GRAVITOELECTROMAGNETISM: FURTHER APPLICATIONS

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ABSTRACT

Spacetime splitting plays the important role of reintroducing into general relativity a sort of "Newtonian terminology" or an "electromagnetic terminology" which results in a better interface of the (spacetime) four geometry with our (space and time) intuition and experience. Here we show how the electromagnetic analogy, gravitoelectromagnetism (GEM), leads to rewriting the entire set of Einstein equations in a Maxwell-like form, once a Komar current for GEM fields is introduced. Motion (of particles, fluid, spinorial field) in the context of GEM is also discussed.

Theiss

1. Gravitoelectromagnetism in brief

Spacetime splitting is accomplished locally by specifying a future pointing unit timelike vector field u which can be thought of as the four velocity field of a family of test observers. It induces a $1 \oplus 3$ orthogonal product structure for each tangent space into the subspace spanned by u (local time direction of the u observers) and LRS_u (the associated local rest space of the u observers).

The orthogonal decomposition can be extended to all tensor spaces and used to define a "measurement process" (associated with the family of test observers u) for tensor fields and tensor equations: the measurement of a tensor field is the collection of spatial tensor fields which result from spatial projection ($P(u) = I + u \otimes u^\flat$ being the projector) into LRS_u of all possible contractions of the tensor field itself by any number of factors of u or $u^\flat$ (where $\flat$ and $\sharp$ indicate the totally covariant and contravariant index-shifted tensors respectively).

Following notation and definitions of [1], measurement of the covariant derivative ${}^{(4)}\nabla u$ produces the kinematical fields of the u congruence: $a(u)$, $\omega(u)$, $\theta(u)$,

$$u^\alpha{}_{;\beta} = -a(u)^\alpha u_\beta - \omega(u)^\alpha{}_\beta + \theta(u)^\alpha{}_\beta \ ,$$

which in turn play the role of gravitoelectromagnetic fields and allow one to study test particle motion in a given gravitational background in analogy with that of charged particles in a given electromagnetic field: $g(u) = -a(u)$ is the gravitolectric (GE) field, $H(u) = 2\vec{\omega}(u)$ (where $\omega(u)$ is the spatial dual of the vorticity $\omega(u)$) is the gravitomagnetic (GM) vector field. But how is it possible to push forward this analogy with electromagnetism? A number of facts which support it can be listed:

- Consider two different families of test observers, u and $\bar{u} = \gamma[u + \nu(\bar{u}, u)]$ in arbitrary relative motion, The transformation laws for the GE and GM vector fields that they measure closely resemble those for the EM fields ($a_{(\mathrm{fw})}$ is defined in [1])

$$\begin{aligned} P(u)g(\bar{u}) &= \gamma^2[g(u) + \tfrac{1}{2}\nu \times_u H(u) - \theta(u) \llcorner \nu - \gamma^2(a_{(\mathrm{fw})} + \nu \times (\nu \times a_{(\mathrm{fw})}))] \\ P(u)H(\bar{u}) &= \gamma^2[H(u) - \nu \times_u g(u) + \mathrm{curl}_u \nu + \nu \times_u (\pounds(u)_u \nu^\flat)^\sharp] \end{aligned}$$

- The measurement of the Einstein equations leads to Maxwell-like equations for the GEM fields. In particular:

  a) The observer four velocity field itself acts like a four potential for the GEM fields and defines an antisymmetric GEM tensor field ${}^{(4)}\mathcal{G}$ (corresponding to the Faraday EM tensor ${}^{(4)}F$)

$$ {}^{(4)}\mathcal{G} \equiv du^\flat = u \wedge g(u) + {}^{*(u)}H(u) \ . $$

  b) The homogeneous Maxwell equations for the EM vector fields, replacing the EM vector fields by the GEM vector fields, lead to the corresponding equations

$$\begin{aligned} \mathrm{div}_u H(u) + g(u) \cdot_u H(u) &= 0 \ , \\ \mathrm{curl}_u g(u) + [\pounds(u)_u + \Theta(u)]H(u) &= 0 \ . \end{aligned}$$

  c) The GEM equations corresponding to the inhomogeneous Maxwell equations are much more complicated due to the additional feature of (written in an observer adapted frame as in [1])

$$\begin{aligned} {}^{(4)}R^{\top}{}_{\top} &= \pounds(u)_u \Theta(u) + \mathrm{Tr}\,\theta(u)^2 + \mathrm{div}_u g(u) - g(u)^2 - \tfrac{1}{2}H(u)^2 \\ &= 8\pi[T^{\top}_{\top} - \tfrac{1}{2}{}^{(4)}T^{\alpha}{}_{\alpha}] \ , \\ 2\,{}^{(4)}R^{\top}{}_{a} &= -2\nabla(u)_b[\theta(u)^b{}_a - \delta^b{}_a \Theta(u)] - [\mathrm{curl}_u H(u)]_a + 2[g(u) \times_u H(u)]_a \\ &= 16\pi\, {}^{(4)}T^{\top}_{a} \ . \end{aligned}$$

  However, after the introduction of the Komar current (defined by the observer congruence itself) for the gravitational field [2]

$$ {}^{(4)}J_{(\mathrm{K})} \equiv (1/4\pi)^* d^{*\,(4)}\mathcal{G} $$

  the correspondence is re-established

$$\begin{aligned} 4\pi \rho_{(\mathrm{K})} &\equiv \mathrm{div}_u g(u) - H(u)^2 \\ &= g(u)^2 - \tfrac{1}{2}H(u)^2 + {}^{(4)}R^{\top}{}_{\top} - [\pounds(u)_u \Theta(u) - \mathrm{Tr}\,\theta(u)^2] \ , \\ 4\pi J_{(\mathrm{K})a} &\equiv [\mathrm{curl}_u H(u)]_a - [g(u) \times_u H(u)]_a - \{[\pounds(u)_u + \Theta(u)]g(u)\}_a \ , \\ &= [g(u) \times_u H(u)]_a - 2\,{}^{(4)}R^{\top}{}_a - 2\nabla(u)_b[\theta(u)^b{}_a - \delta^b{}_a \Theta(u)] \\ &\quad -\{[\pounds(u)_u + \Theta(u)]g(u)\}_a \ . \end{aligned}$$

The difficulties of considering the Komar energy density $\rho_{(K)}$ as the energy of the gravitational field are well known and are reviewed by Cattaneo [3].

- The energy-momentum tensor for the GEM field can be introduced in complete analogy with electromagnetism

$$T^{(G)\alpha\beta} = (1/4\pi)[{}^{(4)}\mathcal{G}^{\alpha\mu}\,{}^{(4)}\mathcal{G}^{\beta}{}_{\mu} + \tfrac{1}{4}g^{\alpha\beta}\,{}^{(4)}\mathcal{G}^{\rho\sigma}\,{}^{(4)}\mathcal{G}_{\rho\sigma}] \, .$$

This permits another definition of an energy for the GEM field in addition to the Komar choice, but again one cannot interpret this as the energy of the gravitational field itself because of the spatial geometry contribution.

2. Motion in the context of GEM

As an application one can study test-particle motion or test-field behavior in the context of GEM: point gyros, fluids, scalar fields, spinorial fields or tensorial fields on a given gravitational background. Once the observer congruence has been specified, one can interpret their behavior in terms of a picture of more familiar physics involving the GE and GM vector fields with obvious advantages for intuition and understanding of effects in terms of the corresponding well known ones in the EM analogy.

For example, in a simplified picture of on a single particle moving along a geodesic, only the GEM force acts; a gyro in a given gravitational background precesses as its spin couples to the GM field; for a test fluid the GEM force adds to the pressure terms and determines the spatial acceleration of the fluid particles; for a Dirac spinor field in a gravitational field the so called Mashhoon or spin-rotation coupling is the coupling of the spin of the field to the GM field.

The details of these applications will be discussed elsewhere.

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References

1. R.T. Jantzen, P. Carini and D. Bini, *Ann. Phys. (N.Y.)* **215** (1992) 1.
2. A. Komar, *Phys. Rev.* **113** (1959) 934.
3. C. Cattaneo, *Ann. Inst. H. Poincaré* **IV-1** (1966) 1.

GRAVITOELECTROMAGNETISM AND INERTIAL FORCES IN GENERAL RELATIVITY

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ABSTRACT

In General Relativity the gravitational field is often described in terms of gravitoelectric and gravitomagnetic fields stressing the analogy between Maxwell's theory of electromagnetism and Einstein's theory of gravitation. Recently a Newtonian analogy has been revived by Abramowicz et al which describes the gravitational field in terms of inertial forces like the centrifugal and Coriolis forces. Here we show how a definition of centripetal acceleration naturally arises within the more general framework of gravitoelectromagnetism, and how it is related to the centrifugal force defined by Abramowicz et al, and how the Newtonian and the Maxwellian analogies are related.

1. Gravitoelectromagnetism and centripetal acceleration

Any description of the gravitational field in terms of spatial fields assumes a decomposition of spacetime into space plus time.[1] In these proceedings[2] "gravitoelectromagnetism" is presented as a very general tool which provides a common framework in which different ways of splitting spacetime and of introducing spatial gravitational forces can be clarified and compared.

It is shown there that the analogy between electromagnetism and gravity at the force level arises when a family of observers "measures" the equation of motion of a test particle. The result of the measurement is a spatial equation of motion which describes the particle under the influence of a Lorentz-like force

$$\begin{aligned} D_{(\mathrm{tem})(U,u)}p^{\alpha}{}_{(U,u)}/d\tau_{(U,u)} &= m\gamma_{(U,u)}[g(u)^{\alpha} + a(\nu_{(U,u)} \times \vec{H}(u))^{\alpha} + b\theta(u)^{\alpha}{}_{\beta}\nu_{(U,u)}{}^{\beta}] \\ &\quad + F_{(U,u)}{}^{\alpha} \equiv F^{(\mathrm{G})}_{(\mathrm{tem})}{}_{(U,u)}{}^{\alpha} + F_{(U,u)}{}^{\alpha}\ , \quad \mathrm{tem=fw,cfw,lie}\ , \end{aligned}$$

where a and b are constants that depend on the choice of spatial intrinsic derivative used to calculate the change of momentum of the test particle.

If one expresses the spatial momentum as the product of its magnitude and unit direction vector, $p_{(U,u)}{}^{\alpha} = ||p_{(U,u)}||\hat{\nu}_{(U,u)}{}^{\alpha}$, one can separate the tangential and the transverse accelerations, where the second defines the centripetal acceleration (from now on let $m = 1$). A reparametrization of the derivative in terms of the spatial arclength $dl_{(U,u)} = ||\nu(U,u)||d\tau_{(U,u)}$ rather than observer proper time leads to the expression

$$F^{(C)}_{(\mathrm{tem})}(U,u)^\alpha = \gamma(U,u)||\nu(U,u)||^2 D_{(\mathrm{tem})}(U,u)\hat{\nu}(U,u)^\alpha / dl_{(U,u)} \ , \qquad \mathrm{tem{=}fw,cfw,lie} \ . \tag{1}$$

(Note that in the Lie case this expression is orthogonal to the unit vector only if the expansion vanishes.) This centripetal acceleration (of each type) is well defined along the worldline of the test particle as follows from the definition of the spatial intrinsic derivative in the companion article.[2]

1.1. Conformal transformation of the spatial metric

In a series of papers Abramowicz et al[3–6] (see 3 for further references) have discussed some properties of the centrifugal force for circular orbits in the static case and have tried to generalize that discussion. Their definition of centrifugal force involves the so-called "optical metric", namely the conformally rescaled spatial metric whose geodesics are light ray paths in the static case.

In gravitoelectromagnetism one can also introduce a conformal spatial metric $\tilde{P}(u)_{\alpha\beta} = L^2 P(u)_{\alpha\beta}$, where L is described below. One can then re-express the total spatial covariant derivative of the unit vector in terms of rescaled (tilde) quantities to get

$$\begin{aligned}\tilde{F}^{(C)}_{(\mathrm{tem})}(U,u)^\alpha &\equiv \gamma(U,u)||\tilde{\nu}(U,u)||^2 \tilde{D}_{(\mathrm{tem})}(U,u)\tilde{\hat{\nu}}(U,u)^\alpha / d\tilde{l}_{(U,u)} \\ &\equiv \gamma(U,u)||\tilde{\nu}(U,u)||[\nabla_{(\mathrm{tem})}(u) + ||\tilde{\nu}(U,u)||\tilde{\hat{\nu}}(U,u)^\beta \tilde{\nabla}(u)_\beta]\tilde{\hat{\nu}}(U,u)^\alpha \\ &= \gamma(U,u)||\nu(U,u)||^2 D_{(\mathrm{tem})}(U,u)\hat{\nu}(U,u)^\alpha / dl_{(U,u)} \\ &\quad -\gamma(U,u)||\nu(U,u)||^2[(P(u)^{\alpha\beta} - \hat{\nu}(U,u)^\alpha \hat{\nu}(U,u)^\beta)\nabla(u)_\beta \ln L \\ &\qquad +(\hat{\nu}(U,u)^\alpha / ||\nu(U,u)||)\nabla_{(\mathrm{tem})}(u) \ln L] \ ,\end{aligned} \tag{2}$$

which defines the "conformal centripetal acceleration."

The last equality allows one to calculate this acceleration if $\tilde{\hat{\nu}}(U,u)$ is defined only on the worldline of the test particle as assumed. This definition differs from the definition of Abramowicz et al. In particular their centripetal acceleration (sign reversal of their centrifugal force) cannot be calculated for a velocity defined only on the worldline of the test particle. In their case one needs to expand smoothly the unit vector to define a vector field near the worldline. Of course this procedure is totally arbitrary and the choice that Abramowcz et al make more or less explicitly in their applications leads to our same results. In more complicated situations the results can differ.

2. Equatorial circular orbit in a stationary axisymmetric spacetime

In order to apply these results to a specific spacetime one must first choose the observer congruence. It is the geometry itself to suggest this choice. Given the spacetime metric ${}^{(4)}g_{\alpha\beta}$ in an appropriate coordinate system, $\{t, x^a\}$, one can take u to be the unit tangent vector m to the timelines $x^a = constant$ defining the "threading point of view" (THD in brief) or one can take u to be the unit vector n normal to the slicing $t = constant$ defining the "hypersurface point of view" (HYP in brief). (The "slicing point of view" is a hybrid case. For more detail see.[1]) If m and n are both timelike a boost will relate the two points of view. In both cases one can define

Lapse	$M = (-{}^{(4)}g_{00})^{1/2}$ (THD)	$N = (-{}^{(4)}g^{00})^{-1/2}$ (HYP)
Shift	$M_a = -{}^{(4)}g_{0a}/{}^{(4)}g_{00}$ (THD)	$N^a = -{}^{(4)}g^{0a}/{}^{(4)}g^{00}$ (HYP)
Spatial Metric	$\gamma_{ab} = {}^{(4)}g_{ab} - {}^{(4)}g_{0a}{}^{(4)}g_{0b}/{}^{(4)}g_{00}$ (THD)	$g_{ab} = {}^{(4)}g_{ab}$ (HYP)

choosing L to be M^{-1} and N^{-1} respectively to define the conformal metrics $\tilde{\gamma}_{ab}$ and $\tilde{g}_{ab}$.

For circular orbits in a stationary axisymmetric spacetime, one has two Killing vectors ∂_t and ∂_ϕ and the spatial velocity of the test particle is along the ϕ–direction (along ∂_ϕ in the HYP but along $\partial_\phi + M_\phi \partial_t$ in the THD). In both points of view, $\hat{v}(U,u)^\phi$ is the reciprocal of the square root of the covariant $\phi\phi$-component of the spatial metric. The relation between the centripetal acceleration and the conformal centripetal acceleration simplifies to

$$F^{(C)}_{(\mathrm{tem})}(U,u)^\alpha = \tilde{F}^{(C)}_{(\mathrm{tem})}(U,u)^\alpha + \gamma(U,u)||\nu(U,u)||^2 g(u)^\alpha \,, \tag{3}$$

where $g(u)^\alpha$ is the gravitoelectric (GE) field. The observed (nongravitational) spatial force $F(U,u)^\alpha$ then equals the centripetal acceleration minus the GE force $\gamma(U,u)g(u)^\alpha$. Using instead the conformal centripetal acceleration in this relationship, the extra term $\gamma(U,u)||\nu(U,u)||^2 g(u)^\alpha$ gets combined with the GE force $\gamma(U,u)g(u)^\alpha$ to yield a new gravitoelectric force $g(u)^\alpha/\gamma(U,u)$ which after multiplication by $\gamma(U,u)$ (they analyze the original 4-force and not its spatial part divided by $\gamma(U,u)$) Abramowicz et al refer to as their gravitational force in the Schwarzschild case. Finally one can calculate explicitly the centripetal acceleration to find in the HYP

$$\gamma(U,n)F^{(C)}_{(\mathrm{lie})}(U,n)_r = -\gamma(U,n)^2||\nu(U,n)||^2[\tfrac{1}{2}\ln(g_{\phi\phi})]_{,r} \,, \tag{4}$$

and an analogous expression for the THD with $\gamma_{\phi\phi}$ replacing $g_{\phi\phi}$, and for the conformal cases with the rescaled quantities replacing the previous ones.

As a specific example consider Minkowski spacetime in rotating cylindrical coordinates, the key example in discussions of centrifugal forces. The nonzero metric components are ${}^{(4)}g_{00} = -(1-\Omega^2 r^2) \equiv -\gamma^{-2}$, ${}^{(4)}g_{0\phi} = \Omega r^2$, ${}^{(4)}g_{\phi\phi} = r^2$, ${}^{(4)}g_{rr} = 1$, from which one can read the various THD and HYP functions. In the HYP case the observed spatial force simply equals the centripetal acceleration and one finds

$$\gamma(U,n)F(U,n)^r = -\gamma(U,n)^2||\nu(U,n)||^2/r \,, \tag{5}$$

while in the THD case one finds (with $\nu(U,m) = \Omega r$)

$$\gamma(U,m)F(U,m)^r = \begin{cases} -\gamma(U,m)^2\gamma^2||\nu(U,m)||^2/r - \gamma(U,m)^2\gamma^2\Omega^2 r - 2\gamma(U,u)^2\gamma^2\Omega||\nu(U,m)|| \\ -\gamma(U,m)^2\gamma^2||\nu(U,m)||^2(1+\Omega^2r^2)/r - \gamma^2\Omega^2 r - 2\gamma(U,m)^2\gamma^2\Omega||\nu(U,m)|| \end{cases} \tag{6}$$

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conformal centripetal [...]

3. Conclusions

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References

1. R.T. Jantzen, P. C[...]
2. R.T. Jantzen, P. C[...] *Grossmann Meetin*[...]
3. M.A. Abramowicz, [...]
4. M.A. Abramowicz, [...] Ellis, A. Lanza, an[...]
5. A.P. Prasanna and[...]
6. S. Iyer and A.P. P[...]

where on the righthand side of the first equation one has three terms: first $\gamma_{(U,m)}$ times the centripetal acceleration, then the GE and the gravitomagnetic (GM) forces (the latter of which is equivalent to the Coriolis force of Abramowicz et al), while for the second equation one has first $\gamma_{(U,m)}$ times the conformal centripetal acceleration and then the redefined GE and the GM forces. The sum of the three contributions in the THD case yields $\gamma_{(U,m)}F_{(U,m)}{}^{r} = -\gamma_{(U,m)}{}^{2}\gamma^{2}(||\nu_{(U,m)}|| + \Omega r)^{2}/r$ which can be transformed from the HYP result through a boost using the usual relativistic addition of velocities.

Consider finally the Schwarzschild case where the THD and HYP case coincides and one gets (for $u = m, n$)

$$\gamma_{(U,u)}F_{(U,u)}{}^{r} = -\gamma_{(U,u)}{}^{2}||\nu_{(U,u)}||^{2}\frac{r-2M}{r^{2}} + \gamma_{(U,u)}{}^{2}\frac{M}{r^{2}} = -\gamma_{(U,u)}{}^{2}||\nu_{(U,u)}||^{2}\frac{r-3M}{r^{2}} + \frac{M}{r^{2}} . \tag{7}$$

Here in the first equality the first and second terms are respectively the centripetal acceleration and GE force while in the second equality they are the conformal centripetal acceleration and the redefined GE force. To go from the first equality to the second, one simply transfers the velocity-dependent part of the GE force (using $\gamma_{(U,u)}{}^{2} = \gamma_{(U,u)}{}^{2}||\nu_{(U,u)}||^{2} + 1$) to the centripetal acceleration. As expected the conformal centripetal force is zero at $r = 3M$ as Abramowicz et al have found.

3. Conclusions

The concept of centripetal acceleration arises naturally within the framework of gravitoelectromagnetism. The spatial intrinsic derivative enables one to clearly define this acceleration along the worldline of a test particle. The use of a conformal transformation of the spatial metric mixes the GE field with the centripetal acceleration, a complication that seems to have meaning only in the static case. The results of applications to some other stationary spacetimes still require interpretation.

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References

1. R.T. Jantzen, P. Carini and D. Bini, 1992, *Ann. Phys.* (N.Y.) **215**, 1.
2. R.T. Jantzen, P. Carini and D. Bini, 1995, *Proceedings of the Seventh Marcel Grossmann Meeting* (in this book)
3. M.A. Abramowicz, 1992, *M.N.R.A.S.* **256**, 710.
4. M.A. Abramowicz, 1993, in *The Renaissance of General Relativity*, eds. G.F.R. Ellis, A. Lanza, and J.C. Miller, (Cambridge University Press, Cambridge).
5. A.P. Prasanna and S.K. Chakrabarti, 1990, *Gen. Relativ. Grav.* **22**, 987.
6. S. Iyer and A.P. Prasanna, 1993, *Class. Quantum Grav.* **10**, L13.