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Gravitoelectromagnetism: Relativity of Splitting Formalisms

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Drawing upon our knowledge of both special relativity and noninertial forces in nonrelativistic mechanics, the "gravitoelectomagnetic" approach to general relativity is seen to be a marriage of both these well known topics on the arena of spacetime. The approach involves splitting spacetime into space plus time, i.e., interpreting spacetime quantities by the introduction of a family of test observers used to measure them.

I. INTRODUCTION

Most of us know special relativity pretty well and are quite happy switching back and forth between the spacetime picture of 4-vector algebra and the space plus time picture of events occurring in space as time elapses, using 3-vector algebra. We have no difficulty using Lorentz transformations to transform 3-dimensional quantities from those measured by one inertial observer to another. We also have no problem extending our splitting algebra to spacetime derivative operators yielding the space plus time equivalent of time derivatives and spatial derivative operators. After all, these are the derivatives we began with before learning special relativity. Figure 1 suggestively compares these two pictures.

Since old habits die hard, there is a strong incentive to push this habit into general relativity. This helps us interpret spacetime information in a curved spacetime using the same intuition that we have about classical 3-dimensional physics. The catch is that one no longer has a privileged class of "global inertial frames" as in special relativity which effectively allows a single inertial observer in flat spacetime to set up a preferred class of global inertial coordinate systems that may be used to

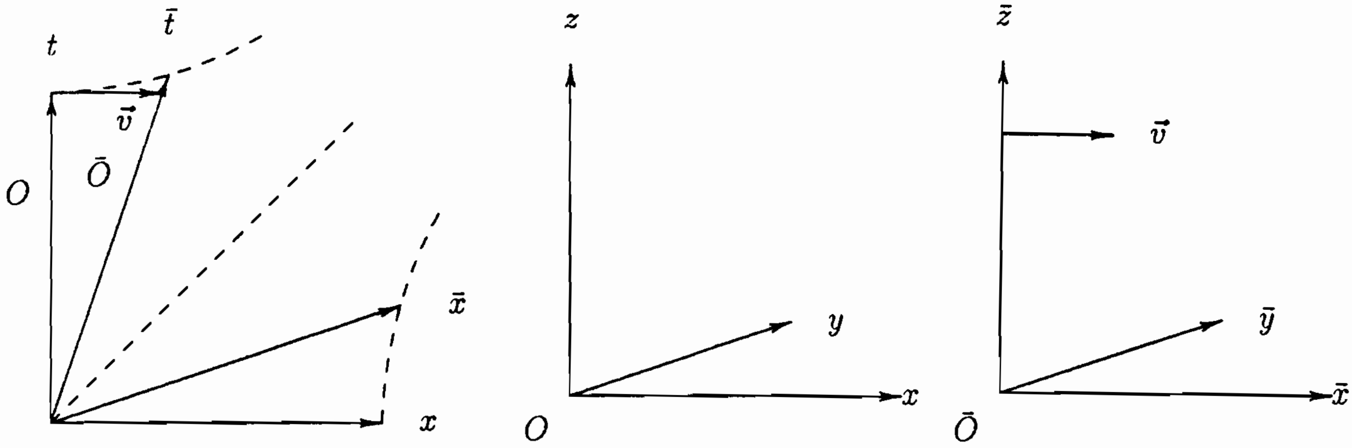


Fig. 1. Spacetime and Space plus Time. Although spacetime is the arena where calculations are simpler, we always interpret them through our space plus time worldview, which depends on the choice of inertial observer.

interpret observations at all other points in spacetime. Essentially, one inertial observer uniquely determines a family of inertial observers filling the spacetime with no relative velocities, and the "observations" of the original observer of an event not on his own worldline are understood to be those of the companion observer who is present.

The only option allowing us to continue to make a spacetime splitting is to give up the global splitting associated with a single preferred observer and settle for the local splitting of each member of a family of observers filling the spacetime and in arbitrary motion in the absence of any preference. Such a splitting takes place in the tangent space to each event in spacetime, describing the locally Minkowskian neighborhood of each observer in the family. If we agree to split each such tangent space based on the 4-velocity of the observer at a given event in the same way that we split flat spacetime globally based on the worldline of a single inertial observer, then all of our familiarity with special relativity can be transferred to general relativity in a rather straightforward way. The only difference is that what we did globally before, with the splitting at every spacetime point determined by the splitting at a single spacetime point, namely any point on the worldline of a chosen inertial observer, must be abandoned in favor of doing the same thing independently at each spacetime point, modulo continuity/differentiability conditions.

There is one catch. Being creatures of habit, we like global splittings, so even though in general there is no preferred way of doing a global splitting, we can just do it arbitrarily. One then has to reconcile the local observer splittings with such a global splitting. This too is not anything particularly deep, and it involves using the linear algebra of nonorthogonal bases independently at each spacetime point (since such a global splitting will in general be nonorthogonal) to represent the orthogonal splitting of the local observers on spacetime.

The new complication in general relativity is that in general one must deal with a family of so called "test observers" in arbitrary motion, and this introduces the well known effects that accompany noninertial (i.e., accelerated) observers even in nonrelativistic physics. However, in the classical example of a "rigidly rotating" family of noninertial observers in nonrelativistic physics, one still has a global correlation between the different members of the family of observers since one has a global (though not physical) Cartesian coordinate system which rotates with the passage of time. In the extension to general relativity one must consider such effects locally at each spacetime point.

The effects of rigid rotation are well known and familiar, as playfully illustrated in Fig. 2. There is the centrifugal (or "merry-go-round") force that a body feels in the rotating frame even if it is not moving with respect to that frame due to the frame's acceleration and the Coriolis ("mobile merry-go-round") force that a body feels if it is in motion with respect to that frame due to the rotation of the frame. Quantitatively in a system rotating with constant angular velocity $\vec{\Omega}$, there is a force on an otherwise free body

$$\ddot{\vec{x}} = \vec{F}/m = \vec{g} + \vec{v} \times \vec{H} \ , \qquad (1)$$

where the "gravitoelectric" force (per unit mass)

$$\vec{g} = -\vec{A} = -\vec{\Omega} \times \vec{V} = -\vec{\Omega} \times (\vec{\Omega} \times \vec{x}) \qquad (2)$$

is the negative of the acceleration field $\vec{A}$ of the rotating observers, whose velocity field is $V = \vec{\Omega} \times \vec{x}$. Similarly the "gravitomagnetic" force is the cross

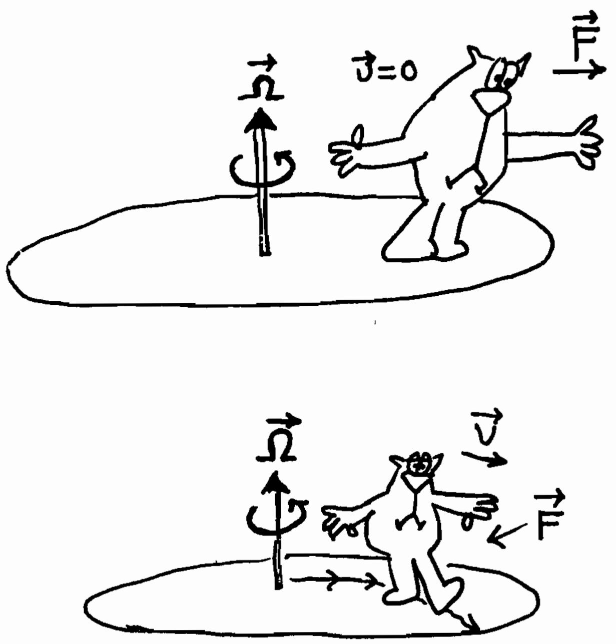


Fig. 2. Noninertial forces experienced in a rotating frame are illustrated with a merry-go-round. For a body at rest on the merry-go-round, there is a radially outward force, and if that body begins to walk on the merry-go-round, a force perpendicular to the direction of motion on the merry-go-round will be felt as well.

product of the body's velocity $\vec{v} = \dot{\vec{x}}$ in the rotating system with the "gravitomagnetic" vector field $\vec{H} = 2\vec{\Omega}$, which is twice the angular velocity vector of the rotating frame.

The "gravitoelectromagnetic" terminology[1] is in direct analogy with the Lorentz force of electromagnetism in a nonrotating system

$$\ddot{\vec{x}} = \vec{F}/q = \vec{E} + \vec{v} \times \vec{B} \ . \qquad (3)$$

Note that the "gravitomagnetic" vector field $\vec{H} = \mathrm{curl}\,\vec{V}$ admits a vector potential $\vec{V}$ in the same way that the magnetic field $\vec{B} = \mathrm{curl}\,\vec{A}$ locally admits a vector potential. Similarly the "gravitoelectric" vector field

$$\vec{g} = -\,\mathrm{grad}[\tfrac{1}{2}\vec{V} \cdot \vec{V}] \qquad (4)$$

admits a scalar potential in this time-independent case just like the conservative electric field $\vec{E}$ in electrostatics.

The velocity field of the rotating observers and half its length squared serve as vector and scalar potentials for the noninertial forces we often call "fictitious" forces in classical mechanics and they are directly interpretable in terms of kinematical properties of the velocity field of the family of noninertial observers used to describe the motion of the body being studied. These forces vanish as soon as we require the members of the family of observers to be inertial.

In a curved spacetime such global frames are not immediately available, so one must analyse the situation in the local rest space of each observer in the family of test observers used to describe the physics in 3-dimensional form. The kinematical properties of the 4-velocity field of these observers in spacetime, with some extra complication, directly generalize the above problem, leading to the introduction of "gravitoelectromagnetic" forces which enter into the force equation when expressed in terms of a family of noninertial observers. Rather than doing a global comparison of their motion with respect to a global inertial frame which does not exist in curved spacetime, it must be a local comparison at each point of spacetime with a suitable locally nonrotating inertial observer with the same 4-velocity. Of course there are new features in curved spacetime which have no analogy in electromagnetism and this has to do with the spatial metric which describes the relative distances between nearby test observers as well as the different proper times that different observers use at the same event in spacetime, both of which depend on the gravitational field. These features, as well as the potentials for the gravitoelectromagnetic vector forces, are contained in the spacetime metric. Taking into account the local proper time complicates slightly the above analogy which relies on a global proper time function on flat spacetime.

Thus we need relativistic definitions of acceleration, gradient, curl, time derivative, etc. This together with the details of the way in which an observer in spacetime measures quantities at a given event will enable us to push our present knowledge about special relativity and noninertial motion to the case of general relativity. This helps explain the so called ADM or three-plus-one approach to general relativity[2] as well as the slightly different Landau-Lifshitz approach,[3] and shows their similarities and differences, both of which involve the structure of a congruence of test observers together with a global time function on spacetime. It also shows how both relate to the Ehlers-Hawking-Ellis[4–7] splitting approach, which is based only on a congruence of test observers with no global time function assumed to be available.

II. WHY BOTHER?

You might say, why invest a lot of time into understanding the details of this way of looking at all the different splitting approaches used in general relativity? After all, relativity physics liberated us from the prison of 3-dimensional language into the arena of spacetime where the true nature of kinematics and dynamics became much simpler to understand. Why try to climb back into the cage of 3-dimensional physics? Wasn't centuries of solitary confinement enough?

Well, many spacetimes and idealized problems we use in understanding gravitational theory practically beg us to do this. A rotating black hole spacetime, for example, has two very different privileged families of test observers, one of which is suited to the ADM picture and the other to the Landau-Lifshitz picture. The one we use depends on the question we want to study. Both turn out to be useful. The Ehlers-Hawking-Ellis picture provides the means to relate these two different pictures to each other, which is important if they both turn out to be needed, as they in fact do. The failure to investigate this "relativity of spacetime splitting formalisms" has kept most of us from having better intuition not only about black holes, or even the relativistic picture of the nonrelativistic problem of rotating coordinate systems (which continues to confuse people even now), but other interesting rotating spacetimes like the Gödel universe.

Even if none of these special spacetimes interests you, if you are interested in any post-Newtonian calculations of more realistic, say isolated self-gravitating systems, then understanding the splitting game helps to make a little more sense out of what is universally done in that field. In recent years many references to "the gravitomagnetic field" have sprung up, but since no one took the time to unambiguously define just what this field was, controversy has blossomed between different schools of thought about just what a "real gravitomagnetic field" is.[8] (See Fig. 3.) People can

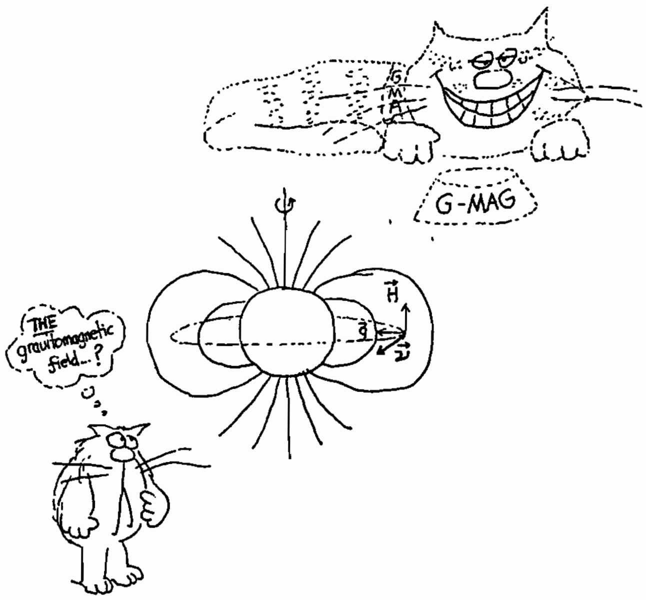
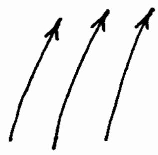
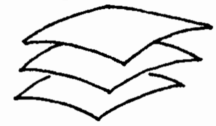
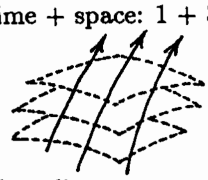
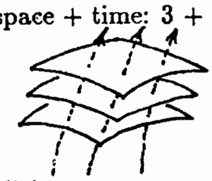
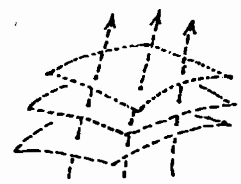


Fig. 3. *"The"* gravitomagnetic field of an isolated rotating self-gravitating body illustrated as the Cheshire cat from *Alice and Wonderland*, whose smile might represent the commonly accepted naive definition in the weak field limit, while the rest of the cat remains obscure otherwise.

Table 1. A characterization of the different points of view that may be adopted in splitting spacetime. Solid lines in diagrams imply the use of the appropriate causality condition while dashed lines indicate the opposite.

	splitting spacetime: 4		
	with causality condition		without causality condition
	1 ⇒ timelike	3 ⇒ spacelike	
partial splitting	time: 1  congruence p.o.v.	space: 3  hypersurface p.o.v.	—
full splitting	time + space: 1 + 3  threading p.o.v.	space + time: 3 + 1  slicing p.o.v.	 reference p.o.v.

agree on a naive definition of the field in this stationary weak-field limit but for strong fields many different definitions are possible depending on the choices one makes for the way in which observers make measurements. All of these different choices can be fit into a general framework and related to each other.

Unfortunately no text on general relativity can spare the space to do justice to this idea of a relativity of splitting formalisms, so we learn general relativity from one school or another but rarely appreciate more than one approach in our working lives even if we are relativists. However, if something is worth doing, it is worth doing right. We have attempted to do this as concisely as possible in a journal article[9] and perhaps a monograph[10] will eventually appear which attempts this in greater detail and with more explanation. Someone else will have to decide if these attempts have been successful.

III. OVERVIEW OF DETAILS WITHOUT THE DETAILS

This is certainly not the appropriate place to try to summarize the details of spacetime splittings. Instead some idea will be given of the different approaches and the general features of a given approach. In order to discuss the generalization of the notions of "space" and "time" that come from splitting spacetime in special relativity to the arena of general relativity, we must reflect a bit on just what they mean to us. First note that the notions of time and space are complementary since a "time line" represents "time elapsing at a point fixed in space" while a "time hypersurface" represents "space at a moment of time". These two different notions of time, the first which focuses on measuring time at a single point of space and the second which is associated with some kind of synchronization of times at different points of space, will be assigned the labels "time" and "space" respectively. These divide the different splitting points of view into two categories, those in which a local time direction is fundamental, and those in which a nonlocal correlation of local times, i.e., space is fundamental. In the time category, the "time lines" must be timelike in order to represent a local time direction at each event in spacetime, while in the space category, the "time hypersurfaces" or "spaces" must be spacelike in order to be associated with a moment of time in the usual sense of a Riemannian space (alternatively, in order that orthogonality define a local time direction). Given this division, one may consider a partial splitting or a full splitting depending on whether any additional structure is assumed. Table 1 establishes this general classification of points of view and the terminology that will be used to describe it.

Given no additional structure, one has only a partial splitting of spacetime, splitting off either the time or the space alone. In the first case, to be called the "*congruence point of view*," one has only a timelike congruence at one's disposal, with a unit timelike tangent vector field u. Spacetime will be assumed to be time-oriented as well as oriented, so one may assume that u is future-pointing. It may then be interpreted as the 4-velocity of a family of test observers whose worldlines are the curves of the congruence, and it determines the *local time direction* at each point of spacetime. The orthogonal complement of this local time direction in the tangent space is the *local rest space* LRS_u of the test observer at that event. It is exactly this structure that one needs for the *measurement process* which will be the same for the full and partial splittings in a given category in Table 1. The orthogonal decomposition of the tensor algebra induced by this decomposition of the tangent space at each event will define the measurement process, modulo a final step in which projection along the local time direction is replaced by contraction with u, yielding a collection of "*spatial tensor fields*" of different rank for each spacetime tensor field that is split.

In general the splitting of the tangent spaces does not extend to the spacetime manifold. Only in the special case that the rotation of u vanishes does such an extension exist and one has a family of orthogonal spacelike hypersurfaces which slice the spacetime, leading to a full splitting in this category to be discussed below. The second case in the category of partial splittings of spacetime, that of a spacelike slicing of spacetime with no additional structure, is essentially equivalent to the special case of a nonrotating congruence since every spacelike slicing admits a family of timelike orthogonal trajectories. These are the integral curves of the (rotation-free) unit normal vector field n to the slicing, which may be assumed to be future-pointing. The accompanying point of view, for the sake of completeness, might be called the "*hypersurface point of view.*" Its measurement process is associated with the normal congruence, taking $u = n$ as the 4-velocity of the family of test observers who do the measuring. The local rest spaces of this family are integrable and coincide with the subspaces of the tangent space which are tangent to the slicing.

A full splitting of spacetime at the manifold level requires both a *slicing* of the spacetime and a congruence, to be referred to as a "*threading*" of the spacetime, together with a compatibility condition that the two families be everywhere transversal (no common directions). Such a structure will be called a "*nonlinear reference frame*" in order to distinguish it first from the terms "reference frame," "frame of reference," "reference system" and "system of reference" that one finds in the literature, second from the connotation of "frame" in the context of a linear frame of vector fields, and third from related terminology which occurs in the discussion of globally constant frames in flat spacetime. A "*parametrized nonlinear reference frame*" will consist of a nonlinear reference frame together with a choice of parametrization of the family of slices. Such a parametrization defines a specific "time function" t on the spacetime which in turn provides an obvious parametrization for each curve in the threading congruence.

In the category of full splittings, the differentiating criterion is the causality condition imposed on the nonlinear reference frame. In the "*slicing point of view*" the slicing is assumed to be spacelike, but no assumption is made about the causality properties of the threading, which serves only as a way of identifying the points on different slices. In the "*threading point of view*," the threading is assumed to be timelike, but no assumption is made about the causality properties of the slicing, which serves only to synchronize in some arbitrary fashion points on different curves in the congruence. If both causality conditions hold, then both points of view hold and one can transform between them. On the other hand it can also be useful in the case that at least one of the two causality conditions holds to not take advantage of that condition and exploit only the structure of the nonlinear reference frame that does not depend on it. This leads to the "*reference point of view*," whose measurement process is associated with the nonorthogonal decomposition of the tangent space into the direct sum of a 1-dimensional subspace tangent to the threading and a three-dimensional subspace tangent to the slicing. One can always relate either the slicing or threading points of view to this acausal approach, which is the way in which they are usually represented in a local coordinate system adapted to the nonlinear reference frame.

The partial splittings may be related to the full splittings in different ways. In the threading point of view one may define a (future-pointing) unit timelike tangent vector field m along the threading congruence, while in the slicing point of view one has the (future-pointing) timelike unit normal vector field n. By making the respective choices $o = m$ and $o = n$ ("o" for "observer") of the 4-velocity of a privileged family of test observers in these two points of view, the identification $u = o$ relates each of them to a corresponding congruence point of view described above, defining for each a measurement process. When both the slicing and threading points of view hold, then a unique boost in each tangent space relates the two timelike unit vectors m and n and this may be extended to a transformation of the measurement process. In the special case of an *orthogonal nonlinear reference frame* (one for which both the causality conditions hold and the slicing and threading are everywhere orthogonal), then $m = n$ and the two points of view coincide.

Evolution is defined first by a choice of a 1-parameter group of diffeomorphisms of the spacetime into itself which in some sense advances into the future (either its orbits are timelike or it pushes certain spacelike hypersurfaces into their future), and second by a choice of transport along its orbits for the spatial fields of the given point of view. For a partial splitting only one congruence is available and it is timelike. In the absense of additional structure one can take u or n respectively in the congruence or hypersurface points of view as the generator of such a group, and choose either spatially-projected ("*spatial*") Lie transport or spatially-projected parallel transport along this congruence. The latter transport of spatial fields coincides with Fermi-Walker transport which defines locally nonrotating axes along a worldline. Each of these choices may be extended to the full splitting in its category but it is the spatial Lie transport along the threading congruence which defines the evolution relative to the nonlinear reference frame, since fields which are "rigidly" attached to this frame do not evolve with this choice. However, unlike Fermi-Walker transport, spatial Lie transport is in general incompatible with orthonormal frames. A compromise between the two kinds of transport leads to *co-rotating Fermi-Walker transport*, which is the closest one can get to attaching an orthonormal frame to the nonlinear reference frame.

Given two different test observer congruences on spacetime, one can consider the transformation which relates the measurements of one to those of the other. This amounts to a Lorentz transformation at each spacetime point which may be used to transform many measurements locally in the same way that they are done with a global Lorentz transformation in special relativity, but for any quantity involving derivatives, derivatives of the Lorentz transformation will also occur in the corresponding transformation law. For example, the electric and magnetic fields measured by two different observer families at a given point will exactly correspond to the usual Lorentz transformation of those quantities, but the transformation laws for the corresponding gravitoelectromagnetic vector fields will have additional terms depending on derivatives of the relative Lorentz transformation, as well as factors related to the local proper time question. It is important to note that the gravitoelectromagnetic fields cannot be defined for a single observer in spacetime, but are functions of a family observers since they locally describe the "frame" set up by the family itself. Of course, given a single observer, one can always imbed that observer locally in a family and define fields for that family.

Thus within the congruence point of view one has a local relativity related to changes in the family of test observers. For a given family one can locally define analogs of all the 3-dimensional quantities of special relativity as well as new ones associated with the spacetime variation of the quantities characterizing the observers themselves. Transformation laws can then be found expressing the relationship between corresponding quantities measured by two different families of test observers. Next, one may apply these transformations to the two observer congruences associated with a single nonlinear reference frame for which both the slicing and threading points of view hold. The threading point of view is just a representation of the congruence point of view associated with the threading congruence, as the hypersurface point of view is a representation of the congruence point of view associated with the congruence of normals to the slicing. The transformation between the two congruence point of view pictures for the observers with respective 4-velocity fields n and m therefore relates the hypersurface point of view to the threading point of view. The slicing point of view, on the other hand, is a hybrid approach where the evolution is defined by the threading while the measurement is defined by the slicing. Additional changes are then required to convert the previous transformation to one between the slicing and threading points of view when evolutionary derivatives are involved. One can then transform between different nonlinear reference frames on the spacetime as well. The transformations between all of these various possibilities then describes the relationships not only between different choices of a given point of view on a fixed spacetime, but also between the different points of view that may be used. It seems reasonable to use the catch word "gravitoelectromagnetism" to refer to all of the mathematics that is involved in this problem.

IV. HISTORICAL BACKGROUND

Armed with this initial vocabulary, the historical background may be sketched in a way that puts the different formalisms into some perspective. The slicing and threading points of view today are introduced to most of us through two leading textbooks, respectively *Gravitation* by Misner, Thorne, and Wheeler[2] and *The Classical Theory of Fields* by Landau and Lifshitz,[3] each of which carefully avoids mention of the "competing" point of view. Both points of view can be traced back to the early forties when the first edition of the Landau-Lifshitz text introduced the threading point of view splitting of the spacetime metric and, in the stationary case, of the spacetime connection to yield spatial gravitational forces, as still described in their latest edition. Soon after, Lichnerowicz[11] introduced the beginnings of the slicing point of view with an article discussing the initial value problem in an orthogonal nonlinear reference frame. The threading point of view apparently dominated during the fifties when much interest was focused on the equations of motion for test particles. Møller discussed a parametrization-dependent definition of spatial gravitational forces for a general spacetime in the first edition of his text *The Theory of Relativity*.[12] This was then refined to a parametrization-independent splitting later in the decade by Zel'manov[13] in the Soviet Union and then independently by Cattaneo[14] in Italy.

Meanwhile the slicing point of view was further developed during the fifties by Choquet-Bruhat,[15] Dirac,[16] and Arnowit, Deser and Misner.[17] This ushered in the new era of domination of the splitting scene by the slicing point of view, pushed by the problem of quantum gravity where Hamiltonian techniques have played a rather important role in an endless quest that has not yet met success. The notation of Arnowit, Deser and Misner, soon labeled by Wheeler's lapse and shift terminology[18] and later effectively propagated by the text of Misner, Thorne and Wheeler has found widespread acceptance. The slicing point of view is also commonly referred to as the "3 + 1" or ADM formalism. The term "1 + 3" formalism with its obvious change in emphasis has been suggested as an alternative label for the threading point of view.

The gravitoelectromagnetic terminology has its origins in Forward's article[19] describing the analogy between electromagnetism and linearized general relativity using the reference point of view, in a slight variation of Møller's formalism. In the late seventies and eighties Thorne and various coauthors developed the actual terminology that is standard today.[1,20–22] In linearized gravity, in the usual weak field slow motion discussions, the corresponding spatial gravitational fields defined in the threading, slicing and reference points of view are very closely related and agree to the lowest order, although not to full post-Newtonian order. The commonly used definitions in fact correspond to the threading point of view in the context of the post-Newtonian nonlinear reference frames that are universally used in this field.

These various splittings of spacetime are particularly interesting in the case of electromagnetism, where all of our intuition is tied to individual electric and magnetic fields, and astrophysical applications can be aided by allowing this intuition to find expression in the context of a splitting. Early work by Ruffini, Damour and collaborators[23–25] revealed the utility of introducing the concept of electric and magnetic fields in studying black hole systems. More recently this has been discussed in great detail from the slicing point of view by Thorne and Macdonald,[26] who summarize the history of the different splittings in general and as applied to Maxwell's equations, and by Thorne et al[27] for application to black hole systems, where the gravitoelectromagnetic jargon is extended to fully nonlinear stationary systems.

V. CONCLUDING REMARKS

An enormous language barrier exists at present between the slicing and threading points of view, preventing those versed in the formalism and notation of one from easily penetrating the other or understanding how the two are related. It is rather straightforward to introduce a common mathematical framework to discuss both approaches on an equal footing. Both of these splitting formalisms can be developed in a completely parallel way as complementary aspects of a single geometrical structure imposed on spacetime (the nonlinear reference frame), aspects which in a close way are related by the same duality that links contravariant and covariant fields on the spacetime manifold. Furthermore, each of these approaches has important ties with the congruence point of view which invariantly describes the geometry of the observer congruence and with the reference point of view which links these discussions to adapted coordinate systems in practice. In the special case of an orthogonal slicing and threading of a spacetime, the various descriptions coincide.

A good example to keep in mind is the exterior of the event horizon in a black hole spacetime, for example, where the Boyer-Lindquist coordinates are valid. These coordinates determine a parametrized nonlinear reference frame which may be used to introduce both a slicing and a threading point of view. The threading point of view is valid outside the ergosphere where the so called "static observers" (or "distantly nonrotating Killing observers") are defined (the threading is timelike), while the slicing point of view is valid outside the event horizon where the "locally nonrotating observers" are defined (the slicing is spacelike). For a nonrotating black hole, these observers coincide, together with the corresponding splitting formalisms.

Associated with any observer congruence is the observer space, namely the space of worldlines of the observers in the family, or in other words the quotient space of the part of spacetime where the observer congruence is defined, modulo the equivalence relation of belonging to the same worldline. Because of stationarity in the black hole case each of these observer spaces in the Boyer-Linquist nonlinear reference frame inherit the structure of a 3-dimensional Riemannian manifold which may be used to understand the geometry of the spacetime in each point of view. In the slicing point of view the Riemannian metric on this space is the time-independent induced metric on the corresponding family of slices in spacetime, while in the threading point of view it is the time-independent projected metric on this family of slices, where the projection is orthogonal to the observer congruence.

In the slicing point of view one may also consider the quotient of the spacetime by the threading congruence, which is in general distinct from the observer congruence; this might be called instead the evolution space. This latter space with the slicing spatial metric was initially taken by Thorne[26] as his mental picture of the "absolute space" around a black hole, but later Thorne et al[27] decided the slicing observer space itself made a better "absolute space" for this problem, pictured as dragged along by the rotation of the hole relative to the "distant stars," in motion relative to the Boyer-Lindquist nonlinear reference frame.

However, the slicing observers do not follow a family of Killing trajectories associated with the stationary axial symmetry of the black hole, while the threading observers do. The result is that the latter observers are somewhat better adapted to the geometry of the spacetime for certain questions, which then appear slightly more complicated when expressed instead in the slicing point of view. By making the threading point of view as easy to apply as the slicing point of view, one has the option to use the one that a particular question might favor rather than the one that one happens to be more familiar with, as well as transform between the two points of view. Furthermore, the machinery one introduces also enables us to better analyze the old question of rotating Cartesian coordinate systems in this new light and see how our intuition about that problem is analogous to the more general problem of stationary axially symmetric spacetimes, including not only black holes but the interesting Gödel spacetime. A preliminary study of this problem is presented in the article by Bini, Carini and Jantzen in this volume.

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