Chapter 1

INERTIAL FORCES: THE SPECIAL RELATIVISTIC ASSESSMENT

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Abstract Rotating observers and circular test particle orbits in Minkowski spacetime are used to illustrate the transport laws and derivative operators needed to define the various "inertial forces" one can introduce using the natural relative observer approach to describing spacetime. Various centripetal accelerations (often called centrifugal forces when multiplied by the mass) are evaluated and compared with the familiar value ν^2/r of nonrelativistic physics.

Keywords: inertial forces, gravitoelectromagnetism, general relativity

1. Introduction

Since the rotational aspects of general relativity are where a clear departure from Newtonian gravitation takes place, they have been a continued source of fascination for relativists since the birth of Einstein's theory of gravity in 1916. While Godel's surprising cosmological solution [1] of 1949 stimulated thinking about rotation and gravitational fields a half century ago, followed by the Kerr black hole solution [2] about 15 years later in 1963, even flat Minkowski spacetime has been a source of controversy since before general relativity ever arrived on the scene regarding rotating frames, starting with Ehrenfest's paradox of 1909 [3]. A recent summary of this history can be found in Rizzi and Ruggiero [4] and Grøn has provided a detailed historical account in this volume [5]. Apart from some outright errors, most of the controversy over the years is due to ambiguity of the questions posed or the quantities being measured, or a misunderstanding of the local Lorentz structure of Lorentzian manifolds.

An early edition of Landau and Lifshitz in the 1940s [6] was the first easily identified introduction of tools for examining curved spacetimes in terms of quantities that can be identified with local rest space and temporal measurements that correspond with our usual space plus time interpretation of kinematics in pre-relativistic and special relativistic mechanics. This (1+3) formalism, based on a family of timelike observers, was developed in the 1950s by among others Cattaneo, Zel'manov and Möller but was later eclipsed by the newer (3+1) version in the 1960s based on spacelike hypersurfaces that quickly found its way into the relativity community through the ADM approach to gravitation, then nicely promoted by the classic Misner, Thorne and Wheeler text *Gravitation* [7] in the 1970s and beyond.

In the second half of the 1980s the old timelike observer based formalism was reawakened by a rediscovery of some known but not really exploited properties of static spacetimes by Abramowicz and collaborators (in an article whose citation link to the past amounts to one earlier article by de Felice, which in turn refers back to Landau and Lifshitz and Cattaneo), who proceeded to generate a large number of papers developing the seeds from that idea (optical geometry). Thus the industry of examining inertial forces in general relativity was born, centered primarily on circular orbits, the natural starting point for theoretical experimentation. This involved a further refinement of the $4 = 1+3$ or $4 = 3+1$ orthogonal splitting of the spacetime tangent space by $3 = 1+2$ (parallel and perpendicular to the direction of relative motion in the local rest space). de Felice, after some scattered preliminary papers, re-

entered this industry in the late 1980s, when Jantzen and Carini, later joined by Bini, stumbled into this topic, soon after joined by Rindler and Perlick, Iyer and Vishweshwara, and Semerak among others in the 1990s. This history is summarized in a review article [8] where relevant references can be found.

All of this discussion is interpretational and any value it may have lies only in helping us understand how properties of given spacetimes are either like or unlike pre-relativistic or special relativistic ideas about kinematics and gravitational field behavior that we may have. In fact, some of these constructions do help us see the ways in which the spacetime geometry affects particle motion and gyro transport along particle trajectories, through its spatial geometry and gravitoelectric and gravitomagnetic fields that the splitting formalisms naturally define. Gravitoelectromagnetism summarizes these splitting methods which bring a natural analogy with electromagnetism into the mix [9, 10, 11]. While not wanting to overstate their importance, studying these questions has given us a handle on trying to make some sense of how global and local rotation is both similar and different but certainly much richer in general relativity compared to its predecessor theories.

The problem of introducing a relativistically correct definition of inertial forces in relativity has been approached from various directions. Abramowicz and coworkers introduced "inductive" definitions, starting from the special situation of a circular orbit around a black hole, while others starting with Landau and Lifshitz have given "deductive" definitions, starting from spacetime splitting techniques employed in at least stationary spacetimes. The former, which generalize the classical notion of inertial forces in certain ways based on the special circumstances, are somewhat unsatisfactory in that they fail to generalize further to more general situations. The optical geometry, while powerful for static spacetimes, is not very useful even for stationary spacetimes because of the disconnect between the optical geometry and null geodesics. On the other hand the most general setting in which the latter definitions arise require a great deal of effort introducing the appropriate formalism which in turn may overshadow the physical content of general formulas, i.e. "sufficient generality" requires a high price too. We outline the problem here and consider applications to circular orbits in Minkowski spacetime.

2. Inertial forces in classical mechanics

Inertial forces first enter our educational lives in the context of circular orbits in a central force field, so this is the natural starting point for

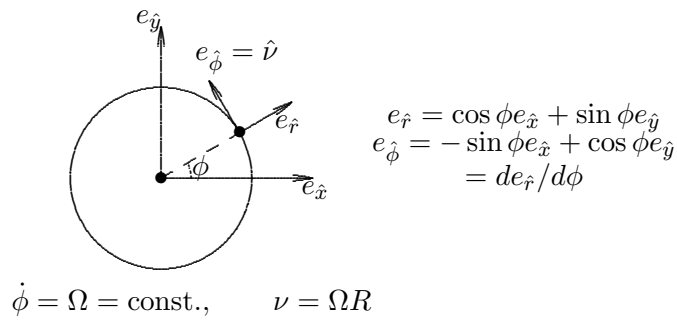

Figure 1.1. Circular motion in classical mechanics.

re-examining them in a more general context. Consider the classical example of a particle in uniform circular motion in the x-y plane with angular velocity Ω along the positive z-axis (counter-clockwise motion, for example, as indicated in Figure 1). The orbit of the particle can be parametrized by the spatial arclength ℓ

$$\ell = R\phi = R\Omega t \ , \tag{1.1}$$

so that its Cartesian coordinate representation is

$$(x, y) = (R\cos(\ell/R), R\sin(\ell/R)) \ . \tag{1.2}$$

The intrinsic (Frenet-Serret) frame consists of the vectors (unit tangent, normal, binormal)

$$\mathbf{t} = e_{\hat{\phi}} = \hat{\nu}, \qquad \mathbf{n} = -e_{\hat{r}}, \qquad \mathbf{b} = e_{\hat{\phi}} \times (-e_{\hat{r}}) = e_{\hat{z}} \tag{1.3}$$

satisfying

$$\frac{d}{d\ell}[\mathbf{t}, \mathbf{n}, \mathbf{b}] = [\kappa\mathbf{n}, -\kappa\mathbf{t} + \tau\mathbf{b}, -\tau\mathbf{n}] = [\boldsymbol{\omega} \times \mathbf{t}, \boldsymbol{\omega} \times \mathbf{n}, \boldsymbol{\omega} \times \mathbf{b}] \tag{1.4}$$

where

$$\boldsymbol{\omega} = \kappa\mathbf{b} + \tau\mathbf{t} = \kappa e_{\hat{z}} \ , \tag{1.5}$$

and the curvature $\kappa = 1/R$ and torsion $\tau = 0$ values easily follow from the relations

$$\frac{de_{\hat{\phi}}}{d\ell} = \frac{1}{R}\frac{de_{\hat{\phi}}}{d\phi} = -\frac{1}{R}e_{\hat{r}} = \kappa\mathbf{n}, \qquad \frac{de_{\hat{r}}}{d\ell} = \frac{1}{R}\frac{de_{\hat{r}}}{d\phi} = \frac{1}{R}e_{\hat{\phi}} = \kappa\mathbf{t} \ . \tag{1.6}$$

The torsion describes the rotation of the frame vectors in the plane orthogonal to the direction of motion, while the curvature describes the

rotation of the direction of motion itself. Its reciprocal, the radius of curvature, in this case is just the radius of the circle.

For a particle of mass m in this circular orbit, the classical notion of centripetal force ($= -$ centrifugal force) is

$$F^{(C)} = -\frac{m\nu^2}{R} e_{\hat{r}} = -m\kappa\nu^2 e_{\hat{r}} = m\nu^2 \frac{de_{\hat{\phi}}}{d\ell} = m\nu^2 \frac{d\hat{\nu}}{d\ell} \ , \qquad (1.7)$$

which must be the value of the force which is responsible for keeping the particle in this orbit. Through the force law it must equal the mass times the constant inward radial (centripetal) acceleration which characterizes the circular motion. Alternatively, from the point of view of the rest frame of the particle, one can conveniently think of a "balancing" of this inward force by an equal but outward centrifugal force obtained by just reversing the sign of the centripetal force and putting it on the other side of the force equation.

This single circular orbit discussion is only the first step in introducing inertial forces. There are two distinct directions in which one can proceed initially. If one considers a rigidly rotating frame with the same angular velocity, then the points fixed in the rotating grid all undergo such circular motion. If one then considers the path of a particle in arbitrary motion, and describes its motion with respect to the rotating frame, then not only does the fictitious centrifugal force of the rotating frame prove useful, but the equivalent Coriolis force arising from its motion relative to the frame itself now plays a role. Alternatively, one can generalize the single circular orbit to an arbitrary trajectory and use the Frenet-Serret machinery to decompose its acceleration into a linear component along the direction of motion and a centripetal component along the normal direction, all in a nonrotating frame. The radius of curvature $1/\kappa$ then takes the place of the circular radius R in the above discussion. In this case one only has a centripetal force responsible for the transverse acceleration (although traditionally the term centripetal is restricted to the case of purely transverse acceleration).

Finally one can combine the two discussions by performing a Frenet-Serret analysis of the trajectory in the rotating frame. One then has the combined effects of the centrifugal and Coriolis forces associated with the noninertial observer motion and the centripetal force associated with the curvature of the trajectory as seen by those observers. No one ever considers such a description in Minkowski spacetime, but if one wants to make some serious generalization to curved spacetime, it pays to think about it. One must also see how to transfer the differential properties of the description to the spacetime tangent space with no underlying flat spacetime that glues them together in the same way that the space-

fixed axes in the Euclidean vector space structure may be identified with a covariant constant orthonormal frame field on the corresponding flat Riemannian 3-manifold, then extended to a flat spacetime by taking the (orthogonal) cross-product with a time line. This preferred global inertial mathematical structure is not available in a general spacetime and is not geometrically relevant to the description with respect to a family of rotating observers.

Interpreting Euclidean space with a choice of origin as a vector space, let $\mathbf{e}_i$ be an orthonormal basis which is rigidly rotating with constant angular velocity $\mathbf{\Omega}$ and let $\dot{f} = df/dt$ be the time derivative, so that $\dot{\mathbf{e}}_i = \mathbf{\Omega} \times \mathbf{e}_i$. Then the time derivative of the position vector

$$(x^i \mathbf{e}_i)^{\cdot} = [\dot{x}^i + V^i]\mathbf{e}_i \tag{1.8}$$

picks up an extra term due to the velocity field $\mathbf{V} = \mathbf{\Omega} \times \mathbf{x}$ of the observers fixed in the rotating frame. The second time derivative picks up two terms

$$(x^i \mathbf{e}_i)^{\cdot\cdot} = [\ddot{x}^i + A^i + (2\mathbf{\Omega} \times \dot{x}^j \mathbf{e}_j)^i]\mathbf{e}_i \ , \tag{1.9}$$

first the acceleration

$$\mathbf{A} = \mathbf{\Omega} \times \mathbf{V} = \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{x}) = \mathbf{\nabla}[-\mathbf{V} \cdot \mathbf{V}/2] \equiv -\mathbf{g} \tag{1.10}$$

of the rigidly rotating observers and second a term arising from the curl of their velocity field

$$\mathbf{H} = 2\mathbf{\Omega} = \mathbf{\nabla} \times \mathbf{V} \ . \tag{1.11}$$

The equations of motion of a particle under the influence of a force $\mathbf{F}$ then take the form

$$m[\ddot{x}^i + A^i + (\mathbf{H} \times \dot{x}^j \mathbf{e}_j)^i] = F^i \ . \tag{1.12}$$

Simply transferring the extra terms from the left hand side to the right hand side of the equation

$$m\ddot{x}^i = m[g^i + (\dot{x}^j \mathbf{e}_j \times \mathbf{H})^i] + F^i \tag{1.13}$$

leads to their interpretation as equivalent to new force terms, referred to as fictitious or inertial or pseudo- forces and which are due to the noninertial motion of the rotating observers: the centrifugal force $m\mathbf{g}$ and Coriolis force $m(\dot{x}^j \mathbf{e}_j \times \mathbf{H})$ entering through the acceleration and curl of the velocity field of these observers. The analogy with the Lorentz force law of electromagnetism in a nonrotating system of coordinates is obvious, suggesting the terminology of a gravitoelectric field $\mathbf{g}$ and a

gravitomagnetic field **H** in this noncovariant flat spacetime discussion, each arising respectively from a scalar and vector potential associated with the observer velocity field.

However, to promote this to a discussion which makes sense in the context of the spacetime splitting of the tangent space to Minkowski spacetime associated with both the rotating and nonrotating observers, one must take into account the fact that their local rest spaces do not coincide and the time and space derivatives should be geometrized. This should be done so that one can apply the results to any family of observers moving arbitrarily (but smoothly) in space at less than the speed of light. (An observer horizon limits the validity of this description where the observer velocity field reaches the speed of light.) The choice of spatial geometry used to describe the flat spacetime situation is also important if one is to generalize to a nonflat scenario. In fact one would need to do this in order to unambiguously introduce a Frenet-Serret description of the left hand side of (1.13) relative to the rotating frame as well as interpret geometrically the operations defining the inertial forces on the right hand side. On the other hand, setting the angular velocity of the observers to zero, centrifugal effects would then be confined to the centripetal acceleration term of the left hand side.

Abramowicz refers to the centrifugal force associated with the noninertial observer in circular motion as Newton's definition and that associated with the Frenet-Serret decomposition as Huygens' definition [12], and chooses the latter (really a centripetal force) as the appropriate one to call centrifugal force in relativity. In reality both aspects are present, and one can smoothly interpolate between both descriptions of the situation in Minkowski spacetime by studying a circular orbit with angular velocity Ω about the z-axis from the point of view of a family of rotating coordinate systems with angular velocity varying from zero to the given value. The endpoint values were conveniently used by Rindler and Perlick [13] to calculate the precession of a spin vector under Fermi-Walker transport around an accelerated circular orbit in Minkowski spacetime as well as around geodesic circular orbits in the Schwarzschild, Kerr and Gödel spacetimes.

Although formally one can continue to use the Euclidean geometry in representing the differential equations of motion of a particle in the rotating frame, this geometry is not directly measurable by the rotating observers. This is where much confusion arises in deciding whether the circumference of a circle in the rotating frame is Lorentz contracted or not. The splitting of the tangent space into a local rest space and local time direction for the rotating observer family is directly connected with a local spatial measurement process. Spatial distances in the local

rest space of a rotating observer may be interpreted as describing the separation of (infinitesimally) nearby observers in the family (identified with points in the tangent space) as determined by halving the light travel time between them and the given observer. This was carefully derived and presented by Landau and Lifshitz in their section discussing distances and time intervals in general relativity [6], and is the basis for considering such splitting schemes in the first place. There is no need to consider measuring the circumference of a circle at "the same time"—simultaneity in fact does not exist. Instead nearby observers around the circle can determine their relative separations by exchanging light signals, from which the total circumference can then be calculated (using a limiting polygonal approximation construction, as in the Taylor-Wheeler discussion of Thomas precession [14]).

A simple figure helps explain the result one must find and is necessary to keep from identifying ruler lengths and nearby observer separations. In fact the paradoxes of special relativity are usually best explained with a spacetime diagram and the present case is no exception. Figure 1 shows 1) the world lines of the ends of a ruler of fixed length $r\Delta\phi$ separating two infinitesimally separated space-fixed observers separated by an angle $\Delta\phi$ on a circle of radius r, 2) the world lines of two rotating observers with the same angular separation, and 3) the world lines of the ends of a ruler of the same fixed length carried by the rotating observers (so that OC is boosted from OA). The rotating ruler appears Lorentz contracted as seen by the nonrotating observers: $||OE|| = ||OC||/\gamma$, but the proper separation between the two rotating observers in their own rest space is instead Lorentz expanded: $||OD|| = \gamma||OC||$, namely tangential lengths expand by exactly the gamma factor of the spatial metric expressed in the rotating coordinate system. The result of the calculation of the total circumference of the circle by the rotating observers is therefore Lorentz expanded, since the local relative distances are Lorentz expanded. The stationary situation saves us from having to think further about the question of time in this indirect measurement.

To summarize, centripetal and centrifugal forces are closely associated notions in Newtonian physics, because of its universal time shared by the reference frame and the rotating particle, but arise in two different contexts: analyzing rotational motion as seen by a nonrotating observer family (centripetal force) or as seen by a rotating observer family (centrifugal and Coriolis forces). The connection between centripetal and centrifugal force is made only when the rotating observer family is corotating with a circularly orbiting particle, so in a more general context of relative motion of the particle and the rotating observers, this direct connection is lost.

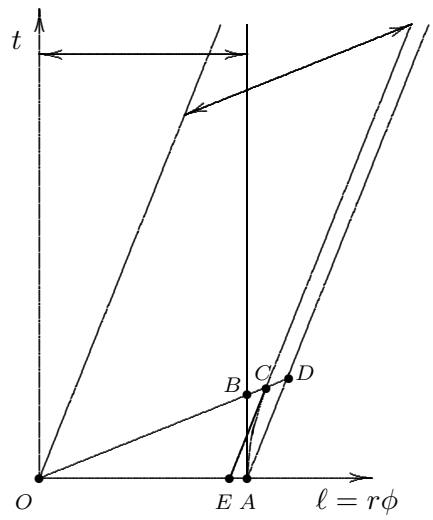

Figure 1.2. The boost between nonrotating and rotating rulers and the Lorentz contraction and expansion of nearby observer separations. The world sheets of the nonrotating ruler and rotating ruler of the same length $r\Delta\phi$ oriented along the tangential direction of the circle (hence related by the boost AC) are indicated by the double arrows. OE is the contracted rotating ruler as seen by the nonrotating observer, while OB is the contracted nonrotating ruler as seen by the rotating observer. The nearby rotating observers separated by the same angle $\Delta\phi$ marking the ends of the nonrotating ruler OA appear to be separated by the length of the nonrotating ruler as seen by the nonrotating observers, but by the Lorentz expanded length OD as seen by those rotating observers themselves. Thus while rulers contract, the relative distance between the corresponding observers expands.

In special and general relativity, the local rest space of the particle is distinct from that of the observer, each with its own local direction of time, so centripetal and centrifugal forces must differ even in the corotating case, and one must re-examine the time derivative itself. One must also keep in mind that the rotating frame description really involves an observer family and and a particle world line and their relative velocity along that world line. Implicit in the discussion is the family of observers fixed in space, simply following the time lines in the associated inertial coordinates on Minkowski spacetime. These space-fixed observers, which are "nonrotating" in a number of senses (individually as world lines, and locally and globally as families), anchor the rotating frame to the inertial properties of that spacetime. Any generalization or geometrization of centrifugal forces will indirectly involve these inertial observers through the choice of local rest space frame along the given world line with respect to which the change of a spatial vector is measured.

To illustrate some of these problems without getting lost in formalism and semantics, it is useful to discuss two simple examples: 1) an accelerated particle in circular motion with angular velocity Ω as seen by the nonrotating observers and by the corotating observers, and 2) a particle fixed in space with respect to the nonrotating observers, seen as moving in a circular orbit by a family of rotating observers having angular velocity Ω. Although one can then formally discuss each of these two situations from the point of view of the particle in its own local rest space in terms of comoving relative Frenet-Serret quantities, for circularly orbiting observers and particles where the relative acceleration is transverse to the direction of relative motion, all interesting accelerations belong to the intersection of the two local rest spaces and nothing new is seen.

How does the family of rotating observers view the test particle trajectory? One must first parametrize the given world line by a measurable parameter. The natural proper time parametrization as measured by the test particle can be converted by the Lorentz dilation gamma factor to correspond to the local observer time of the observers. The observers along the world line measure the relative speed, which can then be used to convert to a spatial arc length parametrization as seen by the observers, which is needed to establish a correspondence with the nonrelativistic Frenet-Serret discussion. But then how does the observer family measure the change in the relative velocity vector in order to express the relative acceleration and force law? Now one needs a fiducial orthonormal spatial frame along the world line (equivalently a way of transporting such a frame) that defines what it means not to change. For the rotating observers, which follow Killing trajectories, there are two choices of transport along the world line which preserve the observer local rest space: spatially projected parallel transport and spatially projected Lie transport (with respect to the observers). These collapse to the same transport with respect to the nonrotating observers.

3. Inertial forces geometrized

To go into the details [9, 10, 11, 15, 16], one must first quantify the relative observer description by introducing definitions for the (relativistic) relative Frenet-Serret analysis. Let the test particle have 4-velocity U and consider a family of test observers, i.e. a congruence of timelike world lines, with 4-velocity u. There arise two relative velocity unit vector fields belonging to the local rest space (LRS) of U and u: $\hat{\nu}(U,u) \in LRS_u$ and $\hat{\nu}(u,U) \in LRS_U$, related by a boost, and defined

only along the particle world line U

$$\begin{aligned} U &= \gamma(U,u)[u + ||\nu(U,u)||\hat{\nu}(U,u)] \ , \\ u &= \gamma(u,U)[U + ||\nu(u,U)||\hat{\nu}(u,U)] \ , \end{aligned} \tag{1.14}$$

where

$$\begin{aligned} &||\nu(U,u)|| = ||\nu(u,U)|| \equiv \nu \ , \\ &\gamma(U,u) = [1 - ||\nu(U,u)||^2]^{-1/2} = \gamma(u,U) \equiv \gamma \end{aligned} \tag{1.15}$$

are the magnitude of the relative velocity and the Lorentz factor respectively. The test particle world line can be parametrized by its proper time τ_U, or by the relative observer proper time or proper length

$$d\tau_U = \gamma^{-1} d\tau_{(U,u)} = (\gamma\nu)^{-1} d\ell_{(U,u)} \ . \tag{1.16}$$

Studying the evolution along of the direction of relative motion $\hat{\nu}(U,u)$ along the particle world line as measured by its covariant derivative along U and projecting it onto the local rest space of the observer u using the spatial projection operator $P(u)$ (in components: $P(u)^\alpha{}_\beta = \delta^\alpha{}_\beta + u^\alpha u_\beta$), one has the Fermi-Walker temporal derivative (spatial with respect to u in the sense that derivatives of spatial fields remain spatial with respect to u)

$$\frac{D_{(\mathrm{fw},U,u)}}{d\ell_{(U,u)}}\hat{\nu}(U,u) = (\gamma\nu)^{-1} P(u) \frac{D}{d\tau_U}\hat{\nu}(U,u) = k_{(\mathrm{fw},U,u)}\hat{\eta}_{(\mathrm{fw},U,u)} \tag{1.17}$$

which naturally defines the "relative centripetal force" in the context of the acceleration evaluation in analogy with (1.7), where the last equality here is just the decomposition of the preceding expression into its direction (unit vector) and magnitude (Fermi-Walker relative curvature). Similarly one can introduce a variation of this derivative in which the temporal part corresponds to the spatially projected Lie derivative along the observer congruence [9]

$$\begin{aligned} \frac{D_{(\mathrm{lie},U,u)}}{d\ell_{(U,u)}}\hat{\nu}(U,u) &= \frac{D_{(\mathrm{fw},U,u)}}{d\ell_{(U,u)}}\hat{\nu}(U,u) + \nu^{-1}\hat{\nu}(U,u) \times \omega(u) \\ &= k_{(\mathrm{lie},U,u)}\hat{\eta}_{(\mathrm{lie},U,u)} \ , \end{aligned} \tag{1.18}$$

where $\omega(u)$ is the vorticity vector of u, thus defining the Lie relative curvature.

Studying instead the evolution along the particle world line of $-\hat{\nu}(u,U)$ (the relative velocity of U with respect to u, as seen by U) and projecting its covariant derivative along U into the local rest space of the particle itself using the projection operator $P(U)$ (namely $P(U)^\alpha{}_\beta = \delta^\alpha{}_\beta + U^\alpha U_\beta$),

one has the Fermi-Walker temporal derivative (spatial with respect to U)

$$\frac{D_{(\mathrm{fw},U)}}{d\ell_{(U,u)}}[-\hat{\nu}(u,U)] = P(U)\frac{D}{d\ell_{(U,u)}}[-\hat{\nu}(U,u)] = \mathcal{K}_{(\mathrm{fw},u,U)}\hat{\mathcal{N}}_{(\mathrm{fw},u,U)} \quad (1.19)$$

which naturally defines the "relative centrifugal force" in the context of the acceleration evaluation.

While the practical meaning of these definitions may not be obvious, this seems to be the natural way of geometrizing inertial forces in relativity. In Eqs. (1.17), (1.17) and (1.19), $k_{(\mathrm{fw},U,u)}$ and $\mathcal{K}_{(\mathrm{fw},u,U)}$ are respectively the Fermi-Walker relative curvatures of the relative motion as seen by u and U respectively. The vanishing of a Fermi-Walker relative curvature defines the notion of *Fermi-Walker relatively straight* curves, while the inverse of the Fermi-Walker relative curvature defines the Fermi-Walker relative curvature radius

$$\begin{aligned} R_{(\mathrm{fw},U,u)} &= ||k_{(\mathrm{fw},U,u)}||^{-1}, \\ \mathcal{R}_{(\mathrm{fw},U,u)} &= ||\mathcal{K}_{(\mathrm{fw},u,U)}||^{-1}. \end{aligned} \quad (1.20)$$

Each of the relative velocity unit vector fields $\hat{\nu}(U,u)$ and $-\hat{\nu}(u,U)$ can be used to introduce a "relative Frenet-Serret" frame and scalars (with the option of using either the Fermi-Walker or Lie relative derivative: let tem = fw, lie) to define spatial frames $(\hat{\nu}(U,u), \hat{\eta}_{(\mathrm{fw},U,u)}, \hat{\beta}_{(\mathrm{fw},U,u)})$ in LRS_u and $(-\hat{\nu}(u,U), \hat{\mathcal{N}}_{(\mathrm{fw},u,U)}, \hat{\mathcal{B}}_{(\mathrm{fw},u,U)})$ in LRS_U satisfying the following relations

$$\begin{aligned} \frac{D_{(\mathrm{tem},U,u)}}{d\tau_U}\hat{\eta}_{(\mathrm{tem},U,u)} &= \gamma\nu[-k_{(\mathrm{tem},U,u)}\hat{\nu}(U,u) + \tau_{(\mathrm{tem},U,u)}\hat{\beta}_{(\mathrm{tem},U,u)}] \ , \\ \frac{D_{(\mathrm{tem},U,u)}}{d\tau_U}\hat{\beta}_{(\mathrm{tem},U,u)} &= -\gamma\nu\tau_{(\mathrm{tem},U,u)}\hat{\eta}_{(\mathrm{tem},U,u)} \ , \end{aligned} \quad (1.21)$$

and

$$\begin{aligned} \frac{D_{(\mathrm{tem},U)}}{d\tau_U}\hat{\mathcal{N}}_{(\mathrm{tem},u,U)} &= \gamma\nu[\mathcal{K}_{(\mathrm{tem},u,U)}\hat{\nu}(u,U) + \mathcal{T}_{(\mathrm{tem},u,U)}\hat{\mathcal{B}}_{(\mathrm{tem},u,U)}] \ , \\ \frac{D_{(\mathrm{tem},U)}}{d\tau_U}\hat{\mathcal{B}}_{(\mathrm{tem},u,U)} &= -\gamma\nu\mathcal{T}_{(\mathrm{tem},u,U)}\hat{\mathcal{N}}_{(\mathrm{tem},u,U)} \ . \end{aligned} \quad (1.22)$$

In (1.21) and (1.20) $\tau_{(\mathrm{tem},U,u)}$ and $\mathcal{T}_{(\mathrm{tem},u,U)}$ are the Fermi-Walker relative torsions. Their vanishing defines the Fermi-Walker or Lie relative flatness of a curve as seen by the observer or particle.

The relative observer decomposition of the force equation $DU/d\tau_U = f(U)$ for a test particle with unit mass $m = 1$ (for simplicity) is obtained

by substituting the decomposition (1.13) of $U = \gamma u + \gamma\nu(U,u)$ and projecting into the local rest space of u. The derivative of the first term leads to gravitoelectric and gravitomagnetic force terms arising from the acceleration and vorticity of the observer 4-velocity analogous to our earlier nonrelativistic discussion of rotating observers. The derivative of the second term can be decomposed into components along the direction of motion and in the transverse directions most easily using the relative Frenet-Serret frame [10]. The transverse acceleration term lying along $\hat{\eta}_{(\mathrm{tem},u,U)}$ is the relative centripetal acceleration seen by the observer and depends on the choice of temporal derivative used in the measurement. There is no other natural way of introducing these quantities that is based only on the geometry of the successive $4 = 1+3$ (time plus space) and $3 = 1+2$ (longitudinal plus transverse) splittings of the spacetime tangent space associated with a pair consisting of an observer congruence and a test particle world line and which does not depend on any special spacetime symmetry. Admittedly in a nonstationary spacetime, these quantities may not be as useful as they seem to be in aiding our interpretation of the geometry of stationary spacetimes. Moreover, the decomposition in the particle local rest space of the relative velocity of the observers is one further step removed from directly measurable quantities and is given here for geometrical completeness.

The equations defining the centripetal or centrifugal forces may be interpreted as the balance of forces along the (relative) transverse normal direction in the respective local rest spaces. The relative centripetal/centrifugal forces for each choice of temporal derivative along the transverse directions $\hat{\eta}_{(\mathrm{tem},U,u)}$ and $\hat{\mathcal{N}}_{(\mathrm{tem},u,U)}$ in the local rest spaces of the observer and the particle are

$$\begin{aligned}
\mathcal{F}^{(C)}_{(\mathrm{tem},U,u)} = \gamma\nu^2 k_{(\mathrm{tem},U,u)} &= [F_{(U,u)} + F^{(G)}_{(\mathrm{tem},U,u)}] \cdot \hat{\eta}_{(\mathrm{tem},U,u)} \ , \\
\mathcal{F}^{(C)}_{(\mathrm{tem},u,U)} = \gamma\nu^2 \mathcal{K}_{(\mathrm{tem},u,U)} &= [f(U) + \mathcal{F}^{(G)}_{(\mathrm{tem},u,U)}] \cdot \hat{\mathcal{N}}_{(\mathrm{tem},u,U)} \ ,
\end{aligned} \tag{1.23}$$

where the spatial gravitational forces in each local rest space are

$$\begin{aligned}
F^{(G)}_{(\mathrm{tem},U,u)} &= -\frac{D_{(\mathrm{tem},U,u)}}{d\tau_U} u = \gamma[g(u) + \epsilon_{(\mathrm{tem})}\nu(U,u) \times_u H(u)] \ , \\
\mathcal{F}^{(G)}_{(\mathrm{fw},u,U)} &= \gamma^{-1} P(U,u) F^{(G)}_{(\mathrm{fw},U,u)} = -\gamma^{-1} \frac{D_{(\mathrm{fw},U)}}{d\tau_U} u \ .
\end{aligned} \tag{1.24}$$

In the observer local rest space gravitational force expression, the gravitoelectric and gravitomagnetic fields are just the sign-reversed acceleration $g(u) = -a(u)$ and twice the vorticity $H(u) = 2\omega(u)$ of the observer 4-velocity u, and represent the inertial forces due to the motion

of the observers analogous to the centrifugal and Coriolis forces of the nonrelativistic discussion, while $\gamma(U,u)F(u,u)$ is the projection of the force $f(u) = ma(u)$ (unit mass for simplicity) into the observer local rest space. In the particle local rest space, only the projection of the spacetime Fermi-Walker derivative is available as a natural temporal derivative operator.

4. Application to rotating observers in Minkowski spacetime

To make this concrete, we return to the problem of an observer family (u) and test particle (U) both in circular motion in Minkowski spacetime

$$\begin{aligned} u &= \gamma_\omega[\partial_t + \omega\partial_\phi], \qquad \gamma_\omega = (1-\omega^2 r^2)^{-1/2} \ , \\ U &= \gamma_\Omega[\partial_t + \Omega\partial_\phi], \qquad \gamma_\Omega = (1-\Omega^2 r^2)^{-1/2} \end{aligned} \tag{1.25}$$

with constant angular velocities ω and Ω referred to nonrotating cylindrical coordinates

$$ds^2 = -dt^2 + dr^2 + r^2 d\phi^2 + dz^2 \ . \tag{1.26}$$

The natural cylindrically symmetric spatial orthonormal frame adapted to the family of nonrotating observers $(e_0 = \partial_t)$ is

$$e_1 = \partial_r \ , \ e_2 = (1/r)\partial_\phi \ , e_3 = \partial_z \ . \tag{1.27}$$

This frame can be boosted into the local rest space of the rotating observers $(E_0 = u)$

$$E_1 = e_1 \ , \ E_2 = \gamma_\omega(e_2 + \omega r e_0) \ , E_3 = e_3 \ , \tag{1.28}$$

which coincides with the relative Frenet-Serret frame up to signs and a permutation

$$\hat{\nu}(U,u) = \mathrm{sgn}(\Omega-\omega)E_2 \ , \ \hat{\eta}_{(\mathrm{fw},U,u)} = -E_1 \ , \ \hat{\beta}_{(\mathrm{fw},U,u)} = \mathrm{sgn}(\Omega-\omega)E_3 \ . \tag{1.29}$$

Similarly from the spacetime point of view, this frame coincides with the Frenet-Serret frame of u (modulo signs and a permutation), along which the frame is Lie dragged (see the appendix).

Solving the transport equations

$$\frac{D_{(\mathrm{tem},U,u)}}{d\ell_{(U,u)}} E_{(\mathrm{tem},U,u)} = 0 \ , \quad \mathrm{tem=fw,lie} \tag{1.30}$$

determines two more geometrically meaningful observer adapted frames related to E_1, E_2, E_3 by a rotation of the first two vectors with angular

velocities $\Omega_{(\mathrm{tem},U,u)}$ about E_3

$$R(\Omega_{(\mathrm{tem},U,u)})\begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} = \begin{pmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} , \tag{1.31}$$

where $(c, s) = (\cos(\Omega_{(\mathrm{tem},U,u)}t), \sin(\Omega_{(\mathrm{tem},U,u)}t))$. A direct calculation leads to the result

$$\Omega_{(\mathrm{fw},U,u)} = -\gamma_\omega \Omega \ , \ \Omega_{(\mathrm{lie},U,u)} = -\gamma_\omega \Omega + \gamma_\omega^2 \gamma_\Omega^{-1} \omega \ . \tag{1.32}$$

The spacetime Fermi-Walker transported frame along u (see the appendix) is of the same form as these with the angular velocity

$$\Omega_{(\mathrm{fw},u)} = -\gamma_\omega \omega \ . \tag{1.33}$$

The speed of the particle relative to the observers is

$$\nu = \frac{r|\Omega - \omega|}{1 - \Omega\omega r^2} \ . \tag{1.34}$$

Evaluating the Fermi-Walker and Lie relative curvatures (see Table 2 of [11]) leads to

$$\kappa_{(\mathrm{fw},U,u)} = \frac{|\Omega|}{r|\Omega - \omega|} \ , \quad \kappa_{(\mathrm{lie},U,u)} = \frac{\gamma_\omega^2}{r} \ . \tag{1.35}$$

Note that the Lie relative curvature is independent of Ω. Since the relative Lie derivative (1.17) used to define it consists of a Lie temporal derivative term [9] which vanishes for the Lie dragged ϕ direction of motion, and a spatial covariant derivative term, only the spatial covariant derivative contributes here. Thus the gamma-squared factor comes only from the spatial geometry of the rotating observer congruence, whose spatial metric is represented by the last three terms in the line element expressed in nonrotating cylindrical coordinates but adapted to the rotating observers

$$ds^2 = -\gamma_\omega^{-2}(dt - \gamma_\omega^2 r^2 \omega d\phi)^2 + dr^2 + \gamma_\omega^2 r^2 (d\phi + \omega dt)^2 + dz^2 \ . \tag{1.36}$$

No matter what the (nonzero) relative speed of the particle, it orbits around a circle of coordinate radius r as seen by the rotating observers. Although the circumferential radius of the circle as seen by the rotating observers is $\gamma_\omega r$, the radius of curvature comes from the rotation of the direction of motion as determined by parallel transport in the spatial geometry, and here one has the spatial geometry effect easily explained with a tangent cone in the embedding diagram of the r-ϕ plane [18].

For this circular motion of both the observers and test particle, all the spatial forces and accelerations are along the radial direction and the relative centripetal force balance equations for a unit mass particle are simply

$$\mathcal{F}^{(C)}_{(\mathrm{tem},U,u)r} = [F_{(U,u)} + F^{(G)}_{(\mathrm{tem},U,u)}]_r \ , \tag{1.37}$$

where the spatial force

$$F_{(U,u)} = -\frac{\gamma_\Omega \Omega^2 r}{\gamma_\omega (1-\Omega\omega r^2)}\partial_r \tag{1.38}$$

arises from the projection and rescaling of the particle 4-acceleration

$$a(U) = \gamma_\Omega^2 \Omega^2 r \partial_r \ . \tag{1.39}$$

The gravitoelectric and gravitomagnetic fields are

$$g(u) = \gamma_\omega^2 \omega^2 r \partial_r \ , \quad H(u) = 2\gamma_\omega^2 \omega \partial_z \ , \tag{1.40}$$

the spatial gravitational forces are

$$\begin{aligned} F^{(G)}_{(\mathrm{fw},U,u)} &= \gamma_\omega \gamma_\Omega \omega \Omega r \partial_r \ , \\ F^{(G)}_{(\mathrm{lie},U,u)} &= \gamma_\omega \gamma_\Omega \omega r[(\Omega-\omega)\gamma_\omega^2 + \Omega]\partial_r \ , \end{aligned} \tag{1.41}$$

and the centripetal accelerations are

$$\begin{aligned} F^{(C)}_{(\mathrm{fw},U,u)} &= -\gamma_\omega \gamma_\Omega \Omega r \frac{(\Omega-\omega)}{1-\Omega\omega r^2}\partial_r \ , \\ F^{(C)}_{(\mathrm{lie},U,u)} &= -\gamma_\omega^3 \gamma_\Omega r \frac{(\Omega-\omega)^2}{1-\Omega\omega r^2}\partial_r \ . \end{aligned} \tag{1.42}$$

There are three interesting cases to discuss:

1 $\omega = 0$: nonrotating observers,

2 $\Omega = 0$: nonrotating particle, and

3 $\Omega = \omega$: observers corotating with the particle.

$\boldsymbol{\omega = 0}$ When the observers are nonrotating, the relative transport angular velocity $\Omega_{(\mathrm{lie},U,u)} = \Omega_{(\mathrm{fw},U,u)} = -\Omega$ reduces to the sign-reversal of the particle angular velocity in order to exactly compensate for the rotation of the cylindrical frame vectors relative to the Cartesian ones.

The spatial gravitational forces vanish $F^{(G)}_{(\mathrm{fw},U,u)} = 0 = F^{(G)}_{(\mathrm{lie},U,u)}$ and the Fermi-Walker and Lie centripetal forces coincide $F^{(C)}_{(\mathrm{fw},U,u)} = F^{(C)}_{(\mathrm{lie},U,u)} = -\gamma_\Omega \Omega^2 r \partial_r$ and balance the spatial force responsible for maintaining the circular orbit.

$\boldsymbol{\Omega} = \mathbf{0}$ When the particle is nonrotating, i.e., fixed in space relative to the nonrotating observers, it is a spacetime geodesic with zero acceleration. The relative Fermi-Walker angular velocity is zero $\Omega_{(\mathrm{fw},U,u)} = 0$ since the cylindrical axes are always located at the same point, so these axes boosted into the local rest space of the rotating observers do not rotate with respect to the Fermi-Walker transported space-fixed axes, while the Lie relative angular velocity $\Omega_{(\mathrm{lie},U,u)} = \gamma_\omega^2 \omega$ acquires an extra gamma factor relative to the similar result for the complementary situation of Fermi-Walker transported frame vectors along a rotating orbit (see the Appendix), apparently due to effects of the spatial geometry as seen by the rotating observers.

The Fermi-Walker forces vanish $F^{(G)}_{(\mathrm{fw},U,u)} = 0 = F^{(C)}_{(\mathrm{fw},U,u)}$ and the Lie forces coincide $F^{(G)}_{(\mathrm{lie},U,u)} = F^{(C)}_{(\mathrm{lie},U,u)} = -\gamma_\omega^3 \omega^2 r \partial_r$, their equality representing the force balance.

$\boldsymbol{\Omega} = \boldsymbol{\omega}$ When the rotating observers are corotating with the orbiting particle, then $\Omega_{(\mathrm{lie},U,u)} = 0$ (the frame is already Lie dragged and so is also relatively Lie dragged along u), while $\Omega_{(\mathrm{fw},U,u)} = -\gamma_\Omega \Omega$. In this latter case one has an extra factor of gamma compared to the angular velocity needed to compensate for the rotation of the cylindrical frame similar to the case of the spacetime Fermi-Walker frame, responsible for the Thomas precession effect [18].

The relative centripetal forces vanish $F^{(C)}_{(\mathrm{fw},U,u)} = 0 = F^{(C)}_{(\mathrm{lie},U,u)}$ and the Fermi-Walker and Lie spatial gravitational forces coincide $F^{(G)}_{(\mathrm{fw},U,u)} = F^{(G)}_{(\mathrm{lie},U,u)} = \gamma_\Omega \Omega^2 r \partial_r$ and balance the spatial force responsible for maintaining the circular orbit.

Finally one could consider the relative centrifugal forces in the particle local rest space, but in this case of circular motion in which all the interesting acceleration fields are in the radial direction, the various force terms only get rescaled by gamma factors.

4.1 Conclusions

If this exercise proves one thing, it is that it makes little sense to speak of "the" centrifugal force or to speak of inertial forces in the context of special or general relativity as though a given situation has only one realization of these nonrelativistic concepts. Instead one has various possiblities depending on how one measures the changes in the relative velocity and which observer family is chosen. In other words, the centrifugal force of our nonrelativistic picture can contribute to either the spatial gravitational force or the relative centripetal force or both. To

keep things as simple as possible, the nonrotating observer congruence eliminates the inertial forces in this flat spacetime example and puts all the action into the relative centripetal force. Indeed in a general stationary axisymmetric spacetime like a black hole spacetime, locally nonrotating observers like the zero angular momentum observers (ZAMOs) probably make the most sense to use in interpreting how the nontrivial gravitational field affects test particle motion, eliminating the Coriolis force and giving some sense to the relative centripetal force as the best representation of "centrifugal force." Indeed the black hole spacetimes have been the prime motivation for considering this whole approach. Apart from the conformal transformation of the spatial metric in the static case, this encompasses the Abramowicz formalism, whose standard presentation is hampered by the confusion about the geometry of differentiation along a world line in spacetime of quantities which are only defined along that world line [10].

Appendix: Adapted spacetime frames

There are two natural spacetime frames which are adapted to the family of rotating observers: the spacetime Frenet-Serret frame defined by the differential properties of each individual world line and the Fermi-Walker transported frame along each world line.

Spacetime Frenet-Serret frame along U

This frame is just the boost of the orthonormalized nonrotating cylindrical coordinate frame which maps ∂_t onto U, explicitly

$$\begin{aligned} E_{(\mathrm{FS},U)0} &= U = \gamma_\Omega[\partial_t + \Omega\partial_\phi] \ , \\ E_{(\mathrm{FS},U)1} &= \partial_r \ , \\ E_{(\mathrm{FS},U)2} &= \gamma_\Omega[\Omega r\partial_t + \frac{1}{r}\partial_\phi] \ , \\ E_{(\mathrm{FS},U)3} &= \partial_z \ , \end{aligned} \tag{A.1}$$

and satisfies the relations

$$\begin{aligned} \frac{D}{d\tau_U}E_{(\mathrm{FS},U)0} &= \kappa E_{(\mathrm{FS},U)1}, \\ \frac{D}{d\tau_U}E_{(\mathrm{FS},U)1} &= \kappa E_{(\mathrm{FS},U)0} + \tau_1 E_{(\mathrm{FS},U)2} \ , \\ \frac{D}{d\tau_U}E_{(\mathrm{FS},U)2} &= -\tau_1 E_{(\mathrm{FS},U)2} + \tau_2 E_{(\mathrm{FS},U)3} \ , \\ \frac{D}{d\tau_U}E_{(\mathrm{FS},U)3} &= -\tau_2 E_{(\mathrm{FS},U)2} \ . \end{aligned} \tag{A.2}$$

The spacetime Frenet-Serret curvature κ (magnitude of the 4-acceleration $DU/d\tau_U$, always nonzero unless $\Omega = 0$) and torsions τ_1 and τ_2 are

$$\kappa = -\gamma_\Omega^2\Omega^2 r \ , \quad \tau_1 = \gamma_\Omega^2\Omega \ , \quad \tau_2 = 0 \ , \tag{A.3}$$

where $\nu = |\Omega r| = |\kappa|/|\tau_1| < 1$ for all the timelike orbits. According to the classification of Synge [17], who systematically studied timelike helices in flat spacetime, the orbit is a helix of type II which is degenerate (it lies in the plane $z = 0$) and of subtype IIc (torsion dominated: $|\kappa| < |\tau_1|$).

Fermi-Walker frame along U

The Fermi-Walker frame along U is obtained from the spacetime Frenet-Serret frame by an additional rotation by the angle $-\gamma_\Omega\Omega t$ of the vectors $E_{(\mathrm{FS},U)1}$ and $E_{(\mathrm{FS},U)2}$, and corresponds to axes in the local rest space of U aligned with three mutually orthogonal gyroscopes [19]

$$\begin{aligned} E_{(\mathrm{FW},U)0} &= U \ , \\ E_{(\mathrm{FW},U)1} &= \cos(\gamma_\Omega\Omega t)\, E_{(\mathrm{FS},U)1} - \sin(\gamma_\Omega\Omega t)\, E_{(\mathrm{FS},U)2} \ , \\ E_{(\mathrm{FW},U)2} &= \sin(\gamma_\Omega\Omega t)\, E_{(\mathrm{FS},U)1} + \cos(\gamma_\Omega\Omega t)\, E_{(\mathrm{FS},U)2} \ , \\ E_{(\mathrm{FW},U)3} &= E_{(\mathrm{FS},U)3} \ . \end{aligned} \tag{A.4}$$

Integrating the Frenet-Serret angular velocity $\omega_{FS} = |\tau_1| = \gamma_\Omega^2|\Omega|$ over an interval of proper time τ_U (the natural Frenet-Serret parametrization) and using the relation

$d\tau_U = 1/\gamma_\Omega dt$ converts the proper time to coordinate time yielding the coordinate angular velocity $\gamma_\Omega \Omega$ in these expressions.

The boost of the inertial frame $(\partial_t, \partial_x, \partial_y, \partial_z)$ which maps ∂_t onto U is instead represented by a rotation with the gamma factor missing, leading to a relative rotation by the angle $(\gamma_\Omega - 1)\phi$, where $\phi = \Omega t$. This is the angular velocity of the Thomas precession [18].

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