Tensor harmonics on the 3-sphere[a)]

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The scalar, vector, and tensor harmonics on the 3-sphere are developed by its identification with SU(2), enabling familiar angular momentum techniques to be employed. The application to spatially homogeneous cosmology is discussed. In this context the classic work of Lifshitz and the recent approach of Hu are bridged. Finally spinor harmonics are introduced.

1. INTRODUCTION

We consider three representations of the Lie group SU(2), whose underlying manifold is the 3-sphere $\mathbf{S}^3$, by dragging along of the tensor algebra over the group by left translation, inverse right translation and conjugation. This leads to a simple way of dealing with the tensor harmonics on $\mathbf{S}^3$ which we identify with SU(2). Specifically, define $\hat{\mathbf{e}}_a = -i\sigma_a$ and $\hat{\mathbf{e}}_4 = 1_2$, where $\{\sigma_a\}$ are the Pauli matrices and 1_2 the two-dimensional identity matrix, and let $\{\hat{e}_\alpha\}$ be the standard basis of R^4 considered as a vector space.[1] We then identity $z^\alpha \hat{e}_\alpha \in \mathbf{S}^3 = \{y^\alpha \hat{e}_\alpha \in R^4 \mid \delta_{\alpha\beta} y^\alpha y^\beta = 1\}$ with $z^\alpha \hat{\mathbf{e}}_\alpha \in \mathrm{SU}(2)$. (Alternatively one may identify $\mathbf{S}^3$ with the unit quaternions of the quaternion algebra induced on R^4 by the identification of $\{\hat{e}_\alpha\}$ and $\{\hat{\mathbf{e}}_\alpha\}$.) As a reminder of this identification we will use the symbol $\mathbf{G}$ to denote $\mathrm{SU}(2) \sim \mathbf{S}^3$ in what follows.

2. STRUCTURE AND GEOMETRY OF G

Let $\mathfrak{g}$ be the three-dimensional vector space of left invariant vector fields on $\mathbf{G}$ (the Lie algebra of $\mathbf{G}$) and $\tilde{\mathfrak{g}}$ its right invariant counterpart. Introduce also the dual vector spaces $\mathfrak{g}^*$ and $\tilde{\mathfrak{g}}^*$ of respectively left and right invariant 1-forms. "$\sim$" may be interpreted as a map which associates to each left invariant tensor field on $\mathbf{G}$ the right invariant field whose value at the identity coincides with that of the original field at the identity.

Let $\{y_\alpha\}$ be standard Cartesian coordinates on R^4 and $\{\partial_\alpha = \partial/\partial y^\alpha\}$ the coordinate frame. The identity $\hat{\mathbf{e}}_4$ of SU(2) corresponds to the north pole of $\mathbf{S}^3$ and has coordinates $\{0, 0, 0, 1\}$. The vector fields $\mathbf{L}_{\alpha\beta} = y^\alpha \partial_\beta - y^\beta \partial_\alpha$ generate the rotations of R^4 about the origin [the identity representation of SO(4, R)] and restrict naturally to vector fields on the orbit $\mathbf{G}$. Let $2e_a = \mathbf{L}_{4a} - \mathbf{L}_{bc}$ and $2\tilde{e}_a = \mathbf{L}_{4a} + \mathbf{L}_{bc}$ be defined on $\mathbf{G}$, where (a, b, c) is a cyclic permutation of $(1, 2, 3)$. $\{e_a\}$ and $\{\tilde{e}_a\}$ are the canonical bases of $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ which agree with the Cartesian derivatives $\{\frac{1}{2}\partial_a\}$ at the identity. [They correspond to the basis $\{\frac{1}{2}\hat{e}_a\}$ of the matrix Lie algebra of SU(2).] They satisfy

$$[e_a, e_b] = C^c{}_{ab} e_c, \quad [\tilde{e}_a, \tilde{e}_b] = -C^c{}_{ab} \tilde{e}_c, \quad [e_a, \tilde{e}_b] = 0, \tag{2.1}$$

with $C^c{}_{ab} = \epsilon_{cab}$. Each basis is a global (analytic) frame on $\mathbf{G}$, having dual frames $\{\omega^a\}$ and $\{\tilde{\omega}^a\}$, respectively, which are the corresponding dual bases of $\mathfrak{g}^*$ and $\tilde{\mathfrak{g}}^*$. The 1-forms $\frac{1}{2}\omega^a$ and $\frac{1}{2}\tilde{\omega}^a$ result from restriction to $\mathbf{G}$ of the 1-forms on R^4 whose components in the Cartesian frame are the same as those of the vector fields $2e_a$ and $2\tilde{e}_a$ on R^4, respectively.

The Euclidean metric $\delta_{\alpha\beta} dy^\alpha \otimes dy^\beta$ restricts to the following bi-invariant metric on $\mathbf{G}$ which is a constant multiple of the Killing metric and whose six-dimensional Killing Lie algebra is $\mathfrak{g} \oplus \tilde{\mathfrak{g}}$:

$$\begin{aligned} \mathbf{g} &= g_{ab} \omega^a \otimes \omega^b = g_{ab} \tilde{\omega}^a \otimes \tilde{\omega}^b, \\ g_{ab} &= \mathbf{g}(e_a, e_b) = \mathbf{g}(\tilde{e}_a, \tilde{e}_b) = \tfrac{1}{4}\delta_{ab}, \\ \mathbf{g}^{-1} &= g^{ab} e_a \otimes e_b = g^{ab} \tilde{e}_a \otimes \tilde{e}_b, \\ g^{ab} &= \mathbf{g}^{-1}(\omega^a, \omega^b) = \mathbf{g}^{-1}(\tilde{\omega}^a, \tilde{\omega}^b) = 4\delta^{ab}. \end{aligned} \tag{2.2}$$

The volume element of the metric is

$$\begin{aligned} \eta &= g^{1/2} \omega^1 \otimes \omega^2 \otimes \omega^3 = g^{1/2} \tilde{\omega}^1 \otimes \tilde{\omega}^2 \otimes \tilde{\omega}^3 \\ &= (3!)^{-1} \eta_{abc} \omega^a \otimes \omega^b \otimes \omega^c, \end{aligned} \tag{2.3}$$

where $g = (4)^{-3}$ is the determinant of the matrix $\mathbf{g}$ whose entries are the components g_{ab}. It is also convenient to have a normalized volume element $\hat{\eta}$ whose integral over $\mathbf{G}$ is unity, obtained from η by dividing out the volume of $\mathbf{G}$. (To each of these volume elements corresponds a bi-invariant Lebesque measure on $\mathbf{G}$.)

Suppose we consider $\binom{p}{q}$-tensor fields on $\mathbf{G}$ with complex valued components in any (analytic) frame which are analytic in the real sense.[2] This vector space $CT^{p,q}(\mathbf{G})$ has a natural Hermitian inner product induced by the metric. If S and T are two such tensor fields with components $S^{a\cdots}{}_{b\cdots}$ and $T^{a\cdots}{}_{b\cdots}$ in the frame $\{e_a\}$, their inner product is

$$\langle S, T \rangle = \int_G \hat{\eta}(S, T), \quad (S, T) = \overline{S^{a\cdots}{}_{b\cdots}} T_{a\cdots}{}^{b\cdots}, \tag{2.4}$$

where the indices are raised and lowered with the metric as usual:

$$T_{a\cdots}{}^{b\cdots} = g_{ac} \cdots g^{bd} \cdots T^{c\cdots}{}_{d\cdots}.$$

Since $(e_a, e_b) = \frac{1}{4}\delta_{ab}$, factors of 4 will often appear. This is so because the natural metric on $\mathbf{G}$ is $4\mathbf{g}$ rather than the induced metric $\mathbf{g}$ and with respect to which $\{e_a\}$ is an orthonormal frame.

Standard formulas[3] may be used to evaluate the components of the metric connection and its Riemann and Ricci tensors in the frame $\{e_a\}$:

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$$\nabla_{e_a} e_b = \Gamma^c{}_{ab} e_c, \quad \Gamma^c{}_{ab} = \tfrac{1}{2} C^c{}_{ab}\,, \qquad (2.5)$$

$$R^{ab}{}_{cd} = \delta^{ab}_{cd}, \quad R^a{}_b = 2\delta^a{}_b.$$

If T is a $\binom{p}{q}$-tensor field, the components of its covariant derivative ∇T in this frame are given by

$$T^{a\cdots}{}_{b\cdots;c} = e_c T^{a\cdots}{}_{b\cdots} + \Gamma^a{}_{cd} T^{d\cdots}{}_{b\cdots} + \cdots$$

$$- \Gamma^d{}_{cb} T^{a\cdots}{}_{d\cdots} - \cdots\,. \qquad (2.6)$$

Replacing $\Gamma^a{}_{bc}$ by $2\Gamma^a{}_{bc} = C^a{}_{bc}$ yields the formula for $(\pounds_{e_c} T)^{a\cdots}{}_{b\cdots}$, where $\pounds_{e_c}$ is the Lie derivative with respect to e_c. These latter components may also be interpreted as the components of the covariant derivative of T with respect to a flat connection ${}^R\nabla$ (with torsion) whose components are ${}^R\Gamma^a{}_{bc} = C^a{}_{bc}$ in the frame $\{e_a\}$ and whose global parallel transport is right translation. $\mathbf{g}$ is covariant constant with respect to this connection. The same formula with $\Gamma^a{}_{bc}$ replaced by 0 holds for $(\pounds_{\tilde{e}_c} T)^{a\cdots}{}_{b\cdots}$ and one may introduce a left connection ${}^L\nabla$ in a similar way.

We use the notation ΔT for the ordinary Laplacian of T:

$$(\Delta T)^{a\cdots}{}_{b\cdots} = - T^{a\cdots}{}_{b\cdots;c}{}^{;c}\,. \qquad (2.7)$$

From the identity

$$T^{c\cdots}{}_{d\cdots;a;b} = (\nabla_{e_b} \nabla_{e_a} T)^{c\cdots}{}_{d\cdots} - T^{c\cdots}{}_{d\cdots;e} \Gamma^e{}_{ba}$$

and the fact that $\Gamma^e{}_{ba} g^{ba} = 0$, it follows that

$$\Delta T = - g^{ab} \nabla_{e_a} \nabla_{e_b} T. \qquad (2.8)$$

Similarly, if Φ is a function,

$$\Delta \Phi = - g^{ab} e_a e_b \Phi. \qquad (2.9)$$

By "index lowering" any $\binom{p}{q}$-tensor field $\hat{T}$ with $r = p + q$ is equivalent to a $\binom{0}{r}$-tensor field T with components $T_{a_1 \cdots a_r}$. Following Lichnerowicz,[5] we define the divergence δT and the DeRham Laplacian $\Delta_{\mathrm{DR}} T$ of T (and hence of $\hat{T}$ by index raising) by the formulas

$$(\delta T)_{a_1 \cdots a_{r-1}} = - T^b{}_{a_1 \cdots a_{r-1};b}$$

$$(\Delta_{\mathrm{DR}} T)_{a_1 \cdots a_r} = (\Delta T)_{a_1 \cdots a_r} + \sum_s T_{a_1 \cdots b \cdots a_r} R^b{}_{a_s}$$

$$- \sum_{s \neq t} T_{a_1 \cdots c \cdots d \cdots a_r} R^c{}_{a_s}{}^d{}_{a_t}. \qquad (2.10)$$

When acting on differential forms this reduces to the usual DeRham Laplacian $\Delta_{\mathrm{DR}} = d\delta + \delta d$. (For a function Φ, $\delta\Phi = 0$, and $\Delta_{\mathrm{DR}} \Phi = \Delta \Phi$.)

3. DRAGGING ACTION

Let $\mathbf{L}_u$, $\mathbf{R}_u$, and $\mathbf{AD}_u = \mathbf{L}_u \circ \mathbf{R}_{u^{-1}}$ denote left translation, right translation, and conjugation by $u \in \mathbf{G}$. (Each of these is an isometry of $\mathbf{g}$ and left and right translations commute.) Use the same symbols to denote the operators which drag along tensor fields by these diffeomorphisms and hence induce linear transformations on the vector spaces $CT^{p,q}(\mathbf{G})$. For example, if Φ, X, σ are a function, vector field and one-form on $\mathbf{G}$ and h a diffeomorphism of $\mathbf{G}$ into itself, the dragged along fields are[6]

$$h\Phi = \Phi \circ h^{-1},$$

$$(hX)(u) = dh(h^{-1}(u)) X(h^{-1}(u)),$$

$$(h\sigma)(u) = dh^{-1}(u) * \sigma(h^{-1}(u)).$$

For higher rank tensor fields these apply to each factor in a tensor product:

$$h(\Phi X \otimes \cdots \otimes \sigma \cdots) = h\Phi\, hX \otimes \cdots \otimes h\sigma \cdots\,.$$

A tensor field T satisfying $hT = T$ is called h-invariant. On a Lie group left (right) invariant tensor fields are invariant under dragging by all left (right) translations and have constant components in a left (right) invariant frame, while bi-invariant tensor fields are invariant under dragging by both left and right translations.

The requirement for a map ρ from a group $\mathbf{G}$ into the group $\mathbf{GL}(V)$ of invertible linear transformations of a vector space V to be a homomorphism and therefore a representation of $\mathbf{G}$ is $\rho_{u_1 u_2} = \rho_{u_1} \circ \rho_{u_2}$, where ρ_u is the value of ρ at u and $\circ$ indicates composition of the linear transformations. This is satisfied by each of the dragging maps $\mathbf{L}$, $\mathbf{R}^{-1}$ and $\mathbf{AD}$ which therefore determine representations of $\mathbf{G}$ on the space $CT^{p,q}(\mathbf{G})$ called, respectively, the left, right, and adjoint $\binom{p}{q}$-tensor dragging representations. [By $\mathbf{R}_u^{-1}$ we mean $(\mathbf{R}_u)^{-1} = \mathbf{R}_{u^{-1}}$. Note that the dragging operator $\mathbf{AD}_u$ equals $\mathbf{L}_u \mathbf{R}_{u^{-1}} = \mathbf{R}_{u^{-1}} \mathbf{L}_u$.] These are unitary representations with respect to the inner product (2.4) since $\hat{\eta}$ is bi-invariant.

Denote the one-parameter group of diffeomorphisms generated by a complete analytic vector field X by $\{X_t \mid t \in R\}$ (the flow of X). The dragging operator X_t, when acting on an analytic tensor field T, has the Lie derivative expansion:

$$X_t T = e^{-t\,\pounds_X} T, \quad \pounds_X T = - (d/dt)\big|_0 X_t T. \qquad (3.1)$$

Tensor fields invariant under the flow of X ($X_t T = T$ for all $t \in R$) have vanishing Lie derivative with respect to X.

From Lie group theory it is well known that for $X \in \mathfrak{g}$:

$$X_t = \mathbf{R}_{\exp(tX)}, \quad \tilde{X}_t = \mathbf{L}_{\exp(tX)},$$

$$(\tilde{X} - X)_t = \mathbf{AD}_{\exp(tX)}, \qquad (3.2)$$

where the exponential map $\exp: \mathfrak{g} \to \mathbf{G}$ may be defined by $\exp(X) = X_1(u_0) = \tilde{X}_1(u_0)$. u_0 denotes the identity of $\mathbf{G}$. The corresponding dragging operators therefore have the expansions:

$$\mathbf{R}_{\exp(tX)} = \exp(-t\pounds_X), \quad \mathbf{L}_{\exp(tX)} = \exp(-t\pounds_{\tilde{X}}),$$

$$\mathbf{AD}_{\exp(tX)} = \exp[-t\pounds_{(\tilde{X}-X)}], \qquad (3.3)$$

valid when acting on analytic tensor fields. As a consequence the action of X and $\tilde{X}$ on a function Φ is given by

$$X\Phi = (d/dt)\big|_0 \Phi \circ \mathbf{R}_{\exp(tX)}, \quad \tilde{X}\Phi = (d/dt)\big|_0 \Phi \circ \mathbf{L}_{\exp(tX)}. \qquad (3.4)$$

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$\mathfrak{g}$ is said to generate the right translations and $\tilde{\mathfrak{g}}$ the left translations. The adjoint diffeomorphisms or conjugations are generated by the Lie algebra with basis $\{\mathbf{L}_{bc}=\tilde{e}_a-e_a\}$, where (a,b,c) is a cyclic permutation of $(1,2,3)$. These are the rotations of $\mathbf{G}$ which leave the identity (north pole) fixed. [The adjoint group acting on SU(2) corresponds to the action on $\mathbf{S}^3$ of the SO$(3,R)$ subgroup of SO$(4,R)$ which leaves $\hat{e}_4$ fixed.] The orbits of this action are 2-spheres of constant y^4 which degenerate to points at the poles. Standard spherical coordinates $\{\chi,\theta,\phi\}$ on $\mathbf{G}$ are conveniently adapted to these orbits; $\{\theta,\phi\}$ are standard spherical coordinates on the 2-sphere of radius $\sin\chi$ for which $y^4=\cos\chi$.

The (linear) adjoint group is the subgroup of $\mathbf{GL}(\mathfrak{g})$ induced by the dragging action of the adjoint group on $\mathfrak{g}$:

$$\mathbf{A}d(u)X=\mathbf{AD}_uX,\quad X\in\mathfrak{g}\,.$$

Its matrix representation with respect to the basis $\{e_a\}$ of $\mathfrak{g}$ is SO$(3,R)$ and provides the standard link between SU(2) and SO$(3,R)$:

$$\mathbf{A}d(u)e_a=e_b\mathcal{R}^b{}_a(u).$$

$\mathcal{R}$ is an SO$(3,R)$-valued function on $\mathbf{G}$ which also provides the transformation between the canonical left and right invariant frames:

$$\tilde{e}_a=e_b\mathcal{R}^{-1b}{}_a,\quad \tilde{\omega}^a=\mathcal{R}^a{}_b\omega^b,$$

$$\mathcal{R}^c{}_a\delta_{cd}\mathcal{R}^d{}_b=\delta_{ab},\quad \mathcal{R}^a{}_c\delta^{cd}\mathcal{R}^b{}_d=\delta^{ab}. \tag{3.5}$$

If $\mathbf{K}_a$ is the matrix whose components are $K_a{}^b{}_c=C^b{}_{ac}$ and if $\delta_{ab}n^an^b=1$, then $\mathcal{R}(\exp\theta n^a\mathbf{K}_a)$ is the matrix of the rotation of R^3 about the origin by an angle θ about the direction specified by the unit vector with Cartesian components n^a. This equality is established by means of the following identities for $X\in\mathfrak{g}$:

$$\pounds_Xe_a=[X,e_a]=X^cC^b{}_{ca}e_b\,,$$

$$\mathbf{AD}_{\exp X}e_a=\mathbf{R}_{\exp(-X)}\mathbf{L}_{\exp(X)}e_a=\mathbf{R}_{\exp(-X)}e_a=e^{tX}e_a.$$

Let us introduce the notation $\mathbf{L}_a=ie_a$, $\tilde{\mathbf{L}}_a=-i\tilde{e}_a$, $\mathbf{J}_a=\mathbf{L}_a+\tilde{\mathbf{L}}_a$ and use the same symbols for the corresponding Lie derivatives,[7] i.e., $\mathbf{L}_aT=\pounds_{\mathbf{L}_a}T=i\pounds_{e_a}T$, etc. Then the first two sets mutually commute and all three satisfy the standard angular momentum commutation relations, i.e., $[\mathbf{L}_a,\mathbf{L}_b]=i\epsilon_{abc}\mathbf{L}_c$. Define also $\mathbf{L}^2=\delta^{ab}\mathbf{L}_a\mathbf{L}_b$, $\tilde{\mathbf{L}}^2=\delta^{ab}\tilde{\mathbf{L}}_a\tilde{\mathbf{L}}_b$, $\mathbf{J}^2=\delta^{ab}\mathbf{J}_a\mathbf{J}_b$, each of which commutes with its corresponding set and with each other, and finally introduce the raising and lowering operators for each set, i.e., $\mathbf{L}_\pm=\mathbf{L}_1\pm i\mathbf{L}_2$. Combining the new notation with our previous formulas we may write:

$$\mathbf{R}^{-1}{}_{\exp(\theta n^ae_a)}=\exp(-i\theta n^a\mathbf{L}_a),$$

$$\mathbf{L}_{\exp(\theta n^ae_a)}=\exp(-i\theta n^a\tilde{\mathbf{L}}_a),$$

$$\mathbf{AD}_{\exp(\theta n^ae_a)}=\exp(-i\theta n^a\mathbf{J}_a). \tag{3.6}$$

We will refer to $\tilde{\mathbf{L}}_a$, $\mathbf{L}_a$, and $\mathbf{J}_a$ as left, right, and total angular momentum. They generate the left, right, and adjoint dragging representations. We will refer to $\mathbf{L}^2$, $\tilde{\mathbf{L}}^2$, and $\mathbf{J}^2$ as their squares. [In fact by the remarks following (2.6), $\mathbf{L}_a=i^R\nabla_{e_a}$ and $\tilde{\mathbf{L}}_a=-i^L\nabla_{\tilde{e}_a}$, and $\tilde{\mathbf{L}}^2$ and $\mathbf{L}^2$ are the Laplacians of the torsion geometries $(\mathbf{G},4g,{}^L\nabla)$ and $(\mathbf{G},4g,{}^R\nabla)$.] Since $\Delta\Phi=-g^{ab}e_ae_b\Phi=-g^{ab}\tilde{e}_a\tilde{e}_b\Phi$ if Φ is a function,

$$\Delta\Phi=4\mathbf{L}^2\Phi=4\tilde{\mathbf{L}}^2\Phi, \tag{3.7}$$

so $\mathbf{L}^2$ and $\tilde{\mathbf{L}}^2$ coincide when acting on functions.

4. REPRESENTATION FUNCTIONS AND SCALAR HARMONICS

Consider the irreducible unitary representations $\{D^J\mid J=0,\frac{1}{2},1,\cdots\}$ of SU(2). Each D^J is a linear transformation-valued function $\mathbf{G}$ satisfying

$$D^J(u_1,u_2)=D^J(u_1)D^J(u_2). \tag{4.1}$$

The basis $\{\mathcal{J}^J_a\}$ of the Lie algebra of the representation D^J determined by the "basis" $\{\mathbf{L}_a\}$ of $\mathfrak{g}$ has the same commutation relations and is defined by

$$\begin{aligned}&\mathcal{J}^J_a=i(d/dt)\big|_0D^J(\exp te_a),\\&D^J(\exp\theta n^ae_a)=\exp(-i\theta n^a\mathcal{J}^J_a).\end{aligned} \tag{4.2}$$

Using (3.4), (4.1), and (4.2) it is easy to compute the following derivatives of the function D^J:

$$\begin{aligned}&\mathbf{L}_aD^J=D^J\mathcal{J}^J_a,\quad -\tilde{\mathbf{L}}_aD^J=\mathcal{J}^J_aD^J,\\&\mathbf{L}^2D^J=\tilde{\mathbf{L}}^2D^J=(\mathcal{J}^J)^2D^J.\end{aligned} \tag{4.3}$$

Let $\{|JM\rangle\mid M=-J,\ldots,J\}$ be the standard orthonormal basis of the $(2J+1)$-dimensional space carrying the representation D^J:

$$\begin{aligned}&\langle JM|JN\rangle=\delta_{MN},\quad \epsilon JM=[J(J+1)-M(M+1)]^{1/2},\\&\mathcal{J}^J_3|JM\rangle=M|JM\rangle,\quad \mathcal{J}^J_+|JM\rangle=\epsilon_{JM}|J\,M+1\rangle,\\&(\mathcal{J}^J)^2|JM\rangle=J(J+1)|JM\rangle,\quad \mathcal{J}^J_-|JM\rangle=\epsilon_{JM-1}|J\,M-1\rangle,\\&D^J|JM\rangle=|JN\rangle D^{JN}{}_M.\end{aligned} \tag{4.4}$$

The matrices of D^J and $\mathcal{J}^J_a$ in this basis are defined by

$$D^{JM}{}_N=\langle JM|D^J|JN\rangle,\quad (\mathcal{J}^J_a)^M{}_N=\langle JM|\mathcal{J}^J_a|JN\rangle.$$

For example, $(\mathcal{J}^J_+)^M{}_N=\delta_{M,N+1}\epsilon_{JN}$. Taking components of (4.3), one finds

$$\begin{aligned}&\mathbf{L}^2D^{JM}{}_N=J(J+1)D^{JM}{}_N=\tilde{\mathbf{L}}^2D^{JM}{}_N,\\&\mathbf{L}_3D^{JM}{}_N=ND^{JM}{}_N,\quad -\tilde{\mathbf{L}}_3D^{JM}{}_N=MD^{JM}{}_N,\\&\mathbf{L}_+D^{JM}{}_N=\epsilon_{JN}D^J{}_{N+1},\quad -\tilde{\mathbf{L}}_+D^{JM}{}_N=\epsilon_{JM-1}D^{JM-1}{}_N,\\&\mathbf{L}_-D^{JM}{}_N=\epsilon_{JN-1}D^{JM}{}_{N-1},\quad -\tilde{\mathbf{L}}_-D^{JM}{}_N=\epsilon_{JM}D^{JM+1}{}_N.\end{aligned} \tag{4.5}$$

By defining $|JM,JM'\rangle=Q^{JMM'}=(2J+1)^{1/2}(-1)^M\times D^{J-M}{}_{M'}$, one finds that left angular momentum acts on the index M of $Q^{JMM'}$ exactly as right angular momentum acts on the index M' of both $Q^{JMM'}$ and $D^{JM}{}_{M'}$, namely in the standard fashion.

In other words the weight J component occurs in

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the decomposition of the left scalar dragging representation $2J+1$ times, once for each value of M', and $|JM, JM'\rangle$ is a standard basis for the subspace of a given M'. An analogous statement holds for the right scalar dragging representation. The behavior of $|JM, JM'\rangle$ under left and right dragging is therefore standard:

$$\mathbf{L}_u|JM, JM'\rangle = |JN, JM'\rangle D^{JN}{}_M(u),$$

$$\mathbf{R}_u^{-1}|JM, JM'\rangle = |JM, JN\rangle D^{JN}{}_{M'}(u). \tag{4.6}$$

Furthermore, from the well-known orthogonality and completeness properties of the representation functions[8] if follows that $\{|JM, JM'\rangle\}$ is an orthonormal basis of $CT^{0,0}(\mathbf{G})$:

$$\langle J_1M_1, J_1M_1'|J_2M_2, J_2M_2'\rangle = \langle Q^{J_1M_1M_1'}, Q^{J_2M_2M_2'}\rangle$$

$$= \delta_{J_1J_2}\delta_{M_1M_2}\delta_{M_1'M_2'}. \tag{4.7}$$

The diagonalization of $\mathbf{J}^2, \mathbf{J}_3, \mathbf{L}^2 = \tilde{\mathbf{L}}^2$ leads to a standard basis of the subspaces of the weight l component of the adjoint scalar dragging representation. This new orthonormal basis $\{Q^J{}_{lm}\}$, where J, l, m refer to $\mathbf{L}^2$, $\mathbf{J}^2$, $\mathbf{J}_3$ and $l = 0, \ldots, 2J$, is obtained by familiar angular momentum addition using Clebsch–Gordon coefficients[9]:

$$Q^J{}_{lm} = C_{JJ}(lm; MM')Q^{JMM'}$$

$$\langle Q^{J_1}{}_{l_1m_1}, Q^{J_2}{}_{l_2m_2}\rangle = \delta_{J_1J_2}\delta_{l_1l_2}\delta_{m_1m_2},$$

$$\mathbf{AD}_u Q^J{}_{lm} = Q^J{}_{ln}D^{Jn}{}_m(u). \tag{4.8}$$

This new basis is orthonormal because of the unitary character of the Clebsch–Gordan transformation. Either basis might be called the scalar harmonics on $\mathbf{G}$. When a distinction is required, we will refer to $\{Q^{JMM'}\}$ as the left–right harmonics and $\{Q^J{}_{lm}\}$ as the adjoint harmonics.

By (3.7) the functions $\{Q^{JMM'}\}$ or $\{Q^J{}_{lm}\}$ for fixed J are n_J^2 eigenvectors of the Laplacian Δ with eigenvalue $\Delta^J = 4J(J+1) = n_J^2 - 1$, where $n_J = 2J+1$. The usual approach[11] is to obtain such eigenvectors by separation of variables in standard spherical coordinates $\{\chi, \theta, \phi\}$ on $\mathbf{G}$, in terms of which the expression for the Laplacian is well known:

$$\Delta = -\sin^{-2}\chi\,\partial/\partial\chi(\sin^2\chi\,\partial/\partial\chi) + \sin^{-2}\chi J^2,$$

$$-J^2 = \sin^{-1}\theta\,\partial/\partial\theta(\sin\theta\,\partial/\partial\theta) + \partial^2/\partial\phi^2. \tag{4.9}$$

Here J^2 is the square of the total angular momentum and it is easy to see that $J_3 = -i\partial/\partial\phi$. [The adjoint group acts on the 2-spheres of constant χ exactly as $SO(3, R)$ acts on the 2-spheres of R^3 centered at the origin.] Since $J^2\mathbf{Y}_{lm} = l(l+1)\mathbf{Y}_{lm}$, where $\mathbf{Y}_{lm}(\theta, \phi)$ are the standard spherical harmonics on the 2-sphere, separation of variables with $\mathcal{Q}^n{}_{lm}(\chi, \theta, \phi) = \Pi_{nl}(\chi)\mathbf{Y}_{lm}(\theta, \phi)$ yields the following χ equation and solution[12]:

$$\{\sin^{-2}\chi\,\partial/\partial\chi(\sin^2\chi\,\partial/\partial\chi) + n^2 - 1$$

$$-\sin^{-2}\chi\, l(l+1)\}\Pi_{nl}(\chi) = 0,$$

$$\Pi_{nl}(\chi) = \sin^l\chi C^{l+1}_{n-(l+1)}(\cos\chi),$$

where $C^l{}_n$ is the Gegenbauer polynomial. $\mathcal{Q}^{nJ}_{lm}$ equals Q^J_{lm} up to a constant factor.

5. VECTOR AND TENSOR HARMONICS

Suppose we denote by $|JM, J'M', s, \binom{p}{q}\rangle$ $\binom{p}{q}$-tensor fields of integral spin s (to be explained shortly) which are simultaneous eigenvectors of $\tilde{\mathbf{L}}^2, \tilde{\mathbf{L}}_3, \mathbf{L}^2, \mathbf{L}_3$ with eigenvalues specified by J, M, J', M', respectively. Simultaneous eigenvectors of $\mathbf{J}^2, \mathbf{J}_3, \tilde{\mathbf{L}}^2, \mathbf{L}^2$ are obtained from these by simple angular momentum addition:

$$\left|lm, JJ', s\binom{p}{q}\right\rangle_{\mathrm{AD}} = C_{JJ'}(lm, MM')\left|JM, J'M', s, \binom{p}{q}\right\rangle \tag{5.1}$$

We shall define the first basis (left–right harmonics) to be a standard orthonormal[13] basis for the left and right $\binom{p}{q}$-tensor dragging representations. The second basis (adjoint harmonics) will then be a standard orthonormal basis for the adjoint $\binom{p}{q}$-tensor dragging representation. It is sufficient, however, to consider only $\binom{0}{p+q}$-tensor fields because of the natural correspondence determined by the bi-invariant metric. (Lie derivatives with respect to $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ commute with "index raising and lowering" since $\mathbf{g}$ is bi-invariant: $\mathbf{L}_a\mathbf{g} = \tilde{\mathbf{L}}_a\mathbf{g} = 0$.) Furthermore, we are only interested in the 1-form case ("vector harmonics") and in the symmetric $\binom{0}{2}$-tensor field case ("tensor harmonics"). We will use the alternative notation $\mathbf{X}^{JM, J'M'}$ and $\mathbf{X}^{JJ'}_{lm}$ for the vector harmonics ($s = 1$) and $\mathbf{T}^{JM, J'M'}_s$ and $\mathbf{T}^{JJ's}_{lm}$ with $s = 0, 2$ for the tensor harmonics. The antisymmetric $\binom{0}{2}$-tensor harmonics ($s = 1$) may be obtained from the vector harmonics by the Hodge star duality operation (which commutes with the DeRham Laplacian). Let $\hat{\sigma}$ denote the vector field associated with the one-form σ (obtained by contracting σ with the contravariant metric $\mathbf{g}^{-1}$ or "raising its index"); for example, $\hat{\omega}^a = 4e_a$. We will have occasion to refer to the vector field harmonics $\hat{\mathbf{X}}^{JM, J'M'}$.

The notion of spin arises in decomposing the adjoint dragging representation in the subspace of either left or right invariant $\binom{0}{r}$-tensor fields. The $\binom{0}{r}$-tensor harmonics are then obtained by coupling the scalar harmonics to spin eigenvectors. The $\binom{0}{r}$-tensor field spin eigenvectors are themselves obtained by decomposing the tensor products of the one-form eigenvectors. Suppose for example we choose the left invariant fields and introduce the spherical basis $\{\omega^A\}$ corresponding to the "Cartesian" basis $\{\omega^a\}$ of $\mathfrak{g}^*$:

$$\omega^{\pm 1} = \mp 2^{-1/2}(\omega^1 \pm i\omega^2), \quad \omega^0 = \omega^3, \quad (\omega^A, \omega^B) = 4\delta_{AB}. \tag{5.2}$$

From the formula $\pounds_{e_a}\omega^b = -C^b{}_{ac}\omega^c$ and from the fact that left angular momentum annihilates ω^b so that $\mathbf{J}_a\omega^A = \mathbf{L}_a\omega^A$, it follows that $\{\omega^A\}$ is a standard spin one basis:

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$$\mathbf{J}^2\omega^A = 2\omega^A, \quad \mathbf{J}_+\omega^A = \epsilon_{1A}\omega^{A+1},$$
$$\mathbf{J}_3\omega^A = A\omega^A, \quad \mathbf{J}_-\omega^A = \epsilon_{1A-1}\omega^{A-1}. \tag{5.3}$$

(The same is true of $\{\tilde{\omega}^A\}$ for which $\mathbf{J}_a\tilde{\omega}^A = \tilde{\mathbf{L}}_a\tilde{\omega}^A$.) We therefore define $\mathbf{X}^{00,1A} = \omega^A$ and obtain all the vector harmonics by right angular momentum coupling of the scalar harmonics to the left invariant spin one basis $\{\omega^A\}$:

$$\mathbf{X}^{JM,J'M'} = C_{J1}(J'M', NA)\,|JM, JN\rangle\,\omega^A,$$
$$J' = J+1, J, J-1. \tag{5.4}$$

From the 1-form basis $\{\omega^A\}$ we may construct the standard second rank spherical basis in the usual fashion yielding spin components with $s = 0, 1, 2$:

$$\omega^{sm} = C_{11}(sm, AB)\,\omega^A \otimes \omega^B,$$
$$\mathbf{J}^2\omega^{sm} = s(s+1)\omega^{sm}, \quad J_3\omega^{sm} = m\omega^{sm},$$
$$(\omega^{sm}, \omega^{s'm'}) = 4^2\delta_{ss'}\,\delta_{mm'}. \tag{5.5}$$

[By successive coupling one may obtain standard left invariant spin bases for $\binom{0}{r}$-tensor fields with $r > 2$.] The explicit expressions are

$$\omega^{22} = \omega^{+1}\otimes\omega^{+1},$$
$$\omega^{21} = 2^{-1/2}(\omega^{+1}\otimes\omega^0 + \omega^0\otimes\omega^{+1}),$$
$$\omega^{20} = 6^{-1/2}(\omega^{+1}\otimes\omega^{-1} + 2\omega^0\otimes\omega^0 + \omega^{-1}\otimes\omega^{+1}),$$
$$\omega^{2-1} = 2^{-1/2}(\omega^{-1}\otimes\omega^0 + \omega^0\otimes\omega^{-1}),$$
$$\omega^{2-2} = \omega^{-1}\otimes\omega^{-1},$$
$$\omega^{11} = 2^{-1/2}(\omega^{+1}\otimes\omega^0 - \omega^0\otimes\omega^{+1}),$$
$$\omega^{10} = 2^{-1/2}(\omega^{+1}\otimes\omega^{-1} - \omega^{-1}\otimes\omega^{+1}),$$
$$\omega^{1-1} = 2^{-1/2}(\omega^0\otimes\omega^{-1} - \omega^{-1}\otimes\omega^0),$$
$$\omega^{00} = 3^{-1/2}(\omega^{+1}\otimes\omega^{-1} - \omega^0\otimes\omega^0 + \omega^{-1}\otimes\omega^{+1}),$$
$$= \mathbf{T}_0{}^{00,00} = -4(3)^{-1/2}g. \tag{5.6}$$

They correspond to the symmetric traceless ($s = 2$), antisymmetric ($s = 1$) and pure trace ($s = 0$) parts of a second rank covariant tensor field.

By right angular momentum coupling we therefore obtain the tensor harmonics:

$$\mathbf{T}_s^{JM,J'M'} = C_{Js}(J'M', Nm)\,|JM, JN\rangle\,\omega^{sm},$$
$$J' = J+s, \ldots, J-s. \tag{5.7}$$

For $s = 0$ this is trivial:

$$\mathbf{T}_0{}^{JM,J'M'} = Q^{JMM'}\omega^{00} = -4(3)^{-1/2}Q^{JMM'}g. \tag{5.8}$$

Both $\{\mathbf{X}^{JM,J'M'}\}$ and $\{\mathbf{T}_s{}^{JM,J'M'}\}$ are orthonormal apart from factors of 4:

$$\langle\mathbf{X}^{J_1M_1,J'_1M'_1}, \mathbf{X}^{J_2M_2,J'_2M'_2}\rangle = 4\delta_{J_1J_2}\delta_{M_1M_2}\delta_{J'_1J'_2}\delta_{M'_1M'_2}$$
$$\langle\mathbf{T}_{s_1}{}^{J_1M_1,J'_1M'_1}, \mathbf{T}_{s_2}{}^{J_2M_2,J'_2M'_2}\rangle = 4^2\delta_{J_1J_2}\delta_{M_1M_2}\delta_{J'_1J'_2}\delta_{M'_1M'_2}$$
$$\times\,\delta_{s_1s_2}. \tag{5.9}$$

However, we could have chosen right invariant spin eigenvectors and left angular momentum coupling. The two approaches are connected by the following result:

$$\mathbf{X}^{1A,00} = C_{11}(00, BC)\,|1A, 1B\rangle\,\omega^C = -\tilde{\omega}^A. \tag{5.10}$$

This is not surprising since $D^{1A}{}_B$ are the components of $\mathcal{R}$ in a spherical basis; the manipulation of (3.5) establishes this result. The minus sign appears since an additional minus sign is required in defining left angular momentum relative to right angular momentum. Using the properties of the Clebsch–Gordan coefficients and the representation functions,[10] one may show that the harmonics constructed by left angular momentum coupling of the scalar harmonics to the right invariant spherical spin bases built from $\{-\tilde{\omega}^a\}$ coincide exactly with the harmonics already defined:

$$\mathbf{X}^{JM,J'M'} = C_{J'1}(JM, NA)\,|J'N, J'M'\rangle\,(-\tilde{\omega}^A),$$
$$\mathbf{T}_s{}^{JM,J'M'} = C_{J's}(JM, Nm)\,|J'N, J'M'\rangle\,\tilde{\omega}^{sm}. \tag{5.11}$$

The vector and tensor harmonics are eigenvectors of the ordinary and DeRham Laplacians. To evaluate the eigenvalues, it is convenient to work in the left invariant frame $\{e_a\}$ and decompose right angular momentum into orbital and spin parts:

$$\mathbf{L}_a = \mathbf{L}_a^{\text{orb}} + \mathbf{S}_a$$
$$(\mathbf{L}_a^{\text{orb}}T)^{b\cdots}{}_{c\cdots} = L_a T^{b\cdots}{}_{c\cdots}$$
$$(\mathbf{S}_a\mathbf{T})^{b\cdots}{}_{c\cdots} = iC^b{}_{ad}T^{d\cdots}{}_{c\cdots} + \cdots - iT^{b\cdots}{}_{d\cdots}C^d{}_{ac} - \cdots. \tag{5.12}$$

Here $\mathbf{T}^{b\cdots}{}_{c\cdots}$ are the components of a $\binom{p}{q}$-tensor field $\mathbf{T}$ in this frame. The square of the right orbital angular momentum is just $\tilde{\mathbf{L}}^2$:

$$(\tilde{\mathbf{L}}T)^{a\cdots}{}_{b\cdots} = \tilde{\mathbf{L}}^2\mathbf{T}^{a\cdots}{}_{b\cdots} = \mathbf{L}^2\mathbf{T}^{a\cdots}{}_{b\cdots}$$
$$= ((\mathbf{L}^{\text{orb}})^2\mathbf{T})^{a\cdots}{}_{b\cdots}, \tag{5.13}$$

while the spin–orbit operator is

$$\mathbf{L}^2 = (\mathbf{L}^{\text{orb}})^2 + \mathbf{S}^2 + 2\mathbf{S}_a\mathbf{L}_a^{\text{orb}}$$
$$2\mathbf{S}_a\mathbf{L}_a^{\text{orb}} = \mathbf{L}^2 - \tilde{\mathbf{L}}^2 - \mathbf{S}^2. \tag{5.14}$$

From (2.5), (2.10), and (5.12) one may verify that the square of the right spin angular momentum is the difference between the ordinary and DeRham Laplacians:

$$\Delta_{\text{DR}} - \Delta = \mathbf{S}^2. \tag{5.15}$$

A similar decomposition $\tilde{\mathbf{L}}_a = \tilde{\mathbf{L}}_a^{\text{orb}} + \tilde{\mathbf{S}}_a$ of left angular momentum using the right invariant frame $\{\tilde{e}_a\}$ leads to a left spin angular momentum which is easily seen to be related to the right spin by $\tilde{\mathbf{S}}_a = \mathbf{S}_b\mathcal{R}^{-1b}{}_a$ which in turn implies $\tilde{\mathbf{S}}^2 = \mathbf{S}^2$.[14] Since $\mathbf{S}_a\mathbf{T}$

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$=\mathbf{L}_a\mathbf{T}=\mathbf{J}_a\mathbf{T}$ if $\mathbf{T}$ is left invariant and $\tilde{\mathbf{S}}_a\mathbf{T}=\tilde{\mathbf{L}}_a\mathbf{T}=\mathbf{J}_a\mathbf{T}$ if $\mathbf{T}$ is right invariant, $\mathbf{S}^2=\tilde{\mathbf{S}}^2$ coincides with $\mathbf{J}^2$ on the invariant tensor fields and is therefore the spin introduced above whose eigenvalues were labeled by s. Functions have spin $s=0$ and vector fields $s=1$ while second rank tensor fields decompose into spins $s=0, 1, 2$.

The covariant derivative and Laplacian may be expressed in terms of these operators using (2.6) and (2.8):

$$\begin{aligned} i\nabla_{e_a} &= \mathbf{L}_a^{\text{orb}}+\tfrac{1}{2}\mathbf{S}_a, \\ \Delta &= -4\delta^{ab}\nabla_{e_a}\nabla_{e_a}=4(\mathbf{L}^{\text{orb}})^2+\mathbf{S}^2+4\mathbf{S}_a\mathbf{L}_a^{\text{orb}} \\ &= 2(\mathbf{L}^2+\tilde{\mathbf{L}}^2)-\mathbf{S}^2, \\ \Delta_{\text{DR}} &= 2(\mathbf{L}^2+\tilde{\mathbf{L}}^2). \end{aligned} \tag{5.16}$$

A spin-s tensor field of left and right angular momentum J and J' respectively is therefore an eigenvector of both Laplacians. Denote the corresponding eigenvalues by $\Delta^{JJ's}$ and $\Delta_{\text{DR}}^{JJ'}$ and define $n_{JJ'}=J+J'+1$, so that $n_{JJ}=n_J$:

$$\begin{aligned} \Delta^{JJ's} &= \Delta_{\text{DR}}^{JJ'}-s(s+1), \\ \Delta_{\text{DR}}^{JJ'} &= 2\{J(J+1)+J'(J'+1)\} \\ &= n_{JJ'}^2-1+(J-J')^2. \end{aligned} \tag{5.17}$$

6. DIFFERENTIAL PROPERTIES

It is worthwhile knowing how the harmonics behave under the operations of taking divergences, exterior derivatives (when appropriate), and symmetrized covariant derivatives. These operations connect the scalar, vector, and tensor harmonics of fixed left and right angular momentum.

Consider the exterior derivative of the scalar harmonics and let $\mathbf{L}_A$ stand for the spherical components of $\mathbf{L}_a$:

$$\begin{aligned} &\mathbf{L}_{\pm 1}=\mp 2^{-1/2}L_\pm, \quad \mathbf{L}_0=\mathbf{L}_3, \\ &id\Phi=\omega^a\mathbf{L}_a\Phi=(-1)^A\omega^{-A}\mathbf{L}_A\Phi. \end{aligned} \tag{6.1}$$

By the definition of the coefficients $C_{J1}(JM, M+A-A)$[15]:

$$\begin{aligned} \mathbf{L}_A Q^{JMM'} &= (-1)^A[J(J+1)]^{1/2} \\ &\quad\times C_{J1}(JM', M'+A-A)Q^{JM,M'+A}, \\ idQ^{JMM'} &= [J(J+1)]^{1/2}C_{J1}(JM', NA)Q^{JMN}\omega^A \\ &= [J(J+1)]^{1/2}\mathbf{X}^{JM,JM'}. \end{aligned} \tag{6.2}$$

This is exactly analogous to a similar situation occurring with the scalar and vector harmonics on R^3.

To compute the exterior derivative of the vector harmonics we use the following formulas:

$$\begin{aligned} (d\mathbf{A})_{bc} &= 2e_{[b}\mathbf{A}_{c]}-\mathbf{A}_d C^d{}_{bc} \\ (*d\mathbf{A})_a &= \tfrac{1}{2}\eta_a{}^{bc}(d\mathbf{A})_{bc}=\epsilon_{abc}(d\mathbf{A})_{bc}=-2i\epsilon_{abc}\mathbf{L}_b\mathbf{A}_c-2\mathbf{A}_a \\ &= ((-2\mathbf{S}_b\mathbf{L}_b^{\text{orb}}-\mathbf{S}^2)\mathbf{A})_a=((\tilde{\mathbf{L}}^2-\mathbf{L}^2)\mathbf{A})_a \\ *d\mathbf{A} &= (\tilde{\mathbf{L}}^2-\mathbf{L}^2)\mathbf{A}. \end{aligned} \tag{6.3}$$

Applying this to the left—right vector harmonics, we obtain

$$\begin{aligned} \delta *\mathbf{X}^{JM,J'M'} &= *d\,\hat{\mathbf{X}}^{JM,J'M'}=[J(J+1)-J'(J'+1)]\mathbf{X}^{JM,J'M} \\ &= (J-J')n_{JJ'}\mathbf{X}^{JM,J'M'}. \end{aligned} \tag{6.4}$$

The same formula holds for the adjoint vector harmonics.

The divergence of $\mathbf{X}^{JM,J'M'}$ is easy to compute using (6.2) and the formula $\Delta\Phi=(\delta d+d\delta)\Phi=\delta\, d\Phi$:

$$\begin{aligned} \delta\mathbf{X}^{JM,J'M'} &= i[J(J+1)]^{-1/2}\delta\, dQ^{JMM'} \\ &= 4i[J(J+1)]^{-1/2}Q^{JMM'}. \end{aligned} \tag{6.5}$$

To evaluate $\delta\mathbf{X}^{JM,J\pm 1M'}$ we need the divergence formula:

$$\delta\mathbf{A}=-g^{ab}e_a\mathbf{A}_b=4i\mathbf{L}_a\mathbf{A}_a=4i(-1)^A\mathbf{L}_A\mathbf{A}_{-A},$$

where $\mathbf{A}_A$ are the spherical components of $\mathbf{A}_a$ defined as in (5.2):

$$\mathbf{A}_a\omega^a=(-1)^A\mathbf{A}_{-A}\omega^A.$$

δ commutes with all three angular momenta (since they generate isometries) so that all harmonics obtained from a divergenceless harmonic by the left and right raising and lowering operators are also divergenceless. The explicit calculation:

$$\mathbf{X}^{JM,J+1\,J+1}=Q^{JMJ}\omega^{+1},$$

$$\delta\mathbf{X}^{JM,J+1\,J+1}=-4i(2)^{-1/2}\mathbf{L}_+Q^{JMJ}=0$$

then establishes $\delta\mathbf{X}^{JM,J+1M'}=0$. The corresponding relation $\delta\mathbf{X}^{JM,J-1,M'}=0$ follows from left—right symmetry [since a similar calculation using (5.11) would show $\delta\mathbf{X}^{J+1M,JM'}=0$]. Thus the vector harmonics are transverse (i.e., divergenceless) for $J\neq J'$ and exact for $J=J'$. This reflects the Hodge decomposition[16] of the space of 1-forms which in our case contains no harmonic elements ($\Delta_{\text{DR}}A=0$).

The divergence of a symmetric second rank covariant tensor field h is given by

$$\begin{aligned} (\delta h)_a &= -g^{bc}e_b h_{ca}=4i\mathbf{L}_b h_{ba} \\ (\delta h)_A &= 4i(-1)^B\mathbf{L}_B h_{-BA}. \end{aligned}$$

Since $\mathbf{T}_2^{JM,J+2\,J+2}=Q^{JMJ}\omega^{+1}\otimes\omega^{+1}$, it follows exactly as in the vector case that $\delta\mathbf{T}_2^{JM,J\pm 2\,M'}=0$, so $\{\mathbf{T}_2^{JM,J\pm 2\,M'}\}$ are transverse tracless harmonics. From (6.2) and the formula $\delta(\Phi\mathbf{g})=-d\Phi$, it follows that

$$\delta\mathbf{T}_0^{JM,JM'}=-4i(3)^{-1/2}[J(J+1)]^{1/2}\mathbf{X}^{JM,JM'}. \tag{6.6}$$

The remaining spin-2 tensor harmonics may be obtained from vector field harmonics by Lie derivation. To simplify notation, let i stand for the four

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indices $JMJ'M'$, δ_{ij} for the product of the four individual Kronecker deltas, n_i for $n_{JJ'}$, and $\Delta^i_{\rm DR}$ for $\Delta^{JJ'}_{\rm DR}$. The vector field harmonics $\hat{\mathbf{X}}^i$ are obtained from the one-form harmonics $\mathbf{X}^i$ by replacing ω^A by $4e_A$ in (5.4). Consider the tensor fields $\mathbf{S}^i = \pounds_{\hat{\mathbf{X}}^i}\mathbf{g}$ with components $2\mathbf{X}^i{}_{(a;b)}$ and let $\mathbf{S}^i_2$ and $\mathbf{S}^i_0$ be the traceless and pure trace parts of $\mathbf{S}^i$. A short calculation using $\mathbf{L}_a\mathbf{g} = 0 = \tilde{\mathbf{L}}_a\mathbf{g}$ shows that

$$\mathbf{L}_a\mathbf{S}^i = \pounds_{\mathbf{L}_a\hat{\mathbf{X}}^i}\mathbf{g}, \quad \tilde{\mathbf{L}}_a\mathbf{S}^i = \pounds_{\mathbf{L}_a\hat{\mathbf{X}}^i}\mathbf{g},$$

so $\mathbf{S}^i$ inherits the angular momentum properties of $\hat{\mathbf{X}}^i$ which in turn are identical with those of $\mathbf{X}^i$. It then follows that $\mathbf{S}^i_0$ and $\mathbf{S}^i_2$ are respectively proportional to $\mathbf{T}^i_0$ and $\mathbf{T}^i_2$. The proportionality factors may be determined up to phase by comparing norms. (Once they are obtained for one i, application of the raising and lowering operators shows that the values are independent of M and M'.)

Since $\mathbf{S}^{ia}{}_a = 2\mathbf{X}^{ia}{}_{;a} = -2\delta\mathbf{X}^i$, $\mathbf{S}^i_0$ vanishes when $J' = J \pm 1$ and so $\mathbf{S}^i$ is itself proportional to $\mathbf{T}^i_2$. When $J' = J$:

$$\begin{aligned}\mathbf{S}^i_0 &= -\tfrac{2}{3}\mathbf{g}\delta\mathbf{X}^i \\ &= (2i/3)[J(J+1)]^{1/2}Q^i\mathbf{g} = 2i(3)^{-1/2}[J(J+1)]^{1/2}\mathbf{T}^i_2.\end{aligned} \tag{6.7}$$

In this case since $d\mathbf{X}^i = 0$:

$$\mathbf{S}^i{}_{ab} = 2\mathbf{X}^i{}_{a;b} = 2i[J(J+1)]^{-1/2}Q^i{}_{;a;b}. \tag{6.8}$$

The divergence of $\mathbf{S}^i$ may be computed with the help of the Ricci identity:

$$\begin{aligned}&\mathbf{X}^i{}_{b;a}{}^{;b} = \mathbf{X}^i{}_{b;}{}^{;b}{}_{;a} + \mathbf{X}^i d\mathbf{R}^d{}_{ba}{}^b = ((2-d\delta)\mathbf{X}^i)_a, \\ &\mathbf{S}^i{}_{ab}{}^{;b} = \mathbf{X}^i{}_{a;b}{}^{;b} + \mathbf{X}^i{}_{b;a}{}^{;b}, \\ &\delta\mathbf{S}^i = (\Delta - 2 + d\delta)\mathbf{X}^i.\end{aligned} \tag{6.9}$$

When $J' = J \pm 1$, $\delta\mathbf{X}^i = 0$ so $\delta\mathbf{S}^i = (\Delta - 2)\mathbf{X}^i = (n_i^2 - 4)\mathbf{X}^i$, while if $J = J'$ then $d\delta\mathbf{X}^i = \Delta_{\rm DR}\mathbf{X}^i$ and $\delta\mathbf{S}^i = 2\Delta\mathbf{X}^i = 2(n_i^2 - 3)\mathbf{X}^i$. Let $\delta\mathbf{S}^i = c^i\mathbf{X}^i$. Then $\{\mathbf{S}^i\}$ are orthogonal

$$\begin{aligned}\langle\mathbf{S}^i, \mathbf{S}^j\rangle &= 2\int^{\mathbf{G}}\hat{\eta}\mathbf{S}^{iab}\mathbf{X}^j{}_{a;b} \\ &= 2\langle\delta\mathbf{S}^i, \mathbf{X}^j\rangle = 2c^i\langle\mathbf{X}^i, \mathbf{X}^j\rangle \\ &= 4^2\delta_{ij}c^i/2.\end{aligned} \tag{6.10}$$

When $J' = J + 1$ it then follows that

$$i\mathbf{S}^i = [(n_i^2 - 4)/2]^{1/2}\mathbf{T}^i_2. \tag{6.11}$$

The phase in (6.11) and (6.12) has been determined by evaluating $\mathbf{S}^{JM,J+1\,J+1}$ and $\mathbf{S}^{JM,JJ}$ explicitly using Clebsch—Gordan coefficient formulas which may be found in Ref. 17. When $J' = J$, knowledge of $\langle\mathbf{S}^i_0, \mathbf{S}^i_0\rangle = 4^2\delta_{ij}[(n_i^2 - 1)/3]^{1/2}$ from (6.7) is sufficient to evaluate $\langle\mathbf{S}^i_2, \mathbf{S}^i_2\rangle$ by orthogonality:

$$\begin{aligned}&\langle\mathbf{S}^i_2, \mathbf{S}^i_2\rangle = \langle\mathbf{S}^i, \mathbf{S}^j\rangle - \langle\mathbf{S}^i_0, \mathbf{S}^i_0\rangle = 4^2\delta_{ij}[2(n_i^2 - 4)/3], \\ &i\mathbf{S}^i_2 = [2(n_i^2 - 1)/3]^{1/2}\mathbf{T}^i_2.\end{aligned} \tag{6.12}$$

The Ricci identity may also be used to evaluate other derivatives. For example

$$\begin{aligned}&\mathbf{S}^i{}_{ac;b}{}^{;c} = \mathbf{S}^i{}_{ac}{}^{;c}{}_{;b} + 3\mathbf{S}^i_{2ab}, \\ &-\mathbf{S}^i{}_{ac;b}{}^{;c} - \mathbf{S}^i{}_{bc;a}{}^{;c} = c^i\mathbf{S}^i{}_{ab} - 6\mathbf{S}^i_{2ab}.\end{aligned} \tag{6.13}$$

Let us adopt the convention that by omitting certain right angular momentum indices on the harmonics we mean the linear span of the harmonics with the remaining indices fixed and the omitted ones assuming all possible values. For example $\mathbf{X}^{JM}$ is the space spanned by $\mathbf{X}^{JM,J'M'}$ with JM fixed. Also by $\mathbf{T}^{JM}$ and $\mathbf{T}^{JM,J'}$ we mean the direct sum of the spaces with $s = 0$ and $s = 2$ only. The Hodge decomposition is then reflected in the orthogonal direct sum:

$$\mathbf{X}^{JM} = \mathbf{X}^{JM,J\pm1} \oplus \mathbf{X}^{JM,J}.$$

$(\mathbf{X}^{JM,J\pm1} = \mathbf{X}^{JM,J+1} \oplus \mathbf{X}^{JM,J-1}$, etc.).

Consider $\mathbf{T}^{JM} = \mathbf{T}_2{}^{JM,J\pm2} \oplus \mathbf{T}_2{}^{JM,J\pm1} \oplus \mathbf{T}^{JM,J}$. The first summand contains the transverse traceless harmonics and the second those which result from Lie derivation of $\mathbf{g}$ by transverse vector field harmonics. The last summand is in an obvious notation $\pounds_{\hat{\mathbf{X}}^{JM,J}}g \oplus Q^{JM}\mathbf{g}$ but the direct sum is not orthogonal. If we decompose $\mathbf{T}^{JM,J}$ into an orthogonal direct sum with $\pounds_{\hat{\mathbf{X}}^{JM,J}}\mathbf{g}$ as the first summand, the second summand must be transverse:

$$\begin{aligned}0 = \langle\mathbf{S}^i, P\rangle &= 2\int^{\mathbf{G}}\hat{\eta}\mathbf{X}^{ia;b}P_{ab} = -2\int^{\mathbf{G}}\hat{\eta}\mathbf{X}^{ia}P_{ab}{}^{jb} \\ &= 2\langle\mathbf{X}^i, \delta P\rangle.\end{aligned}$$

Since $P \in \mathbf{T}^{JM,J}$ implies $\delta P \in \mathbf{X}^{JM,J}$, orthogonality to $\pounds_{\hat{\mathbf{X}}^{JM,J}}\mathbf{g}$ requires that $\delta P = 0$. Given a function Φ, the Ricci identity shows immediately that the following combination of Φ and its second derivatives is divergenceless:

$$(\Phi - \tfrac{1}{2}\Delta\Phi)\mathbf{g} - \tfrac{1}{2}\nabla\nabla\Phi.$$

It is exactly these combinations of the scalar harmonics which span the transverse subspace of $\mathbf{T}^{JM,J}$:

$$\begin{aligned}&p^i = (Q^i - \tfrac{1}{2}\Delta Q^i)\mathbf{g} - \tfrac{1}{2}\nabla\nabla Q^i, \\ &P^{ia}{}_a = (3 - \Delta)Q^i = (4 - n_i^2)Q^i, \\ &\langle P^i, P^j\rangle = 4^2\delta_{ij}[(n_i^2 - 3)(n_i^2 - 4)/32].\end{aligned} \tag{6.14}$$

$\{\mathbf{S}^i, \mathbf{P}^i\}$ is an orthogonal basis of $\mathbf{T}^{JM,J}$ and the following is an orthogonal direct sum reflecting the canonical decomposition of symmetric second rank tensor fields on a Riemannian manifold of constant positive curvature[18]:

$$\mathbf{T}^{JM} = \mathbf{T}_2{}^{JM,J\pm2} \oplus \pounds_{\hat{\mathbf{X}}^{JM}}\mathbf{g} \oplus P^{JM}.$$

7. REALITY PROPERTIES AND PARITY

The reality properties of the harmonics are important if one is interested in real tensor fields. Let $\mathbf{X}^{JM}{}_{NA} = Q^{JMN}\omega^A$ and $\mathbf{T}_s{}^{JM}{}_{Nm} = Q^{JMN}\omega^{sm}$. These are standard orthonormal bases (modulo factors of 4) of the left dragging representations but not the right. The left—right harmonics are related to these left harmonics by the real unitary Clebsch—Gordan transformation which decomposes the right angular

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momentum tensor product representation into irreducible components. The adjoint harmonics are obtained from the latter harmonics by another real Clebsch—Gordan transformation.

The representation functions[10] and therefore our left—right scalar harmonics satisfy

$$\overline{D^{JM}}_{M'} = (-1)^{M-M'} D^{J-M}{}_{-M'}, \quad \overline{Q^{JMM'}} = (-1)^{M+M'} Q^{J-M-N},$$

while the left invariant spherical spin bases satisfy

$$\overline{\omega^A} = (-1)^A \omega^{-A}, \quad \overline{\omega^{sm}} = (-1)^m \omega^{s-m}.$$

Together these imply

$$\overline{X^{JM}}_{NA} = (-1)^{M+N+A} \mathbf{X}^{J-M}{}_{-N-A},$$

$$\overline{\mathbf{T}_s{}^{JM}}_{Nm} = (-1)^{M+N+A} \mathbf{T}_s{}^{J-M}{}_{-N-m}.$$

The sign is always $(-1)^J$[3]. The reality properties of the left—right and adjoint harmonics are therefore

$$\overline{\mathbf{X}^{JM,J'M'}} = (-1)^{M+M'} \mathbf{X}^{J-M,J'-M'},$$

$$\overline{\mathbf{X}^{JJ'}_{lm}} = (-1)^m \mathbf{X}^{JJ'}_{l-m}$$

$$\overline{\mathbf{T}_s{}^{JM,J'M'}} = (-1)^{M+M'} \mathbf{T}_s{}^{J-M,J'-M'},$$

$$\overline{\mathbf{T}^{JJ's}_{lm}} = (-1)^m \mathbf{T}^{JJ's}_{l-m}.$$

One may also introduce the notion of inversion and parity. Let P denote the inverse diffeomorphism and its corresponding dragging operator. P is a discrete isometry of $\mathbf{g}$ satisfying $P^2 = Id$ (the identity diffeomorphism and dragging operator). It inverts $\mathbf{G}$ about the north pole. In Cartesian coordinates restricted to $\mathbf{G}$, $P\{y^a, y^4\} = \{-y^a, y^4\}$ and hence in terms of spherical coordinates, χ remains unchanged while the two-spheres of constant χ behave exactly like the spheres centered at the origin of R^3 under inversion. Q^J_{lm} therefore has the same parity as $\mathbf{Y}_{lm}$.

Let $X \in \mathfrak{g}$ and Φ be a function:

$$(PX)(u)\Phi = X(u^{-1})\Phi \circ P = (d/dt)\big|_0 \Phi \circ P(u^{-1} \exp tX)$$

$$= (d/dt)\big|_0 \Phi((\exp - tX)u) = -\widetilde{X}(u)\Phi.$$

Then it follows that $Pe_\beta = -\widetilde{e}_a$ and $P\omega^a = -\widetilde{\omega}^a$. Since $PD^{JM}{}_{M'} = (-1)^{M-M'} D^{J-M}{}_{-M}$,[10] the scalar harmonics satisfy:

$$PQ^{JMM'} = (-1)^{-2M'} Q^{JM'M} = (-1)^{2J} Q^{JM'M}.$$

With the help of (5.11) the inversion properties of the left—right harmonics are easily deduced:

$$P\mathbf{X}^{JM,J'M'} = (-1)^{2J} C_{J1}(J'M', NA) Q^{JNM'}(-\widetilde{\omega}^A)$$

$$= (-1)^{2J} \mathbf{X}^{J'M'JM},$$

$$P\mathbf{T}_s{}^{JM,J'M'} = (-1)^{2J} \mathbf{T}_s{}^{J'M',JM}.$$

A Clebsch—Gordan coefficient symmetry then yields the inversion properties of the adjoint harmonics:

$$P\mathbf{X}^{JJ'}_{lm} = C_{JJ'}(lm, MM') P\mathbf{X}^{JM,J'M'}$$

$$= (-1)^{2J} C_{JJ'}(lm, MM') \mathbf{X}^{J'M',JM}$$

$$= (-1)^{l+J'-J} C_{JJ'}(lm, M', M) \mathbf{X}^{J'M',JM}$$

$$= (-1)^{l+J'-J} \mathbf{X}^{J'J}_{lm}.$$

The same manipulation shows

$$P\mathbf{T}^{JJ's}_{lm} = (-1)^{l+J'-J} \mathbf{T}^{J'Js}_{lm}, \quad PQ^J{}_{lm} = (-1)^l Q^J_{lm}.$$

For $J < J'$ define the following parity eigenvectors with eigenvalues:

$p(-1)^{l+J'-J}$:

$$\mathbf{X}^{JJ'p}_{lm} = (2)^{-1/2}(\mathbf{X}^{JJ'}_{lm} + p\mathbf{X}^{JJ'}_{lm}),$$

$$\mathbf{T}^{JJ'sp}_{lm} = (2)^{-1/2}(\mathbf{T}^{J'Js}_{lm} + p\mathbf{T}^{J'Js}_{lm}). \tag{7.1}$$

These bases are also orthonormal modulo factors of 4. Also note that (6.4) implies

$$*d\mathbf{X}^{J\,J+1p}_{lm} = -n_{J\,J+1} \mathbf{X}^{J\,J+1-p}_{lm}. \tag{7.2}$$

By stereographic projection from the south pole onto the hyperplane $y^4 = 1$, one may associate a point of R^3 (identified with that hyperplane in the natural way) with each point of $\mathbf{G}$ except the south pole. This is a conformal map. Cartesian or spherical coordinates on R^3 induce coordinates on $\mathbf{G} - \{-\hat{e}_4\}$ which might be called conformal Cartesian and spherical coordinates. The latter coordinates $\{r, \theta, \phi\}$ are related to spherical coordinates by $r = 2\tan(\chi/2)$. P and $\mathbf{J}_a$ coincide exactly with the usual parity and total angular momentum on R^3. The metric is explicitly

$$\mathbf{g} = d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$$= (1 + r^2/4)^{-1}[dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)].$$

Eigenvectors of the vector DeRham Laplacian on $\mathbf{G}$ may be found by separation of variables using the vector spherical harmonics on R^3. This leads to radial equations involving Gegenbauer polynomials.[12] Requiring the eigenvectors to have definite parity produces the adjoint vector harmonics of definite parity. A similar statement holds for the tensor case.[19]

8. WAVE EQUATIONS ON POSITIVE CURVATURE FRIEDMANN SPACE-TIMES

Consider the manifold $\mathbf{M} = R \times \mathbf{G}$ with the following Lorentz metric:

$${}^4\mathbf{g} = a^2(-dt \otimes dt + \mathbf{g}), \tag{8.1}$$

where a is a function of the "time" t. Since we are not here concerned with a specific function a, it suffices to say that a is determined by the Einstein field equations for ${}^4\mathbf{g}$ with a perfect fluid source having the same symmetry group as ${}^4\mathbf{g}$. $(\mathbf{M}, {}^4\mathbf{g})$ is then called a Friedmann space—time.

The vector and tensor harmonics on $\mathbf{S}^3$ were originally developed by Lifshitz in terms of harmonic polynomials on R^4, an approach which does not easily lend itself to explicit calculation. His motivation was the treatment of the perturbations of the Friedmann spacetimes, an elegant and rather classic

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application of the machinery of this paper and to which the reader is referred.[11] More recently Hu and Regge have introduced the idea of exploiting the group properties of $\mathbf{S}^3$ to study perturbations of spatially homogeneous space—times $(\mathbf{M}, {}^4\mathbf{g})$ more general than Friedmann.[20] Their technique essentially amounts to the use of left harmonic expansions; however, the special case of Friedmann was never treated explicitly.

It is also of interest to consider wave equations satisfied by test fields on a Friedmann space—time. A harmonic expansion of these fields then allows the wave equations to be reduced to uncoupled ordinary differential equations for the time dependent expansion coefficients. Let $e_0 = \partial/\partial t$ and $\omega^0 = dt$. Then $\{e_\alpha\}$ is a frame on $\mathbf{M}$ with dual frame $\{\omega^\alpha\}$ and structure functions $C^\alpha{}_{\beta\gamma} = \delta^\alpha_a C^a{}_{bc} \delta^b_\beta \delta^c_\gamma$. The spacetime metric is ${}^4g_{\alpha\beta}\omega^\alpha \otimes \omega^\beta$ with ${}^4g_{0\alpha} = a^2\delta_{0\alpha}$ and ${}^4g_{ab} = a^2 g_{ab}$. It is now a straightforward exercise to evaluate the connection and curvature components and the Laplacians using the formulas of the second section.

The scalar Laplacian is

$$ {}^4\Delta\Phi = a^{-4}(a^2\dot{\Phi})^{\cdot} + a^{-2}\Delta\Phi, \tag{8.2}$$

where $\dot{\Phi} = e_0\Phi$. An expansion of Φ in terms of scalar harmonics reduces the Klein—Gordon equation, for example, to the following ordinary differential equation for the expansion coefficients:

$$\begin{aligned} \Phi &= \sum \Phi^i Q^i, \\ 0 &= [({}^4\Delta + \mu^2)\Phi]^i \\ &= a^{-4}(a^2\dot{\Phi}^i)^{\cdot} + [a^{-2}(n_i^2 - 1) + \mu^2]\Phi^i. \end{aligned} \tag{8.3}$$

The sourceless Maxwell equations may be solved explicitly on a Friedmann spacetime with no knowledge of the function a since they are invariant under conformal scaling of the metric.[21] Let ${}^4A = A_0\omega^0 + A$ with $A = A_a\omega^a$ be the vector potential. In Lorentz gauge:

$$0 = {}^4\delta^4 A = a^{-4}(a^2 A_0)^{\cdot} + a^{-2}\delta A, \tag{8.4}$$

the vector potential satisfies the wave equation ${}^4\Delta_{\mathrm{DR}}{}^4A = 0$.[22] The Latin components of this equation are:

$$0 = ({}^4\Delta_{\mathrm{DR}}{}^4A)_a = a^{-2}(\ddot{A}_a + (\Delta A)_a). \tag{8.5}$$

By introducing the harmonic expansions:

$$\begin{aligned} A_0 &= \sum A_0^i Q^i, \\ A &= \sum (A^{JM,J'M'}\mathbf{X}^{JM,J'M'} + A^{JM,J\pm1\,M'}\mathbf{X}^{JM,J\pm1\,M'}), \end{aligned} \tag{8.6}$$

one obtains the following equations for the coefficients:

$$\begin{aligned} 0 &= a^{-2}(a^2 A_0^i)^{\cdot} + i(n_i^2 - 1)A^i \\ \ddot{A}^i &+ \Delta^i A^i = 0, \end{aligned} \tag{8.7}$$

where $\Delta^i = n_i^2 - 1$ if $J' = J$ and $\Delta^i = n_i^2$ if $J' = J \pm 1$. Let $\omega^i = (\Delta^i)^{1/2}$. The latter equation has the solution $A^i = \mathcal{A}^i e^{i\omega^i t}$, where $\mathcal{A}^i$ is a constant.

A_0 (which incidentally satisfies the scalar wave equation) and the longitudinal $(J' = J)$ part of A are affected by gauge transformations:

$$A_0 \to A_0 + \dot{\lambda}, \quad A \to A + d\lambda\ .$$

To preserve Lorentz gauge λ must satisfy ${}^4\Delta\lambda = 0$. In fact, the gauge dependent parts of ${}^4A$ are removable by a Lorentz gauge preserving transformation with $\dot{\lambda} = -A_0$, exactly as in flat spacetime. In the new gauge A_0 and $A^{JM,J'M'}$ vanish. [$A_0 = 0$ forces $\delta A = 0$ by (8.4).]

Assuming this gauge, the solution may also be expressed in terms of the adjoint harmonics of definite parity:

$$A = \sum A^{JJ+1p}_{lm}\mathbf{X}^{JJ+1p}_{lm}, \quad A^{JJ+1p}_{lm} = \mathcal{A}^{JJ+1p}_{lm}\, e^{in_{JJ+1}t}.$$

The covariant components of the electric field and the contravariant components of the magnetic field density are given by

$$E_a = F_{a0} = -\dot{A}_a, \quad \beta^a = \tfrac{1}{2}\epsilon^{abc}(dA)_{bc} = 4(*dA)_a,$$

where $*$ is the star operation of $\mathbf{g}$ so that (7.2) may be used. The corresponding expansion coefficients

$$E = \sum E^{JJ+1}_{lm}\mathbf{X}^{JJ+1}_{lm}, \quad \beta = \sum \beta^{JJ+1}_{lm}\hat{\mathbf{X}}^{JJ+1}_{lm}\ ,$$

are easily evaluated:

$$E^{JJ+1p}_{lm} = -in_{JJ+1}A^{JJ+1p}_{lm}, \quad \beta^{JJ+1p}_{lm} = -n_{JJ+1}A^{JJ+1-p}_{lm}.$$

The parities of the electric field and magnetic field density of the vector potential $A^{JJ+1p}_{lm}\mathbf{X}^{JJ+1p}_{lm}$ are $p(-1)^{l+1}$ and $p(-1)^l$, respectively. The $p = 1$ and $p = -1$ solutions therefore correspond to the electric and magnetic l-pole radiation of Mashoon.[12]

9. SPINOR HARMONICS

The orthonormal frame $\{\epsilon_\alpha\} = \{a^{-1}e_0,\ 2a^{-1}e_a\}$ may be used to introduce ordinary or Dirac spinor algebras over M. Although spinor fields cannot be decomposed into collections of induced spinor fields on the natural slicing of $M = R \times G$ as can be done with tensor fields, it does make sense to consider a restricted spin connection on G in order to reduce spinor equations on M to ordinary differential equations. We briefly sketch how this may be accomplished.

Let $\{E_q\} = \{E_{1/2}, E_{-1/2}\}$ be the natural basis of C^2 considered as a left invariant spin frame on G associated with the orthonormal frame $\{2e_a\}$.[23] One may extend the action of angular momenta to E_q and hence to any spinor field $\psi = \psi^q E_q$ by

$$\begin{aligned} \tilde{L}_a E_q &= 0 = L_a^{\mathrm{orb}} E_q\ , \\ L_a E_q &= S_a E_q = \tfrac{1}{2}\sigma_a E_q. \end{aligned} \tag{9.1}$$

By (3.6) the natural extensions of the dragging actions are then:

$$L_u E_q = E_q, \quad R_u^{-1}E_q = AD_u E_q = \mathrm{u}E_q, \tag{9.2}$$

where $\mathrm{u} \in \mathrm{Su}(2)$ is the actual matrix of $u \in G$.

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$\{E_q\}$ is a standard spin one-half basis satisfying the $s=\frac{1}{2}$ version of (5.3) with $J_3E_q=qE_q$. This explains our choice of the indices $\{\frac{1}{2},-\frac{1}{2}\}$ rather than the more conventional indices $\{1,2\}$. We may therefore introduce the spinor harmonics by right angular momentum coupling to the scalar harmonics:

$$\mathcal{Y}^{JM,J'M'}=C_{J(1/2)}(J'M',Nq)Q^{JMN}E_q,\quad J'=J\pm\tfrac{1}{2}$$
$$\mathcal{Y}^{JJ'}_{lm}=C_{JJ'}(lm,MM')\mathcal{Y}^{JM,J'M'} \tag{9.3}$$
$$\mathcal{Y}^{JJ+p/2}_{lm}=2^{-1/2}(\mathcal{Y}^{JJ+1/2}_{lm}+p\mathcal{Y}^{J+1/2\,J}_{lm})$$

By considerations similar to (6.3), (6.4), and (7.2) one finds

$$2(\tilde{L}^2-L^2)\mathcal{Y}^{JJ+1/2\,p}_{lm}=-n_{JJ+(1/2)}\mathcal{Y}^{JJ+1/2-p}_{lm}. \tag{9.4}$$

The covariant derivative and ordinary Laplacian may be extended to spinor fields $\psi=\psi^qE_q$ by (5.16), but define a DeRham Laplacian by[5]

$$\Delta_{\rm DR}\psi=(\Delta+\tfrac{1}{4}R)\psi=(\Delta+2S)\psi=[2(\tilde{L}^2+L^2)+S^2]\psi, \tag{9.5}$$

where we have used $R=6$ and $S^2\psi=\frac{3}{4}\psi$. In fact, $\Delta_{\rm DR}$ is the square of σ_aP_a, where $P_a=-i\nabla_{2e_a}$:

$$\sigma_aP_a\psi=-4iS_a(L_a^{\rm orb}+\tfrac{1}{2}S_a)\psi=2(\tilde{L}^2-L^2)\psi,$$
$$\sigma_aP_a\mathcal{Y}^{JJ+1/2\,p}_{lm}=-n_{JJ+(1/2)}\mathcal{Y}^{JJ+1/2-p}_{lm},$$
$$\Delta_{\rm DR}\mathcal{Y}^{JJ+1/2\,p}_{lm}=(\sigma_aP_a)^2\mathcal{Y}^{JJ+1/2\,p}_{lm} \tag{9.6}$$
$$=n^2_{JJ+(1/2)}\mathcal{Y}^{JJ+1/2\,p}_{lm}.$$

Taub has essentially shown[24] that the Dirac equation:

$$(i\gamma^\alpha\nabla_{e_\alpha}+m)\Psi=0$$

may be reduced to the form:

$$(i\partial/\partial t-am\beta)\Phi=\alpha_aP_a\Phi, \tag{9.7}$$

where γ^α, $\beta=\rho_3$, and $\alpha_a=\rho_1\sigma_a$ are the standard Dirac matrices in the notation of Messiah[10] and

$$\Phi=\begin{bmatrix}\phi^1\\ \phi^2\end{bmatrix}=a^{3/2}\Psi$$

is a pair of spinor fields. ϕ^1 and ϕ^2 may each be expanded in terms of the spinor harmonics $\mathcal{Y}^{JJ+1/2\,p}_{lm}$ with time dependent expansion coefficients $\phi^{1JJ+1/2\,p}_{lm}$ and $\phi^{2JJ+1/2\,p}_{lm}$, respectively. Suppressing all indices except p and inserting the expansions into (9.7), one obtains the ordinary differential equations:

$$(id/dt-am)\phi^{1+}=-n\phi^{2-},$$
$$(id/dt-am)\phi^{1-}=-n\phi^{2+},$$
$$(id/dt+am)\phi^{2-}=-n\phi^{1+}, \tag{9.8}$$
$$(id/dt+am)\phi^{2+}=-n\phi^{1-}.$$

For example, in an Einstein spacetime in which a is a constant, one obtains solutions with exponential time dependence and frequencies $\omega=\pm(n^2+a^2m^2)^{1/2}$.

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[1]Latin indices $a,b,c,d\ldots$ assume the values 1,2,3 while Greek indices assume the values 1,2,3,4 when referrring to R^4 and 0,1,2,3 when referring to space—time.

[2]B.L. Beers and R.S. Millman, J. Math. Phys. **16**, 11 (1975).

[3]We observe the conventions of Ref. 4 except for the connection components. If $\mathbf{T}^a{}_{bc}=\omega^a(\mathbf{T}(e_b,e_c))$ are the components of the torsion tensor $\mathbf{T}$ of a connection ∇ in the frame $\{e_a\}$ with dual frame $\{\omega^a\}$ and structure functions $C^a{}_{bc}=\omega^a([e_b,e_c])$ and if $\mathbf{g}=g_{ab}\,\omega^a\otimes\omega^b$ is a covariant constant metric $\nabla\mathbf{g}=0$, then the components of the connection in this frame may be obtained from the formula:

$$2\Gamma_{abc}=g_{\{ab,c\}}+C_{\{abc\}}+\mathbf{T}_{\{abc\}},$$
$$\mathbf{S}_{\{abc\}}=\mathbf{S}_{abc}-\mathbf{S}_{bca}+\mathbf{S}_{cab},\quad g_{ab,c}=e_c\,g_{ab}.$$

The usual index raising and lowering conventions are understood. The components of the Riemann tensor of ∇ in this frame are

$$R^a{}_{bcd}=e_c\Gamma^a{}_{db}-e_d\Gamma^a{}_{cb}-\Gamma^a{}_{eb}C^e{}_{cd}+\Gamma^a{}_{ce}\Gamma^e{}_{db}-\Gamma^a{}_{de}\Gamma^e{}_{cb}.$$

The metric connection of $\mathbf{g}$ has $\mathbf{T}=0$; the components of its Ricci tensor are defined by $\mathbf{R}_{bd}=\mathbf{R}^a{}_{bad}$.

[4]C.W. Misner, K.S. Thorne, J.A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).
[5]A. Lichnerowicz, "Propagateurs, commutateurs, et anticommutateurs en relativité générale, in *Relativity, Groups, and Topology*, edited by C. Dewitt and B.S. Dewitt (Gordon and Breach, New York, 1964).

[6]The differential $dh(u)$: $\mathbf{TG}_u\to\mathbf{TG}_{h(u)}$ and its transpose $dh(u)^*$: $\mathbf{TG}^*_{h(u)}\to\mathbf{TG}^*_u$ of h at u are defined by

$$[dh(u)X(u)]\Phi=X(u)\Phi\circ h$$
$$[dh(u)^*\sigma(h(u))]\ (X(u))=\sigma(h(u))\ (dh(u)X(u))$$

[7]Recall the identity $[\mathcal{L}_X,\mathcal{L}_Y]=\mathcal{L}_{[X,Y]}$.

[8]N.J. Vilenkin, *Special Functions and the Theory of Group Representations* (American Mathematical Society, Providence, R.I., 1968).

[9]The notation $Cj_1j_2(j_3m_3,m_1m_2)$ of Blatt and Weisskopf is more convenient here that $\langle j_1j_2m_1m_2\,|\,j_3m_3\rangle$ as used by Messiah.[10]

[10]A. Messiah, *Quantum Mechanics* (Wiley, New York, 1958) Vol. II.
[11]E.M. Lifshitz and I.M. Khalatnikov, Adv. Phys. **12**, 185 (1963).
[12]B. Mashoon, Phys. Rev. D **8**, 4297 (1973).

[13]Actually these bases will be orthonormal only up to a factor of 4^{p-q}.

[14]To do this one uses the adjoint invariance of the structure constant tensor: $\mathcal{R}^a{}_dC^d{}_{fg}\mathcal{R}^{-1\,s}{}_b\mathcal{R}^{-1\,g}{}_c=C^a{}_{bc}$.

[15]A.E. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University, Princeton, N.J., 1957).
[16]F.W. Warner, *Foundations of Differentiable Manifolds and Lie Groups* (Scott, Foresman and Co., Glenview, Ill. 1971).
[17]L.I. Schiff, *Quantum Mechanics* (McGraw-Hill, New York, 1968).
[18]A.E. Fisher and J.E. Marsden, "Topics in the Dynamics of General Relativity," Enrico Fermi International School of Physics, Varenna, Italy, 1976.

[19]Let $\mathbf{g}\to\rho^2\mathbf{g}$, where $\rho^2=1$. Analytic continuation of ρ and χ to pure imaginary values ($\rho\to i,\chi\to i\chi$) yields the Riemannian metric of a geometry of constant negative curvature on $\mathbf{S}^3-\{-\hat{e}_4\}\sim R^3$. The adjoint harmonics of definite parity may also be analytically continued to harmonics of the new geometry.[11]

[20]B.L. Hu, J. Math. Phys. **15**, 1748 (1974).
[21]L. Infeld and A.E. Schild, Phys. Rev. **70**, 410 (1946).

[22]By introducing a vector potential $F={}^4d^4A$, Maxwell's equations ${}^4dF={}^4\delta F=0$ are reduced to $0={}^4\delta^4d^4A={}^4\Delta_{\rm DR}{}^4A-{}^4d^4\delta^4A$.

[23]J.S. Dowker, Ann. Phys. **71**, 577 (1972).
[24]A.H. Taub, Phys. Rev. **51**, 512 (1937).

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