EXACT POWER LAW SOLUTIONS OF THE EINSTEIN EQUATIONS ☆

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An exact power law metric is discussed which arises when one considers the reduced Einstein equations for certain scale invariant variables associated with a spatially homogeneous or spatially self-similar vacuum or nonvacuum space–time. The metric contains a number of new solutions as well as many known ones.

Recently Rosquist [1] has reformulated the Bogoyavlensky–Novikov qualitative treatment of the Einstein equations for an orthogonal spatially homogeneous perfect fluid [2,3] in a way which extends to the nonorthogonal (tilted) case while remaining compatible with the symmetry transformations of those equations. Although discussed explicitly for Bianchi type VI, the general spatially homogeneous or spatially self-similar case is easily considered [4].

In this treatment, hamiltonian methods in conjunction with the scale invariance of the Einstein equations are used to obtain a reduced system of equations involving only the geometrical variables. Singular points in the interior of the physical domain of this system of first-order differential equations lead to a class of exact power law solutions of the Einstein equations in the same sense as defined by Wainwright for the orthogonal case [5,6]. The form of the metric for this class of "singular point solutions" is determined completely as a function of the variable values which characterize the singular point.

If t is the natural cosmological time for these space–times (proper time along the normal congruence in the spatially homogeneous case and the proper time of the conformally related spatially homogeneous space–time in the spatially self-similar case), then the components of the Riemann, Ricci and Einstein tensors in the natural orthonormal frame (the conformally related spatially homogeneous components in the spatially self-similar case) are polynomials in the constant parameters specifying the metric multiplied by a common factor of t^{-2}. Finding the singular points using the original variables is quite difficult, but one can take the alternative approach of imposing the algebraic conditions on the metric parameters which are necessary to make the Einstein tensor assume the form of the energy–momentum tensor of a perfect fluid. This was in fact done by Dunn and Tupper [7] for a very special type VI_0 orthogonal case (the Taub-like diagonal case, whose solution is the M-asymptote solution of Ellis and MacCallum [8]); they also considered how the metric parameters could be varied to accommodate an additional electromagnetic source. Similar considerations apply to the present case as well.

We consider here the nonsemisimple spatially homogeneous case of Bianchi types I–VII and their spatially self-

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similar generalizations, excluding the exceptional case $^*_1$III in the notation of Eardley [9]. The form of the metric for the singular point solutions of the Rosquist equations is most easily given by specifying the natural orthonormal frame one-forms introduced by Jantzen [10,11]

$$(\omega^{\hat{\alpha}}) = (\mathrm{d}t, \omega''^a)\mathrm{e}^{bx^3}\,, \quad \omega''^a = \mathrm{e}^{\beta^a}\,\omega'^a\,, \quad \omega'^a = S^a{}_b\,\omega^b\,, \tag{1}$$

where S and $\{\omega^a\}$ are explicitly parametrized by the four structure constants $(n^{(1)}, n^{(2)}, n^{(3)}, a)$, $\{\omega^a\}$ being the corresponding left invariant 1-forms on the symmetry group expressed in canonical coordinates $\{x^a\}$ of the second kind. For the case under consideration one may set $n^{(3)} = 0$; the natural orthonormal 1-forms are then explicitly

$$\mathrm{e}^{-bx^3}\begin{pmatrix}\omega^{\hat{0}}\\ \omega^{\hat{1}}\\ \omega^{\hat{2}}\\ \omega^{\hat{3}}\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & t^s\mathrm{e}^{-ax^3}C & -n^{(1)}t^s\mathrm{e}^{-ax^3}S & Nkt\\ 0 & n^{(2)}v^{-1}t^s\mathrm{e}^{-ax^3}S & v^{-1}t^s\mathrm{e}^{-ax^3}C & Mkt\\ 0 & 0 & 0 & kt\end{pmatrix}\begin{pmatrix}\mathrm{d}t\\ \mathrm{d}x^1\\ \mathrm{d}x^2\\ \mathrm{d}x^3\end{pmatrix}, \tag{2}$$

where

$$C = \cos\{\tilde{m}^{(3)}[-x^3 + q\ln(t)]\}\,, \qquad S = (\tilde{m}^{(3)})^{-1}\sin\{\tilde{m}^{(3)}[-x^3 + q\ln(t)]\}\,,$$

$$\tilde{m}^{(3)} = (n^{(1)}n^{(2)})^{1/2}, \quad q = \hat{q}\,\mathrm{e}^{-\alpha^3}, \quad \mathrm{e}^{\alpha^3} = |(n^{(1)})^2 + (n^{(2)})^2|^{1/2}/\sqrt{2}\,,$$

and the limit $\tilde{m}^{(3)} \to 0$ is taken with $\hat{q}$ held fixed. Here b is the parameter characterizing the spatially self-similar models. The other constants (s, k, v, M, N, q) which appear in (2) are to be determined by the field equations. The components of the metric which are off-diagonal with respect to the invariant frame are determined by M, N and q while k and v are related to the anisotropy of the diagonalized metric. It can be shown [4] that the positivity of energy leads to the restriction $0 \leqslant s \leqslant 1$. The extreme value $s = 1$ gives only flat space.

In the coordinates (t, x^1, x^2, x^3), the time coordinate lines are orthogonal to the surfaces of constant time, corresponding to a vanishing shift vector field. The metric can also be expressed in comoving ADM coordinates with nonzero shift and nonorthogonal time lines, in terms of which the time dependence of the metric is explicitly of the power law form. By introducing new spatial coordinates related to the old spatial coordinates x^1 and x^2 by a time dependent transformation, the time dependence of the spatial metric may be successively simplified. First the factors involving q can be eliminated from the spatial metric by absorbing them into the definition of the new spatial coordinates. Then the M and N terms can be absorbed in this way and finally the factors of t^s as well, each simplification leading to the appearance of shift terms with power law time dependence. The time dependence of the spatial metric is then exactly that of the (0, 0, 1) locally rotationally symmetric Kasner solution, also called the Taub asymptote. However, by scaling the first two spatial coordinates by a factor of t which makes the time dependence isotropic (in the sense that the coordinate components of the spatial metric depend on the time only through the conformal factor t^2), the shift terms in the expressions for the frame vectors become time independent. This form of the metric might be called the "quasi-isotropic" form, a terminology introduced by Lifshitz and Khalatnikov [12]. The corresponding expressions for the frame vectors are

$$\mathrm{e}^{-bx^3}\begin{pmatrix}\omega^{\hat{0}}\\ \omega^{\hat{1}}\\ \omega^{\hat{2}}\\ \omega^{\hat{3}}\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & t & 0 & 0\\ 0 & 0 & v^{-1}t & 0\\ 0 & 0 & 0 & kt\end{pmatrix}\begin{pmatrix}\mathrm{d}t\\ \omega^{1'}\\ \omega^{2'}\\ \mathrm{d}x^3\end{pmatrix} + \begin{pmatrix}0\\ \mathcal{N}^1\\ \mathcal{N}^2\\ 0\end{pmatrix}\mathrm{d}t\,, \tag{3}$$

$$s=(2-\gamma)/2\gamma,\quad q_{\pm}=(1-3s)(36s-17)^{-1}\ \{2s\pm[(1-2s)(17-2s)]^{1/2}\},$$
$$k^2=-(3s+3q-1)\{[3s^2+(6q-1)s-q^2-q]\ (s+3q-1)\}^{-1},\quad \bar{M}^2=-32q^2s(3s+3q-1)^{-1}(s-q-1)^{-2},$$
$$\gamma\in(1.0411,10/9):q=q_-,\quad \gamma\in(10/9,1.7169):q=q_+. \tag{9}$$

This family of solutions belongs to a generalization of Wainwright's case 2b to the nonorthogonal case. There also exist similar type VI_h spatially homogeneous solutions. All of these solutions have nonzero fluid vorticity in addition to expansion and shear.

The metric specified by (2) contains as special cases both the Taub and MacCallum asymptotes and arises from the single condition $n^{(3)}=0$. When at most one of the constants $n^{(a)}$ is nonzero, one also has a similar family of power law metrics generalizing the Novikov asymptote and when all three vanish, a family generalizing the Lifshitz–Khalatnikov asymptote and the vacuum Kasner asymptote [5]. These families also contain tilted singular point solutions of the Rosquist equations. The explicit form of the orthonormal one-forms for this case is (disregarding for now the exceptional cases involving logarithms)

$$\omega^{\hat{0}}=e^{bx^3}dt,$$
$$e^{-bx^3}\begin{pmatrix}\omega^{\hat{1}}\\ \omega^{\hat{2}}\\ \omega^{\hat{3}}\end{pmatrix}=\begin{pmatrix}1&-q&N\\0&v_0^{-1}&M\\0&0&1\end{pmatrix}\begin{pmatrix}t^{p_1}&0&0\\0&t^{p_2}&0\\0&0&t^{p_3}\end{pmatrix}\begin{pmatrix}e^{-ax^3}(dx^1+n^{(1)}x^3dx^2)\\ e^{-ax^3}dx^2\\ dx^3\end{pmatrix}. \tag{10}$$

(i) $n^{(1)}\neq 0, a=b=0$: $(p_1,p_2,p_3)=(s,(s+1)/2+j,(s+1)/2-j)$,
(ii) $n^{(1)}=0, |a|+|b|\neq 0$: $(p_1,p_2,p_3)=(s+r/2,s-r/2,1)$,
$a=b=0$: (p_1,p_2,p_3) arbitrary.

The parameters v and v_0 are related by $v=v_0t^r$ where $r=-(1-s)/2-j$ in case (i). For orthogonal models case (i) contains the Novikov asymptote corresponding to Wainwright's case 2a; the field equations then force $j=0$. Case (ii) corresponds to Wainwright's case 2c and contains the Lifshitz–Khalatnikov and Kasner asymptotes when $a=b=0$. This family of metrics can be written in quasi-isotropic form as well. The details of all of these power law metrics will be discussed elsewhere.

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