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LETTER TO THE EDITOR

Harmonic mappings and SU(N) self-dual Yang–Mills fields

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Abstract. The relevance of harmonic mappings to the SU(N) self-dual Yang–Mills field equations is clarified.

Recently an attempt has been made (Xanthopoulos 1981, 1982) to use the theory of harmonic mappings to obtain explicit solutions and solution preserving transformations of the SU(3) self-dual Yang–Mills equations in Yang's R-gauge (Yang 1977, Prasad 1978, Brihaye *et al* 1978). Strictly speaking these equations cannot be formulated as a harmonic mapping between Riemannian manifolds. However it was found (Xanthopoulos 1981) that one can consider an auxiliary system to the Yang–Mills equations by restricting the dependence of the fields to two dimensions. This auxiliary system can be described as a harmonic mapping from the Euclidean space $\mathbb{R}^2$ to the manifold SL(3, $\mathbb{C}$) associated with the space of field variables. Moreover it was shown that the description of the auxiliary system as a harmonic mapping was sufficient for the attempted applications, namely the explicit construction of solutions and of solution preserving transformations of the four-dimensional Yang–Mills equations from the geodesics and the isometries, respectively, of the manifold of fields of the auxiliary system.

In this letter we show that similar considerations can be made for the SU(N) self-dual Yang–Mills equations and prove that the solutions obtained by the method of harmonic mappings are gauge equivalent to the abelian solutions of those field equations.

Adopting the notation of Brihaye *et al* (1978), let the complexified SU(N) Yang–Mills potential one-form in Yang's R-gauge be given by

$$A=-\mathrm{i}D^{-1}(D_{,y}\,\mathrm{d}y+D_{,z}\,\mathrm{d}z)+\mathrm{i}E^{-1}(E_{,\bar y}\,\mathrm{d}\bar y+E_{,\bar z}\,\mathrm{d}\bar z),$$
$$D=LF,\qquad E^{-1}=F\hat L,\qquad \sqrt2\,(y,z)=(x^0-\mathrm{i}x^3,x^2-\mathrm{i}x^1),\tag{1}$$

where F is a diagonal unimodular matrix and $L(\hat L)$ is a lower (upper) triangular matrix with unit diagonal entries. The reality of the gauge fields imposes the constraint $E=(D^\dagger)^{-1}$ on the field variables. By introducing the SL(N, $\mathbb{C}$)-valued field $P=DE^{-1}$, the field equations may be written in the following two simple and equivalent forms

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(Brihaye *et al* 1978):

$$(P^{-1}P_{,y})_{,\bar{y}}+(P^{-1}P_{,z})_{,\bar{z}}=0, \qquad (P_{,\bar{y}}P^{-1})_{,\bar{y}}+(P_{,\bar{z}}P^{-1})_{,z}=0. \tag{2}$$

For real Yang–Mills fields P is the Hermitian matrix $DD^{\dagger}$. Obviously, equations (2) are invariant under multiplication of P from left and right by elements of SL(N, $\mathbb{C}$),

$$P \to P' = VPW^{-1}. \tag{3}$$

The transformation (3) represents a gauge transformation (Yang–Mills fields corresponding to P and P' are gauge equivalent) which preserves the reality of the gauge fields (equivalently the Hermiticity of P) when $V^{\dagger}=W^{-1}$.

In general it does not seem possible to express the equations (2) in terms of a harmonic mapping. However, this can be achieved under the additional condition that two of the coordinates are ignorable, as occurs when P satisfies the conditions $P_{,y}=P_{,\bar{y}}$ and $P_{,z}=P_{,\bar{z}}$. Then the field equations take the form

$$g^{ab}(P^{-1}P_{,a})_{,b}=0, \tag{4}$$

and they can be derived from the Lagrangian density

$$\mathscr{L}=\mathrm{Tr}(g^{ab}P^{-1}P_{,a}P^{-1}P_{,b}) \tag{5}$$

where the components of the flat Euclidean metric g_{ab} in the coordinate system with $y=\bar{y}$, $z=\bar{z}$ are given by $g_{y\bar{y}}=g_{\bar{y}y}=g_{z\bar{z}}=g_{\bar{z}z}=\frac{1}{2}$. Now any solution P of the field equations (4) can be considered as a harmonic mapping from the manifold M with the metric g_{ab} to the manifold M' of the field variables with the metric

$$\gamma=\tfrac{1}{2}\mathrm{Tr}(P^{-1}\,dP\otimes P^{-1}\,\mathrm{d}P). \tag{6}$$

This metric is the natural bi-invariant metric on SL(N, $\mathbb{C}$). It is invariant under independent left and right translations on SL(N, $\mathbb{C}$). Therefore its isometry group is the direct product group SL(N, $\mathbb{C}$)$\otimes$SL(N, $\mathbb{C}$) of the independent left and right translations of SL(N, $\mathbb{C}$) into itself. Obviously, from (3) all these transformations represent gauge transformations of the corresponding Yang–Mills fields.

Another common application of a harmonic mapping description of a system of partial differential equations is in obtaining functionally dependent solutions of the field equations using composite mappings of Riemannian manifolds. More precisely, these solutions are found by obtaining the geodesics of the manifold of field variables with the metric given in (6) and then allowing the affine parameter t of the geodesics to be any solution of Laplace's equation on the base manifold of coordinates

$$g^{ab}t_{;ab}=0. \tag{7}$$

Moreover, by allowing $t=t(y,\bar{y},z,\bar{z})$ to be any solution of the equation

$$t_{,y\bar{y}}+t_{,z\bar{z}}=0, \tag{8}$$

this procedure provides a solution of the Yang–Mills equations (2). In matrix form the general solution of this type is given by

$$P=R\,\mathrm{e}^{st}Q, \tag{9}$$

where R and Q are constant SL(N, C) matrices and s is a constant sl(N, $\mathbb{C}$) matrix. However, because of the gauge freedom (equation (3)), any such solution is equivalent to the solution $P'=\mathrm{e}^{st}$. For real Yang–Mills fields one can assume that t is real and s is Hermitian. Hence P' or s can be diagonalised by constant SU(N, $\mathbb{C}$) matrices. This means that the lower triangular matrix $D = F = e^{1/2st}$, and it is real. Therefore from (1), $A = -\frac{1}{2}is\,dt$ which is in fact gauge equivalent to the zero solution.

The condition $(y = \bar{y},\ z = \bar{z})$ is equivalent to Xanthopoulos's (1981) auxiliary system condition and the metric (6) (specialised on the manifold $SL(3, \mathbb{C})$ and parametrised through P by the field variables F, L and $\hat{L}$) is exactly the metric he obtained for the SU(3) Yang–Mills equations. The isometry group described immediately after equation (6) is the group of transformations he obtained by exponentiating the infinitesimal isometries described by Killing fields and the functionally dependent solutions he obtained are all of the form (9).

In conclusion we have shown that the theory of harmonic mappings—which has been proven to be a very useful tool for generation of exact solutions in many areas of mathematical physics—does not lead to physically distinct or non-trivial solutions of the $SU(N)$ self-dual Yang–Mills field equations.

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