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TRANSITIVELY SELF-SIMILAR SPACE-TIMES

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Space-times admitting a transitive group of self-similarities are discussed. The relation to the conformally related homogeneous space-time is given. Solutions of the Einstein equations with this symmetry are exact power law metrics. The parameters appearing in the metric can be related to the structure constants of the similarity group. As an example, that relation is given explicitly for the Kasner solution.

1. INTRODUCTION

For a long time the use of isometries has been important in the search for new solutions of the Einstein equations. The mathematically simplest case occurs when the isometry group is transitive resulting in homogeneous space-times. Ozsvath[1] and Farnsworth and Kerr[2] found all such models with a perfect fluid source. The homogeneous cosmologies are of considerable physical and mathematical interest; e.g. Gödel's model[3] and other models studied in references 4 and 5. However, being stationary they are not realistic models for the universe. The existence of a transitive isometry group guarantees that the Einstein equations become purely algebraic providing that the source is invariant under the same group. It has been known at least since Eardley's 1974 paper[6] that the field equations remain algebraic under a weaker symmetry condition, namely that the space-time admits a transitive group of self-similarities. The class of transitively self-similar models includes non-stationary universes and is therefore more interesting than the class of homogenous space-times. Non-stationary models admitting a simply transitive similarity group are necessarily spatially homogeneous. In view of the great interest in spatially homogeneous universes in the early and mid seventies, it is therefore somewhat surprising that the systematic study of this class of

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models has not been undertaken earlier. An example of a transitively self-similar model is the expanding and rotating universe found by Rosquist[7]. This model is an example of what Wainwright[8] terms exact power law (EPL) solutions It turns out that the EPL metrics considered by Wainwright are precisely those spatially homogeneous models which admit a self-similarity not lying in the homogeneous hypersurfaces[10]. The connection between EPL metrics and transitive similarity groups has also been discussed independently by Wainwright[9].

2. SPACE-TIMES ADMITTING A SIMPLY TRANSITIVE GROUP OF SELF-SIMILARITIES

Let (M, g) be a space-time with a Lorentzian metric g. A vector field u is said to be a self-similarity (or homothetic motion) if $L_u g_{\alpha\beta} = c g_{\alpha\beta}$ for some constant c. Greek indices run from 0 to 3. The similarities of M form a Lie group H_n. It follows[6] that H_n has a subgroup G_{n-1} which is an isometry group for (M, g). Also, except for a few exceptional cases, g is conformally related to another metric $\tilde{g}$ such that H_n is an isometry group for (M, $\tilde{g}$). Now suppose that H_4 acts transitively on (M, g). Then H_4 has an isometry subgroup G_3 and there is a metric $\tilde{g} = e^{-2\Psi} g$ which is invariant under H_4. Depending on the causal character of the orbits of the isometry group, the space-time is spatially homogeneous, stationary or null hypersurface homogeneous. We discuss primarily SH models although much of the discussion applies to the stationary case with only minor modifications. Let t be the proper time with respect to g of the G_3 homogeneous orbits. Then it can be shown[10] that

$$g = t^2\, \tilde{g}. \tag{1}$$

One may choose a H_4 invariant Lorentz frame $\{\tilde{\Omega}^\alpha\}$ for (M, $\tilde{g}$) with $\tilde{\Omega}^0 = t^{-1}dt$. Then $\{\Omega^\alpha\} = \{t\,\tilde{\Omega}^\alpha\}$ is a Lorentz frame for g. Since $d\tilde{\Omega}^0 = 0$ the H_4 structure constants satisfy $\tilde{C}^0{}_{\alpha\beta} = 0$. In this frame the structure functions, connection and curvature have the form (we use the conventions of Kramer et.al.[16])

$$\begin{aligned} C^\alpha{}_{\beta\gamma} &= t^{-1}\, \hat{C}^\alpha{}_{\beta\gamma}, \\ \Gamma^\alpha{}_{\beta\gamma} &= t^{-1}\, \hat{\Gamma}^\alpha{}_{\beta\gamma}, \\ R^\alpha{}_{\beta\gamma\delta} &= t^{-2}\, \hat{R}^\alpha{}_{\beta\gamma\delta}, \end{aligned} \tag{2}$$

where the carets indicate constant quantities. The structure functions and connections of g and $\tilde{g}$ are in the frames $\{\Omega^\alpha\}$ and $\{\tilde{\Omega}^\alpha\}$ are related by

$$\begin{aligned}
\hat{C}^{\alpha}{}_{\beta\gamma} &= -2\,\delta^{0}_{[\beta}\,\delta^{\alpha}_{\gamma]} + \tilde{C}^{\alpha}{}_{\beta\gamma}, \\
\hat{\Gamma}^{\alpha}{}_{\beta\gamma} &= -g^{0\alpha}\,g_{\beta\gamma} + \delta^{0}_{\beta}\,\delta^{\alpha}_{\gamma} + \tilde{\Gamma}^{\alpha}{}_{\beta\gamma}, \\
\hat{\Gamma}_{\alpha\beta\gamma} &= -2\,\delta^{0}_{[\alpha}\,g_{\beta]\gamma} + \tilde{\Gamma}_{\alpha\beta\gamma}.
\end{aligned} \tag{3}$$

It follows that $C^{0}{}_{\alpha\beta} = 0$. The relation between the curvatures can be written as

$$\begin{aligned}
\hat{R}_{\alpha\beta\gamma\delta} &= 2\,g^{00}\,g_{\alpha[\gamma}\,g_{\delta]\beta} + 4\,K_{[\alpha\beta][\gamma\delta]} + \tilde{R}_{\alpha\beta\gamma\delta}, \\
K_{\alpha\beta\gamma\delta} &= \tilde{\Gamma}^{0}{}_{\alpha\gamma}\,g_{\beta\delta}.
\end{aligned} \tag{4}$$

The Ricci tensors can be related by the somewhat hybrid but compact formula

$$\hat{R}_{\alpha\beta} = 2\,\tilde{\Gamma}^{0}{}_{\alpha\beta} + \tilde{\Gamma}^{0\gamma}{}_{\gamma}\,g_{\alpha\beta} + \tilde{R}_{\alpha\beta}. \tag{5}$$

Finally the scalar curvatures are related by

$$\hat{R} = -6\,g^{00} + 6\,\tilde{\Gamma}^{0\alpha}{}_{\alpha} + \tilde{R}. \tag{6}$$

Note also the identities

$$\begin{aligned}
\Gamma^{0}{}_{\alpha\beta} &= -C_{(\alpha\beta)}{}^{0}, \\
\Gamma^{0\alpha}{}_{\alpha} &= C^{\alpha 0}{}_{\alpha},
\end{aligned} \tag{7}$$

which follow from $C^{0}{}_{\alpha\beta} = 0$. Analogous identities hold for the corresponding quantitites with carets and tildes.

Motivated by this discussion it is natural to define EPL solutions by the existence of a Lorentz frame in which the coordinate dependence of the connection can be written as $\Gamma^{\alpha}{}_{\beta\gamma} \propto t^{-1}$ where t is the proper time of the homogeneous hypersurfaces or an analogous spacelike coordinate in the stationary case. A Lorentz frame with this property will be called an EPL frame.

3. SPATIALLY HOMOGENEOUS MODELS

To discuss spatially homogeneous models we first introduce a standard G_3 invariant (spatial) frame $\{\omega^{a}\}$ (latin indices take the values 1,2,3) as in

Jantzen[11]. The structure constants of G_3 with respect to the standard frame are represented by four numbers $(n^{(a)}, a)$ (do not confuse the index a with the number a) which are restricted by the Jacobi identity $an^{(3)} = 0$. The isometry group G_3 has the Lie algebra g_3 whose special automorphism group is denoted by $SAut(g_3)$. One may choose a 3-dimensional subgroup $\bar{G}$ of $SAut(g_3)$ which can be used to diagonalize the spatial metric. The new diagonal primed frame is given by $\omega'^a = S^a{}_b\omega^b$ where $S = S(\theta^i)$ and the θ^i are time dependent coordinates on $\bar{G}$. In this way the θ^i represent the off-diagonal components of the spatial metric. Changing the origin of the coordinates θ^i corresponds to making constant frame transformations implying that the θ^i are effectively cyclic coordinates and that therefore only the derivatives $\dot{\theta}^i$ play a dynamical role (a dot denoting derivative with respect to t). To be more explicit, only those linear combinations ν^a of the $\dot{\theta}^i$ given by the matrix equation

$$\dot{S}S^{-1} = \varkappa_a \nu^a \tag{8}$$

are relevant to the spatially homogeneous dynamics. Here $\{\varkappa_a\}$ is a basis for $saut(g_3)$, the Lie algebra of $SAut(g_3)$[11,13]. The vector ν^a may be referred to as the automorphism velocity.

The primed spatial metric is written as $g' = e^{2\beta}$ where $\beta = \mathrm{diag}(\beta^1, \beta^2, \beta^3)$. One may introduce a double primed orthonormal spatial frame by $\{\omega''^1 = e^{\beta^1}\omega'^1, \omega''^2 = e^{\beta^2}\omega'^2, \omega''^3 = e^{\beta^3}\omega'^3\}$. Then $\{\omega''^0 = dt, \omega''^a\}$ is a Lorentz frame on the space-time.

The next step is to require that $\{\omega''^\alpha\}$ is an EPL frame. Before doing this let us ask whether we get all EPL solutions in this way, i.e. whether all EPL solutions as defined in the previous section can be put on the form specified by the double primed Lorentz frame. To see this suppose ξ_a are hypersurface adapted generators for G_3, i.e., the ξ_a are defined on M and are parallel to the SH slices. The similarity group H_4 is generated by $\{u = \xi_0, \xi_a\}$ with structure constants $\tilde{C}^\alpha{}_{\beta\gamma}$. Let $\{E_a\}_0$ be a left invariant basis defined on one particular slice S_0 and orthonormal with respect to $\tilde{g}$. Then define $\{E_a\}$ on M by dragging along by u, $L_u E_a = 0$. It follows that

$$L_u[\xi_a, E_b] = C^c{}_{0a}\,[\xi_c, E_b]. \tag{9}$$

This is a 36×36 linear differential system for the 36 components of $[\xi_a, E_b]$. Therefore, since $[\xi_a, E_b] = 0$ on S_0 this relation must hold everywhere on M showing that the E_a are H_4 invariant. The unit normal with respect to $\tilde{g}$ of the SH slicing is invariant under the action of H_4. As in Rosquist and Jantzen[10] let $d\tau$ be the covariant form of the unit normal. Then since $d\tau$ is H_4 invariant, $\{E_0 = d\tau, E_a\}$ is an invariant Lorentz frame with respect to $\tilde{g}$. Let $\{\tilde{\sigma}^0 = d\tau, \tilde{\sigma}^a\}$ be dual to $\{E_\alpha\}$. Then the conformal metric is

$$\tilde{g} = - d\tau \otimes d\tau + \delta_{ab} \tilde{\sigma}^a \otimes \tilde{\sigma}^b. \tag{10}$$

The physical metric then becomes[10]

$$g = - dt \otimes dt + \delta_{ab} \sigma^a \otimes \sigma^b \tag{11}$$

where $\sigma^a = t\tilde{\sigma}^a$. Since $\{\tilde{\sigma}^\alpha\}$ and $\{t^{-1}dt, t^{-1}\omega''^a\}$ are both H_4 invariant Lorentz frames with respect to $\tilde{g}$ they differ by, at most, a constant Lorentz transformation. Hence the frames $\{\sigma^\alpha\}$ and $\{\omega''^0 = dt, \omega''^a\}$ are related by a constant Lorentz transformation showing that no loss of generality is suffered by the assumption that $\{\omega''^\alpha\}$ is an EPL frame.

The off-hypersurface part of the connection in the double primed frame is given in matrix form by

$$(\Gamma''{}_0{}^a{}_b) = - d\beta/dt - \varkappa_a^{*''} \nu^a \propto t^{-1} \tag{12}$$

where

$$\varkappa_a^{*''} = (1/2)(e^\beta \varkappa_a e^{-\beta} + [e^\beta \varkappa_a e^{-\beta}]^T) \tag{13}$$

is the symmetrization of $\varkappa_a$ with respect to the orthonormal frame[14]. It follows that

$$e^\beta = e^{\hat{\beta}} \operatorname{diag}(t^{p_1}, t^{p_2}, t^{p_3}) \tag{14}$$

where $\hat{\beta} = \operatorname{diag}(\hat{\beta}^1, \hat{\beta}^2, \hat{\beta}^3)$ and the p_a are constants which might be called Kasner exponents. The EPL requirement also constrains the off-diagonal components of the metric. For details see Jantzen and Rosquist[12]. The constants $\hat{\beta}^a$, p_a and the constants representing the off-diagonal metric components together specify a given EPL metric. The remaining non-zero components of the connection are $\Gamma''{}^a{}_{bc}$ which are constant linear combinations of $n''^{(a)}$ and a'' leading to

$$n''^{(1)} = e^{\beta^1-\beta^2-\beta^3}\, n^{(1)} \propto t^{p_1-p_2-p_3}\, n^{(1)} \propto t^{-1},$$

$$n''^{(2)} = e^{\beta^2-\beta^3-\beta^1}\, n^{(2)} \propto t^{p_2-p_3-p_1}\, n^{(2)} \propto t^{-1},$$

$$n''^{(3)} = e^{\beta^3-\beta^1-\beta^2}\, n^{(3)} \propto t^{p_3-p_1-p_2}\, n^{(3)} \propto t^{-1}, \tag{15}$$

$$a'' = e^{-\beta^3} a \propto t^{-p_3}\, a \propto t^{-1}.$$

Depending on the Bianchi type, non-zero components of $n^{(a)}$ and a give restrictions on the Kasner exponents p_a. As an example take Bianchi types VIII and IX where a = 0 while all three $n^{(a)}$ are non-zero. This gives $p_1 = p_2 = p_3 = 1$.

4. MATTER AND CONSTANTS OF THE MOTION

All dimensionless quantities which are H_4 invariant are constants. If we assume that the fluid source is invariant, then in particular the ratio p/ρ is constant. Therefore for a perfect fluid the only possible equation of state is $p = (\gamma-1)\rho$. Since this is a common equation of state in cosmological and other applications it is not a serious restriction. A cosmological constant is not compatible with a transitive similarity group. Since Λ has the same dimension as the curvature scalar it follows that Λ, if non-zero, is proportional to t^{-2}, a contradiction.

All class A and orthogonal class B perfect fluid models have the constant of the motion

$$\ell = n\, g^{1/2}\, u^0 \tag{16}$$

where n is defined in terms of the energy density and pressure by $dn/n = d\rho/(\rho+p)$, g is the determinant of the 3-metric in the basis $\{\omega^a\}$ and u^0 is the hypersurface normal component of the fluid 4-velocity. With the equation of state $p = (\gamma-1)\rho$ we have $n = \rho^{1/\gamma}$. From relation (2) we see that $\rho \propto t^{-2}$ whence $n \propto t^{2/\gamma}$. Further u^0 is a constant and $g^{1/2} \propto t^{p_1+p_2+p_3}$. Thus

$$\ell = \hat{\ell}\, t^{-2/\gamma+\Sigma p_a} \tag{17}$$

where $\hat{\ell}$ is a non-zero constant. For models where ℓ is a constant of the motion it follows that

$$\Sigma p_a = 2/\gamma. \tag{18}$$

This is the case for Bianchi types VIII and IX giving $\Sigma p_a = 3 = 2/\gamma$ so that only the unphysical equation of state $p = -\rho/3$ is allowed for EPL solutions of those types. Models with two or less non-zero structure constants $n^{(a)}$ are less severely restricted leading to a rather rich structure of EPL solutions[12].

5. METRIC PARAMETERS AND THE STRUCTURE OF H_4.

We want to relate the constant parameters $\hat{\beta}^a$, p_a etc. to the structure constants of H_4. To do that consider the H_4 invariant basis $\{\tilde{\omega}''^{\alpha} = t^{-1}\,\omega''^{\alpha}\}$. To simplify notation we identify this frame with the frame $\tilde{\Omega}^{\alpha}$ given previously. The structure constants of H_4 in this basis can be calculated by means of the relations $d\tilde{\Omega}^{\alpha} = -\tilde{C}^{\alpha}{}_{\beta\gamma}\tilde{\Omega}^{\beta}\wedge\tilde{\Omega}^{\gamma}$. The non-zero components are

$$\begin{aligned} \tilde{C}^a{}_{bc} &= t\ {}^3C''^a{}_{bc}, \\ \tilde{C}^a{}_{b0} &= (1-p_a)\ \delta^a_b + t\ \varkappa''^a_{c\ b}\ \nu^c \text{ (no sum on a)}, \end{aligned} \tag{19}$$

where ${}^3C''^a{}_{bc} = \varepsilon_{bcd}n''^{ad} - 2\delta^a_{[b}a''_{c]}$ and $n''^{ab} = \mathrm{diag}(n''^{(1)}, n''^{(2)}, n''^{(3)})$ give the relation to (15). As an example consider orthogonal class A models. Then the automorphism velocity may be set equal to zero so that $\tilde{C}^a{}_{b0} = (1-p_a)\delta^a_b$ (no sum on a). As shown in section 4, $\Sigma p_a = 2/\gamma$ implying $\tilde{C}^{\alpha}{}_{0\alpha} = 3-2/\gamma$ which is is non-zero except when $\gamma = 2/3$ corresponding to the equation of state $p = -\rho/3$. Thus for realistic values of γ the trace $\tilde{A}_{\alpha} = (1/2)\tilde{C}^{\beta}{}_{\alpha\beta}$ is always non-zero for this class of models. The condition $\tilde{A}_{\alpha} \neq 0$ is independent of basis. Therefore some of the 4-dimensional groups are excluded as candidates for H_4 in this case. Vacuum solutions correspond to the limit $\gamma \to 2$ and in that case $\tilde{C}^{\alpha}{}_{0\alpha} = 2$.

As a concrete example we take the Kasner solution for which

$$\begin{aligned} &p_1 = -r/N, \qquad p_2 = r(1+r)/N, \qquad p_3 = (1+r)/N, \\ &N = 1 + r + r^2, \qquad 0 < r < 1, \qquad {}^3C^a{}_{bc} = 0. \end{aligned} \tag{20}$$

Let $\{\tilde{E}_{\alpha}\}$ be the basis to which $\{\tilde{\Omega}^{\alpha}\}$ is dual. Then $[\tilde{E}_{\alpha},\tilde{E}_{\beta}] = \tilde{C}^{\gamma}{}_{\alpha\beta}\tilde{E}_{\gamma}$ and in particular $[\tilde{E}_a,\tilde{E}_0] = (1-p_a)\tilde{E}_a$ (no sum). To put the algebra on the canonical form given Patera et.al.[15] we introduce a new basis according to $\{e_1 = \tilde{E}_1,$

$e_2 = \tilde{E}_3$, $e_3 = \tilde{E}_2$, $e_4 = (1-p_1)^{-1}\tilde{E}_0$). Then the algebra is given by $[e_1,e_4] = e_1$, $[e_2,e_4] = ge_2$, $[e_3,e_4] = he_3$ where $g = r^2/(1+r^2)$ and $h = 1/(1+r^2)$. This is the algebra $A^{gh}_{4,5}$ in the classification of Patera et.al.[15]. Distinct values of g and h give give rise to non-isomorphic algebras. Thus to each Kasner solution, given by the value of r, there corresponds a distinct similarity group H_4.

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