm{d}t)\otimes(\bar{\omega}^b + \bar{N}^b_t\,\mathrm{d}t)\,,\\ \bar{\boldsymbol{g}}_t = \boldsymbol{S}_t^{-1T}\boldsymbol{g}_t\boldsymbol{S}_t^{-1}\,,\\ \bar{e}_a = S^{-1b}{}_a e_b\,, \qquad \bar{\omega}^a = S^a{}_b\omega^b - \bar{N}^a\,\mathrm{d}t\,. \end{cases}$$

The matrix $\boldsymbol{S}_t^{-1} = \mathscr{A}_e(h_t)$ satisfies the following ordinary differential equations:

$$(2.14)\qquad \dot{\boldsymbol{S}}_t^{-1}\boldsymbol{S}_t = \mathrm{ad}_e(\mathbf{N}_t) \qquad \text{or} \qquad \dot{\boldsymbol{S}}_t\boldsymbol{S}_t^{-1} = \mathrm{ad}_{\bar{e}}(\mathbf{N}_t)\,,$$

which have as a consequence

$$(2.15)\qquad (\ln|\det\boldsymbol{S}_t|)^{\boldsymbol{\cdot}} = \mathrm{Tr}\,\boldsymbol{S}_t^{-1}\dot{\boldsymbol{S}}_t = \mathrm{Tr}\,\mathrm{ad}_e(\mathbf{N}_t)\,.$$

The manifold M may be reinterpreted in terms of a new product manifold $R\times G$ on which $\bar{e}$ and the restriction of $\{\bar{\omega}^a\}$ to the spatial slices are time independent when then considered as fields on G.

The barred components of the extrinsic curvature and canonical momentum are

$$(2.16)\qquad \begin{cases} 2N_t\bar{K}_{t\,ab} = -\dot{\bar{g}}_{t\,ab} - 2\bar{A}^{\sharp}_{t\,ab}\,,\\ \bar{\pi}_t{}^{ab} = -\bar{g}_t^{\frac{1}{2}}(\bar{K}_t{}^{ab} - \bar{g}_t{}^{ab}\bar{K}_t{}^c{}_c)\,,\\ \bar{A}^{\sharp}_{t\,ab} = \bar{A}_t{}^c{}_{(b}\bar{g}_{t\,a)c}\,, \qquad \bar{\boldsymbol{A}}_t = \mathrm{ad}_{\bar{e}}(\mathbf{N}_t)\,. \end{cases}$$

The relationship between the old and new components of the canonical momentum is

$$(2.17)\qquad \bar{\boldsymbol{\pi}}_t = |\det S_t^{-1}|\boldsymbol{S}_t\boldsymbol{\pi}_t\boldsymbol{S}_t{}^T\,.$$

Note that the transformation law for the components of the canonical momentum under a charge of basis e differs from the natural action of $\mathrm{Aut}_e(g)$ on the canonical co-ordinates π^{ab} of $T^*\mathscr{M}$ by the determinant factor. This is one source of trouble for variational approaches as the next section will show.

3. – Lagrangian/Hamiltonian approach.

The principal difficulty encountered in SH space-times with noncompact spatial slices is that any integral I of a SH integral $\mathscr{I}$ taken over a SH slice will be infinite since the integrand has no spatial dependence and the integral of a left invariant volume element is infinite:

$$I = \int_G \omega^1\wedge\omega^2\wedge\omega^3 \mathscr{I} = \mathscr{I}\int_G \omega^1\wedge\omega^2\wedge\omega^3 = \mathscr{I}\cdot\infty \,. \tag{3.1}$$

Therefore, any formal integrals appearing in the usual discussions of gravitational theory have no meaning here (except in the compact Bianchi type-IX case, where $\int_G \omega^1\wedge\omega^2\wedge\omega^3$ is always a finite constant). However, the SH integrand can be used in an analogous discussion in certain situations provided that caution is used, since there is no guarantee that such a correspondence is valid. For example, the integrand $\mathscr{L}_{\text{ADM}}$ of the ADM Lagrangian ([3]) may be interpreted as a real-valued function on $T\mathscr{M}\times R^+\times\mathfrak{X}(g)$ if one identifies $\{g_{ab}, \dot g_{ab}\}$ appearing in it with the natural co-ordinates on $T\mathscr{M}$:

$$\begin{cases} \mathscr{L}_{\text{ADM}}(\mathbf{g}, \dot{\mathbf{g}}; N, \mathbf{N}) = N g^{\frac12}\big(K^a{}_b K^b{}_a - (K^a{}_a)^2 + {}^3R\big)\,, \\ 2N K_{ab} = -\dot g_{ab} + (\text{Kill}\,\mathbf{N})_{ab}\,, \\ (\text{Kill}\,\mathbf{N})_{ab} = (\mathfrak{L}_{\mathbf{N}}{}^3\mathbf{g})_{ab} = -2\,\text{ad}_e(\mathbf{N})^\flat{}_{ab}\,. \end{cases} \tag{3.2}$$

It is of course understood that the global frame e on G used here is a basis of g and that the space-time metric is of the form (2.13), omitting the barred notation and explicit indication of time dependence.

On the other hand, the integrand $H_{\text{ADM}}(\mathbf{g}, \boldsymbol{\pi}; N, \mathbf{N})$ of the ADM Hamiltonian cannot be interpreted as a real-valued function on $T^*\mathscr{M}\times R^+\times\mathfrak{X}(g)$ upon identifying $\{g_{ab}, \pi^{ab}\}$ appearing in it with the natural canonical co-ordinates on $T^*\mathscr{M}$ since the components of the shift vector field are not necessarily left invariant and they appear explicitly:

$$\begin{cases} H_{\text{ADM}}(\mathbf{g}, \boldsymbol{\pi}; N, \mathbf{N}) = N\mathscr{H} + N^a\mathscr{H}_a\,, \\ \mathscr{H} = g^{-\frac12}\big(\pi^a{}_b\pi^b{}_a - \tfrac12(\pi^a{}_a)^2\big) - g^{\frac12}\,{}^3R\,, \\ \mathscr{H}_a = -2C^b{}_{ac}\pi^c{}_b + 4a_c\pi^c{}_a\,. \end{cases} \tag{3.3}$$

However, the definition

$$\pi^{ab} = \partial\mathscr{L}_{\text{ADM}}/\partial\dot g_{ab} = -g^{\frac12}(K^{ab} - g^{ab}K^c{}_c)\,, \tag{3.4}$$

interpreted as the fibre derivative map (or Legendre transformation) from $T\mathscr{M}$ onto $T^*\mathscr{M}$ determined by the (time-dependent) Lagrangian function $\mathscr{L}_{\text{ADM}}$ on $T\mathscr{M}$, yields the following (time-dependent) Hamiltonian function H on $T^*\mathscr{M}$:

$$H = \pi^{ab}\dot g_{ab} - \mathscr{L}_{\text{ADM}} = N\mathscr{H}^\bullet + P\big(\text{ad}_e(\mathbf{N})\big)\,. \tag{3.5}$$

Here the inverse of the fibre derivative map has been used:

$$\dot{g}_{ab} = 2Ng^{\frac{1}{2}}(\pi_{ab} - \tfrac{1}{2}\, g_{ab}\pi^{c}{}_{c}) + (\mathrm{Kill}\,\mathbf{N})_{ab}\,, \tag{3.6}$$

as well as (3.2) and (2.10). Thus, unless one considers only left invariant shift vector fields, as has been done in all previous discussions of this topic, one is forced to use H and not H_{ADM} as the natural Hamiltonian. Notice that H does not vanish on the constraint subspace of $T^*\mathscr{M}$, where $\mathscr{H} = \mathscr{H}_a = 0$, but equals the momentum function $P(\mathrm{ad}_e(\mathbf{N}))$.

The configuration space for the SH dynamics is just the manifold $\mathscr{M}$ (via the correspondence (2.5) induced by a fixed basis of g), and $T\mathscr{M}$ and $T^*\mathscr{M}$ are, respectively, the velocity and momentum phase spaces. The shift vector field $\mathbf{N}$ enters the dynamics only through the time-dependent matrix $\mathrm{ad}_e(\mathbf{N})$ which together with the lapse may be considered as explicit functions of time appearing in the Lagrangian and Hamiltonian functions. However, there is no guarantee that either $\mathscr{L}_{\mathrm{ADM}}$ or H will generate the correct equations of motion, namely the dynamical Einstein equations written as evolution equations for g_{ab} and π^{ab}. Difficulties arise from two separate quarters.

First, the Einstein force field is a nonpotential force on $\mathscr{M}$ (*i.e.* inexact 1-form on $\mathscr{M}$), only part of which is derivable from the scalar curvature potential function on $\mathscr{M}$:

$$\begin{cases} -\,{}^3G^{ab}\,\mathrm{d}g_{ab} = -\,\mathrm{d}U + Q^{ab}\,\mathrm{d}g_{ab}\,, \\ U = -\,g^{\frac{1}{2}}\,{}^3R\,, \qquad Q^{ab} = -\,2g^{\frac{1}{2}}(a^a a^b - 2a^c\, C^{(a}{}_c{}^{b)})\,. \end{cases} \tag{3.7}$$

Therefore, the nonpotential force Q must be inserted as a driving term in either the Lagrangian or Hamiltonian equations, multiplied by the lapse function of course. When the shift vector field is zero, this is in fact the only modification of those equations required, since the remaining parts of the equations of motion arise from simple partial differentiation of the « kinetic-energy term ». The force Q vanishes identically for the class-A Lie groups, where $2a_b = C^a{}_{ba} = 0$, but not for the class-B Lie groups, where $a_b \neq 0$.

From the Hamiltonian point of view, the second difficulty arises because of the failure of the natural lift to $T^*\mathscr{M}$ of the action of $\mathrm{Aut}_e(g)$ on $\mathscr{M}$ to correspond to the correct transformation for the components of the canonical momentum under a change of spatial frame. The additional term $P(\mathrm{ad}_e(\mathbf{N}))$ in the Hamiltonian does not generate the correct equation of motion for the momentum because it fails to generate the additional scaling of the momentum by the determinant factor $|\det \boldsymbol{S}^{-1}|$. One must, therefore, insert a term into that equation of motion to compensate for this discrepancy. Temporarily reintroducing the barred notation for the nonzero-shift case, one may easily evaluate the time derivative of eq. (2.4) using (2.14) and (2.15):

$$(\dot{\bar{\pi}}_t{}^{ab})^{\boldsymbol{\cdot}} = |\det \boldsymbol{S}_t^{-1}|\boldsymbol{S}_t{}^a{}_c\,\boldsymbol{S}_t{}^b{}_d\,\dot{\pi}_t{}^{ab} + 2\,\mathrm{ad}_{\bar{e}}(\mathbf{N}_t)^{(a}{}_c\bar{\pi}_t{}^{b)c} - \bar{\pi}_t{}^{ab}\,\mathrm{Tr}\,\mathrm{ad}_{\bar{e}}(\mathbf{N}_t)\,. \tag{3.8}$$

The first term is generated by the Hamiltonian $\bar{H} - \bar{P}(\mathrm{ad}_{\bar{e}}(\mathbf{N}_t))$, while $\bar{P}(\mathrm{ad}_{\bar{e}}(\mathbf{N}_t))$ generates the second term. The final term is the one which must be added to the Hamiltonian equation for the momentum. Again by dropping the barred notation and explicit indication of time dependence and introducing the definition

$$\delta\mathbf{N} = -N^a{}_{|a} = -g^{ab}N_{(a|b)} = -\tfrac{1}{2}\,\mathrm{Tr}\,\mathrm{Kill}\,\mathbf{N}\,, \tag{3.9}$$

the trace of the last equation of (3.2) leads to the result

$$\delta\mathbf{N} = \mathrm{Tr}\,\mathrm{ad}_e(\mathbf{N})\,. \tag{3.10}$$

The term in question may then be rewritten as $-\pi^{ab}\,\delta\mathbf{N}$.

Taking into account these two additional forces leads to the correct equations of motion for g_{ab} and π^{ab}:

$$\begin{cases} \dot{g}_{ab} = \{g_{ab}, H\}\,, \\ \dot{\pi}^{ab} = \{\pi^{ab}, H\} + NQ^{ab} - \pi^{ab}\,\delta\mathbf{N}\,. \end{cases} \tag{3.11}$$

One, therefore, has a momentum-dependent force present in addition to the nonpotential component of the Einstein force. The modified Lagrange equations are obtained by equating the negative of the Lagrange derivative of the Lagrangian to the additional force present in the system:

$$-\delta\mathscr{L}_{\mathrm{ADM}}/\delta g_{ab} = -\partial\mathscr{L}_{\mathrm{ADM}}/\partial g_{ab} + (\partial\mathscr{L}_{\mathrm{ADM}}/\partial\dot{g}_{ab})^{\bullet} = NQ^{ab} - \pi^{ab}\,\delta\mathbf{N}\,, \tag{3.12}$$

where π^{ab} is understood to be expressed in terms of $\{g_{ab}, \dot{g}_{ab}\}$ by formula (3.4). Note that, while the force Q vanishes for the class-A Lie groups, the additional shift force vanishes identically only for those groups for which the Lie algebra $\mathrm{aut}_e(g)$ is tracefree, *i.e.* $\mathrm{aut}_e(g) = \mathrm{saut}_e(g)$. This happens only for the semisimple case of Bianchi types VIII and IX where $\mathrm{Aut}_e(g) = \mathrm{SAut}_e(g)$ is a unimodular matrix group.

The presence of the additional shift term in the Lagrangian equations may also be understood directly by considering

$$\boldsymbol{g} = \boldsymbol{S}^T\bar{\boldsymbol{g}}\boldsymbol{S} \tag{3.13}$$

as a time-dependent change of co-ordinates on the configuration space $\mathscr{M}$. The Lagrangian function $\mathscr{L}_{\mathrm{ADM}}$, when expressed in these time-dependent co-ordinates, differs from the Lagrangian density $\bar{\mathscr{L}}_{\mathrm{ADM}}$ expressed in terms of the new frame $\bar{e}$ by a factor $|\det\boldsymbol{S}|$:

$$\mathscr{L}_{\mathrm{ADM}}(\boldsymbol{g}, \dot{\boldsymbol{g}}; N, 0) = |\det\boldsymbol{S}|\,\mathscr{L}_{\mathrm{ADM}}(\bar{\boldsymbol{g}}, \dot{\bar{\boldsymbol{g}}}; N, \mathbf{N})\,. \tag{3.14}$$

Therefore, if one uses $\overline{\mathscr{L}}_{\text{ADM}}$ as a Lagrangian function on $T\mathscr{M}$, one must insert a term in the new components of the Lagrange equations to compensate for the omission of this factor:

$$(3.15)\qquad S^{-1\,a}{}_c S^{-1\,b}{}_d(NQ^{cd} + \delta\mathscr{L}_{\text{ADM}}/\delta g_{cd}) =$$
$$= |\det S|(N\bar{Q}^{ab} + |\det S|^{-1}\,\delta/\delta\bar{g}_{ab}(|\det S|\overline{\mathscr{L}}_{\text{ADM}})) =$$
$$= |\det S|(N\bar{Q}^{ab} + \delta\overline{\mathscr{L}}_{\text{ADM}}/\delta\bar{g}_{ab} - (\ln|\det S|)^{\boldsymbol{\cdot}}\partial\overline{\mathscr{L}}_{\text{ADM}}/\partial\dot{\bar{g}}_{ab}).$$

The barred version of the definition (3.4) of the canonical momentum together with formulae (2.15) and (3.10) show that the additional term in the Lagrange equations is exactly the one appearing in (3.12).

The momentum-dependent force can be easily eliminated from the system by a simple restriction on the shift vector field, namely that it be divergence free:

$$(3.16)\qquad 0 = \delta\mathbf{N} = \operatorname{Tr}\operatorname{ad}(\mathbf{N})\,, \qquad \mathbf{N}\in\mathfrak{X}(g)\,.$$

This corresponds to limiting the diffeomorphism freedom on G to that subgroup of $\mathscr{D}(g)$ whose action on g is equivalent to the action of the special automorphism group $\operatorname{SAut}(g)$ whose Lie algebra $\operatorname{saut}(g)$ is the tracefree subspace of $\operatorname{aut}(g) = \operatorname{der}(g)$. For this subgroup, when referred to a basis e of g, the natural canonical action on $T^*\mathscr{M}$ agrees with the transformation law for $\{g_{ab}, \pi^{ab}\}$ under a change of basis and the compensating term must, therefore, vanish as it indeed does. The restricted group of diffeomorphisms is

$$(3.17)\qquad \mathscr{S}_L(\mathscr{D}(g)) = L(G)\times\operatorname{SAut}(G)$$

with generating Lie algebra

$$(3.18)\qquad s_L(\mathfrak{X}(g)) = \tilde{g}\oplus\operatorname{saut}(G)\,.$$

Here $\operatorname{SAut}(G)$ is the special automorphism group of G or the inverse image of $\operatorname{SAut}(g)$ under the isomorphism $\mathscr{A}:\operatorname{Aut}(G)\to\operatorname{Aut}(g)$, whiles $\operatorname{saut}(G)$ is its generating Lie algebra, namely the inverse image of $\operatorname{saut}(g)$ under the isomorphism $\operatorname{ad}:\operatorname{aut}(G)\to\operatorname{aut}(g)$.

One is left with the dynamical vacuum Einstein equations in the form of a Lagrangian/Hamiltonian system driven by a nonpotential force on the configuration space. The symmetry group of the driven system is $\operatorname{SAut}_e(g)$, which leaves both the Lagrangian and the Hamiltonian invariant as well as the nonpotential force. Its action reflects the freedom available to change the basis e of g while leaving the structure constant tensor components fixed as well as the left invariant volume element $\omega^1\wedge\omega^2\wedge\omega^3$ associated with the basis e. The assumption of simple connectivity guarantees that every such change of basis can be induced by a diffeomorphism of G and that the time-dependent action of $\mathrm{SAut}_e(g)$ can be interpreted in terms of a time-dependent diffeomorphism generated by some shift vector field. If this assumption is dropped, none of the statements at the Lie algebra (*i.e.* g) level change, but one is no longer guaranteed a parallel discussion at the group (*i.e.* G) level.

The discussion is easily extended to the nonvacuum case and to the vacuum and nonvacuum spatially self-similar generalizations of the SH case. In the nonvacuum case one must find a potential function U_s whose addition to the gravitational potential U generates the matter driving term in the dynamical Einstein equations. Such a potential exists for perfect fluids and electromagnetic fields [4,6]. Of course the presence of matter may reduce the symmetry group. In the spatially self-similar case, the group which plays the role of $\mathrm{SAut}_e(G)$ is the isotropy group at $(C^a{}_{bc}, b_a)$ of the natural action of $SL(3, R)$ on the product space of components of the structure constant tensor and of the g^*-valued covector b which together characterize spatially self-similar metrics [6].

For all Bianchi types except I, II and V, $\mathrm{SAut}_e(g)$ is 3-dimensional and the orbits of its action on $\mathscr{M}$ are generically 3-dimensional and almost everywhere transversal to the 3-dimensional diagonal submanifold $\mathscr{M}_\mathrm{D}$, provided that e is a canonical basis of g as defined in [4]. This leads to a parametrization of $\mathscr{M}$ in terms of the diagonal metric matrices and the canonical special automorphism matrix group, *i.e.* « co-ordinates on $\mathscr{M}$ » adapted to the symmetry group of the dynamics. This parametrization nicely reflects one possible splitting of the gravitational variables into « dynamical » and « kinematical » variables and, in fact, leads to a reduction of the constrained Hamiltonian system to an unconstrained one of two gravitational degrees of freedom, even in the presence of matter, with the exception of Bianchi type $\mathrm{VI}_{-\frac{1}{9}}$ [4]. For this type and the other exceptional types I, II and V, the situation is more complicated [4,5].

The utility of the Lagrangian/Hamiltonian approach to the dynamical Einstein equations is that a change of co-ordinates on the configuration space requires only that the Lagrangian/Hamiltonian function and the nonpotential force Q be transformed (together with the supermomentum constraint functions). If such a « change of co-ordinates » is really a parametrization of the space $\mathscr{M}$, then the induced transformation of $T^*\mathscr{M}$ requires the knowledge of the inverse of the differential of the parametrization. This can be avoided by first performing the transformation on $T\mathscr{M}$ where only the differential is required and then using the new form of the Legendre transformation. Furthermore, having set up the problem in terms of a classical mechanical system, one has available a wide range of classical techniques to apply in the study of that system. It also provides a visual model for the dynamics of the system which is not possible if one only has a complicated system of differential equations to work with.

● RIASSUNTO (*)

Si esaminano le difficoltà nell'applicazione di formulazioni lagrangiane o hamiltoniane alla cosmologia spazialmente omogenea, da un punto di vista nuovo e si fornisce una semplice spiegazione delle modifiche necessarie per quelle formulazioni. Un'abbastanza naturale restrizione della libertà di campo vettoriale di spostamento minimizza l'entità delle modifiche richieste, lasciando le equazioni di Einstein dinamiche nella forma di un sistema lagrangiano/hamiltoniano guidato da una forza non potenziale. Il gruppo di simmetria di questo sistema meccanico classico è una rappresentazione del gruppo di diffeomorfismo che corrisponde alla classe ristretta di campi vettoriali di spostamento.

(*) *Traduzione a cura della Redazione.*

Вариационные принципы в космологии.

Резюме (*). — Исследуются трудности, связанные с применением Лагранжева или Гамильтонова формализмов к пространственно однородной космологии. Предлагается простое объяснение необходимых модификаций этих формализмов. Естественное ограничение на свободу векторного поля смещения минимизирует степень требуемых модификаций, приводя к динамическим уравнениям Эйнштейна с непотенциальной силой. Группа симметрий этой системы в классической механике представляет группу диффеоморфизма, которая соответствует ограниченному классу векторных полей смещения.

(*) *Переведено редакцией.*

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