UNIFIED REGULARIZATION OF BIANCHI COSMOLOGY

Kjell ROSQUIST

Institute of Theoretical Physics, University of Stockholm, Vanadisvägen 9, S-11346 Stockholm, Sweden

and

Robert T. JANTZEN

Department of Mathematical Sciences, Villanova University, Villanova, PA 19085, USA and International Center for Relativistic Astrophysics, Dipartimento di Fisica, Università di Roma, 00185 Rome, Italy

NORTH-HOLLAND – AMSTERDAM

PHYSICS REPORTS (Review Section of Physics Letters) 166, No. 2 (1988) 89–124. North-Holland, Amsterdam

UNIFIED REGULARIZATION OF BIANCHI COSMOLOGY

Kjell ROSQUIST*

Institute of Theoretical Physics, University of Stockholm, Vanadisvägen 9, S-11346 Stockholm, Sweden

and

Robert T. JANTZEN

Department of Mathematical Sciences, Villanova University, Villanova, PA 19085, USA
and International Center for Relativistic Astrophysics, Dipartimento di Fisica, Università di Roma, 00185 Rome, Italy

Received December 1987

Contents:

Abstract:
The Einstein equations for a perfect fluid filled spatially homogeneous space-time are expressed in a reduced form which is as close as possible to a compactified regularized first order system of differential equations, while still respecting both the scale invariance and spatial gauge symmetry of those equations. The present work is a generalization to all Bianchi types of the regularization procedure introduced by Rosquist for Bianchi types III and VI. Jantzen's unified Lie algebra automorphism formalism is used to reduce the gravitational phase space to a subspace parametrized by a minimal number of variables. Although motivated by the qualitative theory of differential equations, the reduced spatially homogeneous Einstein system has also proved useful in the search for new exact solutions.

* Supported by the Swedish Natural Science Research Council.

1. Introduction

Spatially homogeneous (SH) universes have played an important role in cosmology for quite some time. Important early contributions were the discoveries of the Kasner metric [Kasner 1921] and the Taub universe [Taub 1951]. Orthogonal models, characterized by having the fluid flow perpendicular to hypersurfaces of homogeneity, were discussed extensively by Ellis and MacCallum [1969] while King and Ellis [1973] covered the nonorthogonal (tilted) case. SH cosmologies have provided a natural framework for studies of the influence of anisotropy on the structure of the cosmic background radiation (e.g. Hawking [1969] and Collins and Hawking [1973], Barrow et al. [1985]), of nucleosynthesis [Matzner et al. 1986] and of singularities. Ellis and King [1974] and later Siklos [1978] discussed SH cosmological singularities accompanied by a Cauchy horizon (whimpers). In recent years SH cosmologies have continued to be useful in the study of inflationary cosmology.

Much of the work on SH cosmology has been directed towards finding the generic behaviour of SH models. The papers by the Russian school [Belinsky et al. 1970, 1983] indicate that SH models generically exhibit a chaotic oscillating regime near the initial singularity. These results were subsequently confirmed by Bogoyavlensky and Novikov [Bogoyavlensky 1976a, b, Bogoyavlensky and Novikov 1973, 1975] and Peresetsky [1977, 1985] who used the qualitative theory of ordinary differential systems (see e.g. [Nemytski and Stepanov 1960]). Collins [1971] initiated the application of these techniques by studying regularized field equations in some special cases where two-dimensional phase plane methods may be used. This followed up earlier work by Ozsváth [1970] who subsequently introduced the idea of reducing the Einstein equations using the symmetries of the dynamics [Ozsváth 1971]. Shikin [1973] also made phase plane analyses of SH models and later Shikin [1976] and Collins and Ellis [1979] applied these methods to a special tilted case.

Following a number of earlier works Jantzen [1979] classified the Lie group automorphisms which are relevant for SH cosmology and used the automorphisms to simplify the field equations. A further development appeared in Rosquist [1984] where Jantzen's methods were employed to facilitate the application of qualitative techniques. In that work the field equations for Bianchi types III and VI were reduced to a regularized system with the smallest possible number of equations by using the scale invariance and automorphism symmetries. At about the same time, Jantzen [1983, 1984] generalized his analysis of the SH models in order to treat the entire space of symmetry types and gauge choices on an equal footing before specializing to a particular symmetry type in a particular choice of gauge. The present work represents a merging of the two sets of ideas.

In their approach Bogoyavlensky [1976a] and Peresetsky [1977, 1985] did not use automorphisms to simplify the equations for the tilted case. As a result they had to work with a 13-dimensional system on an 11-dimensional submanifold. In our approach the automorphisms and the scaling symmetry of the Einstein equations can be used to reduce the system to at most eight equations. The power of our formalism is perhaps best illustrated by the discovery of the tilted exact power law solutions [Rosquist 1983, Jantzen and Rosquist 1986].

The unified approach is essential to get a global picture of SH dynamics and the interrelationships between different symmetry types, special initial data, and generic features of the evolution. However, an ambitious program of this kind unavoidably requires the introduction of a number of successive changes in notation as different geometrical features are incorporated into the development. We regard these passages as essential to the understanding of the structure of the dynamical system. Although our notation may be a bit overwhelming at times, it is important to remember that the ideas which give rise to that notation are rather simple. Too many studies of cosmological models simply adopt an ad hoc calculational approach, disregarding the rich geometry of the dynamics which allows one to see through its complexity and contributing very little to the physical understanding of the problem. The present work hopes to set up the dynamics of these models in a way which takes into account both the gauge symmetries and the scale invariance of the Einstein equations and leads to a form most suitable for the application of the qualitative theory of differential equations. Using the reduced equations derived in the present work, a systematic first order analysis of some orthogonal models has recently been carried out [Uggla and Rosquist 1988].

The primary (unreduced) gravitational phase space used in SH dynamics is the subspace, $\mathcal{P}$, of the 12-dimensional space of spatial metric components and their first derivatives, consisting of those points which correspond to a positive definite spatial metric and a nonnegative energy density. The former condition is a collection of strict inequalities while the latter is a nonstrict inequality. It follows that the physical phase space is neither an open nor a closed set. Accordingly the boundary $\partial\mathcal{P}$ can be divided into a physical and an unphysical part. By using the scale invariance and Lie group automorphisms one arrives at a maximally reduced phase space $\mathcal{P}_{\rm red} \subset \mathcal{P}$ of dimension $d = 11 - r$ where r is the dimension of the automorphism group. The application of the qualitative techniques requires that the field equations be put in the form $\dot{X} = A(X)$ where $X \in \mathcal{P}_{\rm red}$, A is a vector function $A: \mathcal{P}_{\rm red} \to \mathbb{R}^d$ and the dot stands for time derivative. A further requirement is that A be a well-behaved function on the closure $\bar{\mathcal{P}}_{\rm red}$ of the reduced phase space. Ideally, one would like to find a mapping of $\bar{\mathcal{P}}_{\rm red}$ such that the corresponding transformed function A is analytic. In practice one often needs to cover the phase space by more than one coordinate patch each with compact closure and where A is analytic.

The solutions defined by the vanishing of the right hand side, $A(X) = 0$, are of special importance in the qualitative approach. They are the singular (critical, stationary) points of the system and represent asymptotic states of all other solutions. All singular points in the closure of the reduced phase space, $\bar{\mathcal{P}}_{\rm red}$, are dynamically important. One can thus divide those points into physical and unphysical singular points, the latter lying on the unphysical part of the boundary, $\partial\bar{\mathcal{P}}_{\rm red}$. Once a singular point is found approximate solutions may be derived by solving the linearized system $\dot{X} = A^{(1)}X$ where $A^{(1)}$ is the matrix defined by $A = (\partial A/\partial X)_{X=0}$. The type of singular point is determined by the signs of the eigenvalues of $A^{(1)}$.

Although originally motivated by the qualitative theory of differential equations, the reduced system has since proved useful also in the search for new exact solutions. There is one class of exact solutions which arise naturally in the qualitative approach, namely the exact power law (EPL) solutions discussed by Wainwright [1984]. They correspond precisely to the physical singular points, $\{X \in \mathcal{P}_{\rm red} | A(X) = 0\}$. Some new EPL solutions which were discovered in this way can be found in Rosquist [1983] and Rosquist and Jantzen [1985].

The space–time of a SH cosmological model is a 4-dimensional Lorentzian manifold with a metric which can be written in an orthonormal frame $\{\sigma^\alpha\}$ $(\alpha = 0, 1, 2, 3)$ as ${}^4g = \eta_{\alpha\beta}\sigma^\alpha \otimes \sigma^\beta$, where $\eta_{\alpha\beta}$ are the components of the 4-dimensional Lorentz matrix $\eta = \mathrm{diag}(-1, 1, 1, 1)$. Like the geometry of the space–time itself, the geometry of the 6-dimensional gravitational configuration space introduced by DeWitt [1967] also has a Lorentzian signature. We shall use both the 6-dimensional Lorentzian metric of the entire DeWitt manifold and the 3-dimensional Lorentzian metric of its diagonal submanifold. In all these cases we will use the symbol η freely to denote the Lorentzian matrix of any dimension assuming that the dimensionality will be clear from the context. Much of what follows will be aimed toward transforming to orthogonal frames on both the space–time and configuration space in a way compatible with the space–time symmetries and the induced symmetries of the dynamics. These orthogonal frames when normalized lead to natural orthonormal frames on both spaces which are very useful for describing the dynamics. The resulting variables are very closely related to the orthonormal frame approach but with a symmetry compatible choice of frame. Such an approach is used by Hsu and Wainwright [1986] for the case of a fluid flowing orthogonally to the homogeneous hypersurfaces.

The cosmological constant will be assumed to be zero in our discussion. This allows one to exploit the scale invariance of the Einstein equations which is broken by a nonzero cosmological constant. Units will be chosen so that the gravitational constant satisfies $8\pi G = 1$; the Einstein equations then equate the Einstein tensor of the SH space–time to the energy–momentum tensor of the matter content of the model. Lower case latin indices will assume the values 1, 2, 3.

The matter is taken be a perfect fluid with the usual equation of state in cosmological applications, $p = (\gamma - 1)\rho$. The parameter γ is usually restricted to the interval $1 \leq \gamma \leq 2$ to ensure that the pressure is nonnegative and that the velocity of sound does not exceed the velocity of light. The energy momentum tensor is given by

$$T_{\alpha\beta} = (\rho + p)u_\alpha u_\beta + p g_{\alpha\beta} \,. \tag{1.1}$$

Section 2 describes a choice of gravitational variables adapted to the spatial gauge symmetry, equivalent to a transformation from the usual orthogonal gauge to a certain "diagonal gauge" in which both the metric and symmetry constants are in "diagonalized" form. Section 3 discusses the elimination of the fluid variables from the evolution equations by solving the constraint equations. Section 4 sets up the first order form of the evolution equations in terms of the new gravitational variables, using the Hamiltonian approach. Section 5 begins the regularization of those equations by introducing scale invariant momenta and evolution equations, and section 6 treats in detail the fluid contributions which appear in them. Section 7 completes the regularization scheme by introducing scale invariant configuration space variables, leading finally to a reduced regularized system of equations from which the scale degree of freedom has decoupled. Section 8 then discusses the nonsemisimple case in more detail due to the existence of additional symmetries. Finally, the field equations are specialized to the orthogonal case in section 9.

2. Unified symmetry adapted representation of Bianchi cosmologies

We define the space–time manifold as the product $M = \mathbb{R} \times G$ with G a 3-dimensional real Lie group. The manifold M is regarded as a parametrized family of copies of G, $M = \{G(t) | t \in \mathbb{R}\}$, with G acting on the left as a transformation group on M with orbits $G(t)$. These orbits, assumed to be spacelike, are often called the homogeneous hypersurfaces or homogeneous slices, and form a geodesically parallel family of time slices for the space–time. The metric on M is defined by

$${}^4g = -N^2 \, \mathrm{d}t \otimes \mathrm{d}t + g_{ab}(\omega^a + N^a \, \mathrm{d}t) \otimes (\omega^b + N^b \, \mathrm{d}t) \,. \tag{2.1}$$

Here $\{\omega^a\}$ is a left invariant 1-form frame on G dual to the spatial frame $\{e_a\}$, which is a basis of the Lie algebra of the group characterized by a particular choice of the structure constant tensor components C^a_{bc}. The lapse function N must be a function of t only since t is assumed to parametrize the group orbits, thus giving the allowed time gauges, which correspond to different parametrizations of the given family of time slices. The positive definite matrix of metric coefficients g_{ab} will depend on the time alone provided the spatial frame $\{e_a\}$ is invariant under the symmetry group, leading to certain restrictions on the shift vector field [Jantzen 83].

Defining $\omega^0 = \mathrm{d}t$ and $e_0 = \partial/\partial t$ completes the SH spatial frame to a comoving space–time frame, i.e., one in which the spatial frame is dragged along the integral curves of the vector field e_0, which is transversal to the homogeneous hypersurfaces but not necessarily spatially homogeneous or timelike. This space–time frame, adapted as it is to the time slicing, is called a "computational frame" in the terminology of York [1979]. The integral curves of e_0 are the time lines in this geometric approach and correspond to a given choice of spatial gauge. Different choices of the shift vector field compatible with the symmetry lead to different time lines and different SH spatial frames which are dragged along those time lines, but all such spatial frames are related to each other by matrix transformations which depend only on the time. For a given spatial gauge, the constant linear transformations relate the associated comoving spatial frames to each other.

To project orthogonally to the space sections one needs the SH normal vector field $e_\perp = N^{-1}(e_0 - N^a e_a)$ whose integral curves are geodesics. Its covariant form reversed in sign is $\omega^\perp = N\omega^0 = \mathrm{d}\tau$, which in turn defines the differential of the proper time τ. The proper-time time gauge corresponds to $N = 1$. The spatial gauge $N^a = 0$ leads to orthogonal time lines and hence is naturally referred to as orthogonal gauge

$$ {}^4g = -N^2(t)\,\mathrm{d}t \otimes \mathrm{d}t + {}^3g\,, \qquad {}^3g = g_{ab}(t)\,\omega^a \otimes \omega^b\,, \tag{2.2}$$

where ${}^3g$ is the spatial metric. With proper-time time gauge, this is a synchronous reference frame in the sense used by Landau and Lifshitz [1971], and τ is called the synchronous time. The proper time of the fluid coincides with this symmetry adapted synchronous time precisely when the fluid flow is orthogonal to the homogeneous slices (the orthogonal case).

Let $\mathcal{M}$ be the 6-dimensional space of symmetric positive-definite 3-dimensional matrices. For a given synchronous space–time frame, the spatial metric can be identified with a curve in $\mathcal{M}$. It is natural to regard $\mathcal{M}$ as the gravitational configuration space of spatially homogeneous dynamics (cf. [DeWitt 1967]). We denote the diagonal subspace of $\mathcal{M}$ by $\mathcal{M}_\mathrm{D}$. As a basis for $\mathcal{M}$ one may take the natural basis $\{\boldsymbol{e}^a{}_b\}$ of gl(3, ℝ), where $\boldsymbol{e}^a{}_b$ is defined as the matrix with a single nonzero entry equal to unity in the ath column and bth row. A matrix can then be represented as $\boldsymbol{A} = A^a{}_b \boldsymbol{e}^b{}_a$.

Let $\mathfrak{g}$ be the Lie algebra of left invariant vector fields on G. Given a basis $\{e_a\}$ of $\mathfrak{g}$ and its dual basis $\{\omega^a\}$, the group structure is given by the relations

$$ [e_a, e_b] = C^c{}_{ab} e_c \quad \text{or} \quad \mathrm{d}\omega^a = -\tfrac{1}{2} C^a{}_{bc}\,\omega^b \wedge \omega^c\,. \tag{2.3}$$

The structure constant tensor $C^a{}_{bc}$ is antisymmetric in its lower indices and satisfies the Jacobi identity. It is convenient to introduce the 6-dimensional space $\mathscr{C}$ of structure constants of 3-dimensional real Lie algebras,

$$ \mathscr{C} = \{C^a{}_{bc} \mid C^a{}_{(bc)} = 0,\ C^d{}_{[ab} C^e{}_{c]d} = 0\}\,. \tag{2.4}$$

If we choose another basis $\bar{e}_a = (A^{-1})^b{}_a e_b$ of $\mathfrak{g}$, the new structure constants become

$$ \bar{C}^a{}_{bc} = C^d{}_{fg} A^a{}_d (A^{-1})^f{}_b (A^{-1})^g{}_c\,. \tag{2.5}$$

This relation defines a left action by GL(3, ℝ) on $\mathscr{C}$. The isotropy group of this action at a given point in $\mathscr{C}$ is the matrix representation of the automorphism group of the Lie algebra $\mathfrak{g}$ with respect to the basis e_a, denoted by Aut($\mathfrak{g}$); its elements leave the components of the structure constant tensor fixed.

Following Behr the structure constant tensor can be naturally decomposed by taking its dual [Estabrook et al. 1968]

$$\begin{aligned} C^{ab} &\equiv \tfrac{1}{2} C^a{}_{cd}\varepsilon^{bcd} = C^{(ab)} + C^{[ab]} \equiv n^{ab} + \varepsilon^{abc}a_c , \\ C^a{}_{bc} &= \varepsilon_{bcd}n^{ad} + a_f\delta^{fa}_{bc} , \qquad a_f = \tfrac{1}{2}C^a{}_{fa} . \end{aligned} \tag{2.6}$$

The Jacobi identity then reduces to $n^{af}a_f = 0$. When the covector a_f is nonzero, a scalar h is defined by the relation

$$a_f a_g = \tfrac{1}{2} h \varepsilon_{fab}\varepsilon_{gcd} n^{ac} n^{bd} . \tag{2.7}$$

The objects $\boldsymbol{n} = (n^{ab})$ and a_f defined in eq. (2.6) transform under the action (2.5) as

$$\bar{\boldsymbol{n}} = (\det \boldsymbol{A})^{-1} \boldsymbol{A}\, \boldsymbol{n}\, \boldsymbol{A}^{\mathrm{T}} , \qquad \bar{a}_f = (A^{-1})^g{}_f a_g . \tag{2.8}$$

One may use constant orthogonal transformations to reduce the structure constants to standard diagonal form

$$\boldsymbol{n} = \mathrm{diag}(n^{(1)}, n^{(2)}, n^{(3)}) , \qquad a_f = a\delta^3_f , \qquad a^2 = hn^{(1)}n^{(2)} . \tag{2.9}$$

The Jacobi identity is then simply $an^{(3)} = 0$. The subspace of $\mathscr{C}$ corresponding to the standard diagonal form is denoted by $\mathscr{C}_{\mathrm{D}}$. This space consists of two 3-dimensional manifolds which intersect where $a = n^{(3)} = 0$. The subspace $a = 0$ is referred to as class A and the subspace $a \neq 0$ as class B. By appropriate scaling and permutation transformations, the structure constants may be further specialized to the canonical form given in table 1.

Notice that we represent Bianchi type II by a nonzero value of $n^{(1)}$ rather than $n^{(3)}$ which has been the usual practice motivated by local rotational symmetry. The reason for this unorthodox choice is that we want the subset of $\mathscr{C}_{\mathrm{D}}$ corresponding to the canonical values of all the nonsemisimple Bianchi types (i.e. I–VII) to be contained in the subspace of $\mathscr{C}_{\mathrm{D}}$ defined by the condition $n^{(3)} = 0$.

By keeping the parameters $\{n^{(1)}, n^{(2)}, n^{(3)}, a\}$ of the standard diagonal form arbitrary, one obtains a convenient unified treatment of all the Bianchi types [Jantzen 84]. A basis of $\mathfrak{g}$ for which the structure constants are in standard diagonal form will be called a standard diagonal basis. Such a basis is given in [Jantzen 79]. In the context of a SH space–time, such a basis will be called a standard diagonal form

Table 1
Canonical values of the structure constants for each Bianchi type. Note that type III $\equiv$ VI$_{-1}$ and that $a > 0$ in the class B case. A dash signifies that h is undefined

	Class A						Class B			
	I	II	VI$_0$	VII$_0$	VIII	IX	V	IV	VI$_{h\neq 0}$	VII$_{h\neq 0}$
$n^{(1)}$	0	1	1	1	1	1	0	1	1	1
$n^{(2)}$	0	0	−1	1	1	1	0	0	−1	1
$n^{(3)}$	0	0	0	0	−1	1	0	0	0	0
a	0	0	0	0	0	0	1	1	a	a
h	–	–	0	0	0	0	–	–	$-a^2$	a^2

spatial frame. When such a frame is orthogonal, the choice of spatial gauge will be called diagonal gauge.

Once a particular choice of standard diagonal form structure constants has been made, there is still the freedom to change the spatial gauge using the corresponding automorphism matrix group, which does not change those constants. A time independent automorphism of the spatial frame does not change the gauge, i.e., does not change the shift vector field associated with the comoving frame. A time dependent automorphism of the spatial frame does change the spatial gauge and is equivalent to a change in the shift vector field and time lines. Any SH spatial frame can be related to the orthogonal gauge (zero shift) ones by a time dependent automorphism.

It is particularly convenient to transform to a "diagonal gauge" where the spatial metric is diagonal and metric manipulations become much simpler, while curvature formulas simplify since the structure constant tensor which enters them is also in a "diagonalized form". One can always pick a 3-dimensional unimodular subgroup $\hat{G}$ of Aut($\mathscr{g}$) such that the induced action of $\hat{G}$ on $\mathscr{M}$ maps $\mathscr{M}_{\mathrm{D}}$ onto $\mathscr{M}$ [Jantzen 79], i.e., any metric matrix can be diagonalized using this matrix group. Furthermore the induced action of $\hat{G}$ is almost everywhere transversal to $\mathscr{M}_{\mathrm{D}}$. We shall specify $\hat{G}$ explicitly for all Bianchi types in section 4. The choice of $\hat{G}$ to diagonalize the orthogonal gauge metric matrix leads to a unique transformation from orthogonal gauge to a "unimodular" diagonal gauge. When this choice itself is not unique, such a gauge is also not unique.

It is useful to introduce the DeWitt [Jantzen 79] and Euclidean matrix inner products according to

$$\langle \boldsymbol{A}, \boldsymbol{B} \rangle_{\mathrm{DW}} = \mathrm{Tr}(\boldsymbol{A}\boldsymbol{B}) - \mathrm{Tr}\,\boldsymbol{A}\,\mathrm{Tr}\,\boldsymbol{B}\,, \qquad \langle \boldsymbol{A}, \boldsymbol{B} \rangle_{\mathrm{E}} = \mathrm{Tr}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{B})\,. \tag{2.10}$$

Let $\hat{\mathscr{g}}$ be the Lie algebra of $\hat{G}$ and $\{\boldsymbol{\kappa}_a\}$ a 3×3 matrix basis of $\hat{\mathscr{g}}$ with commutation relations

$$[\boldsymbol{\kappa}_a, \boldsymbol{\kappa}_b] = \hat{C}^c{}_{ab}\boldsymbol{\kappa}_c\,. \tag{2.11}$$

Using this basis we define a Euclidean metric E_{ab} on $\hat{\mathscr{g}}$ by

$$E_{ab} = \langle \boldsymbol{\kappa}_a, \boldsymbol{\kappa}_b \rangle_{\mathrm{E}}\,. \tag{2.12}$$

We may now introduce a matrix representation of $\hat{G}$ defined by

$$\boldsymbol{S} = (S^a{}_b) = \mathrm{e}^{\theta^1\boldsymbol{\kappa}_1}\,\mathrm{e}^{\theta^2\boldsymbol{\kappa}_2}\,\mathrm{e}^{\theta^3\boldsymbol{\kappa}_3}\,, \tag{2.13}$$

where $\{\theta^a\}$ are canonical coordinates of the second kind on $\hat{G}$. Under the action of $\hat{G}$, the spatial basis and the metric $\boldsymbol{g}$ transform as

$$e'_a = (S^{-1})^b{}_a e_b\,, \qquad \omega'^a = S^a{}_b\omega^b\,, \qquad \boldsymbol{g} = \boldsymbol{S}^{\mathrm{T}}\boldsymbol{g}'\boldsymbol{S}\,, \tag{2.14}$$

where the primed metric $\boldsymbol{g}' = (g'_{ab})$ is assumed to be diagonal. This assumption fixes the transformation from orthogonal gauge to a particular diagonal gauge in which e'_a is the new spatial frame. In this way offdiagonal metric components are represented by the group coordinates θ^a through the matrix $\boldsymbol{S}$, which is equivalent to a gauge transformation. Initial values of these coordinates are unimportant since one can easily change them by a time independent automorphism of the spatial frame.

The primed metric is conveniently written in terms of the matrix $\boldsymbol{\beta}$ according to

$$\begin{aligned}
&\boldsymbol{g}' = \boldsymbol{D}^2 = e^{2\boldsymbol{\beta}}\,, \qquad \boldsymbol{D} = \mathrm{e}^{\boldsymbol{\beta}} = \mathrm{diag}(D_1, D_2, D_3)\,,\\
&\boldsymbol{\beta} = \mathrm{diag}(\beta^1, \beta^2, \beta^3) = \beta^A \boldsymbol{e}_A\,, \quad (A = 0, +, -)\,,\\
&(\boldsymbol{e}_A) = (\mathbf{1}, \mathrm{diag}(1, 1, -2), \mathrm{diag}(\sqrt{3}, -\sqrt{3}, 0))\,, \qquad \langle \boldsymbol{e}_A, \boldsymbol{e}_B \rangle_{\mathrm{DW}} = 6\eta_{AB}\,,\\
&\beta^1 = \beta^0 + \beta^+ + \sqrt{3}\beta^-\,, \qquad \beta^0 = (1/3)(\beta^1 + \beta^2 + \beta^3)\,,\\
&\beta^2 = \beta^0 + \beta^+ - \sqrt{3}\beta^-\,, \qquad \beta^+ = (1/6)(\beta^1 + \beta^2 - 2\beta^3)\,,\\
&\beta^3 = \beta^0 - 2\beta^+\,, \qquad \beta^- = (\sqrt{3}/6)(\beta^1 - \beta^2)\,.
\end{aligned} \tag{2.15}$$

The parametrization in terms of β^A is called the Misner parametrization and the basis $\{\boldsymbol{e}_A\}$ of the space of diagonal matrices (orthogonal with respect to the DeWitt inner product) is the Misner basis [Misner 1969]. The determinant of the spatial metric is given by $g = \det \boldsymbol{g} = \det \boldsymbol{g}' = \mathrm{e}^{6\beta^0}$. Using a dot to denote time derivatives, the extrinsic curvature in orthogonal gauge is defined by

$$\boldsymbol{K} = (K^a{}_b) = -(2N)^{-1}\boldsymbol{g}^{-1}\,\dot{\boldsymbol{g}} \tag{2.16}$$

and is given in the primed frame by [Jantzen 1979, 1984]

$$\boldsymbol{K}' = \boldsymbol{SKS}^{-1} = -N^{-1}(\dot{\boldsymbol{\beta}} + \boldsymbol{\kappa}_a^{\#\prime}\tilde{\nu}^a)\,, \qquad \boldsymbol{\kappa}_a^{\#\prime} \equiv \tfrac{1}{2}(\boldsymbol{\kappa}_a + \boldsymbol{g}'^{-1}\boldsymbol{\kappa}_a^{\mathrm{T}}\boldsymbol{g}') \tag{2.17}$$

where $\tilde{\nu}^a$ is the automorphism velocity*) introduced in [Jantzen 1979] and defined by the formula

$$\dot{\boldsymbol{S}}\boldsymbol{S}^{-1} = \boldsymbol{\kappa}_a\tilde{\nu}^a\,. \tag{2.18}$$

Note that in the primed frame, the coordinates θ^a and their derivatives occur only in the combination (2.18), which is in fact the matrix of the Lie derivative operator associated with the shift vector field which transforms from the orthogonal gauge to the chosen diagonal gauge. These gauge variables also are absent in spatial curvature formulas since they do not change the structure constants. Thus only the derivatives of the automorphism coordinates (in the combination (2.18)) carry dynamical information, not the coordinates themselves. It is therefore natural to use the automorphism velocities as dynamical variables to represent the offdiagonal degrees of freedom of the spatial metric, which are pure gauge in this context.

Since the primed frame is orthogonal, its normalization leads to a natural standard diagonal form orthonormal spatial frame

$$e''_a = (\mathrm{e}^{-\boldsymbol{\beta}})^b{}_a e'_b\,, \qquad \omega''^a = (\mathrm{e}^{\boldsymbol{\beta}})^a{}_b\,\omega'^b\,, \tag{2.19}$$

which may be completed to a natural space–time orthonormal frame $\{e_\perp, e''_a\}$ with dual frame $\{\omega^\perp, \omega''^a\}$. This orthonormal frame is the one best suited to discuss the dynamics and in fact the variables we eventually use will be constant linear combinations of components taken in this frame.

*) The automorphism velocity has had many names: $\dot{\tilde{\omega}}^a$ [Jantzen 1979, 1982], $\dot{\tilde{\sigma}}^a$ [Jantzen 1983], $\dot{\tilde{W}}^a$ [Jantzen 1984, Jantzen and Rosquist 1986] and $\dot{q}^a$ [Rosquist 1984]. The present notation $\tilde{\nu}^a$ (but without the tilde) was also used in Rosquist and Jantzen [1986].

By definition the matrices $\boldsymbol{\kappa}^{\#\prime}_a$ are symmetrizations of $\boldsymbol{\kappa}_a$ with respect to the primed frame. Defining the inverse E^{ab} of the Euclidean metric E_{ab} we may solve for the automorphism velocity to obtain

$$\tilde{\nu}^a = E^{ab}\,\mathrm{Tr}(\dot{\boldsymbol{S}}\,\boldsymbol{S}^{-1}\,\boldsymbol{\kappa}_b^{\mathrm{T}})\,. \tag{2.20}$$

Note that a component $\tilde{\nu}^a$ of the automorphism velocity cannot be interpreted as a derivative unless the corresponding 1-form $\tilde{W}^a$ defined by the formula

$$\tilde{W}^a = E^{ab}\,\mathrm{Tr}(\mathrm{d}\boldsymbol{S}\,\boldsymbol{S}^{-1}\,\boldsymbol{\kappa}_b^{\mathrm{T}}) \tag{2.21}$$

is exact. The 1-forms $\tilde{W}^a$ on the matrix group $\hat{G}$ are a basis of right invariant 1-forms satisfying

$$\mathrm{d}\tilde{W}^a = \tfrac{1}{2}\hat{C}^a{}_{bc}\tilde{W}^b \wedge \tilde{W}^c\,. \tag{2.22}$$

In addition to the offdiagonal automorphisms admitted by all the Bianchi types, the nonsemisimple types also admit diagonal automorphisms. For all the nonsemisimple types the matrix $\boldsymbol{I}^{(3)} = \mathrm{diag}(1,1,0)$ is an automorphism generator, i.e. $\boldsymbol{I}^{(3)} \in \mathscr{L}(\mathrm{Aut}(\mathfrak{g}))$, the Lie algebra of $\mathrm{Aut}(\mathfrak{g})$. For types III, IV, VI and VII this is the only additional automorphism. The existence of a diagonal automorphism implies that not all of the diagonal metric components are relevant to the dynamics although all of their derivatives are. To see how this works let $\boldsymbol{Q} = \mathrm{e}^{\vartheta \boldsymbol{I}^{(3)}} \in \mathrm{Aut}(\mathfrak{g})$ be the transformation matrix corresponding to the generator $\boldsymbol{I}^{(3)}$. Introducing the barred frame $\bar{e}_a = Q^{-1b}{}_a e_b$ we have

$$\boldsymbol{g}' = \boldsymbol{Q}^{\mathrm{T}}\,\bar{\boldsymbol{g}}\,\boldsymbol{Q}\,, \qquad \bar{\boldsymbol{g}} = \mathrm{e}^{2\bar{\boldsymbol{\beta}}}\,. \tag{2.23}$$

The diagonal metric is represented in the new frame by the matrix

$$\bar{\boldsymbol{\beta}} = \mathrm{diag}(\bar{\beta}^1, \bar{\beta}^2, \bar{\beta}^3) = \boldsymbol{\beta} - \vartheta \boldsymbol{I}^{(3)} = (\beta^0 - 2\vartheta/3)\mathbf{1} + (\beta^+ - \vartheta/3)\boldsymbol{e}_+ + \beta^-\boldsymbol{e}_-\,. \tag{2.24}$$

Of the four variables $(\bar{\beta}^a, \vartheta)$ only three are independent. Therefore we have a certain freedom to choose the form of the matrix $\bar{\boldsymbol{\beta}}$. Two obvious choices suggest themselves. Choosing $\vartheta = 3\beta^+$ makes $\bar{\beta}^+ = 0$, $\bar{\beta}^0 = \beta^0 - 2\beta^+$, while choosing $\vartheta = 3\beta^0/2$ makes $\bar{\beta}^0 = 0$, $\bar{\beta}^+ = -1/2(\beta^0 - 2\beta^+)$. The combination $\beta^0 - 2\beta^+$ is invariant under this gauge transformation, so any other linearly independent combination of the two variables may be assumed to be a gauge variable in this case. Alternatively one may impose a condition such as $\bar{\beta}^1 = 0$ leading to $\vartheta = \beta^1$. Depending on the dimension of the orbit of the diagonal automorphism subgroup further diagonal metric components may become pure gauge in the same sense. In that case the matrix $\boldsymbol{Q}$ would depend on further group parameters and the matrix $\bar{\boldsymbol{\beta}}$ would depend on a correspondingly fewer number of diagonal variables the extreme being represented by type I where one may set $\bar{\boldsymbol{\beta}} = \mathbf{1}$ and none of the metric variables themselves enter the dynamics.

3. Elimination of fluid variables

A perfect fluid source is very special in that the individual fluid variables, four in number, are completely determined by the values of the four components $T^{\alpha}_{\perp}$ of the energy–momentum tensor which enter into the gravitational constraints. In principle one may invert this relationship and use the constraints to obtain the fluid variables as functions of the gravitational variables. For the usual cosmological equation of state $p=(\gamma-1)\rho$, this is equivalent to the solution of a quadratic equation and hence easily done [Bogoyavlensky 1976a, Moncrief 1977]. The spatial energy–momentum tensor may then be re-expressed entirely in terms of the gravitational variables, leaving the evolution equations, namely the spatial components of the Einstein equations, as an entirely geometric system with no constraints other than the requirement that the equivalent fluid variables be physical.

The energy and momentum constraints are

$$\mathcal{H}_{\mathrm{G}} \equiv 2g^{1/2}G^{\perp}_{\ \perp} = 2g^{1/2}T^{\perp}_{\ \perp}\,, \qquad \mathcal{H}^{\mathrm{G}}_{a} \equiv 2g^{1/2}G^{\perp}_{\ a} = 2g^{1/2}T^{\perp}_{\ a} \tag{3.1}$$

where $\mathcal{H}_{\mathrm{G}}$ is the gravitational super-Hamiltonian and $\mathcal{H}^{\mathrm{G}}_{a}$ are the gravitational supermomenta. It is convenient to introduce the following auxiliary functions [Jantzen 1983]

$$\hbar = -(2g^{1/2})^{-1}\mathcal{H}_{\mathrm{G}}\,, \qquad \hbar_a = (2g^{1/2})^{-1}\mathcal{H}^{\mathrm{G}}_{a}\,, \qquad \Delta = \rho(u^{\perp})^2\,. \tag{3.2}$$

The constraint equations then give [Rosquist 1984]

$$\Delta = \tfrac{1}{2}(\hbar + \sqrt{\delta})\,, \qquad \delta = \hbar^2 - 4\gamma^{-2}(\gamma-1)\hbar^a \hbar_a\,. \tag{3.3}$$

The energy density and the spatial components of the energy–momentum tensor can then be expressed in terms of the gravitational variables as

$$\begin{aligned} &\rho = (\gamma-1)^{-1}(\gamma\Delta - \hbar) = \tfrac{1}{2}(\gamma-1)^{-1}[-(2-\gamma)\hbar + \gamma\sqrt{\delta}]\,, \quad (\gamma \neq 1)\,,\\ &\rho = (\hbar^2 - \hbar^a \hbar_a)/\hbar\,, \quad (\gamma = 1)\,,\\ &T^a_{\ b} = (\gamma-1)\hbar\delta^a_{\ b} + (\gamma\Delta)^{-1}[\hbar^a \hbar_b - (\gamma-1)\hbar^c \hbar_c \delta^a_{\ b}]\,. \end{aligned} \tag{3.4}$$

The condition $\rho \geq 0$ is equivalent to $\hbar^2 \geq \hbar^a \hbar_a$ which can also be written in the form $\hbar^a \hbar_a \leq \gamma\Delta\hbar$ which is useful to obtain upper bounds on the matter terms in the regularized system. Note also the relation $\hbar = \rho(1+\gamma\sinh^2\beta)$, where β is the hyperbolic tilt angle defined by $\cosh\beta = u^{\perp}$. It turns out that the Einstein equations fail to be analytic at points where the square root in (3.3) vanishes. In the dust case ($\gamma=1$) there is no problem since then $\Delta = \hbar$. When $\gamma \neq 1$, the inequality $\delta \geq \gamma^{-2}(2-\gamma)^2 \hbar^a \hbar_a$ shows that $\delta = 0$ is equivalent to $\hbar = \hbar^a \hbar_a = 0$ if $\gamma \neq 2$ and to $\hbar^2 = \hbar^a \hbar_a$ if $\gamma = 2$. Therefore the failure of analyticity occurs only where $\rho = 0$.

Although the fluid equations of motion are not needed in this formulation of the problem, they are useful to obtain constants of the motion associated with the fluid. For this purpose it is useful to follow Taub [1969, 1972] in introducing more appropriate fluid variables. In the notation of Misner, Thorne and Wheeler [1973], let $\mu = \gamma\rho^{(\gamma-1)/\gamma}$ and $n = \rho^{1/\gamma}$ be the chemical potential and baryon number density respectively. Taub's circulation 1-form $v_a = \mu u_a$ and the scalar density $l = ng^{1/2}u^{\perp}$ may then be expressed in terms of the gravitational constraints in the form [Jantzen 1983]

$$\begin{aligned} &l = g^{1/2}\Delta^{1/2}\rho^{(2-\gamma)/2\gamma}\,,\\ &v_a = g^{1/2} l^{-1} h_a\,, \end{aligned} \tag{3.5}$$

where Δ and ρ are given by (3.3) and (3.4). The Taub variables (l, v_a) are necessary in a variational approach in which the fluid variables are not eliminated using the constraints. The general case without symmetry is discussed by Bao et al. [1985] using the notation $(e, h, \eta, \mu_a) = (\rho, \mu, 2l, 2lv_a)$.

Rewriting the conservation equations (vanishing divergence of energy–momentum tensor) in terms of these new fluid variables leads to

$$(\ln l)^{\cdot} = 2a_c v^c/v^\perp , \qquad \dot{v}_a = v_c C^c{}_{ba} v^b/v^\perp , \tag{3.6}$$

where $v^\perp = \mu u^\perp = (\mu^2 + v^a v_a)^{1/2}$. There are at least two independent constants of the motion for each Bianchi type. Defining the nonnegative quantity V^2 by $\pm V^2 = n^{ab} v_a v_b$, one sees that both l and V^2 are constants of the motion for all class A types, while V^2/l is a constant of the motion for the class B types. The remaining constant of the motion in the class B case as well as the constant which replaces V^2 when it vanishes identically require more involved discussion [Jantzen 1983]. All of these quantities lead to constants of the motion of the geometric system when re-expressed in terms of the geometrical variables.

4. Field equations

To be able to apply the qualitative theory of ordinary differential equations it is necessary to write the Einstein equations in first order form, i.e. containing only first derivatives. The natural way to achieve this is to use the ADM Hamiltonian formalism [Arnowitt et al. 1962]. In that approach the Einstein system can be derived from a Hamiltonian supplemented by certain constraint equations. In the spatially homogeneous class B case there is a complication in that the Einstein force field acquires a nonconservative part [Jantzen 1979, 1983, MacCallum 1979] making the system non-Hamiltonian. However this circumstance does not cause any problems from the point of view of regularization.

Before writing down the gravitational Hamiltonian it is useful to introduce what might be called Bianchi hyperbolic functions

$$\begin{aligned}
&H_a^\pm(x) = \tfrac{1}{2}\, e^{-\alpha^a}(n^{(b)}\, e^x \pm n^{(c)}\, e^{-x}) ,\\
&T_a(x) = H_a^-(x)/H_a^+(x) , \qquad T_a^*(x) = H_a^+(x)/H_a^-(x) ,\\
&e^{\alpha^a} = 2^{-1/2}[(n^{(b)})^2 + (n^{(c)})^2]^{1/2} , \\
&h_a^\pm = H_a^\pm(\beta^{bc}) , \qquad t_a = T_a(\beta^{bc}) = h_a^-/h_a^+ , \qquad t_a^* = T_a^*(\beta^{bc}) = h_a^+/h_a^- ,\\
&\hat{h}_a^\pm = 2\, e^{\alpha^a} h_a^\pm , \qquad \beta^{ab} = \beta^a - \beta^b ,
\end{aligned} \tag{4.1}$$

where (a, b, c) should be interpreted as a cyclic permutation of $(1, 2, 3)$ when required by the context. These functions are well defined on $\mathscr{C}_{\mathrm{D}}$ except where $\operatorname{rank}(\boldsymbol{n}) < 2$, i.e. for Bianchi types I, II, IV and V, in which case they have direction dependent limits. Any direction may then be picked to represent the dynamics of those models. This multivaluedness does not cause any problems as long as one makes a consistent choice of direction. In this paper we will use a particular limiting sequence for the multivalued cases. Starting from the semisimple case with all components $n^{(a)}$ nonzero, we let $n^{(3)} \to 0$ while keeping $n^{(1)}$ and $n^{(2)}$ nonzero, leading to Bianchi types III, VI and VII where $\boldsymbol{n}$ has rank 2. Next we let $n^{(2)} \to 0$ with $n^{(1)}$ still nonzero taking us to types II and IV. The final step $n^{(1)} \to 0$ leads to types I and V. We shall refer to this limiting sequence as the canonical limit.

The spatially homogeneous gravitational Lagrangian in orthogonal gauge is given by [Jantzen 1979, 1984]

$$L_G = N(\mathcal{T} - U_G) = T - NU_G\,, \qquad U_G = -g^{1/2}R^*\,, \tag{4.2}$$

where R^* is the spatial curvature scalar. When performing the variation, N is to be treated as an independent variable. The gravitational kinetic energy function is given by

$$\begin{aligned} &T = N\mathcal{T} = N\mathcal{G}^{abcd}K_{ab}K_{cd} = N^{-1}g^{1/2}\langle \boldsymbol{K}, \boldsymbol{K}\rangle_{\mathrm{DW}} = (4N)^{-1}\mathcal{G}^{abcd}\dot{g}_{ab}\dot{g}_{cd}\,, \\ &\mathcal{G}^{abcd} = g^{1/2}(g^{a(c}g^{d)b} - g^{ab}g^{cd})\,, \qquad \mathcal{G} = \mathcal{G}^{abcd}\,\mathrm{d}g_{ab} \otimes \mathrm{d}g_{cd}\,, \end{aligned} \tag{4.3}$$

where $\mathcal{G}$ is the DeWitt metric on $\mathcal{M}$ [DeWitt 1967]. Since the coordinates of $\mathcal{M}$, namely g_{ab}, are subindexed, the covariant components of the DeWitt metric have the superindexed form $\mathcal{G}^{abcd}$. We now express the kinetic energy in the primed frame using (2.17) [Jantzen 1979, 1984]

$$\begin{aligned} &T = N\mathcal{T} = N^{-1}g^{1/2}\langle \boldsymbol{K}', \boldsymbol{K}'\rangle_{\mathrm{DW}} = N^{-1}\,\mathrm{e}^{3\beta^0}(6\eta_{AB}\dot{\beta}^A\dot{\beta}^B + \bar{\mathcal{G}}_{ab}\tilde{\nu}^a\tilde{\nu}^b)\,, \\ &\bar{\mathcal{G}}_{ab} = \langle \boldsymbol{\kappa}_a^{\#\prime}, \boldsymbol{\kappa}_b^{\#\prime}\rangle_{\mathrm{DW}}\,. \end{aligned} \tag{4.4}$$

To evaluate $\bar{\mathcal{G}}_{ab}$ we must choose a basis for the matrix Lie algebra $\hat{g}$ of $\hat{G}$. The following offdiagonal basis was introduced in [Jantzen 1984]

$$\boldsymbol{\kappa}_a = \mathrm{e}^{-\alpha^a}\boldsymbol{k}^0_a\,, \qquad \boldsymbol{k}^0_a \equiv -n^{(b)}\boldsymbol{e}^c{}_b + n^{(c)}\boldsymbol{e}^b{}_c\,, \tag{4.5}$$

where the indices (a, b, c) are cyclic permutations of $(1, 2, 3)$. The structure constants of $\hat{g}$ then become

$$\begin{aligned} &\hat{C}^a{}_{bc} = \varepsilon_{bcd}\hat{n}^{ad}\,, \qquad \hat{\boldsymbol{n}} = (\hat{n}^{ab}) = \mathrm{diag}(\hat{n}^{(1)}, \hat{n}^{(2)}, \hat{n}^{(3)})\,, \\ &\hat{n}^{(a)} = n^{(a)}\,\mathrm{e}^{\alpha^a - \alpha^b - \alpha^c} \quad ((a, b, c)\ \text{cyclic permutation})\,. \end{aligned} \tag{4.6}$$

The basis is orthogonal with respect to the Euclidean inner product

$$E_{ab} = \langle \boldsymbol{\kappa}_a, \boldsymbol{\kappa}_b\rangle_{\mathrm{E}} = 2\delta_{ab}\,. \tag{4.7}$$

It is well defined on $\mathcal{C}_{\mathrm{D}}$ except for Bianchi types I, II, IV and V where it has direction dependent limits. In those cases any value can be picked to represent the dynamics as long as it is consistent with the choice made in (4.1). In the basis (3.8) the almost everywhere positive definite matrix $(\bar{\mathcal{G}}_{ab})$ and its inverse are diagonal with components

$$\bar{\mathcal{G}}_{aa} = 2(h^-_a)^2\,, \qquad \bar{\mathcal{G}}^{aa} = \tfrac{1}{2}(h^-_a)^{-2}\,. \tag{4.8}$$

It follows that the $\bar{\mathcal{G}}_{aa}$ are functions of β^{bc} ((a, b, c) cyclic permutation). For canonical values of the structure constants these functions have the zero

$$\bar{\mathscr{G}}_{33}(0)=0 \tag{4.9}$$

in Bianchi types VII–IX and the additional zeroes $\bar{\mathscr{G}}_{11}(0)=\bar{\mathscr{G}}_{22}(0)=0$ in type IX. Additional zeroes occur for certain directions in the limits for the Bianchi types I, II and V. All of the zeroes correspond to singularities in the automorphism transformations where the offdiagonal automorphism generators fail to be transversal to the diagonal submanifold $\mathscr{M}_{\mathrm{D}}$. The gravitational Lagrangian is singular at these zeroes and we will see later that this leads to problems with the regularization procedure. This may be avoided in the Bianchi types I, II and V by a proper choice of the limit. One such limit is the canonical limit defined at the beginning of this section.

Diagonalization of the matrix $(\bar{\mathscr{G}}_{ab})$ diagonalizes the kinetic energy, which may be conveniently written as

$$\begin{aligned} &T=N^{-1}\,\mathrm{e}^{3\beta^0}\,\bar{\mathscr{G}}_{IJ}\tilde{\nu}^I\tilde{\nu}^J\,,\\ &\tilde{\nu}^I=(\dot{\beta}^A,\tilde{\nu}^a)\,,\quad I,J,\ldots=(0,+,-,1,2,3)\,,\\ &(\bar{\mathscr{G}}_{IJ})=\mathrm{diag}(-6,6,6,\bar{\mathscr{G}}_{11},\bar{\mathscr{G}}_{22},\bar{\mathscr{G}}_{33})\,, \end{aligned} \tag{4.10}$$

where we have introduced the collective index family $(I,J,\ldots)$ leading to the compact notation $\{\tilde{W}^I\}=\{\mathrm{d}\beta^A,\tilde{W}^a\}$. Comparison with (4.3) shows that the covariant components of the DeWitt metric in the orthogonal co-frame $\{\mathrm{d}\beta^A,\tilde{W}^a\}$ are given by $\mathscr{G}_{IJ}=4g^{1/2}\bar{\mathscr{G}}_{IJ}$. Note that the covariant form is now subindexed since we are using superindexed coordinates on $\mathscr{M}$. The conjugate momenta corresponding to the velocities $\dot{\beta}^A$ and $\tilde{\nu}^a$ are defined as usual by

$$\begin{aligned} &p_A=\partial T/\partial\dot{\beta}^A=12N^{-1}\,\mathrm{e}^{3\beta^0}\eta_{AB}\dot{\beta}^B\,,\\ &\tilde{P}_a=\partial T/\partial\tilde{\nu}^a=2N^{-1}\,\mathrm{e}^{3\beta^0}\bar{\mathscr{G}}_{ab}\tilde{\nu}^b\,. \end{aligned} \tag{4.11}$$

Since the 1-forms $\tilde{W}^a$ are not exact in general, the momenta $\tilde{P}_a$ are not canonical. Therefore there are extra nonzero Poisson brackets, namely

$$\{\tilde{P}_a,\tilde{P}_b\}=\hat{C}^c{}_{ab}\tilde{P}_c\,. \tag{4.12}$$

Rewriting the kinetic energy in terms of the momenta we obtain

$$\begin{aligned} &T=(24)^{-1}N\,\mathrm{e}^{-3\beta^0}(\eta^{AB}p_Ap_B+6\bar{\mathscr{G}}^{ab}\tilde{P}_a\tilde{P}_b)=N\mathscr{G}^{IJ}p_Ip_J=-(24)^{-1}N\,\mathrm{e}^{-3\beta^0}K\,,\\ &K\equiv-6\bar{\mathscr{G}}^{IJ}p_Ip_J\,,\qquad(p_I)=(p_A,\tilde{P}_a)\,,\\ &(\bar{\mathscr{G}}^{IJ})=\mathrm{diag}(-\tfrac{1}{6},\tfrac{1}{6},\tfrac{1}{6},\bar{\mathscr{G}}^{11},\bar{\mathscr{G}}^{22},\bar{\mathscr{G}}^{33})=(-4g^{1/2}\mathscr{G}^{IJ})\,, \end{aligned} \tag{4.13}$$

where*) K is defined for later use and $(\bar{\mathscr{G}}^{ab})$ is the inverse of the matrix $(\bar{\mathscr{G}}_{ab})$. The kinetic energy T is thus a quadratic form in the momenta p_A and $\tilde{P}_a$ with signature $(-+++++)$. For later use we define

*) The notation $p_a=\tilde{P}_a$ for the right invariant automorphism momenta should not be confused with the notation of [Jantzen 1984] where p_a has a different meaning.

a positive definite diagonal matrix Γ with components $\Gamma^{JJ} = |6\bar{\mathscr{G}}^{JJ}|^{1/2}$ so that $6\bar{\mathscr{G}}^{-1} = \eta\Gamma^2$ where $\bar{\mathscr{G}} = (\bar{\mathscr{G}}_{IJ})$.

The gravitational potential energy function U_G is given by [Jantzen 1979, 1984]

$$U_G = e^{\beta^0}(V^* + 6a^2 e^{4\beta^+}) ,$$

$$V^* = \tfrac{1}{2} e^{-4\beta^0}\Big[\sum_{a=1}^{3} (n^{(a)})^2(D_a)^4 - 2\sum_{a<b} n^{(a)}n^{(b)}(D_aD_b)^2\Big] \tag{4.14}$$

$$= \tfrac{1}{2} e^{4\beta^+}(\hat{h}_3^-)^2 - n^{(3)} e^{-2\beta^+} \hat{h}_3^+ + \tfrac{1}{2}(n^{(3)})^2 e^{-8\beta^+} .$$

The gravitational super-Hamiltonian $\mathscr{H}_G$ is defined by

$$H_G = T + NU_G \equiv N\mathscr{H}_G . \tag{4.15}$$

It is related to the energy–momentum tensor by the energy or super-Hamiltonian constraint $\mathscr{H}_G = 2g^{1/2}T^{\perp}_{\ \perp}$. The vacuum constraint $\mathscr{H}_G = 0$ does not follow from the variation of the gravitational action unless one considers N as an independent variable. This reflects the fact that general relativity is an "already parametrized" theory [Arnowitt et al. 1962] and is the reason for using the prefix "super" before the gravitational Hamiltonian [Misner et al. 1973].

One may interpret the Einstein system as describing motion in the DeWitt geometry of $\mathscr{M}$ driven by a gravitational force field $G \in \mathfrak{X}^*(\mathscr{M})$ and a nongravitational force field $M \in \mathfrak{X}^*(\mathscr{M})$ representing the matter, where $\mathfrak{X}^*(\mathscr{M})$ is the space of 1-form fields on $\mathscr{M}$

$$G = -g^{1/2}\,{}^3G^{ab}\,dg_{ab} , \qquad M = g^{1/2}T^{ab}\,dg_{ab} . \tag{4.16}$$

This system must be supplemented by the constraint equations. The 1-form G is the Einstein force field, where ${}^3G^{ab}$ are the components of the spatial Einstein tensor, and M is the matter force field. Assuming M to be re-expressed in terms of the gravitational variables as discussed in section 3 essentially eliminates the constraints. The components of a given force field F with respect to the frame $(d\beta^A, \tilde{W}^a)$ are given by the formula [Rosquist 1984]

$$F = F^{ab}\,dg_{ab} = 2\,\mathrm{Tr}(\boldsymbol{F}'\boldsymbol{e}_A)\,d\beta^A + 2\,\mathrm{Tr}(\boldsymbol{F}'\boldsymbol{\kappa}_a)\tilde{W}^a \equiv F_A\,d\beta^A + F_a\tilde{W}^a \tag{4.17}$$

where $\boldsymbol{F}'$ is the matrix of mixed components in the primed frame. For the class B Bianchi types it is well known that the Hamiltonian has to be supplemented by an additional gravitational nonpotential force to obtain the Einstein equations [Jantzen 1979, MacCallum 1979]. This happens since the 1-form field G is not exact. Instead one has the relation

$$G = -dU_G + Q$$

where Q is the gravitational nonconservative or nonpotential force. Using the basis (3.8), Q is given by [Jantzen 1979, 1984]

$$Q = 4a\,e^{\beta^0+\beta^+}(6a\,d\beta^+ + e^{\alpha^3}\bar{\mathscr{G}}_{33}\tilde{W}^3) \equiv Q_+\,d\beta^+ + Q_3\tilde{W}^3 . \tag{4.18}$$

It is obvious from this relation that when $a \neq 0$ (class B models), one has $Q \neq 0$ and it is easy to check that $\mathrm{d}Q \neq 0$. Thus Q is a nontrivial force field which cannot be derived from a potential in this case. The dynamical Einstein equations including the driving force terms corresponding to the matter and the class B nonpotential force are given by

$$\begin{aligned} \dot{\beta}^A &= \{\beta^A, H_{\mathrm{G}}\}\,, \\ \dot{\theta}^a &= \{\theta^a, H_{\mathrm{G}}\}\,, \\ \dot{p}_I &= \{p_I, H_{\mathrm{G}}\} + NQ_I + NM_I\,. \end{aligned} \tag{4.19}$$

Since the automorphism coordinates θ^a enter H_{G}, Q and M only through the automorphism velocities, the θ^a-equations decouple from the rest of the system, the remaining variables of which describe the dynamics in the diagonal gauge determined by the choice of $\hat{G}$. Integration of the θ^a-equations is necessary only to transform back to the original orthogonal spatial gauge. The automorphism velocities parametrize the shift vector field which is associated with this gauge transformation.

The decoupled θ^a-equations will be left aside in this paper. They were thoroughly discussed in Jantzen and Rosquist [1986]. We now focus attention on the equations for β^A, p_A and $\tilde{P}_a$. It is convenient to introduce the notation $R_I(p) \equiv \{p_I, H_{\mathrm{G}}\} + NQ_I + NM_I$ for the right hand side of the momentum equations. Evaluation of the Poisson brackets yields the Einstein equations [Rosquist 1984, Jantzen 1984]

$$\begin{aligned} &\dot{\beta}^A = \tfrac{1}{12} N\,\mathrm{e}^{-3\beta^0} \eta^{AB} p_B\,, \qquad \dot{p}_I = R_I(p)\,, \\ &R_0(p) = 3N\mathscr{H}_{\mathrm{G}} - 4NU_{\mathrm{G}} + NM_0\,, \\ &R_\pm(p) = -\tfrac{1}{4} N\,\mathrm{e}^{-3\beta^0}\, \bar{\mathscr{G}}^{ab}_{\pm} \tilde{P}_a \tilde{P}_b - NU_\pm + N\delta^{+}_{\pm} Q_+ + NM_\pm\,, \\ &R_a(p) = \tfrac{1}{2} N\,\mathrm{e}^{-3\beta^0}\, \bar{\mathscr{G}}^{bc} \hat{C}^d{}_{ab} \tilde{P}_c \tilde{P}_d + N\delta^3_a Q_3 + NM_a\,, \\ &\bar{\mathscr{G}}^{ab}_A \equiv \partial \bar{\mathscr{G}}_{ab}/\partial\beta^A\,, \qquad U_A \equiv \partial U_{\mathrm{G}}/\partial\beta^A\,, \\ &U_+ = \mathrm{e}^{\beta^0} V^*_+ + 24a^2\,\mathrm{e}^{4\beta^+}\,, \qquad U_- = \mathrm{e}^{\beta^0} V^*_-\,, \\ &V^*_A \equiv \partial V^*/\partial\beta^A\,, \\ &V^*_+ = 2\,\mathrm{e}^{4\beta^+} (\hat{h}_3^-)^2 + 2n^{(3)}\,\mathrm{e}^{-2\beta^+} \hat{h}_3^+ - 4(n^{(3)})^2\,\mathrm{e}^{-8\beta^+}\,, \\ &V^*_- = 2\sqrt{3}\,\mathrm{e}^{4\beta^+} \hat{h}_3^- (\hat{h}_3^+ - n^{(3)}\,\mathrm{e}^{-6\beta^+})\,. \end{aligned} \tag{4.20}$$

Comparison with (4.18) yields the relation

$$-U_+ + Q_+ = \mathrm{e}^{\beta^0} V^*_+ \tag{4.21}$$

which shows that the class B terms in U_+ and Q_+ cancel in the equation for p_+. In addition to the dynamical equations we must impose the Einstein energy and momentum constraints. Using the constraints to eliminate the fluid variables in favor of the gravitational variables essentially eliminates those constraints, except for inequalities required to make the fluid variables physical. For vacuum solutions the constraints give relations between the gravitational variables which confine the solution to a submanifold of the gravitational phase space.

The special case in which the fluid flow lines are orthogonal to the hypersurfaces of homogeneity is called the orthogonal case. This case is defined by $u^{\perp}=1$ or $v_a=0$ in terms of the fluid variables or by $\mathscr{H}^{\rm G}_a=0$ or $\hbar_a=0$ in terms of the gravitational variables, as follows from (3.5). Using the notation of section 3, one may rewrite the energy constraint in the form

$$H_{\rm G}=-2Ng^{-(\gamma-1)/2}(l/u^{\perp})^{\gamma}[1+\gamma(u^{\perp}-1)]\,. \tag{4.22}$$

In the orthogonal case l is a constant of the motion for all Bianchi types, as follows from (3.6). The lapse choice $N_{\rm BN}=g^{(\gamma-1)/2}$ of Bogoyavlensky and Novikov [Bogoyavlensky and Novikov 1973, 1975, Bogoyavlensky 1976a,b] then reduces the Hamiltonian to the constant of the motion $-2l^{\gamma}$. In contrast with the present approach where the lapse is considered to be an independent variable while the variational equations are derived, they insert this choice of lapse explicitly into the Hamiltonian and then derive new evolution equations which differ from the spatial Einstein equations by a term involving the energy constraint. These new evolution equations are identical with the vacuum equations, as are the momentum constraints, and the presence of the fluid is felt only by allowing the constant value of the Hamiltonian to be negative. This means that the functional form of the solutions in this time gauge are the same for the fluid and vacuum case except that the Hamiltonian constraint on the constants which appear in the general solution of those equations is changed by a constant [Jantzen 1980]. This shows that the orthogonal case is very closely related to the vacuum case. The general fluid case, on the other hand, has essentially different structure from the vacuum case due to the effect of the nonzero supermomenta.

5. The regularization procedure

Having written the Einstein equations in first order form, the next step on the road to regularization is to obtain bounded variables. To this end we will exploit the energy inequality

$$\mathscr{H}_{\rm G}=2g^{1/2}G^{\perp}{}_{\perp}\leq 0 \tag{5.1}$$

which by the energy constraint (3.1) holds for a perfect fluid source if the pressure is positive and with equality holding for vacuum. It is apparent from (4.14) that $U_{\rm G}\geq 0$ for Bianchi types I–VIII which by (4.13) and (5.1) implies the inequality

$$\mathscr{T}=(24)^{-1}\,{\rm e}^{-3\beta^0}(-p_0^2+p_+^2+p_-^2+6\bar{\mathscr{G}}^{11}\tilde{P}_1^2+6\bar{\mathscr{G}}^{22}\tilde{P}_2^2+6\bar{\mathscr{G}}^{33}\tilde{P}_3^2)\leq 0\,. \tag{5.2}$$

It follows that the momenta satisfy the inequalities

$$|p_{\pm}|\leq|p_0|\,,\qquad|(6\bar{\mathscr{G}}^{ab})^{1/2}\tilde{P}_b|\leq|p_0|\,. \tag{5.3}$$

The dynamics of the type I–VIII models is therefore dominated by p_0. In fact the same remarks apply to type IX as well except for a finite region of phase space near the turning point. The following discussion will apply in the type IX case only outside this region.

To obtain bounded diagonal momenta, an obvious choice of new variables is*)

$$r_A = p_A/p_0 \,. \tag{5.4}$$

The new momenta r_A are dimensionless and it is therefore useful to interpret this transformation as a scale transformation

$$p_A \to -\lambda^2 p_A \equiv r_A \tag{5.5}$$

with scale factor $\lambda = |p_0|^{-1/2}$. Scaling all dependent variables according to their dimension and using a dimensionless time variable it follows from the scale invariance of the Einstein equations for zero cosmological constant that the equation for the scale factor itself, i.e. the p_0-equation in the example above, decouples leaving the remaining diagonal momenta $r_\pm$ bounded. To keep the discussion general we do not yet specify the scale factor. However, we always assume that the dimension of the scale factor equals unity. Expressing the diagonal part of the kinetic energy in the scale invariant diagonal momenta yields

$$T_{\mathrm{D}} = (1/24) N g^{-1/2} \lambda^{-4} \eta^{AB} r_A r_B \,. \tag{5.6}$$

A natural choice of lapse function suggested by this expression from the point of view of regularization is

$$N_{\mathrm{R}} = -12 g^{1/2} \lambda^2 \,. \tag{5.7}$$

The factor -12 is chosen to conform with earlier conventions. If $\lambda = |p_0|^{-1/2}$ then N_{R} equals Misner's lapse function $N_{\mathrm{M}} = 12 g^{1/2} p_0^{-1} = 3(\mathrm{Tr}\, \boldsymbol{K})^{-1}$ associated with his scale invariant Ω-time, $\Omega = -\beta^0$. The minus sign means that Ω-time increases towards the initial singularity. Notice that the dimension of the lapse must equal unity in order that the new time variable be dimensionless. The diagonal kinetic energy then takes on the simple regularized form

$$T_{\mathrm{D}} = \tfrac{1}{2} \lambda^{-2} (r_0^2 - r_+^2 - r_-^2) \,. \tag{5.8}$$

The factor λ^{-2} just takes care of the scaling factor which appears in the left hand side of the momentum equations (see (4.20)). The Misner lapse choice leads to $r_0 = 1$ so that

$$T_{\mathrm{D}} = \tfrac{1}{2} \lambda^{-2} (1 - r_+^2 - r_-^2) \,. \tag{5.9}$$

For a diagonal metric, the inequality (5.2) then leads to $r_+^2 + r_-^2 \leq 1$.

We may also use (5.2) to obtain bounded offdiagonal momentum variables. To that end we first introduce new (noncanonical) automorphism momenta by

$$\begin{aligned}
&\pi_I = \Gamma^{IJ} p_J \,, \qquad \boldsymbol{\Gamma} = \mathrm{e}^{\boldsymbol{\gamma}} = \mathrm{diag}(1, 1, 1, \mathrm{e}^{\gamma^1}, \mathrm{e}^{\gamma^2}, \mathrm{e}^{\gamma^3}) \,, \\
&\boldsymbol{\gamma} = (\gamma^{IJ}) = \boldsymbol{\gamma}^{\mathrm{D}} \oplus \boldsymbol{\gamma}^{\mathrm{OD}} \,, \\
&\boldsymbol{\gamma}^{\mathrm{D}} = \mathrm{diag}(\gamma^0, \gamma^+, \gamma^-) \,, \qquad \boldsymbol{\gamma}^{\mathrm{OD}} = \mathrm{diag}(\gamma^1, \gamma^2, \gamma^3) \,, \\
&\gamma^A = 0 \,, \qquad \mathrm{e}^{\gamma^a} = \sqrt{3} (h_a^-)^{-1} \,,
\end{aligned} \tag{5.10}$$

*) Note that r_A was denoted by s_A in Rosquist [1984].

where $\boldsymbol{\gamma}$ has been decomposed into a direct sum of two 3-dimensional matrices representing diagonal and offdiagonal metric indices respectively. The momenta π_I are just certain linear combinations of the components of the momentum tensor density in the natural orthonormal frame associated with diagonal gauge. The corresponding Poisson brackets $\{\pi_I, \pi_J\} = D^K{}_{IJ}\pi_K$ have the noncanonical components

$$\{\pi_a, \pi_b\} = D^c{}_{ab}\pi_c\,, \qquad \{\pi_a, p_\pm\} = \gamma^{ab}_\pm \pi_b\,, \qquad \boldsymbol{\gamma}_A = (\gamma^{ab}_A) \equiv \partial\boldsymbol{\gamma}^{\mathrm{OD}}/\partial\beta^A\,,$$
$$D^c{}_{ab} = \Gamma^{-1cd}\hat{C}^d{}_{ef}\Gamma^{ae}\Gamma^{bf} = \mathrm{e}^{\gamma^a+\gamma^b-\gamma^c}\hat{C}^c{}_{ab} = \varepsilon_{abc}\hat{n}^{(c)}\,\mathrm{e}^{\gamma^a+\gamma^b-\gamma^c}\,. \tag{5.11}$$

The matrices $\boldsymbol{\gamma}_A$ are given by

$$\boldsymbol{\gamma}_0 = \boldsymbol{0}\,, \qquad \boldsymbol{\gamma}_\pm = \mp\sqrt{3}\boldsymbol{e}_\mp \boldsymbol{t}^*\,, \qquad \boldsymbol{t}^* = \mathrm{diag}(t_1^*, t_2^*, t_3^*)\,, \tag{5.12}$$

where we have used the relation

$$\partial\beta^{ab}/\partial\beta^\pm = \pm\sqrt{3}(\boldsymbol{e}_\mp)^c{}_c \quad ((a, b, c) \text{ cyclic permutation})\,. \tag{5.13}$$

Evaluation of these expressions yields

$$\boldsymbol{\gamma}_+ = \mathrm{diag}(-3t_1^*, 3t_2^*, 0)\,, \qquad \boldsymbol{\gamma}_- = \mathrm{diag}(\sqrt{3}t_1^*, \sqrt{3}t_2^*, -2\sqrt{3}t_3^*)\,. \tag{5.14}$$

When $n^{(3)} = 0$ we have the particularly simple expressions

$$\boldsymbol{t}^* = \mathrm{diag}(1, -1, t_3^*)\,,$$
$$\boldsymbol{\gamma}_+ = -3\boldsymbol{I}^{(3)}\,, \qquad \boldsymbol{\gamma}_- = \sqrt{3}\,\mathrm{diag}(1, -1, -2t_3^*)\,. \tag{5.15}$$

The kinetic energy now becomes

$$T = (1/24)N\,\mathrm{e}^{3\beta^0}\eta^{IJ}\pi_I\pi_J \tag{5.16}$$

and the field equations are

$$\dot{\beta}^A = (1/12)N\,\mathrm{e}^{-3\beta^0}\eta^{AB}p_B\,, \qquad \dot{\pi}_I = R_I(\pi)\,,$$
$$R_0(\pi) = 3NH_{\mathrm{G}} - 4NU_{\mathrm{G}} + NM_0$$
$$R_\pm(\pi) = -(1/12)N\,\mathrm{e}^{-3\beta^0}\gamma^{ab}_\pm\pi_a\pi_b - NU_\pm + \delta^+_\pm NQ_+ + NM_\pm$$
$$R_a(\pi) = (1/12)N\,\mathrm{e}^{-3\beta^0}(-D^c{}_{ab}\pi_b\pi_c + \gamma^{aa}_A p_A\pi_a) + \delta^3_a\,\mathrm{e}^{\gamma^3}NQ_3 + \mathrm{e}^{\gamma^a}NM_a\,. \tag{5.17}$$

The functions $R_a(\pi)$ can also be expressed as

$$R_a(\pi) = (1/12)N\,\mathrm{e}^{-3\beta^0}(-\phi^a\pi_b\pi_c + \gamma^{aa}_A p_a\pi_a) + \delta^3_a\,\mathrm{e}^{\gamma^3}NQ_3 + \mathrm{e}^{\gamma^a}NM_a\,,$$
$$\phi^a \equiv d^b - d^c\,, \qquad d^a \equiv D^a{}_{bc} \quad ((a, b, c) \text{ cyclic permutation})\,, \tag{5.18}$$

where the functions ϕ^a are given by

$$\phi^a = 2\sqrt{3}(\hat{h}_b^- \hat{h}_c^-)^{-1}[(n^{(a)})^2 e^{3\hat{\beta}^a} - n^{(b)} n^{(c)} e^{-3\hat{\beta}^a}] \quad ((a, b, c) \text{ cyclic permutation}) \qquad (5.19)$$
$$\hat{\beta}^a \equiv \beta^a - \beta^0 .$$

We now define the dimensionless momenta

$$r_I = -\lambda^2 \pi_I = (r_A, r_a) , \qquad r_A = -\lambda^2 p_A , \qquad r_a = -\lambda^2 \pi_a . \qquad (5.20)$$

These momenta are bounded provided we use the scale factor $\lambda = |p_0|^{-1/2}$. In this case the scale invariant momenta r_I are just certain constant linear combinations of the components of the scale invariant tensor $\boldsymbol{K}(\text{Tr}\, \boldsymbol{K})^{-1}$ with respect to the natural orthonormal frame. The proper choice of dimensionless configuration space variables will be seen to be dependent on the Bianchi type and will therefore be postponed until sections 7 and 8 where the regularized equations for all Bianchi types will be obtained.

To see how the scale transformation affects the equations of motion, we define scale invariant versions of Poisson brackets, Hamiltonian, nonpotential force and matter terms

$$\{\{r_I, r_J\}\} \equiv \lambda^{-2}\{r_I, r_J\} \qquad (5.21)$$
$$\hat{H}_G = -N\lambda^2 \mathscr{H}_G , \qquad \hat{Q}_I = -N\lambda^2 e^{\gamma^I} Q_I , \qquad \hat{M}_I = -N\lambda^2 e^{\gamma^I} M_I .$$

The scale invariant momenta then obey the same commutation relations $\{\{r_I, r_J\}\} = D^K_{IJ} r_K$ as the unscaled momenta π_I but the equations of motion have to be modified to take the time derivative of the scale factor into account. As a result the momentum equations can be written in the scale invariant form

$$\dot{r}_I = \hat{R}_I , \qquad (5.22)$$
$$\hat{R}_I \equiv -N\lambda^2 R_I(\pi) = \{\{r_I, \hat{H}_G\}\} + (\log \lambda^2)^{\cdot} r_I + \hat{Q}_I + \hat{M}_I .$$

The second term on the right hand side is the frictional force discussed by Novikov [1972]. For the scale factor $\lambda = |p_0|^{-1/2}$ we have $(\log \lambda^2)^{\cdot} = -B_0$ where $B_0 \equiv \hat{R}_0$. Since B_0 is negative except near the turning point where the expansion is zero in the type IX models (see section 7), the sign of the Novikov term with respect to Ω-time is opposite to that of ordinary friction, i.e., energy is gained rather than lost. In negative Ω-time (i.e. β^0-time) or fluid proper time the sign of the Novikov term corresponds to that of ordinary friction (in an expanding universe).

Equation (5.22) is valid for an arbitrary lapse. When using the Misner lapse $N_M = 12 g^{1/2} p_0^{-1}$ it is useful to introduce the following notation [Rosquist 1984]

$$E \equiv -24 g^{1/2} \mathscr{H}_G = 48 g \hbar = K - W ,$$
$$K = -24 g^{1/2} \mathscr{T} = -\eta^{AB} p_A p_B - 6 \mathscr{G}^{ab} p_a p_b , \qquad W \equiv 24 g^{1/2} U_G , \qquad (5.23)$$
$$D \equiv 96 g \Delta = E + \sqrt{F} , \qquad F \equiv (96)^2 g^2 \delta = E^2 - 4\gamma^{-2}(\gamma - 1)\Sigma_0 ,$$
$$\Sigma_0 \equiv (24)^2 g g^{ab} \mathscr{H}^G_a \mathscr{H}^G_b = (48)^2 g^2 g^{ab} \hbar_a \hbar_b .$$

All of these functions have their scale invariant counterpart which we denote by the same symbols with a bar on top. For example $\bar{E} \equiv \lambda^4 E = p_0^{-2} E$. Using the dimensionless momenta r_I we have

$$\bar{E} = \bar{K} - \bar{W}, \qquad \bar{K} = 1 - r_+^2 - r_-^2 - r_1^2 - r_2^2 - r_3^2 . \tag{5.24}$$

The function B_0 can then be written [Rosquist 1984]

$$B_0 = -\tfrac{3}{2}(2-\gamma)\bar{K} - \tfrac{1}{2}(3\gamma - 2)\bar{W} + X , \tag{5.25}$$

where the tilt term X is defined by

$$X \equiv (4-3\gamma)\bar{\Sigma}_0(\gamma\bar{D})^{-1} = (\gamma/4)[(4-3\gamma)/(\gamma-1)](\bar{E} - \sqrt{\bar{F}}) , \tag{5.26}$$

or in a form which displays the relation between the vacuum and orthogonal stiff fluid cases

$$B_0 = -\tfrac{3}{2}(2-\gamma)\bar{E} - 2\bar{W} + X . \tag{5.27}$$

6. Matter terms and configuration space variables

We begin this section by evaluating the momentum constraints. The supermomentum is given by [Jantzen 1984]

$$\begin{aligned} \mathcal{H}^{\mathrm{G}}_a &= 2g^{1/2}G^{\perp}{}_a = -2\,\mathrm{Tr}(\boldsymbol{\delta}_a \boldsymbol{\pi}) \\ \boldsymbol{\delta}_a &= \boldsymbol{k}_a - 2a_b\delta^c_a \boldsymbol{e}^b{}_c , \qquad \mathrm{Tr}\,\boldsymbol{\delta}_a = 0 . \end{aligned} \tag{6.1}$$

The matrices $\boldsymbol{\delta}_a$ are given explicitly by

$$\begin{aligned} \boldsymbol{\delta}_1 &= -n^{(2)}\boldsymbol{e}^3{}_2 + n^{(3)}\boldsymbol{e}^2{}_3 - 3a\boldsymbol{e}^3{}_1 = \mathrm{e}^{\alpha^1}\boldsymbol{\kappa}_1 - \mathrm{sgn}(n^{(1)})(3/\sqrt{2})a\boldsymbol{\kappa}_2 , \\ \boldsymbol{\delta}_2 &= -n^{(3)}\boldsymbol{e}^1{}_3 + n^{(1)}\boldsymbol{e}^3{}_1 - 3a\boldsymbol{e}^3{}_2 = \mathrm{e}^{\alpha^2}\boldsymbol{\kappa}_2 + \mathrm{sgn}(n^{(2)})(3/\sqrt{2})a\boldsymbol{\kappa}_1 , \\ \boldsymbol{\delta}_3 &= -n^{(1)}\boldsymbol{e}^2{}_1 + n^{(2)}\boldsymbol{e}^1{}_2 + a\boldsymbol{e}_+ = \mathrm{e}^{\alpha^3}\boldsymbol{\kappa}_3 + a\boldsymbol{e}_+ . \end{aligned} \tag{6.2}$$

The primed gravitational momentum components are

$$\boldsymbol{\pi}' = \boldsymbol{S}\,\boldsymbol{\pi}\,\boldsymbol{S}^{-1} = \tfrac{1}{12}(3p_0\boldsymbol{e}_0 + \eta^{AB}p_A\boldsymbol{e}_B) + \tfrac{1}{2}\tilde{\mathcal{G}}^{ab}\tilde{P}_a\boldsymbol{\kappa}^{\#\prime}_b . \tag{6.3}$$

Using (6.1–6.3) we obtain

$$\mathcal{H}'_a = -2\,\mathrm{Tr}(\boldsymbol{\delta}_a\boldsymbol{\pi}') = -\rho^b{}_a\tilde{P}_b - \delta^3_a a p_+ , \tag{6.4}$$

where we have omitted the superindex G on the primed supermomentum components and the matrix $\boldsymbol{\rho} = (\rho^a{}_b)$ is defined by the relation

$$\boldsymbol{\delta}_a = \rho^b{}_a \boldsymbol{\kappa}_b + a_a \boldsymbol{e}_+ \,. \tag{6.5}$$

The supermomentum components may become linearly dependent if $\det \boldsymbol{\rho} = (1+9h)\, \mathrm{e}^{\alpha^1+\alpha^2+\alpha^3} = 0$ (this expression corrects the expressions given in [Jantzen 1984]). The determinant of $\boldsymbol{\rho}$ vanishes if $\mathrm{rank}(\boldsymbol{n}) < 2$ or if $h = -\frac{1}{9}$, i.e., for types I, II, $\mathrm{VI}_{-1/9}$ and V, leading to linear dependence except for type V where the additional p_+-term in (6.4) ensures linear independence. The linear dependence of the supermomentum components means that certain linear combinations of them vanish. For type I they all vanish while for types II and $\mathrm{VI}_{-1/9}$ there are two independent components.

When computing the matter terms it is useful to introduce the notation $\boldsymbol{\kappa}_I \equiv (\boldsymbol{e}_A, \boldsymbol{\kappa}_a)$. Symmetrization then leaves the diagonal matrices unchanged leading to $(\boldsymbol{\kappa}_I^{\#\prime}) = (\boldsymbol{e}_A, \boldsymbol{\kappa}_a^{\#\prime})$. The matter terms are given by

$$\begin{aligned} M_I &= (1/12) g^{-1/2} \Sigma_I + \delta_I^0 (1/8)(\gamma - 1) g^{-1/2} [E - 2\Sigma_0 (\gamma D)^{-1}] \,, \\ \Sigma_I &= (24)^2 g m_I^{\#ab} \mathcal{H}'_a \mathcal{H}'_b \,, \qquad \boldsymbol{m}_I^{\#} \equiv \boldsymbol{\kappa}_I^{\#\prime} \boldsymbol{g}'^{-1} = \mathrm{e}^{-\gamma^I} \mathrm{e}^{-\boldsymbol{\beta}} \boldsymbol{e}_I^{\#} \mathrm{e}^{-\boldsymbol{\beta}} \,, \end{aligned} \tag{6.6}$$

where we have used the relation

$$\mathrm{e}^{-\beta^b - \beta^c} \boldsymbol{e}_a^{\#} = \mathrm{e}^{-\boldsymbol{\beta}} \boldsymbol{e}_a^{\#} \mathrm{e}^{-\boldsymbol{\beta}} \quad ((a, b, c) \text{ cyclic permutation}) \tag{6.7}$$

and $\boldsymbol{e}_I^{\#} = (e_I^{\#ab}) \equiv \mathrm{e}^{\gamma^I} \mathrm{e}^{\boldsymbol{\beta}} \boldsymbol{\kappa}_I^{\#\prime} \mathrm{e}^{-\boldsymbol{\beta}}$ is an orthogonal basis of the space of symmetric matrices (with respect to the DeWitt inner product) given by

$$\begin{aligned} \boldsymbol{e}_I^{\#} &= (\boldsymbol{e}_A, \boldsymbol{e}_a^{\#}) \,, \qquad \boldsymbol{e}_a^{\#} = -\sqrt{3}(\boldsymbol{e}^b{}_c + \boldsymbol{e}^c{}_b) \quad ((a, b, c) \text{ cyclic permutation}) \,, \\ \langle \boldsymbol{e}_I^{\#}, \boldsymbol{e}_J^{\#} \rangle_{\mathrm{DW}} &= 6\eta_{IJ} \,. \end{aligned} \tag{6.8}$$

It follows that the rescaled matter terms become

$$\begin{aligned} \hat{M}_0 &= \tfrac{3}{2}(\gamma - 1)\bar{E} + (4 - 3\gamma)\bar{\Sigma}_0 (\gamma \bar{D})^{-1} \,, \\ \hat{M}_I &= \hat{\Sigma}_I (\gamma \bar{D})^{-1} \,, \quad (I = +, -, 1, 2, 3) \,, \\ \bar{\Sigma}_I &= \lambda^8 \Sigma_I \,, \quad \hat{\Sigma}_I \equiv \mathrm{e}^{\gamma^I} \bar{\Sigma}_I = (24)^2 \lambda^8 g e_I^{\#bc} \mathcal{H}''_b \mathcal{H}''_c \,, \quad \mathcal{H}''_a = \mathrm{e}^{-\beta^a} \mathcal{H}'_a \,. \end{aligned} \tag{6.9}$$

These formulas suggest that we define rescaled supermomentum components according to

$$\tilde{H}_a \equiv 24 \lambda^4 g^{1/2} \mathcal{H}''_a \,, \tag{6.10}$$

leading to the following expressions for the matter terms

$$\begin{aligned} \hat{\Sigma}_I &= e_I^{\#ab} \tilde{H}_a \tilde{H}_b \,, \qquad \hat{\Sigma}_0 = \tilde{H}_1^2 + \tilde{H}_2^2 + \tilde{H}_3^2 \,, \\ \hat{\Sigma}_+ &= \tilde{H}_1^2 + \tilde{H}_2^2 - 2\tilde{H}_3^2 \,, \qquad \hat{\Sigma}_- = \sqrt{3}(\tilde{H}_1^2 - \tilde{H}_2^2) \,, \\ \hat{\Sigma}_a &= -2\sqrt{3} \tilde{H}_b \tilde{H}_c \quad ((a, b, c) \text{ cyclic permutation}) \,. \end{aligned} \tag{6.11}$$

At this point we need to choose convenient scale invariant configuration space variables. In the nonsemisimple case we take the set (w, x, y) defined by*)

$$w = \lambda e^{\beta^0}, \qquad x = (\lambda e^{\beta^1})^4, \qquad y = e^{-2\sqrt{3}\beta^-} = e^{-\beta^{12}}. \tag{6.12}$$

In the next section we will see that this choice leads to a decoupling of the variable w in the nonsemisimple case. In the semisimple case we use the set (x, y, z) where $z \equiv e^{\beta^{31}}$. An advantage with these variables is that the nonsemisimple class A case arises as the boundary subsystem $z = 0$ thus facilitating comparison of the semisimple and nonsemisimple cases. The variables (x, y, z) are essentially the same as those used by Bogoyavlensky [1976a] in the type IX case.

Evaluation of the rescaled momentum components $\tilde{H}_a$ yields (note redefinition of $\hat{H}_a$ compared with Rosquist [1984])

$$\begin{aligned} &\tilde{H}_a = 4\sqrt{3}x^{1/2}\hat{H}_a, \qquad \hat{H}_1 = f_1^- r_1 + 3ayr_2, \\ &\hat{H}_2 = f_2^- r_2 + 3ayr_1, \quad \hat{H}_3 = f_3^- r_3 + 2\sqrt{3}ayr_+, \end{aligned} \tag{6.13}$$

where we have introduced the notation

$$\begin{aligned} &f_a^{\pm} \equiv 2e^{\alpha^a - \beta^a - 2\beta^1} h_a^{\pm} = n^{(b)}(\zeta^b)^2 \pm n^{(c)}(\zeta^c)^2 \quad ((a, b, c) \text{ cyclic permutation}), \\ &\zeta^a \equiv e^{\beta^a - \beta^1}, \end{aligned} \tag{6.14}$$

leading to the relations

$$\zeta^1 = 1, \qquad y = \zeta^2 = e^{\beta^{21}} = e^{-2\sqrt{3}\beta^-}, \qquad z = \zeta^3 = e^{\beta^{31}} = e^{-3\beta^+ - \sqrt{3}\beta^-}, \tag{6.15}$$

and

$$f_1^{\pm} = n^{(2)}y^2 \pm n^{(3)}z^2, \qquad f_2^{\pm} = n^{(3)}z^2 \pm n^{(1)}, \qquad f_3^{\pm} = n^{(1)} \pm n^{(2)}y^2. \tag{6.16}$$

The matter terms are determined by the functions $\hat{\Sigma}_I$ which become

$$\begin{aligned} &\hat{\Sigma}_0 = 48x(\hat{H}_1^2 + \hat{H}_2^2 + \hat{H}_3^2), \\ &\hat{\Sigma}_+ = 48x(\hat{H}_1^2 + \hat{H}_2^2 - 2\hat{H}_3^2), \qquad \hat{\Sigma}_- = 48\sqrt{3}x(\hat{H}_1^2 - \hat{H}_2^2), \\ &\hat{\Sigma}_a = -96\sqrt{3}x\hat{H}_b\hat{H}_c, \qquad ((a, b, c) \text{ cyclic permutation}). \end{aligned} \tag{6.17}$$

We also compute the functions ϕ^a appearing in the momentum equations (5.18) in terms of the

*) The relation to the variables (u, v) used in [Rosquist 1984] is $u = (\lambda e^{\beta^2})^4 = xy^4$, $v = e^{2\sqrt{3}\beta^-} = y^{-1}$. Both variable sets are adapted to the diagonal automorphism symmetry generated by $I^{(3)}$ while the set (x, y) is also adapted to the additional diagonal automorphism symmetry which is present in Bianchi type II whereas the set (u, v) is not adapted to this additional symmetry. It is not possible to choose variables which are simultaneously adapted to the diagonal automorphism symmetry group for all nonsemisimple types. However by choosing the set (x, y) only type V needs to be treated separately.

variables (x, y, z). To this end it is helpful to introduce the quantities $\eta^a \equiv n^{(a)}(\zeta^a)^2$. We then have the following relations

$$d^a = 2\sqrt{3}\eta^a f_a^-(f_b^- f_c^-)^{-1}\,, \qquad f_a^\pm = \eta^b \pm \eta^c\,,$$
$$\phi^a = 2\sqrt{3}[(\eta^a)^2 - \eta^b\eta^c](f_b^- f_c^-)^{-1}\,. \tag{6.18}$$

7. The reduced regularized system

We are now ready to put together the various pieces which lead to a reduced regularized system. By a judicious choice of lapse the system will be regularized in the sense of being analytic apart from a few exceptional points in the reduced phase space. Our system will at the same time be maximally reduced in the sense of having the fewest possible number of variables which is attainable for each Bianchi type by using the scale group and the automorphism group. The scaling transformation decouples one of the variables and by choosing the scale factor as $\lambda = |p_0|^{-1/2}$, the momentum component p_0 is decoupled while at the same time leaving the remaining orthonormalized momentum components bounded. There remains to be discussed the ranges of the configuration space variables.

To this end we decompose the gravitational potential function as $U_G = U_G^{(0)} + U_G^{(1)}$ where $U_G^{(0)}$ is the nonsemisimple part of U_G obtained by setting $n^{(3)} = 0$. We then obtain the following expressions for the potential function W decomposed in an analogous way

$$\bar{W}^{(0)} = 12x((f_-)^2 + 12a^2y^2)\,, \qquad \bar{W}^{(1)} = 12n^{(3)}xz^2(-2f_+ + n^{(3)}z^2)\,,$$
$$\bar{W}_+^{(0)} = 4\bar{W}^{(0)}\,, \qquad \bar{W}_+^{(1)} = 48n^{(3)}xz^2(f_+ - 2n^{(3)}z^2)\,,$$
$$\bar{W}_-^{(0)} = 48\sqrt{3}xf_+f_-\,, \qquad \bar{W}_-^{(1)} = -48\sqrt{3}n^{(3)}xz^2f_-$$
$$f_\pm \equiv f_3^\pm\,. \tag{7.1}$$

The nonsemisimple models I–VII admit the diagonal automorphism generated by $\boldsymbol{l}^{(3)}$. If one chooses a lapse which is proportional to $e^{3\beta^0}$ then β^0 and β^+ occur in the field equations only in the combination $\beta^0 + \beta^+$. Therefore, using the configuration variables (w, x, y), which apart from scale factors depend on β^0, $\beta^0 + \beta^+$ and β^- respectively, it follows that the w-equation decouples from the system. Thus x and y are the remaining configuration space variables in the reduced system. We first note that both x and y are positive by definition. Second, writing out the expression for $\bar{W}^{(0)}$ in full

$$\bar{W}^{(0)} = 12(n^{(1)})^2x - 24n^{(1)}n^{(2)}xy^2 + 12(n^{(2)})^2xy^4 + 144a^2xy^2 \tag{7.2}$$

shows it to consist of four terms which are all nonnegative if $n^{(1)}n^{(2)} \le 0$. For the nonsemisimple types which in addition have $n^{(1)} \ne 0$ (i.e. types II, III, IV and VI) this implies the inequality $12(n^{(1)})^2x \le 1$. No such restriction can be placed on y. For Bianchi type VI the transformation which interchanges the 1- and 2-axes is a discrete automorphism. Therefore the regions $y \le 1$ and $y \ge 1$ are equivalent up to such a transformation. Since it is only necessary to work with say the region $y \le 1$ the phase space is now fully compactified in the type VI case. For other Bianchi types the phase space needs to be covered by at least one other coordinate patch in which the limit $y \to \infty$ becomes finite. Similar remarks apply whenever $(x, y, z) \in \mathbb{R}^3$ is not confined to a bounded set by the energy constraint.

In the semisimple case the potential function in the variables (x, y, z) becomes

$$\bar{W} = 12x(f_-)^2 - 24n^{(3)}xz^2 f_+ + 12(n^{(3)})^2 xz^4 . \tag{7.3}$$

In this case the variables (x, y, z) will not be confined to a compact set, but they are useful because of the ease with which one can pass to the nonsemisimple case.

The energy condition $\hbar^2 - \hbar^a \hbar_a \geq 0$ takes the scale invariant form $\bar{E}^2 - \hat{\Sigma}_0 \geq 0$. The reduced phase space is therefore given by

$$\mathcal{P}_{\text{red}} = \{x = (x, y, z, r_\pm, r_a) | x, y, z > 0, \bar{E}^2 \geq \hat{\Sigma}_0\} .$$

We can now write down the complete Einstein system:

Decoupled equation: $\dot{p}_0 = B_0 p_0$.

Reduced system in the variables $(x, y, z, r_\pm, r_a)$:

$$\begin{aligned}
&\dot{x} = -2[B_0 + 2(1 - r_+ - \sqrt{3} r_-)]x , \qquad \dot{y} = -2\sqrt{3} r_- y , \qquad \dot{z} = -(3r_+ + \sqrt{3} r_-)z , \\
&\dot{r}_I = B_I - B_0 r_I \quad (I = +, -, 1, 2, 3) , \\
&B_+ = 3t_1^* r_1^2 - 3t_2^* r_2^2 - \tfrac{1}{2}\bar{W}_+ + \hat{Q}_+ + \hat{M}_+ , \\
&B_- = -\sqrt{3}(t_1^* r_1^2 + t_2^* r_2^2 - 2t_3^* r_3^2) - \tfrac{1}{2}\bar{W}_- + \hat{M}_- , \\
&B_1 = -\phi^1 r_2 r_3 + t_1^*(-3r_+ + \sqrt{3} r_-) r_1 + \hat{M}_1 , \\
&B_2 = -\phi^2 r_3 r_1 + t_2^*(3r_+ + \sqrt{3} r_-) r_2 + \hat{M}_2 , \\
&B_3 = -\phi^3 r_1 r_2 - 2\sqrt{3} t_3^* r_- r_3 + \hat{Q}_3 + \hat{M}_3 , \\
&-\tfrac{1}{2}\bar{W}_+ + \hat{Q}_+ = -24x[(f_3^-)^2 + n^{(3)} z^2 (f_3^+ - 2n^{(3)} z^2)] , \\
&\bar{W}_- = 48\sqrt{3} x f_3^- (f_3^+ - n^{(3)} z^2) , \qquad \hat{Q}_3 = 48\sqrt{3} axy f_3^- , \\
&t_a^* = f_a^+ / f_a^- , \\
&\phi^1 = 2\sqrt{3}[(n^{(1)})^2 - n^{(2)} n^{(3)} y^2 z^2](f_2^- f_3^-)^{-1} , \\
&\phi^2 = 2\sqrt{3}[(n^{(2)})^2 y^2 - n^{(3)} n^{(1)} z^2](f_3^- f_1^-)^{-1} , \\
&\phi^3 = 2\sqrt{3}[(n^{(3)})^2 z^2 - n^{(1)} n^{(2)} y^2](f_1^- f_2^-)^{-1} .
\end{aligned} \tag{7.4}$$

The equations of motion for the lapse and scale factors are

$$\dot{\lambda} = -\tfrac{1}{2}B_0\lambda\,, \qquad \dot{N} = -(B_0+3)N\,. \tag{7.5}$$

It is evident from these equations that the function B_0 plays a key role in SH dynamics. Using the inequalities $0 \le \hat{\Sigma}_0/(\gamma\bar{D}) \le \bar{E}/2$, the second part being the scale invariant version of $\hbar^a\hbar_a/(\gamma\Delta) \le \hbar$, it follows that B_0 satisfies the inequalities

$$\begin{aligned} \gamma \le \tfrac{4}{3}:&\ \bar{K}+\bar{W} \le -B_0 \le \tfrac{3}{2}(2-\gamma)\bar{K} + \tfrac{1}{2}(3\gamma-2)\bar{W}\,,\\ \gamma \ge \tfrac{4}{3}:&\ \tfrac{3}{2}(2-\gamma)\bar{K} + \tfrac{1}{2}(3\gamma-2)\bar{W} \le -B_0 \le \bar{K}+\bar{W}\,. \end{aligned} \tag{7.6}$$

Using also $\bar{K} \ge \bar{W}$ implies the following inequality valid for all values of γ [Rosquist 1984]

$$2\bar{W} \le -B_0 \le 2\bar{K}\,. \tag{7.7}$$

As long as $\bar{W} \ge 0$, i.e., for all types except IX in which case $\bar{W}$ is not bounded below, and using $\bar{K} \ge 0$ we find that B_0 satisfies

$$-2 \le B_0 \le 0\,. \tag{7.8}$$

Thus the following inequality is valid for all types

$$B_0 + 2 \ge 0\,. \tag{7.9}$$

Besides being important in the regularization procedure, the Misner lapse also has an immediate physical significance through the relation $\tilde{\theta} = 3/N$ where $\tilde{\theta} \equiv n^{\alpha}{}_{;\alpha}$ ($n \equiv \partial/\partial\tau$) is the expansion of the unit normals.

Setting $a = 0$ to specialize the system to the semisimple case has only the effect of cancelling the terms $\hat{Q}_3$ and the class B term in $\bar{W}$. The nonsemisimple class A case is then obtained by taking the limit $z \to 0$. Therefore the semisimple boundary component $z = 0$ is equivalent to the nonsemisimple class A system.

The system fails to be analytic where the denominators occurring in t^*_a and d^a have zeroes and also at the matter term branch points at $\bar{F} = 0$. The former type of nonanalyticity is caused by the singularities in the automorphism transformations where the action of the automorphism group fails to be transversal to the diagonal submanifold $\mathcal{M}_D$. This happens for canonical type IX values when $f^-_a = 0$, i.e. at the three phase space hypersurfaces defined by the conditions $y = 1$, $z = 1$ and $y = z$. The remaining types with this kind of singularity are types VII and VIII which for canonical values have zero denominators at the hypersurface $y = 1$.

There is one class of solutions to the regularized system which is especially important in the qualitative approach, namely the singular point solutions. They are defined as the solutions to the algebraic system obtained by setting the right hand side of the reduced system equal to zero. These solutions are in fact the EPL solutions considered by Wainwright [1984] but including their tilted analogues as well. Besides being defined as singular point solutions of the reduced system, EPL solutions can also be defined as solutions which admit a homothetic vector field which is transversal to the homogeneous slices [Jantzen and Rosquist 1986, Rosquist and Jantzen 1986]. It therefore follows that the EPL solutions admit a self-similarity group H_4 which acts transitively on the space–time.

To see the equivalence between singular point solutions and the transitively self-similar solutions we write the metric as

$$ {}^4g = -N_M^2\, d\Omega \otimes d\Omega + (e^{2\bar{\beta}})_{ab}(\bar{\omega}^a + \bar{N}^a\, d\Omega) \otimes (\bar{\omega}^b + \bar{N}^b\, d\Omega) , \tag{7.10} $$

where the matrix $\bar{\beta}$ represents the dynamically significant diagonal variables according to (2.24) (and its generalizations to the cases with more than one diagonal automorphism) and where the $\bar{N}^a$ are the shift components corresponding to the barred SH frame as discussed in the appendix. Next we factorize the metric according to

$$ \begin{aligned} {}^4g &= N_M^2\, {}^4g^{(0)} , \\ {}^4g^{(0)} &= -d\Omega \otimes d\Omega + (N_M^{-2}\, e^{2\bar{\beta}})_{ab}(\bar{\omega}^a + \bar{N}^a\, d\Omega) \otimes (\bar{\omega}^b + \bar{N}^b\, d\Omega) . \end{aligned} \tag{7.11} $$

The scale invariant quantities $N_M^{-1}\, e^{\bar{\beta}^a}$ are functions on the reduced phase space (see appendix). The same remark applies to the shift components. Thus for EPL solutions ${}^4g^{(0)}$ is a space–time homogeneous metric. According to (7.5) the quantity $(\log N_M)^{\cdot}$ depends only on the reduced variables which are constants. This means that N_M is exponential in Ω, making $\partial/\partial\Omega$ a homothetic Killing vector field [Defrise-Carter 1975].

Conversely, suppose the space–time is invariant under a homothetic motion which is transversal to the homogeneous slices. One may always choose a frame comoving with some (nontrivial) homothetic Killing vector field as $\partial/\partial\Omega$ so that the metric takes the form (7.11) with $\partial/\partial\Omega$ a Killing vector field of ${}^4g^{(0)}$, using constant automorphisms and scale transformations as necessary. Since ${}^4g^{(0)}$ is entirely specified by the reduced variables, it follows that they must be constants, leading to a singular point solution of the reduced equations.

8. The nonsemisimple system

The field equations for the nonsemisimple models can be obtained by specializing (7.4) to the case $n^{(3)} = 0$ with the result

Decoupled equations:

$$ \dot{w} = -\tfrac{1}{2}(B_0 + 2)w , \qquad \dot{p}_0 = B_0 p_0 . $$

Reduced system in the variables $(x, y, r_+, r_-, r_1, r_2, r_3)$:

$$ \dot{x} = -2[B_0 + 2(1 - r_+ - \sqrt{3} r_-)]x , \qquad \dot{y} = -2\sqrt{3} r_- y , $$

$$ \dot{r}_I = B_I - B_0 r_I \quad (I = +, -, 1, 2, 3) , $$

$$ B_+ = 3(r_1^2 + r_2^2) - 24 x f_-^2 + \hat{M}_+ , $$

$$
\begin{aligned}
&B_- = \sqrt{3}(-r_1^2 + r_2^2 + 2f_+ f_-^{-1} r_3^2) - 24\sqrt{3}xf_+ f_- + \hat{M}_- ,\\
&B_1 = 2\sqrt{3}n^{(1)} f_-^{-1} r_2 r_3 - (3r_+ - \sqrt{3}r_-)r_1 + \hat{M}_1 ,\\
&B_2 = -2\sqrt{3}n^{(2)} y^2 f_-^{-1} r_1 r_3 - (3r_+ + \sqrt{3}r_-)r_2 + \hat{M}_2 ,\\
&B_3 = -2\sqrt{3}r_1 r_2 - 2\sqrt{3}f_+ f_-^{-1} r_- r_3 + 48\sqrt{3}axyf_- + \hat{M}_3 ,\\
&\bar{W} = 12x(f_-^2 + 12a^2y^2) , \qquad f_\pm = n^{(1)} \pm n^{(2)} y^2 ,\\
&\hat{H}_1 = -y^2 r_1 + 3ayr_2 ,\\
&\hat{H}_2 = -r_2 + 3ayr_1 ,\\
&\hat{H}_3 = f_- r_3 + 2\sqrt{3}ayr_+ .
\end{aligned} \tag{8.1}
$$

For the canonical Bianchi type VI values of the structure constants these equations coincide*) with eqs. (3.8) of [Rosquist 1984] apart from the variable change $(u, v) \to (x, y)$ and a relabeling $r_1 \leftrightarrow r_2$. In the type III, IV, VI and VII cases no further reduction is possible for general tilted models. In the Bianchi type I, II and V cases additional diagonal automorphisms exist [Jantzen 1984] which can be used to eliminate one or more of the remaining variables. The reduced systems in those cases are as follows:

Type I: In this case all the supermomentum components vanish, $\hat{H}_a = 0$. Hence all the anisotropic matter terms vanish, $M_\pm = M_a = 0$. Also, type I being a class A model, the nonpotential force is zero, $Q = 0$. From the field equations (4.20) it then follows that $\tilde{P}_a = 0$ is a solution of the offdiagonal momentum equations. Therefore, diagonal initial data are preserved by the evolution equations. Furthermore, since the automorphism group is the entire linear group GL(3, ℝ), one can use (constant) automorphisms to simultaneously diagonalize the spatial metric matrix $\boldsymbol{g}$ and its derivative $\dot{\boldsymbol{g}}$ to obtain diagonal initial data. Thus the most general type I models are diagonal. We still have the freedom to apply the 3-parameter group of diagonal scale transformations. Therefore as discussed in the appendix all of the configuration space variables decouple leaving only $r_\pm$ in a 2-dimensional reduced system according to

Decoupled equations:

$$\dot{w} = -\tfrac{1}{2}(B_0 + 2)w , \qquad \dot{x} = -2[B_0 + 2(1 - r_+ - \sqrt{3}r_-)]x , \qquad \dot{y} = -2\sqrt{3}r_- y .$$

Reduced system in the variables $(r_\pm)$: (8.2)

$$\dot{r}_\pm = -B_0 r_\pm , \qquad B_0 = -\tfrac{3}{2}(2-\gamma)(1 - r_+^2 - r_-^2) ,$$

$$\bar{E} \geq 0 \Leftrightarrow r_+^2 + r_-^2 \leq 1 .$$

There are two sets of singular points, $C_F = \{r_\pm = 0\}$ and $C_K = \{\bar{E} = 0\}$. The point C_F corresponds to

*) There are sign errors in equations (3.9) and (3.10) of Rosquist [1984] which can be corrected by the transformation $\hat{H}_3 \to -\hat{H}_3$.

the flat Friedmann solution while the Kasner circle C_K represents the Kasner solutions. The system can easily be integrated, e.g. using $r = (r_+^2 + r_-^2)^{1/2}$ as the time variable. The resulting non-EPL solutions were first given by Jacobs [1969]. The Jacobs solutions are rays in the $r_\pm$-plane which using the physical time direction start out from the Kasner circle and end up as flat Friedmann models.

Type II: The type II rescaled supermomentum components are $\hat{H}_1 = 0$, $\hat{H}_2 = -r_2$, $\hat{H}_3 = r_3$. It follows that the two anisotropic matter terms $\hat{M}_2$ and $\hat{M}_3$ vanish. There is one additional diagonal generator beside $\boldsymbol{l}^{(3)}$, namely $\boldsymbol{l}^{(2)}$. Because of the $\boldsymbol{l}^{(2)}$ generator the equation for y is decoupled leaving x as the only remaining configuration space variable in the reduced system. A barred frame may then be chosen (see section 2 and appendix) such that the metric is of the form $\bar{\boldsymbol{g}} = \mathrm{diag}(\bar{g}_{11}, 1, 1)$. The full type II automorphism group is 6-dimensional and hence there is a still unused 1-parameter gauge freedom which can be used to simplify the initial data and thereby also simplifying the field equations. This gauge freedom corresponds to the subgroup of the type II automorphism group which leaves $\bar{\boldsymbol{g}}$ invariant and which is isomorphic to O(2), the orthogonal group in two dimensions. This 1-dimensional subgroup $\mathrm{O}_{2,3}$ (using a notation analogous to that of appendix B of [Jantzen 1984]) acts in the $\bar{e}_2\bar{e}_3$-plane. It can be used to simplify the initial data by making one of the offdiagonal momenta vanish. To see how this works we note that $v_2 = v_2' \propto \bar{v}_3$ and v_3 are both constants of the motion [Jantzen 1983]. Hence by applying a constant rotation in the $\bar{e}_2\bar{e}_3$-plane $\bar{v}_2$ may be set equal to zero. In the reduced variables this corresponds to setting $r_2 = 0$. Thus we may set $r_2 = 0$ without loss of generality leaving two nontrivial offdiagonal momenta, r_1 and r_3. Then $\hat{H}_2 = 0$ which together with $\hat{H}_1 = 0$ implies that two velocity components vanish, $u_1 = u_2 = 0$ with u_3 as the only remaining nonzero velocity component. Therefore, interestingly, in the general tilted type II case the number of nontrivial gravitational offdiagonal momenta does not coincide with the number of nontrivial velocity components. Alternatively the velocity component v_3 could be transformed to zero but in the canonical limit this choice does not correspond to the vanishing of a momentum variable so that one would have to use a noncanonical limit to handle that situation. However the field equations for that choice could of course be obtained by the transformation $r_2 \leftrightarrow r_3$. The final reduced system for canonical values of structure constants in the canonical limit becomes

Decoupled equations:

$$\dot{w} = -\tfrac{1}{2}(B_0 + 2)w\,, \qquad \dot{y} = -2\sqrt{3}r_- y\,, \qquad \dot{p}_0 = B_0 p_0\,,$$

Reduced system in the variables $(x, r_\pm, r_1, r_3)$:

$$\begin{aligned}
&\dot{x} = -2[B_0 + 2(1 - r_+ - \sqrt{3}r_-)]x\,,\\
&\dot{r}_I = B_I - B_0 r_I \quad (I = +, -, 1, 3)\,,\\
&B_+ = -24x + \hat{M}_+\,, \qquad B_- = 2\sqrt{3}r_3^2 - 24\sqrt{3}x\,,\\
&B_1 = -(3r_+ - \sqrt{3}r_-)r_1\,, \qquad B_3 = -2\sqrt{3}r_- r_3\,,\\
&\bar{K} = 1 - r_+^2 - r_-^2\,, \qquad \bar{W} = 12x\,,\\
&\hat{\Sigma}_+ = -2\hat{\Sigma}_0 = -96xr_3^2\,.
\end{aligned} \tag{8.3}$$

Type V: The supermomentum components are $\hat{H}_1 = 3yr_2$, $\hat{H}_2 = 3yr_1$, $\hat{H}_3 = 2\sqrt{3}yr_+$. In this case diag$(1, -1, 0)$ is an additional diagonal automorphism generator. This leaves $b \equiv 144xy^2$ as the remaining configuration space variable. Note that the orbits of the type II and V diagonal automorphism groups do not coincide and that as a consequence one cannot use the same reduced configuration variable in the two cases. As in the type II case there is an additional unused offdiagonal automorphism which can be utilized to simplify the initial data. Both v_1 and v_2 are constants of the motion one of which can be set equal to zero. In the canonical limit $v_1' = v_1$ (the automorphism matrix $\boldsymbol{S}$ is the same for Bianchi types II and V in the canonical limit) while $v_2' = \sqrt{2}\theta^3 v_1 + v_2$ showing that it is v_1 which should be set equal to zero. This leads to $\hat{H}_1 = r_2 = 0$ and $\hat{M}^2 = \hat{M}^3 = 0$. Hence the field equations become

Decoupled equations:

$$\dot{w} = -\tfrac{1}{2}(B_0 + 2)w\,, \qquad \dot{y} = -2\sqrt{3}r_-y\,, \qquad \dot{p}_0 = B_0 p_0\,,$$

Reduced system in the variables $(b, r_\pm, r_1, r_3)$:

$$\begin{aligned}
&\dot{b} = -2[B_0 + 2(1 - r_+)]b\,, \\
&\dot{r}_I = B_I - B_0 r_I \quad (I = +, -, 1, 3)\,, \\
&B_+ = 3r_1^2 + \hat{M}_+\,, && \hat{\Sigma}_0 = b(3r_1^2 + 4r_+^2)\,, \\
&B_- = \sqrt{3}(-r_1^2 + 2r_3^2) + \hat{M}_-\,, && \hat{\Sigma}_+ = b(3r_1^2 - 8r_+^2)\,, \\
&B_1 = -(3r_+ - \sqrt{3}r_-)r_1 + \hat{M}_1\,, && \hat{\Sigma}_- = -3\sqrt{3}br_1^2\,, \\
&B_3 = -2\sqrt{3}r_-r_3\,, && \hat{\Sigma}_1 = -12br_+r_1\,, \\
&\bar{K} = 1 - r_+^2 - r_-^2 - r_1^2 - r_3^2\,, && \bar{W} = b \equiv 144xy^2\,.
\end{aligned} \tag{8.4}$$

9. Orthogonal models of the types II–IX

The orthogonal case is defined by the requirement that the fluid 4-velocity is perpendicular to the homogeneous slices. This means that the spatial components of the fluid 4-velocity vanishes, $u^a = 0$, which in terms of the supermomentum constraints is equivalent to setting $\mathscr{H}_a^{\mathrm{G}} = 0$. In the class A case this leads to the diagonal case which in the reduced system is characterized by $r_a = 0$. In all cases the anisotropic matter terms $\bar{M}_I$ $(I = +, -, 1, 2, 3)$ vanish. The vacuum equations are just the orthogonal dust $(\gamma = 1)$ equations on the invariant submanifold $\bar{E} = 0$.

The semisimple case: The metric being diagonalized by the automorphism group stays diagonal due to the supermomentum constraints. The field equations are

Decoupled equation: $\dot{p}_0 = B_0 p_0$.

Reduced system in the variables $(x, y, z, r_\pm)$:

$$\dot{x} = -2[B_0 + 2(1 - r_+ - \sqrt{3}r_-)]x\,, \qquad \dot{y} = -2\sqrt{3}r_- y\,, \qquad \dot{z} = -(3r_+ + \sqrt{3}r_-)z\,,$$

$$\dot{r}_\pm = B_\pm - B_0 r_\pm\,, \qquad B_\pm = -\tfrac{1}{2}\bar{W}_\pm\,, \tag{9.1}$$

$$B_+ = -24x[f_-^2 + n^{(3)}z^2(f_+ - 2n^{(3)}z^2)]\,,$$

$$B_- = -24\sqrt{3}xf_-(f_+ - n^{(3)}z^2)\,,$$

$$\bar{K} = 1 - r_+^2 - r_-^2\,, \qquad \bar{W} = 12x[f_-^2 + n^{(3)}z^2(-2f_+ + n^{(3)}z^2)]\,.$$

Type III, IV, VI($h \neq -\frac{1}{9}$), VII models: The supermomentum constraints in this case lead to $r_1 = r_2 = 0$, $r_3 = -2\sqrt{3}ayf_-^{-1}r_+$. The automorphism group serves to simplify the initial data by bringing the metric to the symmetric case submanifold $\mathcal{M}_{S(3)}$ (see [Jantzen 1984] for the definition of this submanifold), using up two parameters, and to decouple the variables θ^3 and w, using the remaining two automorphism parameters. The one offdiagonal degree of freedom of the metric is represented by r_3 and the field equations become

Decoupled equations:

$$\dot{w} = -\tfrac{1}{2}(B_0 + 2)w\,, \qquad \dot{p}_0 = B_0 p_0\,.$$

Reduced system in the variables (x, y, r_+, r_-):

$$\dot{x} = -2[B_0 + 2(1 - r_+ - \sqrt{3}r_-)]x\,, \qquad \dot{y} = -2\sqrt{3}r_- y\,,$$

$$\dot{r}_\pm = B_\pm - B_0 r_\pm\,, \tag{9.2}$$

$$B_+ = -24xf_-^2\,,$$

$$B_- = -24\sqrt{3}f_+f_-(x - a^2y^2f_-^{-4}r_+^2)\,,$$

$$\bar{K} = 1 - (12a^2y^2 + f_-^2)f_-^{-2}r_+^2 - r_-^2\,, \qquad \bar{W} = 12x(f_-^2 + 12a^2y^2)\,,$$

$$f_\pm = n^{(1)} \pm n^{(2)}y^2\,.$$

Type $VI_{-1/9}$: The supermomentum components are linearly dependent in this case. For canonical values of the structure constants the momentum constraints yield only the two conditions $r_2 = yr_1$ and $r_3 = -(2/\sqrt{3})yf_-^{-1}r_+$. The automorphism group is used to bring the metric to a submanifold which corresponds to the condition $r_2 = yr_1$. The remaining three automorphism parameters are then used to decouple the variables θ^1, θ^3 and w. The field equations are

Decoupled equations:

$$\dot{w} = -\tfrac{1}{2}(B_0 + 2)w\,, \qquad \dot{p}_0 = B_0 p_0\,,$$

Reduced system in the variables (x, y, r_+, r_-, r_1):

$$\dot{x} = -2[B_0 + 2(1 - r_+ - \sqrt{3}r_-)]x\,, \qquad \dot{y} = -2\sqrt{3}r_- y\,,$$

$$\dot{r}_I = B_I - B_0 r_I \quad (I = +, -, 1)\,,$$

$$B_+ = 3f_- r_1^2 - 24xf_-^2\,, \tag{9.3}$$

$$B_- = -\sqrt{3}f_+ r_1^2 + (8\sqrt{3}/3)y^2 f_+ f_-^{-3} r_+^2 - 24\sqrt{3}xf_+ f_-\,,$$

$$B_1 = -4y^2 f_-^{-2} r_+ r_1 - (3r_+ - \sqrt{3}r_-)r_1\,,$$

$$\bar{K} = 1 - \tfrac{1}{3}(4y^2 + 3f_-^2)f_-^{-2}r_+^2 - r_-^2 - f_- r_1^2\,, \qquad \bar{W} = 4x(4y^2 + 3f_-^2)\,,$$

$$f_\pm = 1 \mp y^2\,.$$

Type II: The two independent supermomentum constraints in this case yield $r_1 = r_2 = 0$. The six automorphisms present in this case are used to diagonalize the metric, using three parameters, and to set $r_1 = 0$ using the remaining offdiagonal automorphism and finally to decouple w and y leading to the canonical field equations

Decoupled equations:

$$\dot{w} = -\tfrac{1}{2}(B_0 + 2)w\,, \qquad \dot{y} = -2\sqrt{3}r_- y\,, \qquad \dot{p}_0 = B_0 p_0\,.$$

Reduced system in the variables $(x, r_\pm)$:

$$\dot{x} = -2[B_0 + 2(1 - r_+ - \sqrt{3}r_-)]x\,, \qquad r_\pm = B_\pm - B_0 r_\pm\,, \tag{9.4}$$

$$B_+ = -24x\,, \qquad B_- = -24\sqrt{3}x\,,$$

$$\bar{K} = 1 - r_+^2 - r_-^2\,, \qquad \bar{W} = 12x\,.$$

Type V: The supermomentum constraints give $r_1 = r_2 = r_+ = 0$. The initial data can be simplified by using the four offdiagonal parameters to diagonalize the metric and in addition to set $r_3 = 0$. As before the two diagonal automorphisms serve to decouple the variables w and y leaving the field equations in the canonical form

Decoupled equations:

$$\dot{w} = -\tfrac{1}{2}(B_0 + 2)w\,, \qquad \dot{y} = -2\sqrt{3}r_- y\,, \qquad \dot{p}_0 = B_0 p_0\,,$$

Reduced system in the variables (b, r_-): (9.5)

$$\dot{b} = -2(B_0 + 2)b\,, \qquad \dot{r}_- = -B_0 r_-\,,$$

$$\bar{K} = 1 - r_-^2\,, \qquad \bar{W} = b\,.$$

10. Concluding remarks

Starting from the usual Hamiltonian form of the perfect fluid Einstein equations in orthogonal gauge, geometrical variables have been introduced which are adapted to the action of the combined group of scale transformations of the unit of length and spatial gauge transformations. These variables define an equivalent spatial gauge called "diagonal gauge" in which both the metric and symmetry constants are "diagonalized", leading to a natural orthonormal frame adapted to both the symmetry and the geometry. A reduced system of field equations is obtained for the scale invariant metric variables in this orthonormal frame, using Misner's scale invariant Ω-time as the independent variable. The fluid variables have been eliminated by explicit solution of the gravitational constraints.

This approach contains as essential ingredients both the geometry and interpretation of the Hamiltonian approach in the usual orthogonal gauge as well as the advantages of the orthonormal frame approach, elements which are interwoven through a choice of geometrical variables adapted to the spatial gauge freedom and the geometry rather than to the fluid. The geometric interpretation of the variables themselves and of the various terms which appear in the equations is clear from their relationship to the Hamiltonian approach, which also makes it very easy to see the restrictions on those variables corresponding to additional discrete or continuous space–time symmetries.

The qualitative behavior of the system is most conveniently studied using the form of the equations which this article has derived. However, some problems remain. For Bianchi types VII–IX the use of the diagonalizing automorphisms lead to singularities in the Lagrangian which in the reduced system are manifested by terms which are unbounded at certain submanifolds of the phase space. There are also cases where the physical range of the configuration space variables stretches out to infinity. For both these types of non-compactness further local transformations beyond those discussed in this paper are needed to deal with the problematic regions of phase space.

Once the regularized system has been established the next step in the qualitative analysis is to find the singular points. The number of nonzero eigenvalues (including multiplicities) should then equal the dimension of the system minus the dimension of the set of singular points. The solutions can then be analyzed by power series expansions about the singular points. This leads to a consistent approximation scheme for the Einstein equations. A first step towards a first order analysis of the SH Einstein equations has been taken in Uggla and Rosquist [1988].

The phase space of SH cosmology is neither closed nor open. Accordingly, its boundary has a physical component which is contained in the phase space and an unphysical component which is outside the region of physically admissible values. The physical singular points give rise to the exact power law solutions (vacuum EPL solutions for singular points on the boundary and nonvacuum EPL solutions for singular points in the interior of the phase space) characterized by the existence of an additional homothetic symmetry with orbits transversal to the family of homogeneous hypersurfaces, leading to a transitive group of homothetic transformations of the space–time. For such a space–time, one may choose the spatial frame to be comoving with a nontrivial homothetic Killing vector field, leading to constant values of all scale invariant variables in that frame. Conversely, a singular point of the reduced system leads to a solution which has constant values of all scale invariant variables in an associated diagonal gauge frame, and so determines an EPL solution. Singular points on the unphysical part of the boundary lead to approximate EPL solutions or "power asymptotes". In the case of complex eigenvalues the solutions exhibit a spiralling behavior in the neighborhood of the singular point.

Acknowledgement

The authors wish to thank Claes Uggla for detecting errors in the manuscript.

Appendix. Spatial gauge transformations

Any computational frame $\{\bar{e}_0, \bar{e}_a\}$ adapted to the homogeneous slicing can be related to the SH orthogonal gauge frame $\{e_0, e_a\}$ with $e_0 = Ne_\perp$ by the following transformation

$$\bar{e}_0 = e_0 + \vec{N} , \qquad \bar{e}_a = e_b(A^{-1})^b{}_a , \tag{A.1}$$

where both frames satisfy the comoving condition

$$[e_0, e_a] = 0 = [\bar{e}_0, \bar{e}_a] . \tag{A.2}$$

The second equation, when expressed in terms of the first frame, leads to the result

$$0 = \bar{\omega}^b([\bar{e}_0, \bar{e}_a]) = (-\dot{\boldsymbol{A}}\,\boldsymbol{A}^{-1} + \mathrm{ad}_{\bar{e}}(\vec{N}))^b{}_a , \tag{A.3}$$

or

$$\dot{\boldsymbol{A}}\,\boldsymbol{A}^{-1} = \mathrm{ad}_{\bar{e}}(\vec{N}) ,$$

where the notation $\mathrm{ad}(\vec{N})X = [\vec{N}, X] = \mathscr{L}_{\vec{N}}X$ is used for the inner derivation associated with the shift vector field $\vec{N}$ on the infinite-dimensional Lie algebra of spatial vector fields, and

$$\bar{\omega}^b(\mathrm{ad}(\vec{N})\bar{e}_a) = \mathrm{ad}_{\bar{e}}(\vec{N})^b{}_a \tag{A.4}$$

is its matrix with respect to the spatial frame $\{\bar{e}_a\}$.

This latter spatial frame will also be SH if and only if the matrix $\boldsymbol{A}$ is SH and hence by (A.3) the matrix $\mathrm{ad}_{\bar{e}}(\vec{N})$ must be as well. This means that the derivation $\mathrm{ad}(\vec{N})$ must also act as a derivation of the finite-dimensional Lie algebra $\mathscr{g}$ of SH spatial vector fields. If $\{\bar{E}_Z\}$ is a basis of the finite-dimensional Lie algebra of vector fields acting as derivations of $\mathscr{g}$, chosen to satisfy $[\bar{e}_0, \bar{E}_Z] = 0$ so that it comoves with $\bar{e}_0$, then the shift may be expanded in this basis with SH coefficients $\vec{N} = \bar{\eta}^Z\bar{E}_Z$. In any spatial coordinates which comove with $\bar{e}_0$, the vector fields $\bar{E}_Z$ are independent of time. Explicit expressions for these vector fields may be found as discussed in appendix B of [Jantzen 1984].

For the diagonal gauge parametrization, one may choose the first three basis vectors to satisfy

$$\mathrm{ad}_{\bar{e}}(\bar{E}_a) = \boldsymbol{\kappa}_a , \tag{A.5}$$

and hence the transformation $\boldsymbol{S}$ to the diagonal gauge spatial frame may be accomplished by setting

$$\vec{N} = \bar{\eta}^a\bar{E}_a \tag{A.6}$$

which identifies (A.3) with the definition (2.18).

In the nonsemisimple case one may choose $\bar{E}_4$ so that $\mathrm{ad}_{\bar{e}}(\bar{E}_4) = \boldsymbol{I}^{(3)}$. Setting $\vec{N} = \bar{\eta}^a\bar{E}_a + \bar{\eta}^4\bar{E}_4$ and $\boldsymbol{A} = \exp(\theta^a\boldsymbol{I}^{(3)})\,\boldsymbol{S}$ leads to

$$\dot{\boldsymbol{A}}\,\boldsymbol{A}^{-1} = \bar{\eta}^a \boldsymbol{K}_a + \bar{\eta}^4 \boldsymbol{I}^{(3)} ,$$

$$(\bar{\eta}^a) = ((\exp \theta^4 \boldsymbol{I}^{(3)})^a{}_b \tilde{\nu}^b) = (e^{\theta^4}\tilde{\nu}^1, e^{\theta^4}\tilde{\nu}^2, \tilde{\nu}^3) , \tag{A.7}$$

$$\bar{\eta}^4 = \dot{\theta}^4 .$$

For the choice of variables (w, x, y) one may write the new diagonal metric coefficients as in (2.24) in the form

$$e^{\bar{\beta}} = \lambda^{-1} w^3 (w^{-3} x^{1/4} e^{-\theta^4}, w^{-3} x^{1/4} y\, e^{-\theta^4}, x^{-1/2} y^{-1}) , \tag{A.8}$$

where the overall conformal factor is just $\lambda^{-1} w^3 = -N_M/12$.

By choosing $e^{\theta^4} = w^{-3}$, one makes this an overall conformal factor of the space–time metric, with the remaining part depending only on the reduced variables (x, y, r_A)

$${}^4 g = (N_M)^2 (-d\Omega \otimes d\Omega + (N_M^{-2} e^{2\bar{\beta}})_{ab} (\bar{\omega}^a + \bar{N}^a d\Omega) \otimes (\bar{\omega}^b + \bar{N}^b d\Omega)) , \tag{A.9}$$

since modulo constants $\bar{\eta}^a \sim r_a$ and $\bar{\eta}^4 = -3(\ln w)^{\cdot}$ and the latter depends only on the reduced variables by the equation of motion for w.

A similar argument holds for the case where additional automorphisms are present. One can always choose a spatial gauge in which the scale invariant part of the space–time metric is entirely specified by the minimal set of reduced variables. Therefore it is not necessary to integrate the equations determining the transformation from this special gauge back to the orthogonal gauge.

References

Arnowitt, R., S. Deser and C.W. Misner, 1962, The dynamics of general relativity, in: Gravitation: an Introduction to Current Research, ed. L. Witten (Wiley, New York).

Bao, D., J. Marsden and R. Walton, 1985, Commun. Math. Phys. 99, 319.

Barrow, J.D., R. Juszkiewicz and D.H. Sonoda, 1985, Mon. Not. Not. R. Astron. Soc. 213, 917.

Belinsky, V.A., I.M. Khalatnikov and E.M. Lifshitz, 1970, Adv. Phys. 19, 225.

Belinsky, V.A., I.M. Khalatnikov and E.M. Lifshitz, 1983, Adv. Phys. 31, 639.

Bogoyavlensky, O.I., 1976a, Sov. Phys. – JETP 43, 187.

Bogoyavlensky, O.I., 1976b, Trudy Sem. Petrovsk. 2, 67 [English translation in: Amer. Math. Soc. Transl. 125 (1985) 83].

Bogoyavlensky, O.I. and S.P. Novikov, 1973, Sov. Phys. – JETP 37, 747.

Bogoyavlensky, O.I. and S.P. Novikov, 1975, Trudy Sem. Petrovsk. 1, 7 [English translation in: Sel. Math. Sov. 2 (1982) 2].

Collins, C.B., 1971, Commun. Math. Phys. 23, 137.

Collins, C.B. and G.F.R. Ellis, 1979, Phys. Rep. 56, 65.

Collins, C.B. and S.W. Hawking, 1973, Astron. J. 180, 317.

Defrise-Carter, L., 1975, Commun. Math. Phys. 40, 273.

DeWitt, B.S., 1967, Phys. Rev. 160, 1113.

Ellis, G.F.R. and A.R. King, 1974, Commun. Math. Phys. 38, 119.

Ellis, G.F.R. and M.A.H. MacCallum, 1969, Commun. Math. Phys. 12, 108.

Estabrook, F.B., H.D. Wahlquist and C.G. Behr, 1968, J. Math. Phys. 9, 497.

Hawking, S.W., 1969, Mon. Not. R. Astr. Soc. 142, 129.

Hsu, L. and J. Wainwright, 1986, Classical and Quantum Gravity 3, 1105.

Jacobs, K.C., 1969, Astrophys. J. 153, 661.

Jantzen, R.T., 1979, Commun. Math. Phys. 64, 211.

Jantzen, R.T., 1980, Ann. Phys. (USA) 127, 302.

Jantzen, R.T., 1982, J. Math. Phys. 23, 1137.
Jantzen, R.T., 1983, Ann. Phys. (USA) 145, 378.
Jantzen, R.T., 1984, Spatially homogeneous dynamics: a unified picture, in: Cosmology of the Early Universe, eds R. Ruffini and L.Z. Fang (World Scientific, Singapore) [Corrected version, in: Proc. Int. Sch. Phys. Enrico Fermi Course LXXXVI on Gamow Cosmology, eds F. Melchiorri and R. Ruffini (North-Holland, Amsterdam, 1986)].
Jantzen, R.T. and K. Rosquist, 1986, Classical and Quantum Gravity 3, 281.
Kasner, E., 1921, Am. J. Math. 43, 217.
King, A.R. and G.F.R. Ellis, 1973, Commun. Math. Phys. 31, 209.
Landau, L.D. and E.M. Lifshitz, 1971, The Classical Theory of Fields (Addison-Wesley, Reading, Mass.).
MacCallum, M.A.H., 1979, in: General Relativity: an Einstein Centenary Survey, eds S.W. Hawking and W. Israel (Cambridge University Press, Cambridge).
Matzner, R., T. Rothman and G.F.R. Ellis, 1986, Phys. Rev. D 34, 2926.
Misner, C.W., 1969, Phys. Rev. 186, 1319.
Misner, C.W., K.S. Thorne and J.A. Wheeler, 1973, Gravitation (Freeman, San Francisco) ch. 21.1, Box 30.1.
Moncrief, V., 1977, Phys. Rev. D 16, 1702.
Nemytsky, V.V. and V.V. Stepanov, 1960, The Qualitative Theory of Differential Equations (Princeton University Press, Princeton).
Novikov, S.P., 1972, Sov. Phys. – JETP 35, 1031.
Ozsváth, I., 1970, J. Math. Phys. 11, 2860.
Ozsváth, I., 1971, J. Math. Phys. 12, 1078.
Peresetsky, A.A., 1977, Russ. Math. Notes 21, 39.
Peresetsky, A.A., 1985, in: Topics in Modern Mathematics, Petrovskii Seminar No. 5, ed. O.A. Oleinik (Consultants Bureau, New York).
Rosquist, K., 1983, Phys. Lett. A 97, 145 [Erratum: 100 (1984) 516].
Rosquist, K., 1984, Classical and Quantum Gravity 1, 81.
Rosquist, K. and R.T. Jantzen, 1985, Phys. Lett. A 107, 29.
Rosquist, K. and R.T. Jantzen, 1986, Transitively self-similar space–times, in: Proc. 4th Marcel Grossmann Meeting, ed. R. Ruffini (North-Holland, Amsterdam).
Shikin, I.S., 1973, Sov. Phys. – JETP 36, 811.
Shikin, I.S., 1976, Sov. Phys. – JETP 41, 794.
Siklos, S.T.C., 1978, Commun. Math. Phys. 58, 255.
Taub, A.H., 1951, Ann. of Math. 53, 472.
Taub, A.H., 1969, Proc. 1967 Colloque on "Fluides et champ gravitationnel en relativité générale" No. 170 57 Paris: Centre National de la Recherche Scientifique.
Taub, A.H. and M.A.H. MacCallum, 1972, Commun. Math. Phys. 25, 173.
Uggla, C. and K. Rosquist, 1988, Asymptotic Cosmological Solutions: Orthogonal Bianchi Type I, III, IV, VI and VII Models, Classical and Quantum Gravity, in press.
Wainwright, J., 1984, Gen. Relativ. Gravitation 16, 657.
York Jr, J.W., 1979, in: Sources of Gravitational Radiation, ed. L. Smarr (Cambridge University Press, Cambridge).