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Applications of Gravitoelectromagnetism to Rotating Spacetimes

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The various spacetime splitting formalisms and corresponding gravitoelectromagnetic variables are introduced to study the three best known rotating spacetimes, "rotating Minkowski," Gödel, and Kerr. A careful analysis of the timelike circular geodesics in these spacetimes leads to a better understanding of the effects of rotation in them.

I. INTRODUCTION

"Rotating spacetimes" have captured people's imaginations ever since "rigid" rotations in Minkowski spacetime were considered within the theory of general relativity. Even this simple example which is the foundation of the "fictitious" centrifugal and Coriolis forces in classical physics has led to its share of confusion about rotation in relativity. Gödel's discovery of the spacetime which bears his name[1] certainly added fuel to the fire, which was again stoked by the discovery of the rotating black hole solution of Kerr.[2]

The language of gravitoelectromagnetism (GEM),[3–5] a "relativity of spacetime splitting formalisms," helps understand the effects of rotation in these three classic spacetime examples. By focusing on the the radial force balance equation for circular geodesics,[6,7] one obtains a clearer picture of the action and interrelationships of the various gravitoelectric (GE), gravitomagnetic (GM), and spatial geometry (SG) forces that one may define within each spacetime as well as of the correspondences one may establish between these different spacetimes.

Gravitoelectromagnetism involves the splitting of spacetime into space plus time, accomplished locally by means of an observer congruence, namely a congruence of timelike worldlines with unit tangent vector u which may be interpreted as the four-velocity field of a family of test observers filling the spacetime. The orthogonal decomposition of each tangent space into a local time direction along u and the orthogonal local rest space LRS_u may be used to decompose all spacetime tensor and tensor equations into a space-plus-time representation, i.e., to "measure" them. This leads to a family of "spatial" spacetime tensor fields which represent each spacetime field and a family of spatial equations which represent each spacetime equation. Such a splitting permits a better interface of our three-dimensional intuition and experience with four-dimensional spacetime geometry. In particular, the spatial force equation for a test particle leads one to interpret the acceleration (reversed in sign) and the vorticity of the observer congruence (times 2) as GE and GM spatial force fields, while the "spatial metric" describes the relative geometry of the observer worldlines, also contributing a spatial force to the total force felt by the test particle.

In what follows we consider a full splitting of spacetime by means of a *slicing*, namely a family of hypersurfaces, and a transversal congruence of curves, or a *threading* of spacetime. The structure *slicing plus threading* is what we call a *nonlinear reference frame* and the causality condition imposed on it differentiates between the two splitting formalisms based on it. The *slicing point of view* (ADM)[8,9] assumes the slicing to be spacelike but with no assumption about the causality property of the threading and the *threading point of view* (Landau-Lifshitz)[10] assumes the threading to be timelike without any assumption about the causality property of the slicing. The observer congruence is associated with the field n of unit normals to the slicing in the first case and with the field of unit tangents m to the threading in the second. Evolution is measured with respect to the threading congruence in each point of view, so that in the slicing point of view the observers in general move relative to the threading congruence. Clearly if both causality conditions hold, then both points of view are valid and one can transform between them.[5] A third *reference point of view* is used to describe each of these orthogonal splittings in terms of the nonorthogonal splitting determined by the nonlinear reference frame itself (threading directions and slicing directions). This amounts to the coordinate splitting in coordinates adapted to the nonlinear reference frame. It requires no causality assumption.

II. ROTATING MINKOWSKI, GÖDEL AND KERR SPACETIMES

Consider metrics of the following form, expressed respectively in variables appropriate for the reference, threading, and slicing points of view respectively[5]

$$\begin{aligned} ds^2 &= g_{tt}dt^2 + 2g_{t\phi}dt\,d\phi + g_{\phi\phi}d\phi^2 + g_{\rho\rho}d\rho^2 \\ &\qquad + g_{zz}dz^2 \qquad \text{(reference)}\ , \\ &= -M^2(dt - M_\phi d\phi)^2 + \gamma_{\phi\phi}d\phi^2 + \gamma_{\rho\rho}d\rho^2 \\ &\qquad + \gamma_{zz}dz^2 \qquad \text{(threading)}\ , \qquad (1) \\ &= -N^2dt^2 + g_{\phi\phi}[d\phi + N^\phi dt]^2 + g_{\rho\rho}d\rho^2 \\ &\qquad + g_{zz}dz^2 \qquad \text{(slicing)}\ , \end{aligned}$$

where the various metric functions only depend on the radial coordinate ρ, corresponding to a cylindrically symmetric nonlinear reference frame in a cylindrically symmetric spacetime. The lapses M and N and the shift vector fields $\vec{M} = M^\phi\partial/\partial\phi$ and $\vec{N} = N^\phi\partial/\partial\phi$ relate the observer four-velocity to the nonlinear reference frame and serve as scalar and vector potentials for the GE and GM force fields. The spatial metric serves as a tensor potential for the SG force fields.

Flat spacetime in cylindrical coordinates which rotate with angular velocity Ω about the positive z-axis with respect to the corresponding inertial coordinates ("rotating Minkowski" for short) and the Gödel spacetime (expressed in terms of the parameter Ω equal to the local vorticity of the dust four-velocity field) can be cast in this form and a comparison may be extended to the Kerr solution if attention is restricted to the equatorial plane $z = 0$ in cylindrical coordinates or $\theta = \frac{\pi}{2}$ in Boyer-Lindquist coordinates, or "equatorial Kerr" for short. Since the Boyer-Lindquist coordinates $\{t, r, \theta, \phi\}$ are more familiar, these will be used for the "equatorial Kerr" case, with $\rho \equiv r$ for uniformity of notation. When needed in that case, "$\partial/\partial z$" $= -\rho^{-1}\partial/\partial\theta$ is a unit vector field in the "z" direction at the equatorial plane, with $\gamma_{zz} = 1 = g_{zz}$ there. For rotating Minkowski and

Gödel these conditions hold everywhere. For convenience the parameter Ω is assumed to be positive here, so that the positive ϕ direction is the "co-rotating" direction and the negative ϕ direction the "counter-rotating" one, as in Kerr with the positive parameter a.

1. Evaluating the splitting fields

Restricting the metric Eq. (1) to the equatorial plane specializes it to the form

$$\begin{aligned} ds^2 &= g_{tt}dt^2 + 2g_{t\phi}dt\,d\phi + g_{\phi\phi}d\phi^2 \\ &\quad + g_{\rho\rho}d\rho^2 \qquad \text{(reference)}\,, \\ &= -M^2(dt - M_\phi d\phi)^2 + \gamma_{\phi\phi}d\phi^2 \\ &\quad + \gamma_{\rho\rho}d\rho^2 \qquad \text{(threading)}\,, \\ &= -N^2dt^2 + g_{\phi\phi}[d\phi + N^\phi dt]^2 \\ &\quad + g_{\rho\rho}d\rho^2 \qquad \text{(slicing)}\,. \end{aligned} \tag{2}$$

The various metric variables for each of the cases of interest are listed below with the shorthand notation $a \sim b$ denoting that b is the leading behavior of a in the limit $\rho \to 0$ in the rotating Minkowski and Gödel spacetimes and in the limit $\rho^{-1} \to 0$ in the equatorial Kerr case.

1.1. Reference point of view quantities

$$\text{Minkowski:} \begin{cases} -g_{tt} &= \gamma^{-2} = 1 - \Omega^2\rho^2, \\ g_{t\phi} &= \Omega\rho^2 \\ g_{\phi\phi} &= \rho^2 \\ g_{\rho\rho} &= 1 \end{cases}$$

$$\text{Gödel:} \begin{cases} -g_{tt} &= 1 \\ g_{t\phi} &= \Omega s^2 \sim \Omega\rho^2 \\ g_{\phi\phi} &= s^2[1 - (s/2\mathcal{R})^2] \sim \rho^2 \\ g_{\rho\rho} &= 1 \end{cases} \tag{3}$$

$$\text{Kerr:} \begin{cases} -g_{tt} &= (1 - \frac{2m}{\rho}) \\ g_{t\phi} &= -\frac{2am}{\rho} \\ g_{\phi\phi} &= (\rho^2 + a^2 + \frac{2a^2m}{\rho}) \sim \rho^2(1 + \frac{a^2}{\rho^2}) \\ g_{\rho\rho} &= (1 + \frac{a^2}{\rho^2} - \frac{2m}{\rho})^{-1} \sim 1 - \frac{a^2}{\rho^2} \end{cases} \tag{4}$$

1.2. Threading point of view quantities

$$\text{Minkowski:} \begin{cases} M &= \gamma^{-1} \sim 1 - \frac{1}{2}\Omega^2\rho^2 \\ M_\phi &= \Omega\rho^2\gamma^2 \sim \Omega\rho^2 \\ \gamma_{\phi\phi} &= \gamma^2\rho^2 \sim \rho^2 \\ \gamma_{\rho\rho} &= 1 \end{cases}$$

$$\text{Gödel:} \begin{cases} M &= 1 \\ M_\phi &= \Omega s^2 \sim \Omega\rho^2 \\ \gamma_{\phi\phi} &= S^2 \sim \rho^2 \\ \gamma_{\rho\rho} &= 1 \end{cases} \tag{5}$$

$$\text{Kerr:} \begin{cases} M &= \sqrt{1 - \frac{2m}{\rho}} \sim 1 - \frac{m}{\rho} \\ M_\phi &= -\frac{2am}{\rho - 2m} \sim -\frac{2am}{\rho} \\ \gamma_{\phi\phi} &= \rho^2 + \frac{a^2}{1 - \frac{2m}{\rho}} \sim \rho^2(1 + \frac{a^2}{\rho^2}) \\ \gamma_{\rho\rho} &= (1 - \frac{2m}{\rho} + \frac{a^2}{\rho^2})^{-1} \sim 1 + \frac{2m}{\rho} \end{cases} \tag{6}$$

1.3. Slicing point of view quantities

$$\text{Minkowski:} \begin{cases} N &= 1 \\ N^\phi &= \Omega \\ N_\phi &= \Omega\rho^2 \end{cases}$$

$$\text{Gödel:} \begin{cases} N &= \{[1 + (s/2\mathcal{R})^2]/[1 - (s/2\mathcal{R})^2]\}^{1/2} \\ &\sim 1 + \frac{1}{2}\Omega^2\rho^2 \\ N^\phi &= \Omega/[1 - (s/2\mathcal{R})^2] \sim \Omega \\ N_\phi &= \Omega s^2 \sim \Omega\rho^2 \end{cases} \tag{7}$$

$$\text{Kerr:} \begin{cases} N &= \sqrt{1 - \frac{2m(\rho^2 + a^2)}{(\rho^3 + a^2\rho + 2a^2m)}} \sim 1 - \frac{m}{\rho} \\ N^\phi &= -\frac{2am}{\rho^3 + a^2\rho + 2a^2m} \sim -\frac{2am}{\rho^3} \\ N_\phi &= -\frac{2am}{\rho} \end{cases} \tag{8}$$

The following notation has been used

$$\begin{aligned} \gamma &= (1 - v^2)^{-1/2}\,, \quad v = \Omega\rho\,, \\ s &= 2\mathcal{R}\sinh(\rho/2\mathcal{R}) \sim \rho \\ S &= \mathcal{R}\sinh(\rho/\mathcal{R}) \sim \rho, \end{aligned} \tag{9}$$

where $\mathcal{R} \equiv (\sqrt{2}\Omega)^{-1}$ is Gödel's curvature parameter a. The following relations are useful in working with the Gödel quantities

$$\begin{aligned} c = \cosh(\rho/2\mathcal{R}) &\sim 1 + \tfrac{1}{2}\Omega^2\rho^2 \\ (s/2\mathcal{R})^2 \sim (\rho/2\mathcal{R})^2 &= \tfrac{1}{2}\Omega^2\rho^2 \\ 1 - (s/2\mathcal{R})^2 &\sim 1 - \tfrac{1}{2}\Omega^2\rho^2 \\ S = sc = \mathcal{R}\sinh(\rho/\mathcal{R}) &\sim \rho \\ C = \cosh(\rho/\mathcal{R}) = 1 + 2(s/2\mathcal{R})^2 &\sim 1 + \Omega^2\rho^2 \\ s_{,\rho} = c\,, \qquad S_{,\rho} &= C\,. \end{aligned} \tag{10}$$

Note that in the limit $\rho \to 0$, the Gödel metric and the rotating Minkowski metric have the same leading behavior in these coordinates (with the same vorticity parameter Ω). Clearly this is the best form of the Gödel metric in which to compare with the primary example of rotation in general relativity, namely the rotating Minkowski metric. The Gödel metric is a solution of the Einstein equations with constant dust energy density $\rho_{(\rm matter)}$ and cosmological constant Λ related by[11] $\Lambda = -\Omega^2 = -1/2\mathcal{R}^2 = 4\pi\rho_{(\rm matter)}$.

1.4. Region of validity of points of two points of view

The threading point of view is valid as long as the threading is timelike, i.e., $g_{tt} < 0$ or $M = [-g_{tt}]^{1/2} > 0$. The slicing point of view is valid as long as the slicing is spacelike or equivalently the normal is timelike, i.e., $g^{tt} < 0$ or $N^{-1} = [-g^{tt}]^{1/2} > 0$.

For rotating Minkowski, $N = 1$ so the slicing point of view is valid everywhere, but $M = \gamma^{-1}$ is real and positive only for $\rho < \rho_{(\rm lc)}$, where $\rho_{(\rm lc)} = \Omega^{-1}$ is the radius (of the "light cylinder") at which the threading becomes null and then spacelike as ρ increases. For Gödel, $M = 1$ so the threading point of view is valid everywhere, while $N^{-1} = 0 = g_{\phi\phi}$ at the light cylinder radius

$$\begin{aligned}\rho_{(\rm lc)} &= 2\mathcal{R}\sinh^{-1} 1 = 2\mathcal{R}\ln(1+\sqrt{2}) \\ &= \sqrt{2}\ln(1+\sqrt{2})\Omega^{-1} \approx 1.25\Omega^{-1}\ , \qquad (11)\end{aligned}$$

where the ϕ coordinate lines become null and then timelike as ρ increases, so the slicing point of view is valid only for $\rho < \rho_{(\rm lc)}$.

For Kerr, the threading becomes spacelike for $\rho < \rho_{\rm erg} = 2m$ at the boundary of the ergosphere where the threading is null, so the threading point of view is only valid outside the ergosphere for $\rho > \rho_{\rm erg}$. The slicing becomes null and then timelike as one passes inside the event horizon at $\rho = \rho_{\rm eh} = m + \sqrt{m^2 - a^2} < 2m$ where the ρ coordinate lines become null and then timelike as ρ decreases, so the slicing point of view is valid outside the event horizon. Both points of view hold outside the ergosphere, which contains the event horizon, while only the slicing point of view holds in the ergosphere itself.

In the rotating Minkowski and Gödel cases, the slicing observers are the family of observers which are locally nonrotating with respect to the observers at the axis of symmetry. In the latter case these observers are forced to rotate by the global rotation of the spacetime as one reaches the light cylinder at $\rho = \rho_{(\rm lc)}$, analogous to the locally nonrotating slicing observers in the Kerr case where they are forced to rotate with the hole as one reaches the event horizon from larger radii. These latter observers are nonrotating with respect each other and locally gyro-fixed directions, while to the observers the threading observers are nonrotating with respect to "inertial observers" at spatial infinity.

In the Minkowski case the slicing spatial metric is flat, but the threading spatial metric is inhomogeneous and has negative curvature which goes infinite at the Killing horizon $\rho = \rho_{(\rm lc)}$. In the Gödel case the threading spatial metric is flat along the z direction which is orthogonal to the ρ-ϕ 2-spaces of constant negative curvature $-1/\mathcal{R}^2$ corresponding to a radius of curvature $\mathcal{R}$, while the slicing spatial 2-metric is inhomogeneous, i.e., not of constant curvature, with a singularity at $\rho = \rho_{(\rm lc)}$ where a signature change occurs. In the cylindrical coordinates a radial spatial geometry force is necessary due to the curvature of the circular ϕ coordinate lines, leading to a radially outward spatial geometry force to compensate. The negative constant curvature geometry contributes an additional outward radial force compared to a flat 2-geometry, that remains even in conformally flat coordinates in this 2-space in the threading case.

1.5. Threading and slicing gravitomagnetic field

The threading gravitomagnetic vector field, namely twice the the vorticity vector of m, has only a component along the z-axis. Since $\gamma_{zz} = 1$, this is the magnitude apart from a possible sign. Its formula

$$\begin{aligned} H(m)^z &= 2\omega(m)^z = [M\,\mathrm{curl}_m \vec{M}]^z \\ &= [M^{*(m)}d(m)\bar{\bar{M}}]^z \qquad (12) \\ &= M(\gamma_{\rho\rho}\gamma_{\phi\phi})^{-1/2}M_{\phi,\rho} \end{aligned}$$

upon evaluation gives respectively

$$\begin{aligned} \text{Minkowski:} &\quad H(m)^z = 2\Omega\gamma^2 \sim 2\Omega\ , \\ \text{Gödel:} &\quad H(m)^z = 2\Omega \qquad (13) \\ \text{Kerr:} &\quad H(m)^z = \frac{2am}{\rho^3(1-2m/\rho)}\ . \end{aligned}$$

The slicing gravitomagnetic field is also aligned with the z-axis and has the physical component (since $g_{zz} = 1$)

$$\begin{aligned} H(n,e_0)^z &= N^{-1}[\mathrm{curl}_n \vec{N}]^z \\ &= N^{-1}(g_{\rho\rho}g_{\phi\phi})^{-1/2}N_{\phi,\rho} \end{aligned} \quad (14)$$

with values

$$\begin{aligned} \text{Minkowski:}\quad & H(n,e_0)^z = 2\Omega\ , \\ \text{Gödel:}\quad & H(n,e_0)^z = 2\Omega c/[1-(s/2\mathcal{R})^2]^{1/2} \\ & \sim 2\Omega\ . \\ \text{Kerr:}\quad & H(n,e_0)^z = \frac{2am}{\rho^3}\ . \end{aligned} \quad (15)$$

The threading gravitomagnetic vector field describes the local rotation of the nearby observers with respect to orthogonal spatial axes of a given observer whose directions are fixed by gyros. The vorticity of the observer congruence is zero in the slicing point of view, where the gravitomagnetic field instead describes a hybrid effect due to the motion of the observers with respect to the nonlinear reference frame. Roughly speaking, as one follows a threading curve (corresponding to sitting at a point of the evolution "space" defined by the nonlinear reference frame), the axes of the sequence of slicing observers passing by this point appear to rotate when the slicing GM field is nonzero.

1.6. Threading and slicing gravitoelectric field

The gravitoelectric field is just minus the acceleration of the observer congruence. The threading and slicing gravitoelectric fields are both radial and the corresponding one-forms (indicated by the $\flat$ notation) are

$$\begin{aligned} \vec{g}(m)^\flat &= -d(m)\ln M = -M^{-1}M_{,\rho}d\rho\ , \\ \vec{g}(n)^\flat &= -d(n)\ln N = -N^{-1}N_{,\rho}d\rho\ . \end{aligned} \quad (16)$$

The threading gravitoelectric field vanishes for the Gödel spacetime where the threading observers are geodesic, but not for the accelerated observers of the rotating Minkowski case, which rotate with respect to the global inertial frame of the slicing observers at the axis of symmetry. Similarly the slicing observers are geodesic in the rotating Minkowski case leading to vanishing slicing gravitoelectric field, but the Gödel slicing observers are accelerated as they rotate in opposition to the global geodesic rotation of the universe. In the Kerr case, both sets of observers are accelerated in order to resist the gravitational attraction of the source and remain at a fixed radius.

The threading fields are

$$\begin{aligned} \text{Minkowski:}\quad & g(m)_\rho = \Omega^2\rho\gamma^2 \sim \Omega^2\rho\ , \\ \text{Gödel:}\quad & g(m)_\rho = 0\ , \\ \text{Kerr:}\quad & g(m)_\rho = -\frac{m}{\rho(\rho-2m)} \sim -\frac{m}{\rho^2}\ , \end{aligned} \quad (17)$$

and the slicing fields are

$$\begin{aligned} \text{Minkowski:}\quad & g(n)_\rho = 0\ , \\ \text{Gödel:}\quad & g(n)_\rho = -\Omega^2 S/[1-(s/2\mathcal{R})^4] \\ & \sim -\Omega^2\rho\ , \\ \text{Kerr:}\quad & g(n)_\rho = \\ & -\frac{m[(\rho^2+a^2)^2-4a^2m\rho]}{\rho(\rho^2+a^2-2m\rho)(\rho^3+a^2\rho+2a^2m)} \\ & \sim -\frac{m}{\rho^2}\ . \end{aligned} \quad (18)$$

2. Equatorial plane timelike circular geodesics

In the Gödel metric the "moving" circular geodesics must counter-rotate with respect to the vorticity as in the rotating Minkowski spacetime. The key difference is that the nonlinear reference frame has accelerated threading observers in the rotating Minkowski case but geodesic ones in the Gödel case, while these properties are interchanged for the slicing observers. In rough terms, the rotating Minkowski space case has a centrifugal force which is missing in Gödel in the threading point of view. Exactly the opposite is true in the slicing point of view, where the Gödel spacetime has a centrifugal force which is missing in the Minkowski case.

Also only the rotating Minkowski case does not admit a pair of oppositely rotating circular geodesics due to the absence of any "real forces," i.e., both the GE and GM forces vanish in the slicing point of view. The counter-rotating circular geodesics are just the points "fixed in space," i.e., with respect to the inertial frame of the observer at the axis of symmetry. In the Gödel case, the corotating circular geodesics are used as the threading curves themselves, and thus are at rest with respect to the nonlinear reference frame. The GM field is responsible for the existence of this pair of oppositely rotating circular geodesics. In the

Kerr spacetime the GE field is instead responsible for the existence of the pair of co-rotating and counter-rotating circular geodesics, while the GM field introduces an asymmetry in their angular velocities.

Studying the radial force equation for timelike geodesics helps to see the relationship between these three cases and understand the basis of the previous remarks. The Lagrangian for these geodesics in the equatorial plane is

$$\begin{aligned} L &= \tfrac{1}{2}[-|g_{tt}|t'^2 + 2g_{t\phi}\phi' t' + g_{\phi\phi}\phi'^2 + g_{\rho\rho}\rho'^2] \\ &= \tfrac{1}{2}[-M^2(t' - M_\phi\phi')^2 + \gamma_{\phi\phi}\phi'^2 + \gamma_{\rho\rho}\rho'^2] \\ &= \tfrac{1}{2}[-N^2 t'^2 + g_{\phi\phi}(\phi' + N^\phi t')^2 + g_{\rho\rho}\rho'^2], \end{aligned} \quad (19)$$

where f' indicates the derivative of f with respect to an affine parameter along the geodesic. Similarly $\dot{f} = f'/t'$ will indicate the coordinate time derivative of f along the geodesic.

There are two conserved momenta $\tilde{p}_\alpha = \partial L/\partial x^{\alpha\prime}$ for such geodesics in this nonlinear reference frame. These are the angular momentum $\tilde{p}_\phi$ associated with the axial symmetry, and the energy $\mathcal{E} = -\tilde{p}_t$ associated with the time translation. These latter two have the representations

$$\begin{aligned} p_\phi &= g_{\phi\phi}\phi' + g_{t\phi}t' \\ &= t'[\gamma_{\phi\phi}\dot{\phi} + (1 - M_\phi\dot{\phi})M^2 M_\phi] \\ &= t'[g_{\phi\phi}(\dot{\phi} + N^\phi)] , \end{aligned} \quad (20)$$

and

$$\begin{aligned} \mathcal{E} &= |g_{tt}|t' - g_{t\phi}\phi' \\ &= t'[1 - M_\phi\dot{\phi}]M^2 \\ &= t'[N^2 - (\dot{\phi} + N^\phi)N_\phi] . \end{aligned} \quad (21)$$

The radial force balance equation for circular geodesics with $\rho' = 0$ is

$$\begin{aligned} 0 &= (g_{\rho\rho}\rho')' \\ &= \partial L/\partial\rho \\ &= t'^2[\underbrace{-\tfrac{1}{2}|g_{tt}|_{,\rho}}_{GE} + \underbrace{g_{t\phi,\rho}\dot{\phi}}_{GM} + \underbrace{\tfrac{1}{2}g_{\phi\phi,\rho}\dot{\phi}^2}_{SG}] \\ &= t'^2[\underbrace{(-\tfrac{1}{2}M^2)_{,\rho}(1 - M_\phi\dot{\phi})^2}_{GE} + \\ &\quad + \underbrace{M^2(1 - M_\phi\dot{\phi})M_{\phi,\rho}\dot{\phi}}_{GM} \\ &\quad + \underbrace{\tfrac{1}{2}\gamma_{\phi\phi,\rho}\dot{\phi}^2}_{SG}] \\ &= t'^2[\underbrace{(-\tfrac{1}{2}N^2)_{,\rho}}_{GE} + \underbrace{N_{\phi|\rho}(\dot{\phi} + N^\phi)}_{GM} \\ &\quad + \underbrace{\tfrac{1}{2}g_{\phi\phi,\rho}\dot{\phi}(\dot{\phi} + N^\phi)}_{SG}] , \end{aligned} \quad (22)$$

where

$$N_{\phi|\rho} = N_{\phi,\rho} - \tfrac{1}{2}g_{\phi\phi,\rho}N^\phi . \quad (23)$$

The three terms on each right hand side represent respectively the radial gravitoelectric, gravitomagnetic, and spatial geometry force terms in the reference, threading, and slicing points of view. The latter two are regroupings with respect to the reference decomposition. In the threading point of view $\dot{\phi}$ is the coordinate relative angular velocity of the geodesics with respect to the observers, which instead has the value $\dot{\phi} + N^\phi$ in the slicing point of view.

One can also re-express the three terms in the reference decomposition of the radial force balance equation in terms of the slicing and threading points of view

$$\begin{aligned} 0 &= +\underbrace{(-\tfrac{1}{2}M^2)_{,\rho}}_{RGE} + \underbrace{(M^2 M_\phi)_{,\rho}\dot{\phi}}_{RGM} \\ &\quad + \underbrace{\tfrac{1}{2}(\gamma_{\phi\phi} - M^2 M_\phi M_\phi)_{,\rho}\dot{\phi}^2}_{RSG} \\ &= +\underbrace{(-\tfrac{1}{2}N^2)_{,\rho} + N^\phi N_{\phi|\rho}}_{RGE} + \underbrace{N_{\phi,\rho}\dot{\phi}}_{RGM} + \underbrace{\tfrac{1}{2}g_{\phi\phi,\rho}\dot{\phi}^2}_{RSG} . \end{aligned} \quad (24)$$

This helps one see the correspondence with the simpler reference point of view equation.

Since the radial force balance equation for circular geodesics is a quadratic equation in the coordinate time angular velocity $\dot{\phi}$, it is easily solved in general yielding

$$\dot{\phi} = [g_{\phi\phi,\rho}]^{-1}[-g_{t\phi,\rho} \pm \sqrt{g_{t\phi,\rho}{}^2 + |g_{tt}|_{,\rho}g_{\phi\phi,\rho}}] , \quad (25)$$

which is a function of ρ. The explicit solutions in the three cases of interest are

$$\begin{aligned} \text{Minkowski:} &\quad \dot{\phi} = -\Omega , \\ \text{Godel:} &\quad \dot{\phi} = 0 \quad \text{or} \\ &\quad \dot{\phi} = -2\Omega/[1 - 2(s/2\mathcal{R})^2] \sim -2\Omega , \\ \text{Kerr:} &\quad \dot{\phi} = \omega_\pm \equiv \frac{\pm\sqrt{m/\rho^3}}{1 \pm a\sqrt{m/\rho^3}} \\ &\quad \sim \pm\sqrt{m/\rho^3}[1 \mp a\sqrt{m/\rho^3}] . \end{aligned} \quad (26)$$

Since $a \geq 0$, the absolute value of the co-rotating angular velocity $|\omega_+|$ is less than the counter-rotating one $|\omega_-|$ in the Kerr case, leading to the counter-rotation of the meeting points, with coordinate-time angular velocity

$$\dot{\phi} = \omega_+ + \omega_- = -\frac{2am/\rho^3}{1 - a^2 m/\rho^3} \ . \quad (27)$$

Note that the leading behavior of this counter-rotational angular velocity agrees with the leading behavior of minus the magnitude of the gravitomagnetic vector field (in any point of view), similar to the counter-rotating circular geodesic angular velocity in the Gödel case.

The circular geodesics are timelike when the Lagrangian function, just the kinetic energy, is negative. For the rotating Minkowski case, one finds $2L = -\mathcal{E}^2 < 0$, i.e., they are always timelike. For the Gödel case, one finds

$$2L = -\mathcal{E}^2[1 - (S/\mathcal{R})^2]/[1 + 2(s/2\mathcal{R})^2] \ . \quad (28)$$

This vanishes for

$$\rho = \mathcal{R} \sinh^{-1} 1 = \tfrac{1}{2}\rho_{(\mathrm{lc})} \ , \quad (29)$$

so the geodesics are timelike only for $\rho < \frac{1}{2}\rho_{(\mathrm{lc})}$. In the Kerr case the counter-rotating circular geodesics are timelike only for $\rho > \rho_- > 2\sqrt{3}m$ while the co-rotating ones are timelike for $\rho > \rho_+ < 2\sqrt{3}m$.

2.1. Rotating Minkowski: radial force balance equation

The radial force balance equation is the radial reference splitting of the geodesic equation; for flat spacetime, one has in the reference point of view

$$\begin{aligned} 0 &= \underbrace{-\Gamma^\rho{}_{tt}}_{GE} \quad \underbrace{-2\Gamma^\rho{}_{t\phi}\dot{\phi}}_{GM} \quad \underbrace{-\Gamma^\rho{}_{\phi\phi}\dot{\phi}^2}_{SG} \\ &= \Omega^2\rho \quad +2\Omega\rho\dot{\phi} \quad +\rho\dot{\phi}^2 \\ &= \Omega^2\rho \quad -2\Omega^2\rho \quad +\Omega^2\rho = 0 \end{aligned} \quad (30)$$

where the last line indicates the case of counter-rotating circular geodesics which result from an equal balance of the inward GM force and the two equal outward SG and GE forces, which occurs for $\dot{\phi} = -\Omega < 0$.

The rotating Minkowski radial force balance equation in the threading point of view is

$$\begin{aligned} 0 &= \underbrace{(-\tfrac{1}{2}M^2)_{,\rho}(1 - M_\phi\dot{\phi})^2}_{GE} + \underbrace{M^2(1 - M_\phi\dot{\phi})M_{\phi,\rho}\dot{\phi}}_{GM} \\ &\quad + \underbrace{\tfrac{1}{2}\gamma_{\phi\phi,\rho}\dot{\phi}^2}_{SG} \\ &= \rho\Omega^2[1 - \Omega\rho^2\gamma^2\dot{\phi}]^2 + 2\Omega\rho\gamma^4[1 - \Omega\rho^2\gamma^2\dot{\phi}]\dot{\phi} \\ &\quad + \rho\gamma^4\dot{\phi}^2 \\ &= \Omega^2\rho\gamma^4 - 2\Omega\rho\gamma^4 + \Omega^2\rho\gamma^4. \end{aligned} \quad (31)$$

This only differs by the gamma factor from the previous result for the circular geodesics.

The slicing point of view is

$$\begin{aligned} 0 &= \underbrace{(-\tfrac{1}{2}N^2)_{,\rho}}_{GE} + \underbrace{N_{\phi|\rho}(\dot{\phi} + N^\phi)}_{GM} \\ &\quad + \underbrace{\tfrac{1}{2}g_{\phi\phi,\rho}\dot{\phi}(\dot{\phi} + N^\phi)]}_{SG} \\ &= 0 + \rho\Omega(\dot{\phi} + \Omega) + \rho\dot{\phi}(\dot{\phi} + \Omega) \\ &= 0 + 0 + 0. \end{aligned} \quad (32)$$

Here all the velocity dependent slicing forces vanish since the circular geodesics are at rest with respect to the slicing observers, and are just the worldlines of these inertial observers.

2.2. Gödel: radial force balance equation

For the Gödel spacetime the radial force balance equation becomes

$$\begin{aligned} 0 &= \underbrace{-\Gamma^\rho{}_{tt}}_{GE}\underbrace{-2\Gamma^\rho{}_{t\phi}\dot{\phi}}_{GM}\underbrace{-\Gamma^\rho{}_{\phi\phi}\dot{\phi}^2}_{SG} \\ &= 0 + 2\Omega S\dot{\phi} + S[1 - 2(s/2\mathcal{R})^2]\dot{\phi}^2 \\ &= 0 - 4\Omega^2 S/[1 - 2(s/2\mathcal{R})^2] \\ &\quad + 4\Omega^2 S/[1 - 2(s/2\mathcal{R})^2] = 0 \\ &\sim 0 - 4\Omega^2\rho + 4\Omega^2\rho = 0 \ . \end{aligned} \quad (33)$$

where the last two lines indicate the case of counter-rotating circular geodesics which result from an equal balance of the inward GM force and the outward SG force.

The radial force balance equation in the threading point of view is

$$\begin{aligned} 0 &= \underbrace{(-\tfrac{1}{2}M^2)_{,\rho}(1 - M_\phi\dot{\phi})^2}_{GE} + \underbrace{M^2(1 - M_\phi\dot{\phi})M_{\phi,\rho}\dot{\phi}}_{GM} \\ &\quad + \underbrace{\tfrac{1}{2}\gamma_{\phi\phi,\rho}\dot{\phi}^2}_{SG} \\ &= 0 + 2\Omega S[1 - \Omega s^2\dot{\phi}]\dot{\phi} + SC\dot{\phi}^2 \\ &= 0 - 4\Omega^2 SC/[1 - 2(s/2\mathcal{R})^2]^2 \\ &\quad + 4\Omega^2 SC/[1 - 2(s/2\mathcal{R})^2]^2 = 0 \\ &\sim 0 - 4\Omega^2\rho + 4\Omega^2\rho = 0 \ , \end{aligned} \quad (34)$$

and in the slicing point of view one has

$$0 = \underbrace{(-\tfrac{1}{2}N^2)_{,\rho}}_{GE} + \underbrace{N_{\phi|\rho}(\dot\phi + N^\phi)}_{GM} + \underbrace{\tfrac{1}{2}g_{\phi\phi,\rho}\dot\phi(\dot\phi + N^\phi)}_{SG} \quad (35)$$

where the three terms have the values

$$\begin{aligned} GE: &\quad \frac{-\Omega^2 S}{[1-(s/2\mathcal{R})^2]^2} \sim -\Omega^2\rho \\ GM: &\quad \frac{S\Omega}{1-(s/2\mathcal{R})^2}[\dot\phi + \frac{\Omega}{1-(s/2\mathcal{R})^2}] \\ &\quad \sim -\Omega^2\rho \\ SG: &\quad S(1-2(s/2\mathcal{R})^2)\dot\phi[\dot\phi + \frac{\Omega}{1-(s/2\mathcal{R})^2}] \\ &\quad \sim +2\Omega^2\rho\,. \end{aligned} \quad (36)$$

For both the reference and threading points of view, all force terms vanish for $\dot\phi = 0$ representing the worldlines of the inertial threading observers. In the slicing point of view the inward GE and outward GM forces balance each other.

2.3. Equatorial Kerr: radial force balance equation

For Kerr spacetime, in the reference point of view, the radial force balance equation is

$$\begin{aligned} 0 &= \underbrace{-\Gamma^\rho{}_{tt}}_{GE} \quad \underbrace{-2\Gamma^\rho{}_{t\phi}\dot\phi}_{GM} \quad \underbrace{-\Gamma^\rho{}_{\phi\phi}\dot\phi^2}_{SG} \\ &= -\tfrac{m}{\rho^2} \quad +\tfrac{2am}{\rho^2}\dot\phi \quad +(\rho - \tfrac{a^2m}{\rho^2})\dot\phi^2 \\ &\sim -\tfrac{m}{\rho^2} \quad +\tfrac{2am}{\rho^2}\dot\phi \quad +\rho\dot\phi^2\,. \end{aligned} \quad (37)$$

In the threading point of view it becomes

$$\begin{aligned} 0 &= \underbrace{(-\tfrac{1}{2}M^2)_{,\rho}(1-M_\phi\dot\phi)^2}_{GE} + \underbrace{M^2(1-M_\phi\dot\phi)M_{\phi,\rho}\dot\phi}_{GM} \\ &\quad + \underbrace{\tfrac{1}{2}\gamma_{\phi\phi,\rho}\dot\phi^2}_{SG} \\ &= -\frac{m}{\rho^2}(1+\frac{2am}{\rho-2m}\dot\phi)^2 + \frac{2am}{\rho(\rho-2m)}(1+ \\ &\quad \frac{2am}{\rho-2m}\dot\phi)\dot\phi + (\rho - \frac{a^2m}{(\rho-2m)^2})\dot\phi^2 \end{aligned} \quad (38)$$

and in the slicing point of view

$$0 = \underbrace{(-\tfrac{1}{2}N^2)_{,\rho}}_{GE} + \underbrace{N_{\phi|\rho}(\dot\phi + N^\phi)}_{GM} + \underbrace{\tfrac{1}{2}g_{\phi\phi,\rho}\dot\phi(\dot\phi + N^\phi)]}_{SG} \quad (39)$$

with explicit values

$$\begin{aligned} GE: &\quad -\frac{m((a^2+\rho^2)^2 - 4a^2m\rho)}{(\rho^3+a^2\rho+2a^2m)^2} \\ GM: &\quad \frac{2am(2\rho^3+a^2m+a^2\rho)}{\rho^2(\rho^3+a^2\rho+2a^2m)}(\dot\phi \\ &\quad -\frac{2am}{\rho^3+a^2\rho+2a^2m}) \\ SG: &\quad (\rho - \frac{a^2m}{\rho})\dot\phi(\dot\phi - \frac{2am}{\rho^3+a^2\rho+2a^2m}). \end{aligned} \quad (40)$$

The leading behavior of the three terms in both the threading and slicing points of view agrees with the leading behavior of the corresponding terms in the reference point of view.

III. DISCUSSION OF RESULTS AND CONCLUDING REMARKS

We now summarize and discuss the previous results in order to focus attention on the common features and the differences between the various spacetimes in the various splitting approaches.

The threading GM field, namely twice the the vorticity vector of m, the four-velocity of the threading observers, has only a component along the z-axis. The threading GE field vanishes for the Gödel spacetime where the threading observers are geodesic, but not for the radially inward accelerated observers of the rotating Minkowski case. Similarly the slicing observers are geodesic in the rotating Minkowski case leading to vanishing slicing GE field, but in the Gödel spacetime the slicing observers are accelerated radially outward to resist the global rotation of the spacetime by counter-rotating, leading to a radially inward slicing GE field. The equatorial Kerr slicing and threading observers are both accelerated radially outward to oppose the attraction of the central mass, leading to inward GE fields which allow circular orbits even in the absence of rotation.

The spatial metric in the slicing point of view is the flat Euclidean metric so no spatial geometry force arises in Cartesian coordinates, but in cylindrical coordinates a radially outward spatial geometry force is necessary to compensate for the inward radial acceleration which allows counter-rotating circular orbits (corresponding to points "fixed" in nonrotating space). The spatial geometry in the remaining cases and points of view is not flat leading to "real" spatial geometry forces in the radial direction. Note that in the Gödel spacetime, the homogeneous threading spatial metric is the product of a 1-dimensional flat space with a 2-dimensional space of constant negative curvature, while in rotating Minkowski the corresponding 2-dimensional space is inhomogeneous and of negative curvature, like the slicing spatial metric in the Gödel case. The equatorial Kerr case is similar, in all cases leading to an outward radial spatial geometry force for the circular geodesics. By introducing conformally Cartesian coordinates in the equatorial plane in a given point of view, one can attempt to determine the excess spatial geometry force beyond that due to the curvature of the ϕ coordinate lines.

The analysis of the timelike circular geodesics in the Gödel spacetime shows that the circular geodesics must counter-rotate with respect to the vorticity as in the rotating Minkowski spacetime, a fact hard to find clearly stated in any existing work on the Gödel spacetime. It is exactly this same effect due to the GM field which is responsible for the counter-rotation of the meeting points of the co-rotating and counter-rotating circular geodesics in the Kerr spacetime, while the circular orbit itself is allowed by the inward GE field of Kerr.

The threading shift vector field in rotating Minkowski spacetime and the slicing shift in the Gödel spacetime are both in the positive ϕ direction and both increase with increasing ρ (so that the corresponding GM field is in the positive z direction), leading to horizons at which the corresponding point of view fails to hold outside of the

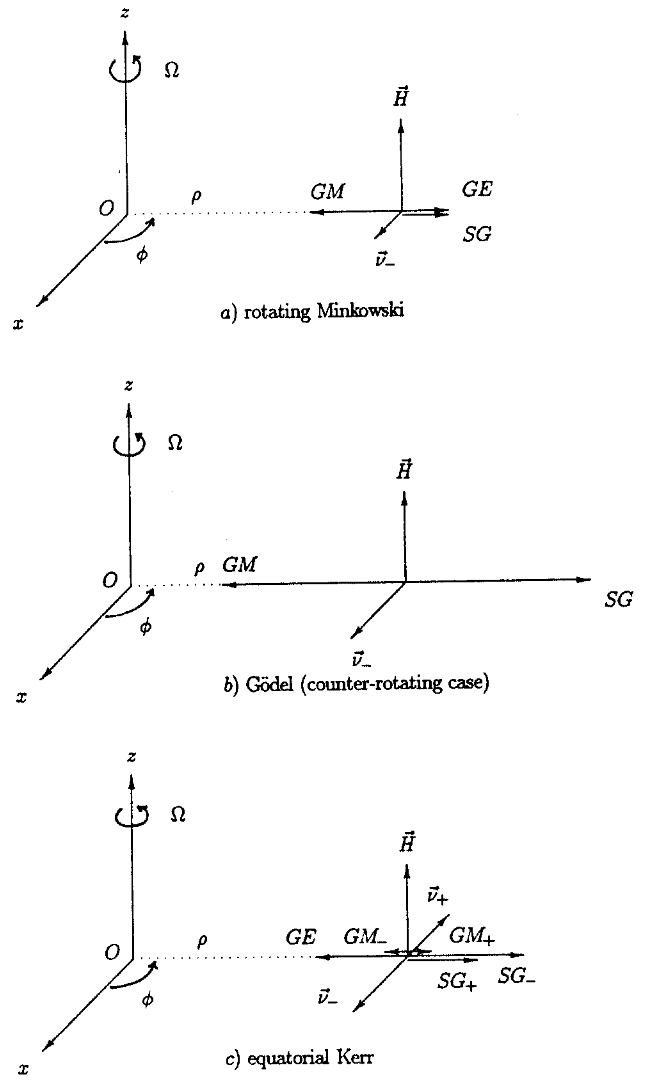

Fig. 1. The threading/reference point of view picture of the radial force balance for circular geodesics.

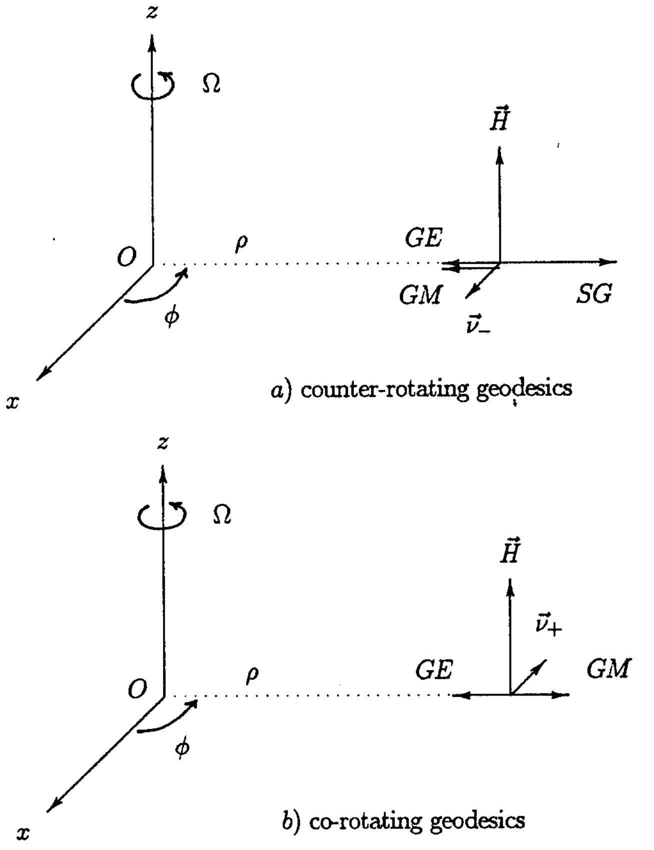

Fig. 2. The slicing point of view picture of the radial force balance for circular geodesics in the Gödel spacetime.

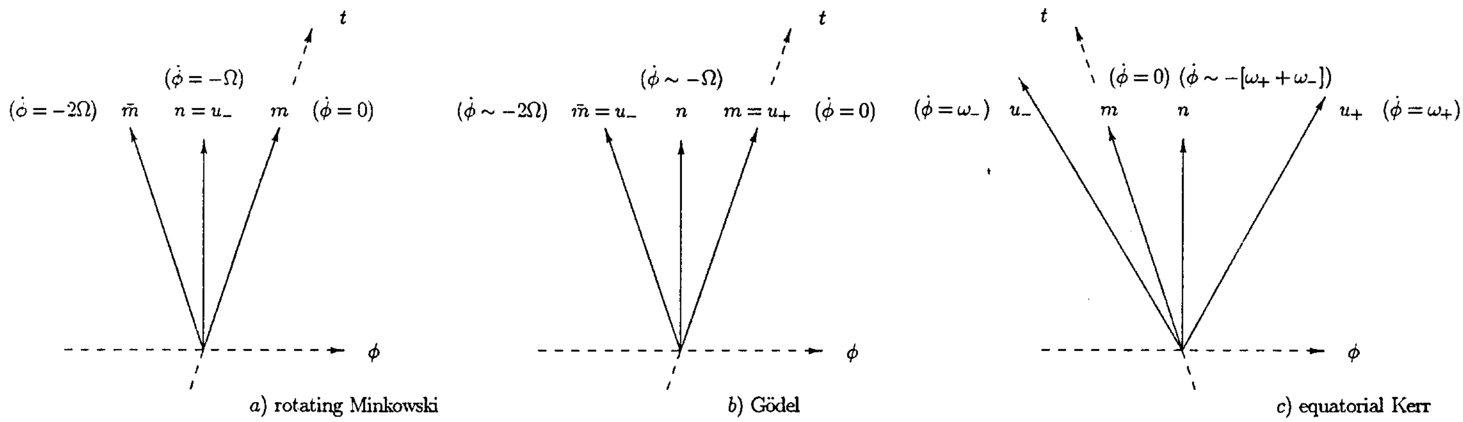

Fig. 3. The t-ϕ plane of the tangent space indicating the four-velocities of the observers and of the circular geodesics.

radius at which the threading and slicing respectively change their causal nature. The rotating Minkowski spacetime has a threading horizon at $\rho = \rho_{(\mathrm{lc})}$ where the threading becomes null and then spacelike for larger ρ values. The Gödel spacetime has a slicing horizon at $\rho = \rho_{(\mathrm{lc})}$, where the normal trajectories to the slicing become null and then spacelike. It is at this point that the family of observers which is locally nonrotating with respect to the axis of symmetry of the nonlinear reference frame can no longer oppose being dragged along by the spacetime geometry.

For Kerr the situation is reversed. In each point of view the shift increases with decreasing radius (but is in the counterclockwise direction so the GM field is still in the positive z direction). Thus as one moves inward first the threading becomes null and then spacelike as one passes the boundary of the ergosphere, and then the slicing becomes null and then spacelike as one passes the event horizon. The addition of the GM field adds an outward radial force to the inward GE force for the co-rotating circular geodesics, thus increasing the equilibrium circular velocity and slightly reducing the radial SG force. Just the opposite situation describes the counter-rotating circular geodesics, where the additional radially inward GM force decreases the equilibrium circular velocity. The difference in absolute value of their two oppositely signed angular velocities leads to the counter-rotation of their meeting points by an angular velocity which at large distances from the ergosphere approaches the value of the GM field and of the relative angular velocity of the slicing and threading observers.

Many of these remarks are better visualized with the help of the suggestive Fig. 1, which illustrates the radial force balance in a spatial diagram qualitatively representing the threading or reference points of view, indicating only the counter-rotating geodesic in the Gödel case. For a given sufficiently small radius where one can compare the Minkowski and Gödel cases, the velocity of the counter-rotating circular geodesic is doubled in Gödel compared to Minkowski, leading to a doubled GM force and a SG force four times as large to compensate for this doubling and the absense of an equal GE force.

Figure 2 illustrates the slicing point of view for the both the co-rotating and counter-rotating circular geodesics in the Gödel case. In the slicing point of view the slicing velocity is halved, and the GM force involves another factor of $\frac{1}{2}$, so the GM force is reduced by a factor of 4 compared to the threading point of view for the counter-rotating geodesics. It is balanced by two equal contributions from the outward GE and SG forces. The co-rotating circular geodesics have no SG force since $\dot\phi = 0$, leading to a balance of the same inward GE force by the outward GM force.

Figure 3 is a t-ϕ plane tangent space diagram indicating the four-velocities of the two observers and the circular geodesics. In the rotating Minkowski diagram $\bar{m}$ indicates the four-velocity of the counter-rotating observers with the opposite angular velocity (and acceleration) of the threading observers relative to the slicing observers. In the Gödel case $\bar{m}$ indicates the counter-rotating geodesics, which have the opposite angular velocity compared to the threading observers with respect to the slicing observers in the limit $\rho \to 0$. In the Kerr case no particularly useful such four-velocity $\bar{m}$ arises.

A more extensive discussion of these spacetimes in "Cartesian-like" coordinates helps to drive home the relationships between these three spacetimes and the way in which rotation ("gravitomagnetism") manifests itself in various combinations with acceleration ("gravitoelectricity") and spatial geometry effects in each of the possible splitting representations of the gravitational field. Understanding such relationships helps to develop a better intuition for gravitational physics in more realistic situations and provides a little better foundation for some of the things which are done in approximation schemes.

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