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Perfect Fluid Sources for Spatially Homogeneous Spacetimes*

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Spatially homogeneous perfect fluid spacetimes are studied from a point of view which emphasizes the spatial geometry and the action of that subgroup of the spatial gauge group of the three-plus-one formulation of general relativity which is compatible with the spatial homogeneity. The specializations of the dynamics which correspond to the existence of additional spacetime symmetries are classified. An unconstrained set of gravitational and fluid variables is obtained by elimination of the gravitational constraints using an approach which obtains the gravitational evolution equations from a suitably modified Lagrangian/Hamiltonian formalism. A slightly different choice of variables is then described which allows one to take full advantage of the spatial gauge group and of the 1-parameter group of scale transformations of the unit of length.

1. INTRODUCTION

During the past two decades, spatially homogeneous spacetimes have attracted a great deal of attention as mathematically tractable models allowing the study of various ideas of gravitational theory ranging from rather abstract notions to those with important physical implications for the actual universe. However, certain aspects of these investigations have not been very systematic in character. In particular, many features of special cases have gone unrecognized as having more generality, as is revealed by a more complete understanding of the way in which spatial homogeneity acts in simplifying the mathematical treatment of such spacetimes. By properly exploiting the common properties of spatially homogeneous spacetimes of all of the Bianchi symmetry types, one gains a better understanding of each individual Bianchi type not only in the context of spatial homogeneity but as far as more general ideas of gravitational theory are concerned. This has been illustrated in a number of earlier papers [1–6].

The most frequently used source for the gravitational field in cosmology, apart from no source at all (vacuum spacetimes), is a perfect fluid, whose energy-momentum tensor T is determined by the pressure p, the energy density ρ, and the 4-velocity u whose integral curves are the flow lines of the fluid. (The notation of Misner *et al.* [7, chap. 22] is followed here.) Since shocks are inconsistent with global

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spatial homogeneity, a spatially homogeneous perfect fluid is adiabatic, with the specific entropy s constant along the flow lines. Since it is also independent of the flow lines on each hypersurface of homogeneity, it is constant on the whole spacetime; i.e., a spatially homogeneous perfect fluid is automatically isentropic. Thus only a single thermodynamic variable serves to characterize the fluid, from which the remaining thermodynamic variables can be derived. This can be taken to be the "baryon number density" n.

Given an equation of state, i.e., ρ as a function of n or p as a function of ρ, the isentropic form of the first law of thermodynamics

$$(\rho + p)^{-1}\, d\rho = d \ln n \tag{1.1}$$

may be used to determine the other thermodynamic variables in terms of n. One useful such variable is the "chemical potential" $\mu = (\rho + p)\, n^{-1}$ which satisfies in the isentropic case

$$(\rho + p)^{-1}\, dp = d \ln \mu. \tag{1.2}$$

For example, the following equation of state is used almost exclusively in cosmology,

$$p = (\gamma - 1)\, \rho, \tag{1.3}$$

where $\gamma \in [1, 2]$ is a constant, in which case

$$\rho = n^{\gamma}, \qquad p = (\gamma - 1)\, n^{\gamma}, \qquad \mu = \gamma n^{\gamma - 1}. \tag{1.4}$$

The case of vanishing pressure, when the perfect fluid is called dust, corresponds to $\gamma = 1$ in which case $\rho = n$ and $\mu = 1$, while the so called "stiff perfect fluid" corresponds to $\gamma = 2$. The value $\gamma = 4/3$ describes a radiation gas.

The kinematical properties of a spatially homogeneous perfect fluid involve the 4-velocity u and may be somewhat more transparent in an approach using an orthonormal or "nearly" orthonormal frame adapted to the fluid, as illustrated in papers by Ellis and various coworkers [8–12]. Similarly, use of the Newmann–Penrose formalism by Siklos easily answered certain questions about singularities in spatially homogeneous perfect fluid spacetimes [13]. However, the field equations are extremely complicated in these formulations and do not easily lend themselves to any intuitive understanding. Since the spacetime geometry is considerably more complicated than the fluid source, it pays instead to simplify the description of that geometry as much as possible. Furthermore, the spacelike orbits of the spatial homogeneity isometry group provide a natural slicing of a spatially homogeneous spacetime making such spacetimes the most logical candidates for the application of the three-plus-one approach to gravitation, an approach which describes the evolution of the spatial geometry of a hypersurface as it moves through spacetime ("dynamics"). The existence of a preferred slicing ultimately relates every treatment of a spatially homogeneous spacetime to the three-plus-one approach.

A powerful tool in analyzing the dynamics of spatially homogeneous spacetimes which can lead to important simplications are Lagrangian or Hamiltonian methods.

These were first advocated by Misner [14] and extensively applied by Ryan [15–19] who introduced an "Ω-time" Hamiltonian formalism coupled with an $SO(3,R)$-adapted parametrization of the gravitational configuration space $\mathscr{M} \subset GL(3,R)$ of matrices of inner products on R^3, for all the Bianchi types of spatial homogeneity.

This configuration space, called "minisuperspace" by Misner [20], is tied to the space of spatially homogeneous metrics on the spacelike hypersurfaces of homogeneity through a choice of a comoving frame adapted both to the slicing of the spacetime by those hypersurfaces and to the symmetry itself. Such a "comoving ADM frame" for this slicing (or "computational frame" [21]) consists of a spatial frame tangent to the slicing which is dragged along by the remaining off-surface element of the frame, or "ADM generator" for the slicing. The frame is adapted to the symmetry by requiring that the spatial frame be spatially homogeneous, implying that the normal and tangential parts of spatially homogeneous spacetime fields relative to the natural slicing form a collection of spatially homogeneous spatial fields whose components in the spatial frame are themselves spatially homogeneous, i.e., depend only on the parametrization of the slicing. This class of comoving ADM frames is the largest one compatible with the spatial homogeneity in the sense that spacetime tensor equations for spatially homogeneous fields, when decomposed into normal and tangential parts with respect to the spatially homogeneous slicing and expressed in a frame of this class, reduce to ordinary differential equations in the spatial components of the collection of spatial fields obtained from the normal-tangential decomposition of the original spacetime fields. The matrix of components of the spatial metric on the family of spatially homogeneous hypersurfaces, for example, will be a curve in the gravitational configuration space $\mathscr{M}$, and the Einstein equations become ordinary differential equations for this curve.

Given a parametrization of the spatially homogeneous slicing, i.e., a choice of the spatially homogeneous lapse function, the choice of a symmetry adapted comoving ADM frame amounts first to a choice of basis $e = \{e_a\}$ of the Lie algebra $\mathfrak{g}$ of the 3-dimensional isometry group G associated with the spatial homogeneity (here assumed to be simply connected), i.e., a set of structure constant tensor components $C^a{}_{bc}$. This determines a spatially homogeneous spatial frame on an initial hypersurface of homogeneity. Next, a choice of shift vector field must be made such that this spatial frame remains spatially homogeneous as it is dragged along to the other hypersurfaces of homogeneity. The lapse and shift together determine the ADM generator completing the spatial frame to a comoving ADM frame. The shift must therefore be confined to the Lie algebra of the largest group of spatial diffeomorphisms which preserves the spatial homogeneity of spatial fields. This Lie group, denoted here by $\mathscr{D}(\mathfrak{g})$, is the spatial gauge group of the class $F_G(M)$ of symmetry adapted comoving ADM frames on the spacetime manifold M.

Its linear action on the components of spatially homogeneous spatial fields with respect to a given element of $F_G(M)$ generated from a basis e of $\mathfrak{g}$ is represented by the group $\mathrm{Aut}_e(\mathfrak{g})$, the matrix representation of the automorphism group $\mathrm{Aut}(\mathfrak{g})$ of $\mathfrak{g}$ with respect to the basis e, equivalently defined as the subgroup of $GL(3,R)$ which leaves the structure constant tensor components $C^a{}_{bc}$ invariant under the natural action of $GL(3, R)$. This latter group therefore plays the same role in the ordinary differential equations obtained from spatially homogeneous tensor equations that the 3-dimensional diffeomorphism group plays in the partial differential equations obtained from general tensor equations on a spacetime [22, 23]. However, the reduction to finite dimension enables one to complete many of the programs which the infinite-dimensional setting prevents. The best example is the isolation of the true gravitational degrees of freedom [21, 23–25] and its relation at the lapse and shift level to the work of Smarr and York [21, 26], both of which are intimately connected with the Moncrief decomposition [23, 27]. This is accomplished in the spatially homogeneous case by using a parametrization of the "minisuperspace" adapted to the orbits of the gauge group $\mathrm{Aut}_e(\mathfrak{g})$, the orbit space $\mathscr{M}/\mathrm{Aut}_e(\mathfrak{g})$ being the true analogue of superspace [28].

The importance of the group $\mathrm{Aut}_e(\mathfrak{g})$ was recognized by Collins and Hawking in their treatment of Bianchi type VII_h and VII_0 spacetimes [29]. In addition to its role in the determination of the spatial frame at later times obtained from a spatial frame at some initial time, which by continuity involves only the identity component $\mathrm{Aut}_e(\mathfrak{g})^+$, the full automorphism group is relevant in the choice of the initial spatial frame, since for a given set of structure constant tensor components $C^a{}_{bc}$, the basis e of $\mathfrak{g}$ is only determined modulo the action of $\mathrm{Aut}(\mathfrak{g})$ on $\mathfrak{g}$. This freedom is important in simplifying initial data.

For Bianchi types I and IX, the group $SO(3, R)$ is, respectively, contained in and is the group $\mathrm{Aut}_e(\mathfrak{g})$ and the Ryan approach may be successfully applied. The surprising simplifications which occur are directly related to the fact that $\mathrm{Aut}_e(\mathfrak{g})$ is the finite-dimensional analogue of the spatial diffeomorphism gauge group in the three-plus-one approach to general relativity. For the remaining Bianchi types of spatial homogeneity, however, $SO(3, R)$ is not related to the group $\mathrm{Aut}_e(\mathfrak{g})$ and the utility of the Ryan approach is destroyed. It may be salvaged by introducing a parametrization of the configuration space $\mathscr{M}$ adapted to the symmetry, namely, a parametrization of the spatial metric component matrix, where $SO(3, R)$ is replaced by a suitable 3-dimensional subgroup of $\mathrm{Aut}_e(\mathfrak{g})$. This must be accompanied by a choice of structure constant tensor components in "standard diagonal form" [1] to simplify the description of the spatial geometry. The result is that the discussion of the remaining Bianchi types closely parallels that of the Bianchi type I and IX cases, with the accompanying simplifications.

Variational methods which use a Lagrangian or Hamiltonian to obtain the Einstein equations cannot be blindly applied to the spatially homogeneous system since the imposition of symmetry does not always commute with the derivation of the field equations [30]. In particular, if one wishes to derive the evolution equations from a variational principle using the most general shift vector field compatible with the spatial homogeneity, problems arise for all Bianchi types except the semisimple types VIII and IX, although a slight restriction on the shift (that it be divergencefree [3]) limits them to the class B Bianchi types. These problems are simply resolved by realizing that the spatially homogeneous Einstein equations expressed in a symmetry adapted comoving frame with restricted shift are equivalent to a constrained

Lagrangian/Hamiltonian system in the canonical gravitational variables which is driven by the source variables and a nonpotential force arising in the class B case from the failure of the spatial Einstein tensor force field to be conservative (an exact 1-form on $\mathscr{M}$). The Lagrangian for this system is just the standard ADM Lagrangian density expressed in the symmetry adapted comoving ADM frame, from which the Hamiltonian is easily derived in the usual way (Legendre transformation) rather than by the somewhat complicated process of writing the formal spatially homogeneous Einstein action in first order form.

The simplest formulation is the Lagrangian/Hamiltonian formulation described in Ref. [1] employing zero shift vector field and unit lapse function so that the classical mechanical time of the system coincides with the cosmological proper time function. This means that the ADM generator is fixed to be the unit vector field $e_\perp$ normal to the spatially homogeneous slicing. A different choice of lapse may be made later by reparametrization of the classical mechanical time which is most naturally accomplished by reducing the Hamiltonian system by the super-Hamiltonian constraint. This leads to the Misner–Ryan Ω-time parametrization, for example. The reduction of the Hamiltonian system by the supermomentum constraint (allowed only by the use of a symmetry adapted parametrization of the configuration space) may be interpreted as equivalent to the introduction of a nontrivial shift vector field of a new symmetry adapted comoving ADM frame which simplifies the components of tensors associated with the spatial geometry. This has been discussed in detail in Ref. [1] and is a straightforward (though suitably modified) application of the three-plus-one approach to general relativity.

However, the natural action of the automorphism group $\mathrm{Aut}_e(\mathfrak{g})$ on this Lagrangian/Hamiltonian system is not a faithful representation of the remaining spatial gauge freedom since the Lagrangian function is by definition a scalar on the velocity phase space. While the component of the Lagrangian density is invariant under the action of a constant element of $\mathrm{Aut}_e(\mathfrak{g})$, the Lagrangian function scales by the determinant of this element. By limiting the compatible diffeomorphism group to that subgroup whose linear action on the components of spatially homogeneous fields is represented by the special automorphism group $\mathrm{SAut}_e(\mathfrak{g})$ (the subgroup for which the distinction between tensors and tensor densities is irrelevant), one avoids this difficulty. This limitation corresponds exactly to the restriction on the shift vector field which eliminates an additional velocity-dependent force in the evolution equations present for all Bianchi type groups except the semisimple types VIII and IX, where $\mathrm{SAut}(\mathfrak{g})$ is the identity component of $\mathrm{Aut}(\mathfrak{g})$ [3].

The utility of the group $\mathrm{SAut}_e(\mathfrak{g})$ in simplifying the Einstein equations was recognized by Lukash but used only in the qualitative analysis of special Bianchi type VII_0 and VII_h spacetimes [31] as well as the general type IX spacetimes [32]. In the latter case a specialization of the orthogonal parametrization dates back implicitly to the work of Gödel [33] and later emerged in work by Ozsváth [34]. In fact the present approach to the dynamics of spatially homogeneous cosmology grew out of the work of Ozsváth, who derived the reduced Hamiltonian system presented by Gödel without any details. However, because he failed to understand the role played by the special automorphism group, Ozsváth was unable to extend his results to the general case. This was done in Ref. [1] for an arbitrary spatially homogeneous source.

There remains the appropriate choice of source variables such that the source super-Hamiltonian considered as a function on the gravitational configuration space gives rise to the source driving force appearing in the evolution equations. It is not always possible to avoid a nonpotential component of this force. Furthermore, the equations satisfied by the source variables themselves must be discussed. This has been done for a classical Dirac spinor field, where a nonpotential component of the source driving force necessarily arises in the class B case while the correct source equations also follow from the Hamiltonian in the class A case [6]. In the case of a purely electromagnetic source, no nonpotential component of the source driving force arises with the obvious choice of variables [35]. The aim of the present article is to discuss the perfect fluid in this context. It is found that no nonpotential component of the source driving force need arise [35].

Since this article is intended as a sequel to Ref. [1], the notation and principal results of that work are only briefly sketched here, avoiding as much as possible needless repetition of formulas involving only the gravitational variables. This is followed by manipulation of the fluid equations of motion suggesting the proper choice of fluid variables. The equations of motion of these variables are then interpreted in terms of the action of the linear adjoint group $\mathrm{Ad}(G) \subset \mathrm{Aut}(\mathfrak{g})$ which makes the fluid constants of the motion obvious. Next a classification is given of the specializations which are possible for the combined equations of motion, following from the existence of additional spacetime symmetry. Section 5 introduces new gravitational and fluid variables adapted to the fluid constants of the motion and to the gauge action of a suitable subgroup of $\mathrm{SAut}_e(\mathfrak{g})$. Among these variables is found a minimal unconstrained set sufficient to describe the dynamics in the Lagrangian/Hamiltonian approach. For the semisimple Bianchi types where $\mathrm{SAut}_e(\mathfrak{g})$ is the identity component of $\mathrm{Aut}_e(\mathfrak{g})$, the unconstrained gravitational variables correspond to the two degrees of freedom associated with the conformal 3-geometry in accordance with the work of York [21, 24] and the elimination of the spatial gauge degrees of freedom is equivalent to the imposition of the appropriate analogue of the minimal distortion condition of Smarr and York [26]. For all of the Bianchi types this choice of variables is equivalent to working in "diagonal gauge," characterized by a diagonal spatial metric matrix and a nontrivial shift vector field which maintains the diagonality condition.

Section 6 discusses a way to exploit the 1-parameter symmetry group of scale transformations of the unit of length. For the nonsemisimple Bianchi types this is then modified by introducing a new diagonal metric parametrization in order to also take full advantage of the larger automorphism group rather than just a 3-dimensional proper unimodular subgroup which enables the momentum contraints to be eliminated. In this context a maximal automorphism subgroup acting freely almost everywhere on $\mathscr{M}$ together with the 1-parameter group of uniform scale transformations lead to a metric parametrization which allows the Einstein equations to be

TABLE I

Categorization of the Bianchi types, together with the dimensions of the associated automorphism and special automorphism groups and of their orbits on the space $\mathscr{M}$ of metric matrices.

		Class A	Class B	$\mathrm{Aut}_e(g)$	$\mathrm{SAut}_e(g)$	$\mathscr{O}_{\mathrm{Aut}_e(g)}$	$\mathscr{O}_{\mathrm{SAut}_e(g)}$
Abelian	Nonsemisimple	I		9	8	6	5
Nonabelian	Nonsemisimple	II	V	6	5	5	4
Nonabelian	Nonsemisimple	VI_0, VII_0	IV, $\mathrm{VI}_{h\neq 0}$, $\mathrm{VII}_{h\neq 0}$	4	3	4	3
Nonabelian	Semisimple	VIII, IX		3	3	3	3

integrated in terms of the fewest number of gravitational variables. This turns out to be equivalent to imposing the appropriate analogue of the minimal strain condition of Smarr and York [26] on the shift vector field, together with a class of lapse choices fixed by the effective action of the uniform scale transformation group within the class of minimal strain frames. Again the single degree of freedom corresponding to the conformal 3-geometry for most of the nonsemisimple Bianchi types plays a crucial role. Finally, remarks are made concerning several other alternative approaches.

While reading this article it is helpful to keep in mind the chart (Table I) of Bianchi types, the details of which are more fully explained in subsequent sections. The Bianchi types are organized according to whether they are abelian or nonabelian, semisimple or nonsemisimple, and of class A or class B, the latter division corresponding to the vanishing or nonvanishing of the trace $C^f{}_{af}$ of the Lie algebra structure constant tensor components. The division by rows into four categories is made according to the properties of the associated Lie algebra automorphism and special automorphism groups and their action on the space $\mathscr{M}$ of metric matrices. The columns containing numbers list the dimensions of these groups for each category as well as the generic dimensions of their orbits on $\mathscr{M}$.

For the lower two categories of Table I, both the automorphism and special automorphism groups act freely on their generic orbits (i.e., the orbit and group dimensions coincide) allowing the "gauge degrees of freedom" to be parametrized directly by these groups. For the upper two categories, the generic orbits are of smaller dimension than the corresponding groups due to the existence of continuous isotropy subgroups. A further complication arises for these categories since no subgroup exists which sweeps out these orbits in a one-to-one manner, the consequences of which are described in Section 6.

The importance of these facts lies in the possibility of parametrizing the spatial metric matrix (g_{ab}) in the following way,

$$g_{ab} = R^c{}_a R^d{}_b (e^{\beta_D})_{cd},$$

where β_D is at least a diagonal matrix while the matrix $(R^a{}_b)$ is an arbitrary element of $\mathrm{SAut}_e(\mathfrak{g})$ (Section 5) or $\mathrm{Aut}_e(\mathfrak{g})$ (Section 6) for the lower two categories and of

TABLE II

The number of linearly independent components of the gravitational supermomentum.

I	0
II, $VI_{-1/9}$	2
All others	3

arbitrarily selected freely acting subgroups of these two groups for the upper two categories. These choices of variables lead to considerable simplification of the Einstein equations. For Bianchi type IX, where $\mathrm{SAut}_e(\mathfrak{g}) = \mathrm{Aut}_e(\mathfrak{g}) = SO(3, R)$, this is just the well known Misner–Ryan parametrization, where $(R^a{}_b)$ is an orthogonal matrix and β_D an arbitrary diagonal matrix.

A second categorization of the Bianchi types may be made according to the number of linearly independent components of the gravitational supermomentum as listed in Table II. For Bianchi types I, II, and $V_{-1/9}$, the supermomentum constraints are degenerate, which leads to further variation in the description of the dynamics beyond that associated with the categories of Table I. In both tables it is the magic number 3 characterizing the number of spatial coordinate degrees of freedom in general relativity which signals a correspondence with the state of affairs for general asymptotically flat or spatially compact spacetimes. The presence of numbers other than 3 reflects new features of the finite dimensional case which require individual discussion by category.

2. Spatial Homogeneity

A spatially homogeneous spacetime $(M, {}^4g)$ is a spacetime on which a 3–dimensional Lie group G acts simply transitively on spacelike hypersurfaces (hypersurfaces of homogeneity or spatially homogeneous hypersurfaces) as an isometry group of the spacetime metric [36]. (The Kantowski–Sachs case [19] in which only a 4-dimensional Lie group acts transitively on spacelike hypersurfaces will not be considered here.) In the interest of simplicity, the spacetime manifold M will be identified with the product manifold $R \times G$ and the tensor algebras over each component manifold may be identified with subalgebras of the tensor algebra over the product manifold without causing too much confusion. The assumption that G is simply connected is also made to avoid complications arising from nontrivial topology [37]. This guarantees that every automorphism of the Lie algebra $\mathfrak{g}$ of left invariant vector fields on G comes from an automorphism of G itself [39].

The natural coordinate t on the real line R induces a time function t on the spacetime whose values parametrize the family of spatially homogeneous hypersurfaces. A basis $e = \{e_a\}$ of the Lie algebra $\mathfrak{g}$ of left invariant vector fields on G with dual basis $\{\omega^a\}$ of the dual space $\mathfrak{g}^*$ of left invariant 1-forms and with structure constant tensor components $C^a{}_{bc} = \omega^a([e_b, e_c])$ leads to a symmetry adapted comoving ADM frame $\{e_\alpha\} = \{e_0 = \partial/\partial t, e_a\} \in F_G(M)$ with dual frame $\{\omega^\alpha\} = \{\omega^0 = dt, \omega^a\}$. The structure functions for this frame are $C^\alpha{}_{\beta\gamma} = \omega^\alpha([e_\beta, e_\gamma]) = \delta^\alpha{}_a \delta^b{}_\beta \delta^c{}_\gamma C^a{}_{bc}$. In this article, e_0 is chosen to be the unit normal $e_\perp$ to the family of spatially homogeneous hypersurfaces (let $\omega^\perp = \omega^0$) and $\{e_\alpha\}$ is referred to as a normal frame. The time function t is then the proper time measured along the normal congruence.

The left action of G on $M = R \times G$ is therefore taken to be the natural one, namely, left translation on each copy of G. The Lie algebra $\tilde{\mathfrak{g}}$ of right invariant vector fields on G, considered as a subalgebra of the Lie algebra $\mathfrak{X}(M)$ of vector fields on M, generates this action and its elements are Killing vector fields of $(M, {}^4g)$. Spatially homogeneous fields fields on M, i.e., spacetime fields which are invariant under this action, are simply those fields whose components in a normal frame $\{e_\alpha\} \in F_G(M)$ depend only on the time t. The spatial fields obtained from spatially homogeneous spacetime fields by the normal-tangential decomposition may then be easily interpreted as time-dependent left invariant fields on G. In particular, the class of spatially homogeneous spatial frames on M corresponds to time-dependent bases of the Lie algebra $\mathfrak{g}$. The spacetime Killing vector fields associated with spatial homogeneity are time-independent elements of $\tilde{\mathfrak{g}}$. If $\{\tilde{e}_a\}$ is the basis of $\tilde{\mathfrak{g}}$ which coincides with $\{e_a\}$ at the identity of G and if $\{\tilde{\omega}^a\}$ is the dual basis of $\tilde{\mathfrak{g}}^*$, then $\tilde{C}^a{}_{bc} \equiv \tilde{\omega}^a([\tilde{e}_b, \tilde{e}_c]) = -C^a{}_{bc}$.

The semidirect product Lie group $\mathscr{D}(\mathfrak{g}) = R(G) \times_s \mathrm{Aut}(G) = L(G) \times_s \mathrm{Aut}(G)$ of translations and automorphisms of G is the largest subgroup of the group $\mathscr{D}(G)$ of diffeomorphisms of G which maps the space of left invariant fields into itself under dragging along [2]. It is therefore the spatial gauge group of the class $F_G(M)$ of symmetry adapted comoving ADM frames. Its action on G is generated by the semidirect sum Lie algebra $\mathfrak{X}(\mathfrak{g}) = \mathfrak{g} \oplus_s \mathfrak{aut}(G) = \tilde{\mathfrak{g}} \oplus_s \mathfrak{aut}(G)$, where $\tilde{\mathfrak{g}}$ generates the left translations, $\mathfrak{g}$ the right translations, and $\mathfrak{aut}(G)$ the automorphisms. The Lie bracket action of $\mathfrak{aut}(G)$ on $\mathfrak{g}$ generates the automorphisms of $\mathfrak{g}$, i.e., the map $\mathrm{ad}: \mathfrak{aut}(G) \to \mathfrak{aut}(\mathfrak{g})$ defined by $\mathrm{ad}(\xi) X = [\xi, X] = \pounds_\xi X$ for $\xi \in \mathfrak{X}(\mathfrak{g})$ and $X \in \mathfrak{g}$ is an isomorphism. The same action of $\mathfrak{g}$ on itself generates the group $\mathrm{Ad}(G) \subset \mathrm{Aut}(\mathfrak{g})$ of inner automorphisms of $\mathfrak{g}$ with Lie algebra $\mathrm{ad}(\mathfrak{g})$ (this group is also called the linear adjoint group), while the Lie algebra $\tilde{\mathfrak{g}}$ commutes with $\mathfrak{g}$. Thus the Lie bracket action of $\mathfrak{X}(\mathfrak{g})$ on $\mathfrak{g}$ generates only the automorphisms of $\mathfrak{g}$. Let the subscript e indicate the representation of a linear map or Lie group or Lie algebra of linear maps on $\mathfrak{g}$ with respect to the basis e, a notation already introduced with $\mathrm{Aut}_e(\mathfrak{g})$. This latter group represents the action on the components of left invariant fields in the frame e on G induced by the dragging action of $\mathscr{D}(\mathfrak{g})$.

Let $F_{G,C}(M)$ be the set of frames in $F_G(M)$ characterized by a fixed set of structure constant tensor components $C^a{}_{bc}$ and let $F^\perp_{G,C}(M) \subset F_{G,C}(M)$ be the subset of normal frames. This latter space is in a 1–1 correspondence with $\mathrm{Aut}(\mathfrak{g})$ since any basis of $\mathfrak{g}$ differing from a particular one e but having the same structure constant tensor components is related to e by an automorphism. Any element $\{\bar{e}_\alpha\}$ of $F_{G,C}(M)$ is related to a normal frame in $F^{\perp}_{G,C}(M)$ by lapse and shift fields corresponding to the freedom to reparametrize the spatially homogeneous slicing and "rethread" that slicing [21] (choose new time coordinate lines). The lapse function N must be spatially homogeneous, depending only on t, but the time-dependent shift vector field $\vec{N} \doteq N^a e_a = \bar{N}^a \bar{e}_a \in \mathfrak{X}(\mathfrak{g})$ is not necessarily spatially homogeneous, and generates a time-dependent automorphism $\mathbf{S}(t) \in \mathrm{Aut}_e(\mathfrak{g})$ which maps the fixed basis e onto a time-dependent one $\bar{e}$

$$\bar{e}_0 = N(t)\, e_0 + \vec{N}, \qquad \bar{e}_a \doteq S^{-1b}{}_a(t)\, e_b. \tag{2.1}$$

The comoving condition $[\bar{e}_0, \bar{e}_a] = 0$ relates $\mathbf{S}$ to $\vec{N}$

$$N\mathbf{S}^{-1}\dot{\mathbf{S}} = \mathrm{ad}_e(\vec{N}), \qquad N\dot{\mathbf{S}}\mathbf{S}^{-1} = \mathrm{ad}_{\bar{e}}(\vec{N}). \tag{2.2}$$

A dot is used to denote the time derivative d/dt of a function only of the proper time t. Note that since $\tilde{\mathfrak{g}}$ lies in the kernel of the map ad: $\mathfrak{X}(\mathfrak{g}) \to \mathfrak{aut}(\mathfrak{g})$, a given curve $\mathbf{S}(t)$ only determines the shift modulo a curve in $\tilde{\mathfrak{g}}$, i.e., a time-dependent Killing vector field of the spatial metric.

The time-dependent automorphism $\mathbf{S}(t)$ may be written $\mathbf{S}_0(t)\,\mathbf{S}(t_0)$, where $\mathbf{S}_0(t) \equiv \mathbf{S}(t)\,\mathbf{S}(t_0)^{-1} \in \mathrm{Aut}_e(\mathfrak{g})^+$ lies in the identity component of $\mathrm{Aut}_e(\mathfrak{g})$. The factor $\mathbf{S}_0(t)$ represents the shift freedom which only involves the identity component since it continuously relates spatial frames on each element of the spatially homogeneous slicing to a spatial frame on a given initial value hypersurface $t = t_0$. The factor $\mathbf{S}(t_0)$, however, involves the entire automorphism group and leads to initial data on that hypersurface which differs only by a spatial diffeomorphism from initial data for which $\mathbf{S}(t_0) = \mathbf{1}$. This fact is important in simplifying initial data.

The spacetime metric expressed in a normal frame and in a general frame $\{\bar{e}_\alpha\}$ with dual frame $\{\bar{\omega}^\alpha\}$ is

$$\begin{aligned} {}^4g &= -dt \otimes dt + g_{ab}\,\omega^a \otimes \omega^b \\ &= -N^2\, d\bar{t} \otimes d\bar{t} + \bar{g}_{ab}(\bar{\omega}^a + \bar{N}^a\, d\bar{t}) \otimes (\bar{\omega}^b + \bar{N}^b\, d\bar{t}), \end{aligned} \tag{2.3}$$

where the reparametrized time function $\bar{t}$ is defined up to a constant by $\bar{\omega}^0 = d\bar{t} = N(t)^{-1}\, dt$. The component matrices $\mathbf{g} = g_{ab}\mathbf{e}^b{}_a$ and $\bar{\mathbf{g}} = \bar{g}_{ab}\mathbf{e}^b{}_a$ depend only on time and are therefore curves in the space $\mathscr{M}$ which are related by the time-dependent automorphism $\mathbf{S}(t)$, while the spatial metric ${}^3g = g_{ab}\,\omega^a \otimes \omega^b$ is a time-dependent left invariant metric on G. $\{\mathbf{e}^a{}_b\}$ is the natural basis of $\mathfrak{gl}(3, R)$. It is convenient to introduce a matrix notation for the mixed components of second rank spatial tensors or tensor densities other than the metric by defining $\mathbf{M} = M^a{}_b\mathbf{e}^b{}_a$ and $\bar{\mathbf{M}} = \bar{M}^a{}_b\mathbf{e}^b{}_a$ for a field M with respective components $M^a{}_b$ and $\bar{M}^a{}_b$. (One exception to this rule is made for the density $\mathbf{n} = n^{ab}\mathbf{e}^b{}_a$ introduced below.) Thus the extrinsic curvature component matrix is given by

$$\mathbf{K} = -\tfrac{1}{2}\mathbf{g}^{-1}\dot{\mathbf{g}} = -(2N)^{-1}\,\mathbf{g}^{-1}\overset{\circ}{\mathbf{g}}, \tag{2.4}$$

where $\overset{\circ}{f} \equiv df/d\bar{t} = N\dot{f}$ for any function f only of time. This raises the question of how to transform time derivatives.

All spatially homogeneous spatial tensors and tensor densities transform by $\mathbf{S}(t)$ according to the corresponding representation of $GL(3, R)$. If $\sigma: GL(3, R) \to GL(V)$ denotes a given tensor or tensor density representation of $GL(3, R)$ with the associated Lie algebra representation $\sigma': \mathfrak{gl}(3, R) \to \mathfrak{gl}(V)$ which may be defined by

$$\sigma(\mathbf{S})^{-1} \dot{\sigma(\mathbf{S})} = \sigma'(\mathbf{S}^{-1}\dot{\mathbf{S}}), \tag{2.5}$$

then the transformation of components of a field "of type σ" may be written

$$\bar{F} = \sigma(\mathbf{S}) \cdot F, \tag{2.6}$$

while the time derivative transforms as follows:

$$\begin{aligned} \overset{\circ}{\bar{F}} &= N\sigma(\mathbf{S}) \cdot \dot{F} + \sigma'(\mathrm{ad}_{\bar{e}}(\vec{N})) \cdot \bar{F} \\ &= \sigma(\mathbf{S}) \cdot (N\dot{F} + \sigma'(\mathrm{ad}_e(\vec{N})) \cdot F). \end{aligned} \tag{2.7}$$

Thus if $\bar{F}$ is constant one has

$$\dot{F} = -\sigma'(\mathrm{ad}_e(N^{-1}\vec{N})) \cdot F. \tag{2.8}$$

As is well known [10, 39], in three dimensions the structure constant tensor C, a $\binom{1}{2}$-tensor over $\mathfrak{g}$, may be decomposed into a covector $\operatorname{tr}\mathrm{ad} \in \mathfrak{g}^*$ and a symmetric second rank contravariant tensor density n which together determine a scalar h [29]. In component form one has

$$\begin{aligned} &C^a{}_{bc} = \varepsilon_{bcd} n^{ad} + a_f \delta^{fa}_{bc} = \varepsilon_{bcd} C^{ad}, \\ &C^{ad} = \tfrac{1}{2} C^a{}_{bc} \varepsilon^{bcd}, \qquad C^{(ab)} = n^{ab}, \qquad C^{[ab]} = a_c \varepsilon^{cab}, \\ &a_a a_b = \tfrac{1}{2} h \varepsilon_{acd} \varepsilon_{bfg} n^{cf} n^{dg}, \\ &0 = a_f n^{fa} = a_f C^{fa} = a_f C^f{}_{bc}. \end{aligned} \tag{2.9}$$

The final line reflects the Jacobi identity. It is useful to introduce the adjoint matrices

$$\mathbf{k}_a = \mathrm{ad}_e(e_a) = C^b{}_{ac} \mathbf{e}^c{}_b, \qquad \operatorname{Tr}\mathbf{k}_a = 2a_a \tag{2.10}$$

and some related tracefree matrices

$$\boldsymbol{\delta}_a = \mathbf{k}_a - 2a_c \delta^b{}_a \mathbf{e}^c{}_b = \delta_a{}^b{}_c \mathbf{e}^c{}_b. \tag{2.11}$$

The Bianchi types are simply the orbits of the natural action of $GL(3, R)$ on the 6-dimensional space of possible structure constant tensor components. In the present article the components are always assumed to have the canonical values specified in

Ref. [1] for each Bianchi type, unless stated otherwise. These canonical components are in the following "standard diagonal form":

$$\mathbf{n} \equiv n^{ab}\mathbf{e}^b{}_a = \mathrm{diag}(n^{(1)}, n^{(2)}, n^{(3)}), \qquad a_b = a\delta^3{}_b, \\ an^{(3)} = 0, \qquad a^2 = hn^{(1)}n^{(2)}. \tag{2.12}$$

Any basis e of $\mathfrak{g}$ whose structure constant tensor components are canonical will be called a canonical basis. The corresponding class of symmetry adapted comoving ADM frames will also be referred to as canonical, as will the associated matrix automorphism groups, subgroups and Lie algebras.

A Bianchi type is said to be of class A if a_b vanishes (in which case the linear adjoint group is unimodular) and of class B if not [10]. The vanishing of a_b means that the divergence of any left invariant vector field on G with respect to a left invariant metric $g_{ab}\omega^a \otimes \omega^b$ vanishes. This is apparent from the first of the following formulas for the divergence of a left invariant vector field X and symmetric tensor T

$$X^a{}_{;a} = -2a_a X^a, \\ T_{ab}{}^{;b} = \delta_a{}^b{}_c T^c{}_b = \mathrm{Tr}\, \boldsymbol{\delta}_a \mathbf{T}. \tag{2.13}$$

Interpreting $\{g_{ab}\}$ as the component "coordinate functions" on the space $\mathscr{M}$, introduce the scalar curvature potential function $U_G = -g^{1/2}\,{}^3R$ and the two 1-forms ${}^3G = -g^{1/2}\,{}^3G^{ab} dg_{ab}$ (Einstein force field) and $Q = Q^{ab} dg_{ab} = 2g^{1/2} a^c \delta_c{}^{(ab)} dg_{ab}$ (nonpotential force field), where ${}^3R$ and ${}^3G^{ab}$ are the components of the scalar curvature and Einstein tensor of the spatial metric ${}^3\mathrm{g} = g_{ab}\omega^a \otimes \omega^b$ in the spatial frame e and $g = \det \mathbf{g}$. Letting a prime denote the derivative of a 1-parameter family of metrics $g_{ab}(\lambda)$ evaluated at $g_{ab} = g_{ab}(0)$, one has the following variational formula which holds for the components of an arbitrary metric on G:

$$(g^{1/2}\,{}^3R)' = -g^{1/2}\,{}^3G^{ab} g'_{ab} + g^{1/2}(g'_{ab}{}^{;ab} - g'^a{}_{a;b}{}^{;b}). \tag{2.14}$$

For left invariant metrics, $g'^a{}_a$ is constant and the associated term vanishes, leaving the double divergence which is easily evaluated using (2.13):

$$g^{1/2} g'_{ab}{}^{;ab} = -Q^{ab} g'_{ab}. \tag{2.15}$$

Thus one has the result that for class B Bianchi types where both Q and dQ are in general nonvanishing, the Einstein force field has a nonpotential component

$${}^3G = -dU_G + Q. \tag{2.16}$$

This nonpotential component must be included in the Lagrangian and Hamiltonian equations.

In a normal symmetry adapted comoving frame, the gravitational Lagrangian is just the standard ADM Lagrangian and the associated Hamiltonian is the gravitational super-Hamiltonian. The fluid spatial energy-momentum tensor and nonpotential force when nonzero act as driving forces in the equations of motion, which are equivalent to the evolution equations for the spatial metric. These equations are subject to the super-Hamiltonian and supermomentum constraints. The evolution and constraint equations together are equivalent to the Einstein equations

$$G^{\alpha}{}_{\beta} = kT^{\alpha}{}_{\beta} \tag{2.17}$$

expressed in the normal frame. The gravitational Lagrangian and Hamiltonian, the Legendre transformation from the velocity phase space $(T\mathscr{M})$ to the momentum phase space $(T^*\mathscr{M})$, the gravitational equations of motion, and the constraints are given by the following formulas:

$$\begin{aligned}
L_G &= g^{1/2}(\mathrm{Tr}\,\mathbf{K}^2 - \mathrm{Tr}^2\mathbf{K}) - U_G,\\
\pi^{ab} &= \partial L_G/\partial \dot{g}_{ab}, \qquad \boldsymbol{\pi} = \pi^a{}_b \mathbf{e}^b{}_a = -g^{1/2}(\mathbf{K} - \mathbf{1}\,\mathrm{Tr}\,\mathbf{K}),\\
\mathscr{H}_G &= g^{1/2}(\mathrm{Tr}\,\boldsymbol{\pi}^2 - \tfrac{1}{2}\mathrm{Tr}^2\boldsymbol{\pi}) + U_G,\\
\delta L_G/\delta g_{ab} &= -Q^{ab} - kg^{1/2}T^{ab},\\
\dot{g}_{ab} &= \{g_{ab}, \mathscr{H}_G\}, \qquad \dot{\pi}^{ab} = \{\pi^{ab}, \mathscr{H}_G\} + Q^{ab} + g^{1/2}T^{ab},\\
\mathscr{H}^G_a &= -2\,\mathrm{Tr}\,\boldsymbol{\delta}_a\boldsymbol{\pi}, \qquad \mathscr{H}^M_a = -2kg^{1/2}T^{\perp}{}_a,\\
\mathscr{H}_M &= -2kg^{1/2}T^{\perp}{}_{\perp},\\
\mathscr{H} &= \mathscr{H}_G + \mathscr{H}_M = 0, \qquad \mathscr{H}_a = \mathscr{H}^G_a + \mathscr{H}^M_a = 0.
\end{aligned} \tag{2.18}$$

In the next section it is shown that by a proper choice of fluid variables, the matter super-Hamiltonian $\mathscr{H}_M$ generates the fluid spatial energy-momentum tensor driving force. The total Hamiltonian for the gravitational equations of motion is then just the total super-Hamiltonian $\mathscr{H}$, leaving only the nonpotential force as an additional driving term in the equations of motion for class B Bianchi types.

It is worth noting that the kinetic energy function on either $T\mathscr{M}$ or $T^*\mathscr{M}$ is just the one which generates the equations of motion for the explicitly known geodesics of the DeWitt metric [40] on $\mathscr{M}$

$$\mathscr{G} = \mathscr{G}^{abcd} dg_{ab} \otimes dg_{cd} = g^{1/2}(-g^{ab}g^{cd} + g^{a(c}g^{d)b})\, dg_{ab} \otimes dg_{cd}, \tag{2.19}$$

namely,

$$\mathscr{E} = \tfrac{1}{4}\mathscr{G}^{abcd}\dot{g}_{ab}\,\dot{g}_{cd} = 4\mathscr{G}^{-1}_{abcd}\pi^{ab}\pi^{cd}, \tag{2.20}$$

where $\{g_{ab}, \dot{g}_{ab}\}$ and $\{g_{ab}, \pi^{ab}\}$ are the natural lifted "coordinates" on $T\mathscr{M}$ and $T^*\mathscr{M}$ arising from the "coordinates" $\{g_{ab}\}$ on $\mathscr{M}$. The Legendre transformation from $T\mathscr{M}$ to $T^*\mathscr{M}$ is simply "index lowering" with this metric, apart from a conventional factor of 2:

$$\pi^{ab} = \tfrac{1}{2}\mathscr{G}^{abcd}\dot{g}_{cd}, \qquad \dot{g}_{ab} = 2\mathscr{G}^{-1}_{abcd}\pi^{cd}. \tag{2.21}$$

The general linear group $GL(3,R)$ acting on $\mathcal{M}$ on the left in the natural way

$$\mathbf{g}\in\mathcal{M}\to f_{\mathbf{A}}(\mathbf{g})=\mathbf{A}^{-1T}\mathbf{g}A^{-1}, \quad \mathbf{A}\in GL(3,R) \tag{2.22}$$

is a group of homothetic motions of $(\mathcal{M},\mathcal{G})$ with isometry sugroup $SL(3,R)$. For Bianchi type I, where the spatial curvature vanishes, $SL(3,R)$ is a symmetry group of the perfect fluid source dynamics and in the dust case the solution curves are just timelike geodesics of the DeWitt metric which reduce to null geodesics in the vacuum case. For nonzero pressure the geodesics of the restriction of the DeWitt metric to the unimodular submanifold of $\mathcal{M}$ are relevant, enabling the entire content of the Einstein equations to be reduced to the super-Hamiltonian constraint involving only the degree of freedom g which may be solved by quadrature for the proper time as a function of g [35].

For each $\mathbf{A}=A^a{}_b\mathbf{e}^b{}_a\in\mathfrak{gl}(3,R)$, interpreted as the matrix of mixed components of a spatial tensor field with respect to the spatial frame e, one can introduce a new matrix representing the mixed components of its symmetrization with respect to the spatial metric $g_{ab}\,\omega^a\otimes\omega^b$, which is used to raise and lower indices

$$\mathbf{A}^{\#}=A^{\#a}{}_b\mathbf{e}^b{}_a=g^{ad}g_{c(d}A^c{}_{b)}\,\mathbf{e}^b{}_a\equiv \mathrm{SYM}_{\mathbf{g}}(\mathbf{A}). \tag{2.23}$$

Because of the natural identification $T\mathcal{M}\sim\mathcal{M}\times S_2$, where $S_2\subset\mathfrak{gl}(3,R)$ is the set of symmetric matrices, one sees that $-2A^{\#}{}_{ab}\mathbf{e}^b{}_a\in S_2\sim T\mathcal{M}_{\mathbf{g}}$ corresponds to the value of the generating vector field at $\mathbf{g}$ of the action on $\mathcal{M}$ of the 1-parameter subgroup generated by $\mathbf{A}$

$$d/d\lambda|_0 f_{\exp\lambda\mathbf{A}}(\mathbf{g})=-2A^{\#}{}_{ab}\mathbf{e}^b{}_a. \tag{2.24}$$

The DeWitt norm of this tangent vector, omitting the factor of 4,

$$\mathcal{G}^{abcd}A^{\#}{}_{ab}A^{\#}{}_{cd}=g^{1/2}\langle\mathbf{A}^{\#},\mathbf{A}^{\#}\rangle_{DW}, \tag{2.25}$$

is compactly written in terms of the DeWitt inner product $\langle\mathbf{A},\mathbf{B}\rangle_{DW}=\operatorname{Tr}\mathbf{AB}-\operatorname{Tr}\mathbf{A}\operatorname{Tr}\mathbf{B}$ on $\mathfrak{gl}(3,R)$ which coincides with the trace inner product $\langle\mathbf{A},\mathbf{B}\rangle=\operatorname{Tr}\mathbf{AB}$ on $\mathfrak{gl}(3,R)$ when restricted to $\mathfrak{sl}(3,R)$. As another example, interpreting the extrinsic curvature matrix (2.4) as a matrix-valued function on $T\mathcal{M}$, the kinetic energy function on $T\mathcal{M}$ is simply

$$\mathcal{E}=g^{1/2}\langle\mathbf{K},\mathbf{K}\rangle_{DW}. \tag{2.26}$$

On the other hand the trace inner product on $\mathfrak{gl}(3,R)$ is associated with the natural positive definite metric $g^{a(c}g^{d)b}dg_{ab}\otimes dg_{cd}$ on $\mathcal{M}$ which corresponds to the local inner product of symmetric second rank tensors

$$\begin{aligned} h=h_{ab}\omega^a\otimes\omega^b\to\mathbf{gh}=h_{ab}\mathbf{e}^b{}_a\in S_2\sim T\mathcal{M}_{\mathbf{g}},\\ g^{ac}g^{bd}h_{ab}h_{cd}=\langle\mathbf{h},\mathbf{h}\rangle. \end{aligned} \tag{2.27}$$

As discussed in Ref. [2] it is this inner product which is appropriate for the decomposition of the tangent space $T\mathcal{M}_{\mathbf{g}}$ into subspaces tangent to the vacuum supermomentum constraint surfaces and the gauge orbits and the subspaces which are orthogonal to each of these.

The symmetrization map $\mathrm{SYM}_{\mathbf{g}}$ is useful primarily in expressing the extrinsic curvature matrix in a general comoving frame $\{\bar{e}_\alpha\} \in F_G(M)$. Using Eqs. (2.4)–(2.7) for representation (2.22), one finds

$$-N\bar{\mathbf{K}} = \tfrac{1}{2}\bar{\mathbf{g}}^{-1}\mathring{\bar{\mathbf{g}}} + \mathrm{SYM}_{\bar{\mathbf{g}}}(\mathring{\mathbf{S}}\mathbf{S}^{-1}) = -N\mathbf{S}\mathbf{K}\mathbf{S}^{-1}. \tag{2.28}$$

Since the trace inner product is conjugation invariant, the kinetic energy function may also be written $\mathscr{T} = g^{1/2}\langle \bar{\mathbf{K}}, \bar{\mathbf{K}}\rangle_{DW}$.

3. Fluid Equations

For a general isentropic perfect fluid on M with energy density ρ, pressure p and 4-velocity vector field u, the components of the energy-momentum tensor T are

$$T^\alpha{}_\beta = (\rho + p)\, u^\alpha u_\beta + p\delta^\alpha{}_\beta. \tag{3.1}$$

The fluid equations of motion follow from the conservation equations

$$T_\alpha{}^\beta{}_{;\beta} = 0. \tag{3.2}$$

Following Taub [30, 41], introduce the circulation 1-form v by $v_\alpha = \mu u_\alpha$, its restriction to the slicing ${}^3v = v_a\omega^a$, the natural fluid ADM generator $\vec{t}$ by $t^\alpha = \mu^{-1}u^\alpha$ and the quantity $l = ng^{1/2}u^\perp$ which is the component of a scalar density on each hypersurface in a given slicing. The projections of the conservation equations along u and orthogonal to u may then be written in the form

$$\operatorname{div} nu = 0 = \pounds_{\vec{t}}\, v. \tag{3.3}$$

Thus in a slicing of the spacetime generated from an initial spacelike slice by dragging along by the vector field $\vec{t}$, the components $\bar{v}_\alpha$ of v in a comoving ADM frame $\{\bar{e}_0 = \vec{t}, \bar{e}_a\}$ adapted to this slicing are time independent. The same is true of the component $\bar{l}$ since in such a frame $\bar{u}^a = 0$ and $0 = (n\bar{u}^\beta)_{;\beta} = N^{-1}\bar{g}^{-1/2}\vec{t}(\bar{l})$, where N is the lapse function for the frame. Note that $\bar{v}_0 = v(\vec{t}) = 1$ in such a frame, so only the spatial components $\bar{v}_a$ carry information about the fluid.

In a spatially homogeneous spacetime with $\{\bar{e}_\alpha\} \in F_G(M)$ and where the perfect fluid is spatially homogeneous, $\bar{l}$ and $\bar{v}_a$ are therefore constants. The corresponding components in a normal frame are related by the appropriate transformation involving $\mathbf{S}$ (scalar density and covector, respectively), where now $\mathbf{S} \in \mathrm{Ad}_e(G)$ is an inner automorphism of $\mathfrak{g}$ since the shift for the ADM frame comoving with the fluid is left invariant or spatially homogeneous

$$\begin{aligned} N^{-1}\vec{N} &= (v^{\perp})^{-1} v^a e_a \in \mathfrak{g}, \\ v^{\perp} &\equiv (\mu^2 + v^a v_a)^{1/2}. \end{aligned} \tag{3.4}$$

According to Eq. (2.8), the equations of motion for l and ${}^3v$ may therefore be written

$$\begin{aligned} \dot{l} &= \mathrm{Tr}\, \mathrm{ad}_e((v^{\perp})^{-1} v^a e_a)\, l, \\ {}^3\dot{v} &= \mathrm{ad}_e((v^{\perp})^{-1} v^a e_a)^* \, {}^3v, \end{aligned} \tag{3.5}$$

where the asterisk indicates the coadjoint representation, or in component form

$$\begin{aligned} (\ln l)^{\dot{}} &= 2a_c v^c (v^{\perp})^{-1}, \\ \dot{v}_a &= v_c C^c{}_{ba} v^b (v^{\perp})^{-1}. \end{aligned} \tag{3.6}$$

Let $\mathscr{S} = [0, \infty) \times [0, \infty) \times R^3$ be the fluid configuration space in which (n, l, v_a) assume values. The full configuration space is then $\mathscr{M} \times \mathscr{S}$, on which $\mathrm{Aut}_e(\mathfrak{g})$ acts on the left in the following way

$$\begin{aligned} (\mathbf{g}, n, l, v_a) &\mapsto (\mathbf{S}_0{}^{-1T} \mathbf{g} \mathbf{S}_0^{-1}, n, l\,|\det \mathbf{S}_0^{-1}|, v_b S_0{}^{-1b}{}_a), \\ &\mathbf{S}_0 \in \mathrm{Aut}_e(\mathfrak{g}). \end{aligned} \tag{3.7}$$

Any adjoint invariant function of l and v_a (i.e., a function invariant under the action of the linear adjoint group $\mathrm{Ad}_e(G) \subset \mathrm{Aut}_e(\mathfrak{g})$ of inner automorphisms of $\mathfrak{g}$) will be a constant of the motion since it coincides with its value in the frame comoving with the fluid, where it is a function of the constants $\bar{l}$ and $\bar{v}_a$. Since n^{ab} are the components of a tensor density over $\mathfrak{g}$, the quantity V^2 defined by $V = |n^{ab} v_a v_b|^{1/2}$ is a (oriented) scalar density and hence has the same equation of motion as l. Thus the adjoint invariant quantity $V^2 l^{-1}$ is a constant for all Bianchi types while V^2 and l are individually constants for the class A types, where the linear adjoint group is unimodular. For later use let $\varepsilon V^2 = n^{ab} v_a v_b$ define the sign ε of $n^{ab} v_a v_b$ when $V \neq 0$, setting $\varepsilon = 0$ when $V^2 = 0$.

Displaying the fluid equations in a more explicit form helps to reveal further constants of the motion for several Bianchi types. In a canonical basis e of $\mathfrak{g}$ one has

$$\begin{aligned} v^{\perp}\dot{l} &= 2alv^3, \\ v^{\perp}\dot{v}_1 &= (av_1 + n^{(2)}v_2)\, v^3 - n^{(3)} v_3 v^2, \\ v^{\perp}\dot{v}_2 &= (av_2 - n^{(1)}v_1)\, v^3 + n^{(3)} v_3 v^1, \\ v^{\perp}\dot{v}_3 &= -a(v_1 v^1 + v_2 v^2) + n^{(1)} v_1 v^2 - n^{(2)} v_2 v^1. \end{aligned} \tag{3.8}$$

For Bianchi type IV ($\mathbf{n} = \text{diag}(1, 0, 0)$ and $a = 1$), when the constant $l^{-1/2}v_1$ vanishes, the quantity $l^{-1/2}v_2$ is then constant since it transforms like a scalar under the adjoint action when $v_1 = 0$. For Bianchi type V ($\mathbf{n} = 0, a = 1$), both $l^{-1/2}v_1$ and $l^{-1/2}v_2$ are constants of the motion, but since in this case $\text{Aut}_e(\mathfrak{g})$ contains a subgroup $GL(2)$, corresponding to arbitrary linear transformations of the subspace span $\{e_1, e_2\}$, one may always set any linear combination of these constants to zero without loss of generality.

For Bianchi types VI_0 and VI_h, where $\mathbf{n} = q \text{ diag}(1, -1, 0)$ and $a = (-h)^{1/2} q$ with $q = 1$ for canonical components and $a = h = 0$ for type VI_0, the subspace span $\{e_1, e_2\}$ undergoes either pure boosts (type VI_0) or boosts accompanied by dilations (type VI_h) under the action of $\text{Ad}_e(G)$ [1]. Therefore the "lightcone basis" $\bar{e}$

$$\bar{e}_1 = 2^{-1/2}(e_1 + e_2), \qquad \bar{e}_2 = 2^{-1/2}(e_1 - e_2), \qquad \bar{e}_3 = e_3, \tag{3.9}$$

for which $\bar{\mathbf{n}} = q(\mathbf{e}^2{}_1 + \mathbf{e}^1{}_2)$, is more convenient in certain circumstances than the canonical basis e [2,10]. The new components $(\bar{v}_a) = (2^{-1/2}(v_1 - v_2), 2^{-1/2}(v_1 + v_2), v_3)$ have the equations of motion

$$\begin{aligned} v^{\perp}\dot{\bar{v}}_1 &= (a + q)\, \bar{v}_1 \bar{v}^3, \\ v^{\perp}\dot{\bar{v}}_2 &= (a - q)\, \bar{v}_2 \bar{v}^3, \\ v^{\perp}\dot{\bar{v}}_3 &= -(a + q)\bar{v}_1 \bar{v}^1 - (a - q)\bar{v}_2 \bar{v}^2. \end{aligned} \tag{3.10}$$

Note that both $\bar{v}_1$ and $\bar{v}_2$ are "invariant relation quantities" [39]; that is, they have the property of being either always zero or always nonzero. For Bianchi type VI_h, where $\lambda = qa^{-1}$ is well defined, $\bar{v}_1$ and $\bar{v}_2$ transform as scalar densities of weights $\frac{1}{2}(1 + \lambda)$ and $\frac{1}{2}(1 - \lambda)$, respectively. Therefore $l^{-1/2(1+\lambda)}\bar{v}_1$ and $l^{-1/2(1-\lambda)}\bar{v}_2$ are both adjoint invariant and hence constants of the motion, the product of which yields the constant $l^{-1}\varepsilon \bar{V}^2$ already discussed. Setting $q = 0$ yields the Bianchi type V limit.

In fact two independent fluid constants of the motion exist for all of the class B Bianchi types. From the formulas of Section II of Ref. [1] setting $n^{(3)} = 0$, one sees that the canonical adjoint matrix (i.e., an arbitrary element of the canonical linear adjoint group $\text{Ad}_e(G)$) may be parametrized in the following way for all the nonsemisimple Bianchi types except types I and II,

$$\mathbf{R}(\theta) = e^{\theta^1 \mathbf{e}^3{}_1 + \theta^2 \mathbf{e}^3{}_2} e^{\theta^3 \mathbf{k}_3} = \begin{bmatrix} c_3 e^{a\theta^3} & -n^{(1)} s_3 e^{a\theta^3} & \theta^1 \\ n^{(2)} s_3 e^{a\theta^3} & c_3 e^{a\theta^3} & \theta^2 \\ 0 & 0 & 1 \end{bmatrix}, \tag{3.11}$$

with the added constraint that $\theta^2 = -\theta^1$ for Bianchi type $\text{III} \equiv \text{VI}_{-1}$. The following notation is used with $m^{(3)} = (-n^{(1)}n^{(2)})^{1/2}$ and formulas with $m^{(3)} = 0$ understood as the limit as $m^{(3)} \to 0$:

$$\begin{aligned} c(x) &= \cosh m^{(3)}x, \qquad s(x) = m^{(3)-1} \sinh m^{(3)}x, \\ t(x) &= s(x)/c(x), \qquad c_3 = c(\theta^3), \qquad s_3 = s(\theta^3). \end{aligned} \tag{3.12}$$

Let $t^{-1}(y) = x$ for $y = t(x)$ define the inverse function t^{-1}. This discussion may be extended to Bianchi type II with the constraint $\theta^2 = 0$ by using the noncanonical structure constant tensor cmponents $\mathbf{n} = \text{diag}(1, 0, 0)$; for type I, $\mathbf{R}(\theta) = \mathbf{1}$ and both l and v_a are trivially adjoint invariant.

Setting $\mathbf{S}_0{}^{-1} = \mathbf{R}(\theta)$ in (3.7), one finds

$$(l, v_1, v_2, v_3) \to (e^{2a\theta^3}l, e^{a\theta^3}(c_3 v_1 + n^{(2)}s_3 v_2),\\ e^{a\theta^3}(-n^{(1)}s_3 v_1 + c_3 v_2), v_3 + \theta^1 v_1 + \theta^2 v_2). \tag{3.13}$$

From this one obtains the result for $n^{(1)} \neq 0$ that

$$t^{-1}(v_2/(n^{(1)}v_1)) \to t^{-1}(v_2/(n^{(1)}v_1)) - \theta^3. \tag{3.14}$$

It then follows that $l_B \equiv l \exp(2at^{-1}(v_2/(n^{(1)}v_1)))$ is another constant of the motion linearly independent of $l^{-1}\varepsilon V^2 = l^{-1}(n^{(1)}v_1{}^2 + n^{(2)}v_2{}^2)$ for all class B types except type V, where v_2/v_1 (or v_1/v_2) is itself a constant of the motion. In the class A limit $a \to 0$, the constant l_B reduces to l. For Bianchi type IV the combination $(2a)^{-1}(l^{-1/2}v_1)\ln(le^{2av_2/v_1})$ of the two constants of the motion reduces to the conditional constant $l^{-1/2}v_2$ when the constant $l^{-1/2}v_1$ vanishes.

For Bianchi types I and II, v_a and v_3, respectively, are constants of the motion but which are required to vanish by the supermomentum constraint as discussed below. For type II when $v_3 = 0$, both v_1 and v_2 are adjoint invariant and hence constants of the motion. As for Bianchi type V, they are equivalent under the action of the automorphism group.

The components of the rotation vector field associated with the fluid flow and the norm of this spacelike vector field, unlike the expansion and shear, may be expressed as functions on $\mathscr{M} \times \mathscr{S}$.

$$\begin{aligned}\omega^\alpha &= -\tfrac{1}{2}\eta^{\alpha\beta\gamma\delta}u_\beta u_{\gamma;\delta}\\ &= (2\mu^2 v^\perp g^{1/2})^{-1}(\varepsilon V^2 v^\alpha + \mu^2 v_f C^{fa}\delta^\alpha{}_a),\\ \omega &= (\omega^\alpha\omega_a)^{1/2} = (2\mu v^\perp g^{1/2})^{-1}(V^4 + \mu^2 v_f v_g C^{fa}C^{gb}g_{ab})^{1/2}.\end{aligned} \tag{3.15}$$

The unit alternating tensor is defined by $\eta_{\alpha\beta\gamma\delta} = (-{}^4g)^{1/2}\varepsilon_{\alpha\beta\gamma\delta}$ with $\varepsilon_{0123} = 1$.

Expressing the various components of the energy-momentum tensor in terms of l and v_a leads to the formulas

$$\begin{aligned}\mathscr{H}_M &= -2kg^{1/2}T^\perp{}_\perp = 2klv^\perp - 2kpg^{1/2},\\ \mathscr{H}^M_a &= -2kg^{1/2}T^\perp{}_a = -2klv_a,\\ g^{1/2}T^{ab} &= lv^a v^b(v^\perp)^{-1} + pg^{1/2}g^{ab}.\end{aligned} \tag{3.16}$$

Recall that p and μ are assumed to be explicit functions of n. The fluid super-Hamiltonian is a function on $\mathscr{M} \times \mathscr{S}$. If d is the exterior derivative on $\mathscr{M}$ only, one immediately sees the formula

$$-d\mathscr{H}_M = kg^{1/2}T^{ab}dg_{ab}. \tag{3.17}$$

Only by considering l as an independent variable and not as an explicit function of g, n, and v_a does the fluid super-Hamiltonian generate the correct driving term in the gravitational equations of motion. The definition of l in terms of g, n, and v_a may instead be considered as an equation implicitly determining n in terms of g, v_a, and l so that an equation of motion for n need not be considered [15]. For the class A Bianchi types l is a constant and one recovers the Ozsváth formulation in the dust case [34].

For Bianchi types I, II, and VI$_{-1/9}$, the components of the gravitational supermomentum are degenerate since the matrices $\{\boldsymbol{\delta}_a\}$ are not linearly independent and this imposes constraints on the fluid spatial 1-form ${}^3v$. In a canonical frame, $\boldsymbol{\delta}_a = 0$, $\boldsymbol{\delta}_3 = 0$ and $\boldsymbol{\delta}_1 + \boldsymbol{\delta}_2 = 0$, respectively, for these three types. The supermomentum constraint therefore imposes $v_a = 0$, $v_3 = 0$, and $v_1 + v_2 = 0$, respectively, as a comparison of (2.18) and (3.9) shows. For the first two types, $v_b C^{ba}$ then vanishes, so that $V^2 = |v_b C^{ba} v_a|$ and therefore ω also vanish, leading to rotationfree flow. This also implies $\dot{v}_a = 0$ which makes v_1 and v_2 constants for Bianchi type II. This is also evident from the fact that $v_b C^{ba} = 0$ or equivalently, $\mathrm{ad}(X)^* \, {}^3v = 0$ for all $X \in g$ means that ${}^3v$ is invariant under the coadjoint action of G and therefore the components v_a coincide with their constant values $\bar{v}_a$ in a frame comoving with the fluid.

The equations of motion for ${}^3v$ may be written

$$v^{\perp}\dot{v}_a = -\varepsilon_{abc} v^b (v_d C^{dc}). \tag{3.18}$$

In order for $\dot{v}_a$ to vanish, so that $v_a = \bar{v}_a$ are constants, v_a must be an eigenvector of the matrix $C^a{}_b = C^{ad} g_{db}$

$$v_a C^{ab} = \lambda v^b, \qquad \lambda = \varepsilon V^2 (v^c v_c)^{-1}. \tag{3.19}$$

The rotation vector then has the simpler expression

$$\begin{aligned} \omega^\alpha &= \varepsilon\omega\mu^{-1}(v^a v_a)^{-1/2}\,(\delta^\alpha{}_0 v_c v^c + \delta^\alpha{}_c v^c v^{\perp}), \\ \omega &= V^2 (2\mu g^{1/2})^{-1}\,(v^a v_a)^{-1/2}. \end{aligned} \tag{3.20}$$

The time independence of v_a imposes an algebraic constraint on the metric matrix leading to what will be described below as a "symmetric case." In particular, $v_a = 0$ is a solution of the equation of motion, in which case the fluid velocity u coincides with the unit normal $e_{\perp}$ and the metric is subject to the vacuum supermomentum constraints.

4. Additional Symmetry

The maximal isometry group G' of a spatially homogeneous spacetime $(M, {}^4g)$ may be larger than the group $L(G)$ of left translations acting naturally on $M = R \times G$. If the orbits of G' are M itself, the spacetime is then spacetime homogeneous [19, 42, 43]. The present section will instead consider the case where the orbits of G' are also the spatially homogeneous hypersurfaces. Then the isotropy group G'_y of G' of any spacetime point y will be nontrivial. (Recall that $G'_y \subset G$ is the subgroup which leaves y fixed.) The isomorphic linear isotropy group $dG' \subset GL(TM_y)$, which is the linear transformation group induced on the tangent space TM_y by the dragging action of G'_y at the fixed point y, is then isomorphic to a subgroup of the orthogonal group (since orthonormal spatial frames are mapped among themselves by such isometries) and thus consists of reflections and rotations of the spatial tangent space.

Any such spacetime isometries will leave the spatial metric, extrinsic curvature, and energy-momentum tensor of the source invariant on each spatially homogeneous hypersurface. When the energy-momentum tensor does not completely determine the source, the source itself may not be invariant under these isometries. An electromagnetic field may undergo a duality rotation and a complex field may undergo a constant phase transformation, for example, but a perfect fluid is necessarily invariant since it is entirely determined by its energy-momentum tensor.

Schmidt [44] has shown the converse, namely, that any additional isometry of initial data on a spatially homogeneous hypersurface induces an additional isometry of the spacetime. This isometry is a "time-independent" spatial isometry in the sense that it is a diffeomorphism of G considered as a diffeomorphism of $M = R \times G$ by the natural inclusion $\mathscr{D}(G) \subset \mathscr{D}(R \times G)$; this is true since the normal congruence is invariant under any isometry whose orbits are the spatially homogeneous hypersurfaces. In particular, the isotropy group at a point P of the partially Hamiltonian phase space $T^*\mathscr{M} \times \mathscr{S}$ with coordinates $(g_{ab}, \pi^{ab}; n, l, v_a)$ under the following action of the automorphism group $\mathrm{Aut}_e(\mathfrak{g})$,

$$(g_{ab}, \pi^{ab}; n, l, v_a) \mapsto (S^{-1c}{}_a S^{-1d}{}_b\, g_{cd}, |\det \mathbf{S}^{-1}| S^a{}_c S^b{}_d \pi^{cd}; n, |\det \mathbf{S}^{-1}| l, v_b S^{-1b}{}_a), \tag{4.1}$$

is just the matrix representation of the linear isotropy group associated with a naturally isomorphic subgroup of $\mathrm{Aut}(G)$ which leaves the initial data P invariant. This latter subgroup, when considered as a subgroup of $\mathscr{D}(M)$, is therefore an additional isometry subgroup of the spacetime with initial data P; it is in fact an isotropy subgroup at each point of the normal geodesic which threads together the identity of G in each copy of G in M. The isotropy group at any other normal geodesic is a conjugate subgroup of the isotropy group at the "identity" normal geodesic. Continuous isotropy groups of dimension 1 and 3 are associated with local rotational symmetry [9] and isotropy, respectively, while the discrete isotropy groups are associated with what have been called "symmetric case" and "diagonal case" spacetimes as well as another less familiar class of spacetimes which occurs for Bianchi types VI_0 and VI_h.

Action (4.1) of $\mathrm{Aut}_e(\mathfrak{g})$ on $T^*\mathscr{M}$ is not the lift of action (2.22) on $\mathscr{M}$ under which π^{ab} transforms as a tensor rather than a tensor density [3]. Under the lifted action, which is a canonical action, the natural symplectic 2-form $d\pi^{ab} \wedge dg_{ab}$ of $T^*\mathscr{M}$ is invariant. This distinction is unimportant only for the special automorphism group $\mathrm{SAut}_e(\mathfrak{g})$ and makes the situation slightly more complicated for nonunimodular automorphisms from the Lagrangian/Hamiltonian point of view.

As discussed by Schmidt [44], not every isotropy of the spatial metric arises from the automorphism group. Bianchi [45], for example, showed that every Bianchi type $\mathrm{III} \equiv \mathrm{VI}_{-1}$ and type V spatial metric has an additional 1-parameter and 3-parameter isotropy group, respectively. In the first case the 1-parameter isotropy group at the identity of G does not consist of automorphisms, while in the second case two independent such 1-parameter subgroups are not automorphisms [35]. Similarly for isotropic spatial metrics of Bianchi types VII_0 and VII_h, two independent 1-parameter isotropy subgroups of the identity of G are unrelated to automorphisms. Such isotropies induce space-dependent rotations of an orthonormal basis of $\mathfrak{g}$ and are relevant to spacetime isotropies only in the case of local rotational symmetry for Bianchi type III and isotropy in the remaining cases. There also exist various discrete isotropies which are not related to automorphisms. In what follows, only those isotropies will be discussed which are necessary to distinguish the various classes of spatially homogeneous spacetimes whose maximal isometry groups G' are isomorphic. The class for which $G' = L(G)$ is referred to as the general case; the remaining classes are special cases which will be assigned individual names.

To enumerate the relevant discrete automorphisms, it is helpful to introduce the discrete subgroup of $O(3, R)$ which contains the reflections and permutations. Define $\mathbf{r}^n_{(a)} = \exp(\frac{1}{2}n\pi\mathbf{k}^{IX}_a) \in SO(3, R)$, where $\mathbf{k}^{IX}_a = -\varepsilon_{abc}\mathbf{e}^c{}_b$ are the elements of the standard canonical basis of the Lie algebra of the canonical Bianchi type IX linear adjoint group $SO(3, R)$, and also define $\mathbf{s}_{(a)} = \mathbf{1} - 2\mathbf{e}^a{}_a$. The matrix $\mathbf{r}^2_{(a)}$ represents reflection about the origin in the plane orthogonal to the ath axis and $\mathbf{s}_{(a)}$ represents a reflection of this axis itself. If (a, b, c) is a cyclic permutation of $(1, 2, 3)$, then the matrix $\mathbf{q}_{(a)} = \mathbf{s}_{(b)}\mathbf{r}^1_{(a)} = \mathbf{r}^1_{(a)}\mathbf{s}_{(c)}$ interchanges the bth and cth axis.

Now consider the intersection of the orthogonal group with the canonical automorphism group. For $\mathbf{0} \in O(3, R) \cap \mathrm{Aut}_e(\mathfrak{g})$, one has

$$\mathbf{n} = (\det \mathbf{0}^{-1})\, \mathbf{0}\mathbf{n}\mathbf{0}^T, \quad a_b = a_c O^{-1c}{}_b.$$

Since $\mathbf{n}$ is diagonal, the unimodular matrices $\{\mathbf{r}^2_{(a)}\}$ all leave $\mathbf{n}$ invariant and hence $\{\mathbf{1}, \mathbf{r}^2_{(a)}\}$ is a discrete automorphism subgroup in the class A case. In the class B case, only $\mathbf{r}^2_{(3)}$ also leaves the covector $(0, 0, a)$ invariant so one has the smaller group $\{\mathbf{1}, \mathbf{r}^2_{(3)}\}$. Let (A, B, C) be a fixed cyclic permutation of $(1, 2, 3)$. If $n^{(B)} = n^{(C)}$, then the 1-parameter subgroup of $SO(3, R)$ generated by $\mathbf{k}^{IX}_A$ leaves $\mathbf{n}$ invariant and hence is an automorphism subgroup in the class A case; in the class B case, invariance of the covector $(0, 0, a)$ requires $(A, B, C) = (3, 1, 2)$. On the other hand if $n^{(B)} = -n^{(C)}$, then the nonunimodular matrix $\mathbf{q}_{(A)}$ leaves $\mathbf{n}$ invariant only when $n^{(A)} = 0$. This is nontrivial only for Bianchi type VI_0 $(a = 0)$ and VI_h $(a \neq 0)$, where $\mathbf{n} = \mathrm{diag}(1, -1, 0)$ and $\mathbf{q}_{(3)}$ is an automorphism overlooked by Schmidt [44]. The case of equal values $n^{(B)} = n^{(C)}$ occurs for Bianchi types II, V, VII_0, VII_h and VIII with $A = 3$ and for Bianchi types I and IX with each value of A. Finally for Bianchi types I $(a = 0)$ and V $(a = 1)$, where $\mathbf{n} = 0$, $O(3, R) \cap \mathrm{Aut}_e(\mathfrak{g})$ is, respectively, $O(3, R)$ and its $O(2, R)$ subgroup "$O(2)_3$" which leaves the third axis of R^3 fixed.

These orthogonal matrix automorphism subgroups turn out to be sufficient to describe all possible spacetime isotropies arising from $\mathrm{Aut}(G)$ as will be seen below. With this end in mind, a sequence of natural submanifolds of $\mathscr{M}$ may be introduced such that one of these matrix subgroups is a subgroup of the isotropy group of action (2.22) of $O(3, R)$ on $\mathscr{M}$ common to every point of a given such submanifold. Again let (A, B, C) be a cyclic permutation of $(1, 2, 3)$. Define the symmetric case submanifolds $\mathscr{M}_{S(A)} = \{\mathbf{g} \in \mathscr{M} \mid g_{AB} = g_{AC} = 0\}$, the diagonal submanifold $\mathscr{M}_D = \mathscr{M}_{S(1)} \cap \mathscr{M}_{S(2)} \cap \mathscr{M}_{S(3)}$, the Taub submanifolds $\mathscr{M}_{T(A)} = \{g \in \mathscr{M}_D \mid g_{BB} = g_{CC}\}$ and the isotropic submanifold $\mathscr{M}_I = \{\mathbf{g} = e^{2\beta^0}\mathbf{1} \mid \beta^0 \in R\} = \mathscr{M}_{T(1)} \cap \mathscr{M}_{T(2)} \cap \mathscr{M}_{T(3)}$. The common isotropy subgroups of $SO(3, R)$ for the submanifolds $\mathscr{M}_{S(A)}, \mathscr{M}_D, \mathscr{M}_{T(A)}$ and $\mathscr{M}_I$ are, respectively, $I^+_{S(A)} = \{\mathbf{1}, \mathbf{r}^2_{(A)}\}$, $I^+_D = \{\mathbf{1}, \mathbf{r}^2_{(a)}\}$, $I^+_{T(A)} = I^+_D \times \{\exp \theta \mathbf{k}^{IX}_A \mid \theta \in [0, 2\pi)\}$ and $I^+_I = SO(3, R)$. The corresponding subgroups of $O(3, R)$ are $I_{S(A)} = I^+_{S(A)} \cup \{-\mathbf{1}, \mathbf{s}_{(A)}\}$, $I_D = I^+_D \cup \{-\mathbf{1}, \mathbf{s}_{(a)}\}$, $I_{T(A)} = I_D \times \{\exp \theta \mathbf{k}^{IX}_A \mid \theta \in [0, 2\pi)\}$ and $I_I = O(3, R)$.

The additional canonical automorphism matrix $\mathbf{q}_{(3)}$ for Bianchi types VI_0 and VI_h leads to another interesting submanifold of $\mathscr{M}$ called the Taublike symmetric case submanifold $\mathscr{M}_{TS(3)} = \{\mathbf{g} \in \mathscr{M}_{S(3)} \mid \mathbf{q}^T_{(3)} \mathbf{g} \mathbf{q}_{(3)} = \mathbf{g}\}$. In the noncanonical basis $\bar{e}_a = A^{-1b}{}_a e_b$ with $\mathbf{A} = \mathbf{r}^{1/2}_{(3)}$ introduced in (3.9), this automorphism is diagonalized

$$\bar{\mathbf{q}}_{(3)} = \mathbf{A}\mathbf{q}_3\mathbf{A}^{-1} = \mathbf{s}_{(1)}$$

and the submanifold $\mathscr{M}_{TS(3)}$ is mapped onto $\mathscr{M}_D$ while the submanifold $\mathscr{M}_{T(3)} = \mathscr{M}_{TS(3)} \cap \mathscr{M}_D$ is invariant under this transformation. In the noncanonical frame one has $\bar{\mathbf{n}} = \mathbf{A}\mathbf{n}\mathbf{A}^T = \mathbf{e}^1{}_2 + \mathbf{e}^2{}_1$, which is an example of an alternative choice of canonical structure constant tensor components discussed by Ellis and MacCallum [10] for Bianchi types for which $\mathbf{n}$ has a pair of eigenvalues of differing sign and which is used by Bianchi in his original classification [45]. (Note that $g_{ab}n^{ab} = 0$ for $\mathbf{g} \in \mathscr{M}_{TS(3)}$.) The isotropy subgroup common to all points of $\mathscr{M}_{TS(3)}$ is $I_{TS(3)} = \mathbf{A}^{-1} I_D \mathbf{A}$ with $I_{TS(3)} \cap \mathrm{Aut}_e(\mathfrak{g}) = \{\mathbf{1}, \mathbf{r}^2_{(3)}, \mathbf{q}_{(3)}, \mathbf{q}_{(3)}\mathbf{r}^2_{(3)}\}$.

Similarly on the fluid configuration space $\mathscr{S}$, one can define a corresponding sequence of submanifolds by $\mathscr{S}_{S(A)} = \mathscr{S}_{TS(A)} = \{(n, l, v_a) \in \mathscr{S} \mid v_a = \delta^A_a v_A\}$ (no sum on A) and $\mathscr{S}_D = \mathscr{S}_{T(A)} = \mathscr{S}_I = \{(n, l, 0) \in \mathscr{S}\}$. Finally one may define such a sequence of submanifolds $[T\mathscr{M}_X]_{\mathrm{inc}} \subset T\mathscr{M}$ and $[T\mathscr{M}_X]^*_{\mathrm{inc}} \subset T^*\mathscr{M}$, where $X \in \{S(a), TS(a), D, T(a), I\}$. Here $[T\mathscr{M}_X]_{\mathrm{inc}}$ is just the tangent bundle $T\mathscr{M}_X$ of $\mathscr{M}_X$ considered as a submanifold of $T\mathscr{M}$ by inclusion, while $[T\mathscr{M}_X]^*_{\mathrm{inc}}$ is its image in $T^*\mathscr{M}$ by the Legendre map which is equivalent to the natural isomorphism between $T\mathscr{M}$ and

$T^*\mathcal{M}$ provided by the DeWitt metric $\mathcal{G}$ on $\mathcal{M}$ ("index lowering"), apart from a conventional factor of 2. Since $I_X \subset O(3, R)$ leaves every point of $\mathcal{M}_X$ fixed, it also leaves every point of $[T\mathcal{M}_X]_{\text{inc}}$ fixed. Furthermore, since $O(3, R)$ acting on $\mathcal{M}$ by (2.22) is an isometry group of the DeWitt metric, I_X is also the common isotropy subgroup on $[T\mathcal{M}_X]^*_{\text{inc}}$.

Thus I_X is an isotropy subgroup common to all points of the submanifold $[T\mathcal{M}_X]^*_{\text{inc}} \times \mathcal{S}_X$ of the Hamiltonian initial data space. If I_X intersects $\text{Aut}_e(\mathfrak{g})$, then the intersection is the matrix representation of the linear isotropy group of a spacetime isotropy subgroup of $\text{Aut}(G) \subset \mathscr{D}(M)$ for the spacetimes corresponding to initial data P lying in this submanifold. Since this will be true on any spatially homogeneous hypersurface, the solution curve through the initial data point P will remain in this submanifold; i.e., one has a driven Hamiltonian system of smaller dimension or a "special case." Of course the same is true for the image of any of these submanifolds by an automorphism, but since one may interpret such a transformation as simply a different choice of canonical normal symmetry adapted comoving ADM frame on the same spacetime, it is sufficient to consider only the submanifolds described above. On the other hand I_X may contain linear isotropies of the spatial metric not associated with isotropies arising from automorphisms, with a similar effect on the dynamics. These must also be considered in classifying the special cases.

For the class A Bianchi types, one has a symmetric case $\mathcal{M}_{S(A)}$ for each value of the index A, but at least two if not all of these cases are equivalent since they may be transformed into each other by an automorphism. Only Bianchi types II, VI_0, VII_0 and VIII have two inequivalent symmetric cases $\mathcal{M}_{S(A)}$ depending on whether the corresponding structure constant tensor component $n^{(A)}$ is vanishing or nonvanishing in the first three types or positive or negative in the last type. For the remaining class A types I and IX it is sufficient to consider only the case associated with $\mathcal{M}_{S(3)}$. For the class B Bianchi types, only the symmetric case $\mathcal{M}_{S(3)}$ exists due to the smaller group of discrete automorphisms. In every symmetric case $\mathcal{M}_{S(A)}$, the single nonvanishing component v_A of the fluid spatial current 1-form is a constant and the vanishing or nonvanishing of the rotation is equivalent to the vanishing or nonvanishing of the component $n^{(A)}$ by (3.4). Only for class B types does a nontrivial fluid equation of motion remain, namely, the equation of motion for l. For class A types, the system reduces to a Hamiltonian system in the gravitational variables containing the fluid constants of the motion as parameters. This was first realized by Gödel [33] in his treatment of the Bianchi type IX symmetric case, which he so named due to the existence of the discrete reflection symmetry. In the class A dust case, the symmetric case system differs from the corresponding vacuum case only in that the gravitational super-Hamiltonian and single nonvanishing component of the gravitational supermomentum are allowed to be nonzero constants [4].

For the Bianchi types VI_0 and VI_h, one also has a Taublike symmetric case $\mathcal{M}_{TS(3)}$. This is a diagonal case with respect to the noncanonical basis $\bar{e}$ of $\mathfrak{g}$ introduced above, although $v_3 \neq 0$. The modifier "Taublike" is used since any $\mathrm{g} \in \mathcal{M}_{TS(3)}$ may be mapped to a point of the Taub submanifold $\mathcal{M}_{T(3)}$ by an automorphism. By setting the parameter $q = n^{(1)} = -n^{(2)}$ to zero in the type VI_h case, one obtains a Bianchi type V Taublike symmetric case which is a diagonal case with respect to the basis $\bar{e}$ which is still canonical, although $v_3 \neq 0$.

For each class A type, one has a diagonal case ($v_a = 0$) since the discrete isotropy group I_D belongs to $\mathrm{Aut}_e(\mathfrak{g})$. For Bianchi type I, since the fluid spatial current 1-form is required to vanish by the supermomentum constraint, one may in fact transform any initial data point to diagonal initial data [1] using the automorphism group $GL(3, R)$ and hence the general case is equivalent to the diagonal case. Similarly for the Bianchi type II symmetric case $\mathscr{M}_{S(3)}$, the supermomentum constraint requires $v_3 = 0$ and hence $v_a = 0$, but arbitrary initial data for this case may be transformed to diagonal initial data using the automorphism group [1] so this symmetric case is equivalent to the diagonal case. The same statement holds for the Bianchi type V symmetric case. In fact since one may always choose a new spatial frame using the automorphism subgroup $O(2)_3$ for which either $v_1 = 0$ or $v_2 = 0$, and since this condition is preserved in time because $l^{-1/2}v_1$ and $l^{-1/2}v_2$ are constants of the motion, the general Bianchi type V case is characterized by only two nonvanishing components of the spatial circulation 1-form.

For each class A type one also has a Taublike case $\mathscr{M}_{T(3)}$, corresponding to local rotational symmetry in all types but VI_0 (the common automorphism isotropy group is $I_D \times I_{T(3)}$) and to additional discrete symmetry in the latter case (the common automorphism isotropy group is $I_D^+ \times \{\mathbf{1}, \mathbf{q}_{(3)}\}$).

The Bianchi type I and VII_0 Taublike cases are equivalent since for $\mathbf{g} \in \mathscr{M}_{T(3)}$, metrics of these two Bianchi types coincide, provided one identifies the group manifold G of these types in an obvious way [6]. The same statement is true for the class B Bianchi types V and VII_h, for which the equivalent Taublike cases $\mathscr{M}_{T(3)}$ associated with local rotational symmetry exist, although v_3 may be nonvanishing and the common automorphism isotropy subgroup is not $I_D \times I_{T(3)}$ but $I_{S(3)} \times I_{T(3)}$. For Bianchi types I and IX, all Taublike cases exist but are equivalent under the automorphism group. The Taub submanifolds take their name from the most famous nontrivial exact spatially homogeneous solution, Taub–Nut spacetime, which arises from the integration of the Bianchi type IX vacuum Taublike case. For Bianchi type $\mathrm{III} = \mathrm{VI}_{-1}$ when referred to the noncanonical frame $\bar{e}$, the Taublike case $\mathscr{M}_{T(2)}$ exists corresponding to local rotational symmetry not associated with automorphisms [35]. These models are just the negative curvature analogues of the Kantowski–Sachs models.

Finally for Bianchi types "$\mathrm{I} \cap \mathrm{VII}_0$", "$\mathrm{V} \cap \mathrm{VI}_h$", and IX one has the isotropic case $\mathscr{M}_I$ with $v_a = 0$ corresponding to the spatially isotropic Friedmann or Robertson-Walker spacetimes of constant vanishing, negative, and positive spatial curvature, respectively.

What has been described is essentially a stratification of the "driven Hamiltonian phase space" $T^*\mathscr{M} \times \mathscr{S}$ determined by the existence of additional discrete and continuous spacetime symmetries. These symmetries allow the reduction of the system on those strata of a given symmetry type or higher to a driven Hamiltonian system of smaller dimension than the general case, as in the Marsden–Weinstein reduction of a purely Hamiltonian system with continuous symmetry [46].

5. Symmetry Adapted Variables

It should be clear that the gauge action of $\mathrm{Aut}_e(\mathfrak{g})$ on the configuration space $\mathscr{M}\times\mathscr{S}$ is extremely important in understanding the dynamics of spatially homogeneous spacetimes. In particular, great simplification can be achieved by choosing new variables, adapted to the orbits of the action of this group and its subgroup $\mathrm{Ad}_e(G)$ which is associated with the fluid constants of the motion. Since the Lagrangian/Hamiltonian approach is also a great help in simplifying the dynamics, special attention should be given to the special automorphism matrix group $\mathrm{SAut}_e(\mathfrak{g})$ which is the subgroup of $\mathrm{Aut}_e(\mathfrak{g})$ whose action on the gravitational velocity phase space corresponds via the Legendre map to a canonical action on the momentum phase space [3]. Furthermore, the canonically conjugate variables ($g^{1/2}$, $\frac{2}{3}g^{-1/2}\mathrm{Tr}\,\pi=\frac{4}{3}\,\mathrm{Tr}\,\mathbf{K}$), which are "intrinsic" and "extrinsic" time functions, respectively [25, 47], are singled out by the Einstein equations [48] and as will be seen below, it is the special automorphism group which allows one to split off these variables from the complementary "almost canonically conjugate" [24, 25] variables ($\tilde{g}_{ab}=g^{-1/3}g_{ab}$, $g^{1/3}\pi^{TFab}=g^{1/3}\ (\pi^{ab}-\frac{1}{3}\,g^{ab}\pi^c_c)$), namely, the conformal metric and the shear (density) of the normal congruence. This separation was shown by York to be essential to the solution of the initial value problem as well as to the geometrization of certain useful coordinate conditions on spatially compact and asymptotically flat spacetimes and finds important application in spatially homogeneous spacetimes as well [2].

The essential idea is to use action (2.22) of $GL(3,R)$ on $\mathscr{M}$ to parametrize the manifold $\mathscr{M}$, adapting it to the abelian subgroup of scale transformations $\mathrm{Diag}(3,R)^+=\{\exp\boldsymbol{\beta}\,|\,\boldsymbol{\beta}\text{ diagonal}\}=\mathscr{M}_D$ which generates the diagonal submanifold $\mathscr{M}_D$ from the point $\mathbf{1}\in\mathscr{M}$ and to a 3-dimensional special automorphism matrix subgroup $\hat{G}\subset\mathrm{SAut}_e(\mathfrak{g})$ with off-diagonal generators $\{\boldsymbol{\kappa}_a\}$ which maps $\mathscr{M}_D$ onto $\mathscr{M}$. The diagonal submanifold is important because of the algebraic simplifications which occur as well as the fact that it is $\mathscr{M}_D$ or a submanifold of it which is in a 1–1 correspondence with the isometry classes of spatially homogeneous metrics [1]. The "off-diagonal" degrees of freedom on the other hand are necessarily associated with the gauge action of $\mathrm{SAut}_e(\mathfrak{g})$. The following parametrization [1, 6] mapping $\mathscr{M}_D\times\hat{G}$ onto $\mathscr{M}$ nicely separates these variables:

$$
\begin{aligned}
&\mathbf{g}=(e^{\boldsymbol{\beta}}\mathbf{S})^T e^{\boldsymbol{\beta}}\mathbf{S}=\mathbf{S}^T e^{2\boldsymbol{\beta}}\mathbf{S}\equiv\mathbf{S}^T g'\mathbf{S},\\
&\boldsymbol{\beta}=\mathrm{diag}(\beta^1,\beta^2,\beta^3)=\beta^A\mathbf{e}_A=\beta^0\mathbf{e}_0+\beta^+\mathbf{e}_++\beta^-\mathbf{e}_-,\\
&\{\mathbf{e}_0,\mathbf{e}_+,\mathbf{e}_-\}=\{\mathbf{1},\mathrm{diag}(1,1,-2),\sqrt{3}\,\mathrm{diag}(1,-1,0)\}.
\end{aligned}
\tag{5.1}
$$

The variables $\{\beta^A\}$ parametrizing $\mathscr{M}_D$ were introduced by Misner [20]. An explicit parametrization of the matrix $\mathbf{S}\in\hat{G}$ is not necessary, but it is convenient to choose a basis $\{\boldsymbol{\kappa}_a\}$ of the Lie algebra $\hat{\mathfrak{g}}$ of $\hat{G}$ such that the 1-parameter subgroup generated by the element κ_a maps $\mathscr{M}_D$ onto $\mathscr{M}_{S(a)}$. Thus if one introduces canonical coordinates of the second kind on $\hat{G}$ corresponding to the parametrization

$$\mathbf{S} = e^{\theta^1 \kappa_1} e^{\theta^2 \kappa_2} e^{\theta^3 \kappa_3}, \tag{5.2}$$

then each submanifold $\mathscr{M}_{S(a)}$ corresponds to the case where the single nonvanishing parameter is θ^a, while the diagonal submanifold corresponds to the vanishing of all of these parameters. They are therefore conveniently adapted to the various possible additional discrete spacetime symmetries discussed in the previous section. Both discrete and continuous additional symmetries motivate the choice of the diagonal basis $\{\mathbf{e}_A\}$ since the submanifold $\mathscr{M}_{TS(3)}$ corresponds to $\beta^- = 0 = \theta^1 = \theta^2$, while $\beta^- = 0 = \theta^a$ corresponds to $\mathscr{M}_{T(3)}$ and $\beta^+ = \beta^- 0 = \theta^a$ to $\mathscr{M}_I$. Of course the relevance of diagonal and off-diagonal spaces and of the various submanifolds of $\mathscr{M}$ to spatially homogeneous spacetimes depends crucially on the fact that the canonical basis e of $\mathfrak{g}$ assumed here has structure constant tensor components in standard diagonal form.

As discussed in Ref. [6], parametrization (5.1) may be interpreted in terms of two successive time-dependent transformations of the canonical basis e of $\mathfrak{g}$

$$e'_a = S^{-1b}{}_a e_b, \qquad e''_a = (e^{-\boldsymbol{\beta}})^b{}_a e'_b \tag{5.3}$$

which first orthogonalize and then normalize the spatial frame. Since $\mathbf{S} \in \mathrm{Aut}_e(\mathfrak{g})$, the first spatial frame has the same structure constant tensor components and may be completed to a symmetry adapted comoving ADM frame $\{e'_0 = e_0 + N'^b e'_b, e'_a\}$ corresponding to a nonzero choice of shift vector field determined algebraically by (2.2) from the "automorphism" velocities $\{\dot{\sigma}^a\}$ defined by

$$\dot{\mathbf{S}}\mathbf{S}^{-1} = \kappa_a \dot{\sigma}^a \tag{5.4}$$

which are related to the rotation of the orthonormal spatial frame $\{e''_a\}$ along the normal congruence [6]. ($\dot{\sigma}^a$ are designated by $\dot{\omega}^a$ in previous papers; see the appendix for their interpretation.) The orthonormal spatial frame is useful when a variational approach is not used to obtain the gravitational equations of motion. The diagonal matrix $\mathbf{g}' = e^{2\boldsymbol{\beta}}$, satisfying $\mathbf{g}'^{-1} = e^{-2\boldsymbol{\beta}}$ and $g = g' = \det \mathbf{g}' = e^{3\beta^0}$, is the metric component matrix with respect to the orthogonal primed frame which is a convenient frame for calculations.

The choice of parametrization (5.1) for the zero shift case is equivalent to choosing an initial orthogonal spatial frame with the same structure constant tensor components as $\{e_a\}$ and a shift vector field which preserves the orthogonality condition during the evolution. Provided one expresses the fluid equations of motion and the constraint equations in the primed spatial frame, the variables $\mathbf{S}$ appear only through the automorphism velocities through the presence of equivalent shift terms. In almost all cases the momentum constraints may be used to solve for these velocities in terms of the primed fluid and gravitational variables, thereby eliminating those constraints and three gravitational degrees of freedom. Since the equivalent shift components are then known, the spacetime metric may be expressed in the comoving primed frame as in (2.3), without integrating the equations of motion (2.2) for S which determine the transformation relating the primed frame to a normal one. It is logical to refer to the choice of such a comoving frame as "diagonal gauge."

The appropriate choice of fluid variables is made in two steps. First one must extend the automorphism transformation from $\mathscr{M}$ to $\mathscr{M} \times \mathscr{S}$ by choosing as new variables the components of the fluid current 1-form and the density l with respect to the primed spatial frame

$$v'_a = v_b S^{-1b}{}_a, \qquad l' = l, \qquad v^{\perp} = (\mu^2 + g'^{ab} v'_a v'_b)^{1/2}. \tag{5.5}$$

Using Eqs. (2.2) and (2.5)–(2.7) with $N = 1$ one finds the new equations of motion

$$\begin{aligned} (v'_a)^{\cdot} &= v'_f[(v^{\perp})^{-1} C^f{}_{ga} v'^g - \kappa_g{}^f{}_a \dot{\sigma}^g], \\ (\ln l)^{\cdot} &= 2a_a v'^a (v^{\perp})^{-1}. \end{aligned} \tag{5.6}$$

In the general case the solution of the supermomentum constraints enables one to replace the automorphism velocities by explicit expressions involving the primed fluid and gravitational variables.

Next one would like to take advantage of the fluid constants of the motion. For the class A Bianchi types these constants are $l = l'$ and $\varepsilon V^2 = n^{ab} v_a v_b = n^{ab} v'_a v'_b$, where the submanifolds of constant values of l and εV^2 on $\mathscr{S}$ are the orbits of the linear adjoint group $\mathrm{Ad}_e(G)$. One should therefore adapt the coordinates to these orbits. For the semisimple types IX and VIII, where the group $\mathrm{Aut}_e(\mathfrak{g})^+ = \mathrm{Ad}_e(G) = \mathrm{SAut}_e(\mathfrak{g})$ is, respectively, $SO(3, R)$ and $SO(2, 1)$, one may define when $V \neq 0$

$$v'_a = V \hat{v}_a \tag{5.7}$$

and then parametrize the "unit vector" $\hat{v}_a$ using the action of $\mathrm{Ad}_e(G)$

$$\begin{aligned} \text{IX: } (\hat{v}_a) &= (\sin\lambda_1 \cos\lambda_2, \sin\lambda_1 \sin\lambda_2, \cos\lambda_1), \\ \text{VIII: } (\hat{v}_a) &= (\tfrac{1}{2}(e^{\lambda_1} + \varepsilon e^{-\lambda_1}) \cos\lambda_2, \tfrac{1}{2}(e^{\lambda_1} + \varepsilon e^{-\lambda_1}) \sin\lambda_2, \pm\tfrac{1}{2}(e^{\lambda_1} - \varepsilon e^{-\lambda_1})). \end{aligned} \tag{5.8}$$

$\{V, \lambda_1, \lambda_2\}$ may be taken as new coordinates in place of $\{v_a\}$. The type VIII parametrization for v'_a rather than $\hat{v}_a$ is valid for the case $\varepsilon = V = 0$; alternatively one may simply use $v'_3 = \pm\frac{1}{2}e^{\lambda_1}$ and λ_2 as the independent variables in this case. The $\pm$ sign is necessary for $\varepsilon = 0, -1$ to permit $\hat{v}_3 < 0$, although the sign of $\hat{v}_3$ is unimportant due to the existence of a reflection automorphism.

Similarly for Bianchi types VII_0 and VI_0 when $V \neq 0$

$$\begin{aligned} \mathrm{VII}_0\text{: } (v'_a) &= (V\cos\lambda_2, V\sin\lambda_2, v'_3), \\ \mathrm{VI}_0\text{: } (v'_a) &= (\pm\tfrac{1}{2}V(e^{\lambda_1} + \varepsilon e^{-\lambda_1}), \pm\tfrac{1}{2}V(e^{\lambda_1} - \varepsilon e^{-\lambda_1}), v'_3). \end{aligned} \tag{5.9}$$

The $\pm$ sign is again unimportant in the type VI_0 case; when $V = 0$, then $|v'_1| = |v'_2|$ and one can take v'_1 and v'_3 as the independent variables.

The class B case is slightly more complicated since l is no longer a constant of the motion unless $v_a = 0$; however, the constant of the motion $l^{-1}\varepsilon V^2 = l^{-1}(n^{(1)}(v'_1)^2 + n^{(2)}(v'_2)^2)$ motivates the following parametrization:

$$(v'_a) = (l^{1/2}w_1, l^{1/2}w_2, v'_3). \tag{5.10}$$

For Bianchi type IV, $\{l, w_1, w_2, v'_3\}$ may be assumed as new variables with $|w_1| = l^{-1/2}V$ a constant of the motion; when $w_1 = 0$ then w_2 becomes a constant of the motion. For Bianchi type VII$_h$, setting $(w_1, w_2) = w(\cos\lambda_2, \sin\lambda_2)$, then $\{l, w, \lambda_2, v'_3\}$ are new variables with $w = l^{-1/2}V$ a constant of the motion. For Bianchi type VI$_h$, setting $(w_1, w_2) = w(\frac{1}{2}(e^{\lambda_2} + \varepsilon e^{-\lambda_2}), \frac{1}{2}(e^{\lambda_2} - \varepsilon e^{-\lambda_2}))$ when $V \neq 0$ yields new variables $\{l, w, \lambda_1, v'_3\}$ for which $|w| = l^{-1/2}V$ is a constant of the motion; in the case $V = 0$, then $|w_1| = |w_2|$ and $\{l, w_1, v'_3\}$ may be taken as the independent fluid variables.

For each of the class A and class B Bianchi types so far discussed, one may easily derive equations of motion for the new fluid variables from (5.6); for Bianchi type IX these equations have been given by Ryan (see Eq. (12.27) of Ref. [19]). The remaining Bianchi types I, II, and V are exactly those for which the subgroup $\hat{G}$ and hence parametrization (5.1) are not uniquely determined and they will be discussed below. First one must identify the appropriate subgroup $\hat{G} \subset \mathrm{SAut}_e(\mathfrak{g})$.

For the semisimple Bianchi types VIII and IX, the equalities $\mathfrak{aut}_e(\mathfrak{g}) = \mathfrak{saut}_e(\mathfrak{g}) = \mathfrak{ad}_e(\mathfrak{g})$ hold, while for the nonsemisimple types one has the following vector space direct sum decomposition of the canonical automorphism matrix Lie algebra,

$$\mathfrak{aut}_e(g) = \mathrm{span}\{\mathbf{E}\} \oplus \mathfrak{saut}_e(g), \tag{5.11}$$

where $\mathbf{E}$ is given by $\mathbf{1}$ for Bianchi type I, by diag(1, 1, 2) for Bianchi type II and by $\mathbf{I}^{(3)} \equiv \mathrm{diag}(1, 1, 0)$ for all the other nonsemisimple types. For these types the diagonal matrix $\mathbf{E}$ corresponds to a diagonal automorphism degree of freedom. By choosing an alternate basis of the Lie algebra of diagonal matrices which contains $\mathbf{E}$ as a basis element, one can adapt parametrization (5.1) to this additional automorphism degree of freedom. This has been done for the case $\mathbf{E} = \mathbf{I}^{(3)}$ by Siklos [49] and will be discussed below. In the Bianchi type I case the parametrization involving the variables $\{\beta^A, \mathbf{S}\}$ is already adapted to both $\mathbf{E}$ and the diagonal automorphism subgroup $\mathrm{Diag}(3, R)^+$.

Except for Bianchi types I, II and V, $\mathfrak{aut}_e(\mathfrak{g})$ is a 3-dimensional Lie algebra of off-diagonal matrices and one may set $\hat{G} = \mathrm{SAut}_e(\mathfrak{g})^+$. For Bianchi types II and V, introduce the $SL(3, R)$ subgroups $SL(2)_3 = \{\mathbf{A} \in SL(3, R) | A^a{}_3 = A^3{}_a = \delta^a{}_3\}$, $T(2)_3 = \{\mathbf{1} + \theta^1\mathbf{e}^3{}_1 + \theta^2\mathbf{e}^3{}_2 | (\theta^1, \theta^2) \in R^2\}$ and its transpose $T(2)_3{}^T$ (the semidirect product group $T(2)_3 \times_s SL(2)_3$ is the natural matrix representation of the group of special affine transformations of R^2); the group $\mathrm{SAut}_e(\mathfrak{g})^+$ is given, respectively, by $T(2)_3 \times_s SL(2)_3^+$ and $T(2)_3{}^T \times_s SL(2)_3^+$. For these two types the 3-dimensional group $SL(2)_3^+$ maps $\mathscr{M}_D$ onto $\mathscr{M}_{S(3)}$ and therefore one has to arbitrarily choose a 1-dimensional subgroup in order to reduce $\mathrm{SAut}_e(\mathfrak{g})^+$ to a 3-dimensional group $\hat{G}$ with the desired properties. It is convenient to choose the orthogonal subgroup

$SO(2)_3 = \{\exp \theta(\mathbf{e}^1{}_2 - \mathbf{e}^2{}_1) | \theta \in R\} \subset SL(2)_3^+$, and set $\hat{G} = T(2)_3 \times_s SO(2)_3$ and $\hat{G} = T(2)_3{}^T \times_s SO(2)_3$, respectively. For Bianchi type I, $\mathrm{SAut}_e(\mathfrak{g}) = SL(3, R)$ and the choice of G for any of the other Bianchi types would suffice; it is convenient again to choose the orthogonal subgroup $SO(3, R)$ as in the Bianchi type IX case. However, the arbitrariness of this choice introduces spurious degrees of freedom if the initial data is not chosen carefully [1]; as discussed in the previous section the general Bianchi type I case is equivalent to the diagonal case, as are the Bianchi type II and V symmetric cases $\mathscr{M}_{S(3)}$.

For Bianchi types II and V, recall that (v_1, v_2) and $l^{-1/2}(v_1, v_2)$ are, respectively, constants of the motion, but which now undergo a time-dependent rotation under the transformation to the new frame e'. However, this leaves the Euclidean norm of these 2-vectors invariant so it is appropriate to choose the parametrization of v'_a given for Bianchi types VII_0 and VII_h, respectively. Note that the choice of primed variables unavoidably leads to the loss of one fluid constant of the motion in the sense that only one functionally independent combination of the two constants of motion can be expressed in terms of the prime variables only; any other functionally independent combination necessarily depends explicitly on the transformation between the primed and unprimed frame. The same thing happens to the additional constant of the motion l_B described in Section 3 for the remaining class B types as well as to the conditional constant of the motion for type IV. For example, if l'_B denotes the same function of the primed components as l_B is of the unprimed components, and if one uses parametrization (5.2) for $\mathbf{S}$ (just set $a = 0$ in (3.11)), then one finds $l_B = l'_B e^{-2a\theta^3}$; hence one needs to know θ^3 explicitly to evaluate the constant of the motion l_B in the primed frame. This is exactly analogous to the loss of two of the four constants of the motion associated with a frame comoving the fluid when one passes to a normal frame as described in Section 3. It seems that the more closely adapted a frame is to the spatial geometry, the less closely it is adapted to the fluid.

Having specified the group $\hat{G} \subset \mathrm{SAut}_e(\mathfrak{g})$ for all of the Bianchi types it is necessary to specify a basis $\{\boldsymbol{\kappa}_a\}$ of its Lie algebra $\hat{\mathfrak{g}}$ such that the 1-parameter groups associated with the basis elements generate the three symmetric case submanifolds $\mathscr{M}_{S(a)}$ from the diagonal submanifold $\mathscr{M}_D$. A suitable such basis for the class A and class B Bianchi types, respectively, is given by

$$\begin{aligned} \text{class } A\colon \{\boldsymbol{\kappa}_a\} &= \mathbf{k}_a + \delta^I{}_Z \mathbf{k}^{IX}{}_a + \delta^{II}{}_Z \delta^3{}_a \mathbf{k}^{IX}{}_3, \\ \text{class } B\colon \{\boldsymbol{\kappa}_a\} &= \{\mathbf{e}^3{}_1, \mathbf{e}^3{}_2, \mathbf{k}^0{}_3 + \delta^V{}_Z \mathbf{k}^{IX}{}_3\}. \end{aligned} \tag{5.12}$$

Here $\delta^Z_{Z'}$ is the Bianchi type Kronecker delta, $\mathbf{k}^0{}_b$ is the matrix obtained from $\mathbf{k}_b$ by setting the structure constant a to zero and $\mathbf{k}^{IX}{}_a = -\varepsilon_{abc}\mathbf{e}^c_b$ are the canonical Bianchi type IX adjoint matrices. Recall that $\mathbf{k}^0{}_a = -n^{(b)}\mathbf{e}^c{}_b + n^{(c)}\mathbf{e}^b{}_c$ for each cyclic permutation (a, b, c) of $(1, 2, 3)$ and that $\mathbf{k}^0{}_a = \mathbf{k}_a$ for class A types [1].

In the class B case, $\boldsymbol{\kappa}_1$ and $\boldsymbol{\kappa}_2$ should be interchanged in order that the basis element $\boldsymbol{\kappa}_a$ always generate $\mathscr{M}_{S(a)}$ from $\mathscr{M}_D$ as described after Eq. (5.1); however, this is unimportant in the class B case and will be ignored. Alternatively, to facilitate comparison with the class A types, one could introduce the following class B basis (excluding type V)

$$\kappa_a = \mathbf{k}^0{}_a - \delta^{IV}{}_Z \delta^1{}_a \mathbf{e}^3{}_2, \tag{5.13}$$

but to conform with previous papers this will not be done. (See note added in proof.)

The components $\hat{C}^a{}_{bc}$ of the structure constant tensor of $\hat{\mathfrak{g}}$ in the basis $\{\kappa_a\}$, defined by

$$[\kappa_a, \kappa_b] = \hat{C}^c{}_{ab}\kappa_c, \tag{5.14}$$

are important for the equations of motion (see the appendix). For both (5.12) and (5.13) one finds that $\hat{C}^a{}_{bc} = \varepsilon_{bcd} n^{ad} \equiv C^{0a}{}_{bc}$ for all Bianchi types except I, where $\hat{C}^a{}_{bc} = C^q{}_{bc}$ (IX), and types II and V, where $\hat{C}^a{}_{bc} = C^a{}_{bc}$ (VII_0) and type IV for (5.13), where $\hat{C}^2{}_{31} = 1 = -\hat{C}^2{}_{13}$ are the only nonvanishing components; the Roman numeral in parentheses indicates the canonical components of that Bianchi typed.

As in Eq. (2.23), define the "symmetrized" matrices by

$$\mathscr{K}'{}_a = \tfrac{1}{2}(\kappa_a + e^{-2\beta}\kappa_a{}^T e^{2\beta}) \equiv \mathrm{SYM}_{g'}(\kappa_a). \tag{5.15}$$

$\{\mathbf{e}_A, \mathscr{K}'{}_a\}$ are the matrices of mixed primed components of symmetric tensor fields on the Riemannian manifold $(G, {}^3g)$ which when interpreted as elements of $T\mathscr{M}_{g'}$ (the tangent space at $g' = e^{2\beta} \in \mathscr{M}_D$) form an orthogonal basis with respect to the DeWitt metric

$$\langle \mathbf{e}_A, \mathbf{e}_B \rangle_{DW} = 6\eta_{AB}, \qquad \langle \mathscr{K}'{}_a, \mathscr{K}'{}_b \rangle_{DW} = \mathscr{G}_{ab}. \tag{5.16}$$

Both η_{AB} and $\mathscr{G}_{ab}$ are diagonal with $-\eta_{00} = \eta_{++} = \eta_{--} = 1$ and having inverses $\eta^{AB} = \eta_{AB}$ and $\mathscr{G}^{-1ab}$. $\mathscr{G}_{ab}$ are the components of a nonnegative quadratic form which depend only on $\beta^{\pm}$. Zeros of these diagonal components correspond to singularities in parametrization (5.1) and are associated with orthogonal subgroups of $\hat{G}$ which become isotropy subgroups on the singular subspaces of $\mathscr{M}$. Explicit expressions for $\mathscr{G}_{ab}$ may be found in Refs. [1, 6].

Evaluating the primed components of the extrinsic curvature matrix and the kinetic energy function on the velocity phase space $T\mathscr{M}$ one finds

$$\begin{aligned} -\mathbf{K}' &= \dot{\beta} + \mathscr{K}'{}_a \dot{\sigma}^a, \\ \mathscr{E} &= e^{3\beta^0}\langle \mathbf{K}', \mathbf{K}' \rangle_{DW} = e^{3\beta^0}(6\eta_{AB}\dot{\beta}^A\dot{\beta}^B + \mathscr{G}_{ab}\dot{\sigma}^a\dot{\sigma}^b). \end{aligned} \tag{5.17}$$

Computing the momenta p_A and P_a conjugate to the velocities $\dot{\beta}^A$ and $\dot{\sigma}^a$ in the usual way from $\mathscr{E}$ alone (since the fluid Lagrangian is independent of the gravitational velocities), one finds the kinetic energy function on the momentum phase space

$$\begin{aligned} p_A &= \partial\mathscr{E}/\partial\dot{\beta}^A = 12e^{3\beta^0}\eta_{AB}\dot{\beta}^B, \qquad P_a = \partial\mathscr{E}/\partial\dot{\sigma}^a = 2e^{3\beta^0}\mathscr{G}_{ab}\dot{\sigma}^b, \\ \dot{\beta}^A &= (12)^{-1}\, e^{-3\beta^0}\eta^{AB}p_B, \qquad \dot{\sigma}^a = \tfrac{1}{2}e^{-3\beta^0}\mathscr{G}^{-1ab}P_b, \\ \mathscr{E} &= \tfrac{1}{4}e^{-3\beta^0}(\tfrac{1}{6}\eta^{AB}p_A\,p_B + \mathscr{G}^{-1ab}P_aP_b). \end{aligned} \tag{5.18}$$

Similarily one may evaluate the gravitational potential and the appropriate components of the nonpotential force, both of which depend only on the scale variables, while the fluid potential is

$$\mathscr{H}_M = 2klv^{\perp} - 2kpe^{3\beta^0}, \qquad v^{\perp} = (\mu^2 + g'^{ab}v'_a v'_b)^{1/2}. \tag{5.19}$$

Since $\mathscr{E}$ is a diagonal quadratic form in the new velocities or momenta, the equations of motion take a very simple form. Note that **S** does not appear explicitly in the Lagrangian or Hamiltonian when expressed in terms of $\{\beta^A, \dot{\beta}^A, \dot{\bar{\sigma}}^a, v'_a\}$ or $\{\beta^A, p_A, P_a, v'_a\}$, but one must recall that $v'_a = v_b S^{-1b}{}_a$ to obtain the correct equations of motion for the automorphism variables. This is discussed in the appendix. The equations of motion for the automorphism velocities or momenta are first order and do not explicitly involve **S**.

Thus the gauge symmetry of the dynamics has enabled the appearance of the special automorphism variables to be limited to their velocities $\dot{\bar{\sigma}}^a$ or equivalently to their momenta P_a. These velocities or momenta, together with the scale variables and primed fluid variables, are in fact all that are required to specify completely the spacetime metric and fluid variables in a symmetry adapted comoving ADM frame $\{e'_a\}$ with unit lapse function and a shift vector field determined by the automorphism velocities. However, in all cases except Bianchi types I, II, V and $\mathrm{VI}_{-1/9}$, the primed components of the supermomentum constraints may be explicitly solved for the automorphism velocities or momenta in terms of the remaining degrees of freedom, allowing them to be entirely eliminated from the dynamics. In those degenerate cases where an automorphism velocity is not determined in this way, its first order equation of motion must be retained instead.

The primed components of the supermomentum constraints have been evaluated in Refs. [1, 6]; however, $\mathscr{H}^G{}_{a'}$ should be replaced by $\frac{1}{2}\mathscr{H}^G{}_{a'}$ in the formulas of Ref. [1]. In terms of the momenta one has

$$\mathscr{H}_{a'} = -\rho^b{}_a P_b - a_a p_+ - 2klv'_a,$$

class A (except I, II): $\boldsymbol{\rho} = \mathbf{1} = \boldsymbol{\rho}^{-1}$,

$$\mathrm{I}: \boldsymbol{\rho} = 0, \qquad \mathrm{II}: \boldsymbol{\rho} = \mathbf{I}^{(3)};$$

class B: $\boldsymbol{\rho} = -3a\mathbf{I}^{(3)} - \mathbf{k}^0{}_3 + (1 - \delta^V{}_Z)\,\mathbf{e}^3{}_3$,

$$\det \boldsymbol{\rho} = (9a^2 + n^{(1)}n^{(2)})(1 - \delta^V{}_Z) = a^2(9 + h^{-1})(1 - \delta^V{}_Z),$$
$$(\det \boldsymbol{\rho})\boldsymbol{\rho}^{-1} = -3a\mathbf{I}^{(3)} + \mathbf{k}^0{}_3 + \mathbf{e}^3{}_3 \qquad (\text{except V}, \mathrm{VI}_{-1/9}). \tag{5.20}$$

Thus in all cases except Bianchi types I, II, V and $\mathrm{VI}_{-1/9}$ one has

$$P_a = -\rho^{-1b}{}_a(2klv'_a + a_a p_+) = 2e^{3\beta^0}\mathscr{G}_{ab}\dot{\bar{\sigma}}^b, \tag{5.21}$$

which is easily inverted to obtain the velocities. Except for P_3 in the class B case which depends on p_+, these expressions may be substituted into the Hamiltonian without affecting the remaining equations of motion, leaving behind only an effective potential; in the class B case the substitution of the expression for P_3 can be performed only after the equation of motion for β^+ is derived. For Bianchi type V one has instead

$$(P_1, P_2, p_+) = 2kl(3a)^{-1}(v'_1, v'_2, -3v'_3), \tag{5.22}$$

so P_1 and P_2 leave behind an effective potential but the first order equation of motion for P_3 remains, while the β^+ degree of freedom is governed by the first order equation of motion for β^+ since p_+ is known. For Bianchi type II

$$(P_1, P_2) = -2kl(v'_1, v'_2) \tag{5.23}$$

so here too P_1 and P_2 leave behind an effective potential while P_3 must be evolved using its equation of motion. For Bianchi type I, the supermomentum constaints simply force $v'_a = 0$ leaving all three automorphism momenta to be evolved by their equations of motion; however, as argued above it is sufficient to consider the diagonal case for a perfect fluid source, and one may set $P_a = 0$.

For Bianchi type $\mathrm{VI}_{-1/9}$, it is convenient to use the noncanonical basis $\bar{e}_a = (r^{1/2}_{(3)})^{-1b}{}_a e_b$ introduced in Eq. (3.9) together with an explicit parametrization of $\bar{\mathbf{S}}$ (a bar indicates noncanonical components)

$$\begin{aligned}
\bar{\mathbf{g}} &= (\mathbf{r}^{1/2}_{(3)})^{-1T}\,\mathbf{g}(\mathbf{r}^{1/2}_{(3)})^{-1} = \bar{\mathbf{S}}^T\bar{\mathbf{g}}'\bar{\mathbf{S}},\\
\bar{\mathbf{S}} &= \exp\theta^3\bar{\boldsymbol{\kappa}}_3\exp(\theta^1\boldsymbol{\kappa}_1 + \theta^2\boldsymbol{\kappa}_2) \in \mathrm{Aut}_{\bar{e}}(\mathfrak{g}),\\
\bar{\boldsymbol{\kappa}}_3 &= (\mathbf{r}^{1/2}_{(3)})^{-1}\,\boldsymbol{\kappa}_3(\mathbf{r}^{1/2}_{(3)}), \qquad \bar{v}'_a \doteq v'_b(r^{1/2}_{(3)})^{-1b}{}_a.
\end{aligned} \tag{5.24}$$

Let $\dot{\theta}^a$ and p_a be the coordinate automorphism velocities and conjugate momenta, which turn out to satisfy $(p_1, p_2, p_3) = (e^{\theta^3}\bar{P}_1, e^{-\theta^3}\bar{P}_2, \bar{P}_3)$. The supermomentum constaints then take the form

$$(\bar{P}_1, 0, \bar{P}_3) = (2kl\bar{v}'_1, 2kl\bar{v}'_2, -2kl\bar{v}'_3 - ap_+). \tag{5.25}$$

The fluid constraint $\bar{v}'_2 = 0$ makes θ^2 a cyclic variable and since the nonpotential force has no component along θ^2, p_2 is a constant of the motion. The momenta $\bar{P}_1$ and $\bar{P}_3$ are determined but the first order equation of motion for $\bar{P}_2 = e^{\theta^3}p_2$ must be retained and is clearly equivalent to a first order equation for θ^3. Since it is not necessary to integrate the equation of motion for θ^3, the constant of the motion p_2 is of no use in further reducing the number of nontrivial equations of motion.

In all cases the remaining degrees of freedom are subject only to the super-Hamiltonian constraint. As described in Ref. [1], this constraint is most naturally eliminated by choosing $\beta^0 \equiv -\Omega$ as the new time function, corresponding to the new lapse choice $N = -12e^{3\beta^0}(p_0)^{-1}$. This leads to a reduced Hamiltonian which generates the equations of motion for the remaining unconstrained gravitational degrees of freedom, exactly as discussed in Ref. [25], where this process is called the "intrinsic time" reduction. Briefly one makes the definitions

$$\begin{aligned} \mathscr{H} + h &= \tfrac{1}{24} e^{-3\beta^0}(-{p_0}^2 + {I_h}^2) = 0, \\ -I_h &= \mp [24 e^{3\beta^0}(\mathscr{H} + h) + {p_0}^2]^{1/2}, \end{aligned} \tag{5.26}$$

where I_0 is the result of solving the super-Hamiltonian constraint for $-p_0$, the minus sign applying when p_0 is negative (expansion) and the plus sign when p_0 is positive (contraction). One then has a reduced Hamiltonian system with Hamiltonian function I_0; for example,

$$d\beta^{\pm}/d\beta^0 = \{\beta^{\pm}, I_0\}, \qquad dp_{\pm}/d\beta^0 = \{p_{\pm}, I_0\} + NQ_{\pm}, \tag{5.27}$$

where the lapse is defined by

$$N = dt/d\beta^0 = \partial I_h/\partial h|_{h=0}. \tag{5.28}$$

This choice of time fails at points of maximum expansion or minimum contraction and only differs in sign from the Misner–Ryan Ω-time [19].

Thus one has a reduced Hamiltonian system for the two degrees of freedom $\beta^{\pm}$ driven by the fluid variables l and v'_a and whatever automorphism momenta were not determined by the supermomentum constraints, both types of variables satisfying first order equations of motion, while the single thermodynamic variable n is determined implicitly by the defining relation for l. In the semisimple case where no automorphism momenta remain, the variables $\beta^{\pm}$ parametrize the conformal 3-geometry [1] so one has an explicit illustration of York's idea that the true degrees of freedom of the gravitational field reside in the conformal 3-geometry [21]. Furthermore, the equivalent shift vector field picked out by the solution of the supermomentum constraints (and unique modulo spatial Killing vector fields) is exactly the appropriate analogue of the minimal distortion shift vector field introduced by Smarr and York [2, 26] to eliminate spatial gauge degrees of freedom without explicitly solving the supermomentum constraints. For the remaining Bianchi types the close correspondence with the general theory is unfortunately broken.

6. Alternative Formulations

The approach described above is not the only way to formulate the dynamics of spatially homogeneous perfect fluid spacetimes. One can abandon the Lagrangian/Hamiltonian approach, for example. The driven Lagrangian or Hamiltonian equations of motion for the symmetry adapted gravitational variables are equivalent to the spatial part of the Einstein equations (2.17) expressed in the orthonormal frame $\{e''_\alpha\} = \{e_0, e''_a\}$. The diagonal components of these equations evolve the scale variables while the off-diagonal components evolve the special automorphism variables [6, 35]. The evolution equations determined by instead using the spatial part of the Ricci tensor form of the Einstein equations

$$R''^a{}_b = kT^{TR\,''a}{}_b \equiv k(T''^a{}_b - \tfrac{1}{2}T''^\alpha{}_\alpha \delta^a{}_b) \tag{6.1}$$

agree with the original equations of motion except for the variable β^0 whose equation of motion changes by a term proportional to the super-Hamiltonian [4]. However, if one chooses gravitational variables not including the scale degree of freedom represented by β^0, more of the equations of motion differ by terms involving the super-Hamiltonian and it is sometimes more convenient to use the Ricci form of the evolution equations [4, 49].

Whatever means are chosen to formulate the evolution equations, two further kinds of symmetries remain to be exploited. Independent of spacetime symmetry, the equations of general relativity are invariant under the 1-parameter group of scale transformations of the unit of length under which each field undergoes a uniform scaling with weight given by the dimension of the field. Although in general this symmetry is trivial, for the spatially homogeneous spacetimes, it plays a role on an equal footing with the symmetry group Aut($\mathfrak{g}$). The consequences of scale invariance for spatially homogeneous dynamics were first developed by Novikov [50] and used by himself and Bogoyavlensky [5] in their studies of the initial singularity. Secondly, for the nonsemisimple Bianchi types, the automorphism group is larger than the 3-dimensional subgroup $\hat{G} \subset \mathrm{SAut}_e(\mathfrak{g})$ used in the parametrization (5.1), and its intersection with the diagonal scale group $\mathrm{Diag}(3, R)^+ \subset GL(3, R)$ can be exploited by choosing a basis of the Lie algebra $\mathrm{diag}(\mathfrak{gl}(3, R))$ adapted to the additional diagonal automorphisms and using this basis in the parametrization $g' = e^{2\beta}$. This was described for a large class of nonsemisimple types by Siklos [49], who also took advantage of a modified scale invariance described below. For the abelian Bianchi type I, the entire scale group including the uniform scale transformation subgroup is an automorphism subgroup and no changes are required.

Under the 1-parameter group of constant scale transformations of the unit of length, each physical geometric object field Φ of dimension q undergoes the rescaling

$$\Phi \to e^{q\alpha}\Phi, \tag{6.2}$$

where $\alpha \in R$ is the group parameter. The spacetime metric is of dimension 2 and the transformation ${}^4g \to e^{2\alpha}\,{}^4g$ takes the following form in a normal comoving ADM frame with proper time t:

$$(t, g_{ab}, \pi^{ab}) \to (e^\alpha t, e^{2\alpha} g_{ab}, \pi^{ab}). \tag{6.3}$$

If on the other hand one introduces a nontrivial lapse function by $dt = N d\bar{t}$, then the coordinate $\bar{t}$ is invariant provided that N is of dimension 1. The dimensions of the fluid quantities (ρ, p, u_α, n, μ, l, v_a) for the equation of state (1.3) are, respectively, (−2, −2, 1, $-2\gamma^{-1}$, $2(\gamma^{-1}-1)$, $3-2\gamma^{-1}$, $2\gamma^{-1}-1$). Note that the orthonormal components u''_α, $l'' = e^{-3\beta^0} l$ and v''_a have dimensions 0, $-2\gamma^{-1}$, and $2(\gamma^{-1}-1)$, respectively. Similarly the covariant Einstein and Ricci tensors have dimension zero but their orthonormal components have dimension −2.

Suppose one considers a nontrivial lapse function N and associated time function $\bar{t}$. The ADM Lagrangian is

$$\begin{aligned} L_N &= 2Ng^{1/2}(G^{\perp}{}_{\perp} - kT^{\perp}{}_{\perp}) \\ &= Ng^{1/2}(\langle \mathbf{K}, \mathbf{K}\rangle_{DW} + {}^3R) - N\mathscr{H}_M, \qquad \mathbf{K} = -(2N)^{-1}\,\mathbf{g}^{-1}\overset{\circ}{\mathbf{g}}, \end{aligned} \tag{6.4}$$

and the evolution equations and super-Hamiltonian constraint become

$$\begin{aligned} -\bar{\delta}L_N/\bar{\delta}g_{ab} &= Ng^{1/2}(G^{ab} - kT^{ab}) + NQ^{ab} = NQ^{ab}, \\ \partial L_N/\partial N &= -\mathscr{H} = 0, \end{aligned} \tag{6.5}$$

where the notation $\bar{\delta}/\bar{\delta}g_{ab}$ reminds one to use the new time derivative "°". Next suppose N is chosen to be an explicit function of $\mathbf{g}$ and set $\mathscr{L} = L_{N(\mathbf{g})}$. Because of the identity

$$\bar{\delta}\mathscr{L}/\bar{\delta}g_{ab} = \bar{\delta}L_N/\bar{\delta}g_{ab}|_{N=N(\mathbf{g})} - \mathscr{H}\partial N/\partial g_{ab} \tag{6.6}$$

and the super-Hamiltonian constraint, $\mathscr{L}$ may be used as the Lagrangian to generate the evolution equations. The Hamiltonian associated with L_N is $H_N = N\mathscr{H}$. Here N may in fact be chosen to be an explicit function of $\mathbf{g}$ and $\boldsymbol{\pi}$, i.e., $N = N(\mathbf{g}, \boldsymbol{\pi})$, since the equations of motion are again only changed by terms which are proportional to the super-Hamiltonian. (Similar arguments are given in [54].) Provided that the explicit function N in each case is chosen to have dimension 1, the Lagrangian and Hamiltonian will both have dimension 2. For example, in terms of parametrization (5.1), both

$$N_{(1)} = (\dot{\beta}^0)^{-1} = ((\ln g^{1/6})^{\dot{}})^{-1} = -(\tfrac{1}{3}\,\mathrm{Tr}\,\mathbf{K})^{-1} = -(\tfrac{1}{6}g^{-1/2}\,\mathrm{Tr}\,\boldsymbol{\pi})^{-1} = -12e^{3\beta^0}p_0{}^{-1} \tag{6.7}$$

and $N_{(2)} = e^{\beta^0} = g^{1/6}$ have dimension 1 and the scale transformations of the spatial metric are simply translations by α of the parameter β^0. The first choice of lapse is associated with the logarithmic volume time $\bar{t} = \beta^0 = -\Omega$ (or negative Ω-time), while the second is the generalization of the "conformal time" in the Friedmann models since $e^{2\beta^0}$ becomes an overall conformal factor of the spacetime metric.

By defining $\Phi = e^{q\beta^0}\underset{\sim}{\Phi}$ for all fields of dimension q, one obtains a new set of scale invariant fields $\underset{\sim}{\Phi}$. The equations of motion for the degrees of freedom other than β^0 may be rewritten in terms of the scale invariant variables, provided one uses the identity

$$\overset{\circ}{\underset{\sim}{\Phi}} = e^{-q\beta^0}\overset{\circ}{\Phi} - q\overset{\circ}{\beta}{}^0\,\underset{\sim}{\Phi} \tag{6.8}$$

to add an additional term to each time derivative of a field Φ of dimension q. The remaining gravitational variables $(\beta^{\pm}, \mathbf{S})$ and the velocities $(\overset{\circ}{\beta}{}^A, \overset{\circ}{\omega}{}^a = N\dot{\omega}^a)$ have zero dimension but the canonical momenta $(p_A, P_a) = e^{2\beta^0}(\underset{\sim}{p}_A, \underset{\sim}{P}_a)$ have dimension 2 so the gravitational evolution equations when rewritten in terms of the scale invariant quantities include a new term equivalent to a force proportional to the momentum of the system with proportionality factor $-2\mathring{\beta}^0$; for example,

$$\begin{aligned} -\bar{\delta}\mathscr{L}/\bar{\delta}\beta^{\pm} &= \underset{\sim}{N}\underset{\sim}{Q}_{\pm} - 2\mathring{\beta}^0\underset{\sim}{p}_{\pm}, \\ \mathring{\beta}^{\pm} = \partial\underset{\sim}{H}/\partial\underset{\sim}{p}_{\pm}, \qquad \mathring{\underset{\sim}{p}}_{\pm} &= -\partial\underset{\sim}{H}/\partial\beta^{\pm} + \underset{\sim}{N}\underset{\sim}{Q}_{\pm} - 2\mathring{\beta}^0\underset{\sim}{p}_{\pm}. \end{aligned} \tag{6.9}$$

The new "pseudoforce" is exactly analogous to a frictional force [50, 51] at least when β^0 is increasing with the time $\bar{t}$ and the force acts against the momentum.

The scale invariant velocity $\mathring{\beta}^0$ has the value 1 for the lapse choice N_1 for which $\bar{t} = \beta^0$ (and $\underset{\sim}{N}_1 = -12\underset{\sim}{p}_0^{-1}$). The reduced Hamiltonian for the $\beta^{\pm}$ degrees of freedom is just $\underset{\sim}{I}_0 = -\underset{\sim}{p}_0$ expressed as a function of the remaining variables by the super-Hamiltonian constraint. Since p_0 has the same dimension as the Hamiltonian H, the reduced Hamiltonian equations rewritten in scale invariant form with Hamiltonian $\underset{\sim}{I}_0^{*} = \underset{\sim}{p}_0$ have the same form as the second line of (6.9).

On the other hand for the lapse choice $N_{(2)} = e^{\beta^0} = (12)^{-1} I_0 N_{(1)} (\underset{\sim}{N}_{(2)} = 1)$, the relation $\mathring{\beta}^0 = -(12)^{-1}\underset{\sim}{p}_0$ holds and the scale invariant Hamiltonian is $\underset{\sim}{H} = (24)^{-1}(-\underset{\sim}{p}_0{}^2 + \underset{\sim}{I}_0{}^2)$. The equations of motion for the degrees of freedom other than β^0 may therefore be derived from the Hamiltonian $(24)^{-1}\underset{\sim}{I}_0{}^2$ of familiar form rather than the square root Hamiltonian $\underset{\sim}{I}_0$ for the other lapse choice. Thus the two obvious lapse choices correspond essentially to using the square root reduced Hamiltonian or its square in scale invariant form. For the choice $N_{(2)}$ one must still integrate (by quadrature) the equation $\mathring{\beta}^0 = (12)^{-1}\underset{\sim}{I}_0$ for the variable β^0. This choice is better suited to the point of maximum expansion $\mathring{\beta}^0 = 0$ which occurs in Bianchi type IX.

For the nonsemisimple Bianchi types where the identity component $\mathrm{Diag}(3, R)^+ \cap \mathrm{Aut}_e(\mathfrak{g})$ of the diagonal automorphism subgroup is nontrivial, one can choose a new basis of the Lie algebra of diagonal matrices adapted to the subspace of diagonal automorphism generators and its orthogonal complement with respect to the trace inner product. Using this basis in parametrization (5.1) and letting $\mathbf{g}'''$ denote the factor of $\mathbf{g}'$ associated with the complementary variables, one may interpret the parametrization in terms of an additional scaling of the canonical basis e' by the diagonal automorphism factor in the parametrization to obtain a new canonical basis e'''. Since by construction the complementary variables describe directions in $\mathscr{M}$ which are orthogonal (with respect to the inner product (2.27)) to the orbits of $\mathrm{Aut}_e(\mathfrak{g})$, the class of equivalent shift vector fields associated with the frame e''' consists of true minimal strain shift vector fields [2], the natural analogues of the minimal strain shifts introduced by Smarr and York for general asymptotically flat or spatially compact spacetimes [26]. Describing the Einstein equations and fluid equations of motion using the frame e''' is therefore equivalent to imposing the true minimal strain condition on the shift vector field; this is naturally called "minimal strain gauge." (For the semisimple Bianchi types the true minimal strain and minimal distortion shift vector fields coincide with the equivalent shift determined by the solution of the supermomentum constraints [52].)

However, the uniform scale transformations act as translations in both the diagonal automorphism parameter(s) and in the complementary parameter space. It is therefore useful to consider instead only the component of the uniform scale translation which acts in the space of complementary variables, leaving the diagonal automorphism parameter space fixed. The above discussion of scale invariant variables will then take a slightly different form as will become clear below, but enabling one to obtain a system of equations involving the modified scale invariant quantities. In this way one can avoid the appearance of some of the diagonal parameters in the equations of motion but only at the expense of adding additional terms involving their velocities.

In the previous section the diagonal automorphism generator $\mathbf{E}$ associated with the nonunimodular part of $\mathrm{Aut}_e(\mathfrak{g})$ was introduced, having the value $\mathbf{1}$ for Bianchi type I, $\mathrm{diag}(1,1,2)$ for Bianchi type II and $\mathbf{I}^{(3)} \equiv \mathrm{diag}(1,1,0)$ for the remaining nonsemisimple types. For all Bianchi types except I, II, and V, no more independent diagonal automorphism generators exist, but $\mathbf{e}_-$ generates such an automorphism for these exceptional types, as does $\mathbf{e}_+$ or $\mathbf{I}^{(3)}$ in the type I case. In each case $\mathbf{e}_-$ and $\mathbf{E}$ are orthogonal with respect to the trace inner product $\langle , \rangle$.

In the nonabelian semisimple case introduce the matrix $\mathbf{F}$ which represents the component of the uniform scale generator $\mathbf{e}_0 = \mathbf{1}$ which is orthogonal to the diagonal automorphism generators

$$\mathbf{F} = \mathbf{1} - \langle \mathbf{1}, \mathbf{E} \rangle \langle \mathbf{E}, \mathbf{E} \rangle^{-1} \mathbf{E}. \tag{6.10}$$

This has the value $\frac{1}{3}\,\mathrm{diag}(1,1,-1)$ for Bianchi type II and $\mathbf{e}^3{}_3$ for the remaining nonabelian nonsemisimple types. Then $\{\mathbf{e}_-, \mathbf{F}, \mathbf{E}\}$ is an orthogonal basis of $\mathrm{diag}(\mathfrak{gl}(3,R))$ with respect to the trace inner product.

The unit matrix and any matrix such as $\mathbf{F}$, which differs from it by an automorphism generator, generates the matrix representation of a 1-parameter group of projective automorphisms of $\mathfrak{g}$ under which the structure constant tensor components undergo a uniform scaling of weight -1 relative to the canonical parameter

$$\mathbf{A} = e^{\alpha(1+\mathbf{X})}, \qquad \mathbf{X} \in \mathfrak{aut}_e(\mathfrak{g}); \qquad A^a{}_d C^d{}_{fg} A^{-1f}{}_b A^{-1g}{}_c = e^{-\alpha} C^a{}_{bc}. \tag{6.11}$$

The matrix representation $\mathrm{PAut}_e(\mathfrak{g})$ of the projective automorphism group of $\mathfrak{g}$ is in fact generated by the direct sum Lie algebra $\mathfrak{paut}_e(\mathfrak{g}) = \mathrm{span}\{\mathbf{1}\} \oplus \mathfrak{aut}_e(\mathfrak{g})$.

For all of the nonabelian nonsemisimple types $\mathfrak{g}_P = \hat{\mathfrak{g}} \oplus_s \mathrm{span}\{\mathbf{E}, \mathbf{F}\} \subseteq \mathfrak{paut}_e(\mathfrak{g})$ is a semidirect sum Lie algebra, with $\{\boldsymbol{\kappa}_\alpha\} = \{\boldsymbol{\kappa}_0 \equiv \mathbf{E}, \boldsymbol{\kappa}_a\}$ being a basis of $\mathfrak{aut}_e(g) = \hat{\mathfrak{g}} \oplus_s \mathrm{span}\,\{\mathbf{E}\}$ for the types other than II and V and of the proper Lie subalgebra $\hat{\mathfrak{g}} \oplus_s \mathrm{span}\,\{\mathbf{E}\}$ of $\mathfrak{aut}_e(\mathfrak{g})$ for the latter types. Adding the automorphism generator $\mathbf{e}_-$ to the basis $\{\boldsymbol{\kappa}_\alpha\}$ does not lead to a Lie subalgebra of $\mathfrak{aut}_e(\mathfrak{g})$ for these latter two types; the group generated by $\{\boldsymbol{\kappa}_\alpha\}$ is in fact a choice of one element from one of three conjugacy classes of proper Lie subgroups of $\mathrm{Aut}_e(\mathfrak{g})$ which act freely almost everywhere on $\mathscr{M}$ (i.e., act without isotropy groups). No larger such subgroup exists for these two types. Similarly the basis $\{\mathbf{e}_0, \boldsymbol{\kappa}_a\}$ generates the largest such subgroup containing $\hat{G} = SO(3,R)$ for Bianchi type I since the Lie brackets with $\{\mathbf{e}_\pm\}$ do not close; a maximal such subgroup for this case is instead generated by the basis $\{\mathbf{F}, \mathbf{E}, \boldsymbol{\kappa}_a\}$ for any of the other nonsemisimple Bianchi types. For this reason the Bianchi type I case will not be explicitly discussed below; Lie algebra contraction of the results for these other types cover this case.

Parametrization (5.1), choosing instead the diagonal parametrization,

$$\begin{aligned} \mathbf{g}' &= e^{2\boldsymbol{\beta}}, \qquad \boldsymbol{\beta} = \beta^- \mathbf{e}_- + \zeta\mathbf{F} + \lambda\mathbf{E}, \\ 3\beta^0 &= \zeta \operatorname{Tr} \mathbf{F} + \lambda \operatorname{Tr} \mathbf{E}, \qquad 6\beta^+ = \zeta\langle \mathbf{e}_+, E\rangle + \lambda\langle \mathbf{e}_+, \mathbf{E}\rangle, \end{aligned} \tag{6.12}$$

may be interpreted in terms of two further scale transformations of the spatial frame e':

$$\begin{aligned} e_a''' &= e_b'(e^{-2\lambda \mathbf{E}})^b{}_a, \qquad \mathbf{g}''' = e^{2(\beta^-\mathbf{e}_- + \zeta\mathbf{F})}, \\ e_a'''' &= e_b'''(e^{-2\zeta\mathbf{F}})^b{}_a, \qquad \mathbf{g}'''' = e^{2\beta^-\mathbf{e}_-}. \end{aligned} \tag{6.13}$$

The additional β^- scaling leads to the orthonormal spatial frame e''. The spatial frame e''' is canonical and may be completed to a comoving frame with ADM generator $e_0''' = Ne_\perp + \vec{N}$ satisfying

$$\begin{aligned} \operatorname{ad}_{e'''}(N^{-1}\vec{N}) &= (e^{\lambda\mathbf{E}}\mathbf{S})^{\cdot}\, (e^{\lambda\mathbf{E}}\mathbf{S})^{-1} = \dot{\Sigma}^\alpha \boldsymbol{\kappa}_\alpha, \\ \dot{\Sigma}^0 = \dot{\lambda}, \qquad \dot{\Sigma}^a &= (e^{\lambda \mathbf{I}^{(3)}})^a{}_b\, \overset{\circ}{\sigma}{}^b, \qquad \overset{\circ}{\Sigma}{}^\alpha \equiv N\dot{\Sigma}^\alpha. \end{aligned} \tag{6.14}$$

For Bianchi types II and V and in the Bianchi type I limit, where β^- is also an automorphism degree of freedom, one could consider an intermediate canonical frame obtained from e''' by scaling away β^-; this latter frame is the minimal distortion frame referred to above (except for type I, where it is instead the orthonormal spatial frame e''). However, once the equations of motion are solved for the remaining variables associated with this frame including $\overset{\circ}{\beta}{}^-, \beta^-$ would have to be found by quadrature in order to determine the corresponding shift vector field for the frame precisely because the Lie brackets of $\mathbf{e}_-$ with $\{\boldsymbol{\kappa}_\alpha\}$ do not close. (The failure of $\{\mathbf{e}_-, \mathbf{F}, \boldsymbol{\kappa}_\alpha\}$ to generate a Lie subalgebra of $\operatorname{Aut}_e(\mathfrak{g}) = GL(3, R)$ for Bianchi type I, where $\{\mathbf{F}, \boldsymbol{\kappa}_\alpha\}$ is defined by any other nonsemisimple Bianchi type, leads to a similar statement for the type I canonical frame e'' relative to e''''; however, this is only relevant to sources other than perfect fluids which excite the off-diagonal degrees of freedom.) Therefore for uniformity of discussion definition (6.13) for e''' is used, but the possibility of eliminating β^- from the equations of motion for these special types should be kept in mind.

The spatial frame e'''' is not canonical but has time-dependent structure constant tensor components $C''''^a{}_{bc} = e^{-\zeta} C^a{}_{bc}$ according to (6.10) and (6.11). The velocities analogous to (6.14) and which are associated with the 5-dimensional group G_P generated by $\mathfrak{g}_P = \operatorname{span}\{\mathbf{F}, \boldsymbol{\kappa}_\alpha\}$ are still important though

$$\begin{aligned} (e^{\zeta\mathbf{F}+\lambda\mathbf{E}}\mathbf{S})^{\cdot}\, (e^{\zeta\mathbf{F}+\lambda\mathbf{E}}\mathbf{S})^{-1} &= \dot{\zeta}\mathbf{F} + \dot{\Gamma}^\alpha \boldsymbol{\kappa}_\alpha, \\ \dot{\Gamma}^0 = \dot{\Sigma}^0, \qquad \dot{\Gamma}^a &= (e^{-\zeta\langle \mathbf{1},\mathbf{E}\rangle\langle \mathbf{E},\mathbf{E}\rangle^{-1}\mathbf{I}^{(3)}})^a{}_b\, \dot{\Sigma}^b, \qquad \overset{\circ}{\Gamma}{}^\alpha = N\dot{\Gamma}^\alpha. \end{aligned} \tag{6.15}$$

Parametrization (6.12) with (5.1) represents the parametrization of $\mathscr{M}$ in terms of this 5-dimensional group acting freely (almost everywhere) on its orbits and the 1-dimensional submanifold $\{g'''' = e^{\beta^- e_-} | \beta^- \in R\}$ of $\mathscr{M}$ which parametrizes these orbits. For the Bianchi types other than I, II and V this submanifold also parametrizes the conformal 3-geometry [1] and the group G_P is just the identity component of $\text{PAut}_e(\mathfrak{g})$. It will be shown below that only the velocities $\{\overset{\circ}{\zeta}, \overset{\circ}{\Gamma}{}^a\}$ associated with this group together with the remaining gravitational degree of freedom associated with β^- need appear in the Einstein equations. These "projective automorphism velocities" appear in the component matrix of the extrinsic curvature with respect to the frame e''''

$$-N\mathbf{K}'''' = \overset{\circ}{\beta}{}^- \mathbf{e}_- + \overset{\circ}{\zeta}\mathbf{F} + \overset{\circ}{\lambda}\mathbf{E} + \text{SYM}_{\mathbf{g}''''}(\kappa_a)\,\overset{\circ}{\Gamma}{}^a \tag{6.16}$$

and in the kinetic energy function (5.15)

$$N\mathscr{T} = N^{-1} e^{3\beta^0}[\langle \overset{\circ}{\boldsymbol{\beta}}, \overset{\circ}{\boldsymbol{\beta}} \rangle_{DW} + \langle \text{SYM}_{\mathbf{g}''''}(\kappa_a), \text{SYM}_{\mathbf{g}''''}(\kappa_b) \rangle \overset{\circ}{\Gamma}{}^a \overset{\circ}{\Gamma}{}^b]. \tag{6.17}$$

The uniform scale transformations $(\beta^0, \beta^\pm) \to (\beta^0 + \alpha, \beta^\pm)$ have become $(\beta^-, \zeta, \lambda) \to (\beta^-, \zeta + \alpha, \lambda + \langle 1, \mathbf{E} \rangle \langle \mathbf{E}, \mathbf{E} \rangle^{-1} \alpha)$ in terms of the new parameters. In particular the combination $\lambda - \langle 1, \mathbf{E} \rangle \langle \mathbf{E}, \mathbf{E} \rangle^{-1} \zeta$ which appears in the rescaling (6.14)–(6.15) of the special automorphism velocities $\overset{\circ}{\sigma}{}^1$ and $\overset{\circ}{\sigma}{}^2$ is a scale invariant quantity. It is exactly this rescaling which shifts the λ and ζ dependence of $\mathscr{G}_{11}$ and $\mathscr{G}_{22}$ in the kinetic energy expression (5.15) to the velocities $\overset{\circ}{\Gamma}{}^1$ and $\overset{\circ}{\Gamma}{}^2$ in the kinetic energy expression (6.17).

Translation by α of ζ alone reflects the effect of the uniform scale transformation on objects referred to the class of canonical frames e''' in which the matric matrix $\mathbf{g}'''$ is of the form (6.13). Passing to the frame e'''' merely shifts the ζ dependence from the metric matrix to the structure constant tensor since $C''''^a{}_{bc} = e^{-\zeta} C^a{}_{bc}$, so the components of fields and equations expressed in this latter frame still contain ζ explicitly. However, under translations in ζ the components of geometric objects transform by weights which agree with the corresponding orthonormal component dimensions since the β^- scaling does not affect the dimension. Thus if Φ'''' are the components of a geometric object whose orthonormal components have dimension q'', one obtains a new set of scale invariant quantities by defining $\Phi'''' = e^{q''\zeta} \underset{\sim}{\Phi}''''$. As in (6.8) the new time derivatives are given by

$$\overset{\circ}{\underset{\sim}{\Phi}}{}'''' = e^{-q''\zeta} \overset{\circ}{\Phi}{}'''' - q'' \overset{\circ}{\zeta} \underset{\sim}{\Phi}''''. \tag{6.18}$$

However, one must make a distinction between the Lagrangian which is treated as a scalar and the component of the ADM Lagrangian density which transforms as a density

$$\begin{aligned} L_N &= 2Ng^{1/2}(G^\perp{}_\perp - kT^\perp{}_\perp) = e^{\lambda \text{Tr}\,\mathbf{E} + \zeta(\text{Tr}\,\mathbf{F} - 1)} \underset{\sim}{L}_{\underset{\sim}{N}}, \\ \underset{\sim}{L}_{\underset{\sim}{N}} &\doteq \underset{\sim}{N}(\underset{\sim}{G}^\perp{}_\perp - k\underset{\sim}{T}^\perp{}_\perp). \end{aligned} \tag{6.19}$$

Here $\underset{\sim}{N} \equiv e^{-\zeta}N$ is assumed to be scale invariant (and independent of λ and ζ); for example, one again has the two obvious choices $\underset{\sim}{N}_{(1)} = e^{-\zeta}(\dot{\zeta})^{-1}$ and $\underset{\sim}{N}_{(2)} = 1$. The scale invariant Lagrangian $\underset{\sim}{L}_{\underset{\sim}{N}}$ or its corresponding Hamiltonian may again be used to generate scale invariant equations of motion for the degrees of freedom other than λ and ζ provided that one adds an additional force proportional to the scale invariant momentum with coefficient $-(\lambda \operatorname{Tr} \mathbf{E} + \zeta(\operatorname{Tr} \mathbf{F} - 1))$ as in (6.9).

Note that since the velocities $(\boldsymbol{\beta}, \overset{\circ}{\tilde{\Gamma}}{}^a)$ are scale invariant, the conjugate momenta must scale in the same way as the Lagrangian. The Lagrangian equations of motion then require that the product of the lapse and the nonpotential force also scale in this irregular way. This Lagrangian scaling factor has the value $e^{2\lambda}$ for all the nonsemisimple Bianchi types except type II, where it is instead $e^{4\lambda-(2/3)\zeta}$. To obtain the Lagrangian or Hamiltonian equations of motion for the velocities $\overset{\circ}{\tilde{\Gamma}}{}^a$ or their conjugate momenta, one must first generalize the equations of the appendix from $\hat{G}$ to the larger group G_P. The scale invariant equations of motion for $\overset{\circ}{\lambda}$ and $\overset{\circ}{\zeta}$ can only be obtained by rescaling the equations of motion obtained from the unscaled Lagrangian. Only the scale invariant fluid variables and the gravitational variables $(\beta^+, \overset{\circ}{\beta}{}^-, \overset{\circ}{\zeta}, \overset{\circ}{\tilde{\Gamma}}{}^a, \underset{\sim}{N})$, or the corresponding set with velocities replaced by their conjugate scale invariant momenta, appear in the gravitational equations of motion and constraints.

The same statement holds for the scale invariant fluid equations of motion. To obtain these one first passes from the frame e' to e''' using Eqs. (2.7) and (6.14), obtaining the result

$$
\begin{aligned}
l''' &= e^{-\lambda \operatorname{Tr} \mathbf{E}} l, \qquad v_a{}''' = v_b{}'(e^{-\lambda \mathbf{E}})^b{}_a, \\
(\ln l''')^\circ &= 2N a_a v'''^a (v^\perp)^{-1} - \overset{\circ}{\lambda} \operatorname{Tr} \mathbf{E}, \\
(v_a{}''')^\circ &= v_f{}'''(N(v^\perp)^{-1} C^f{}_{ga} v'''^g - \kappa_\alpha{}^f{}_a \overset{\circ}{\tilde{\Sigma}}{}^\alpha).
\end{aligned}
\tag{6.20}
$$

Since E has nonzero trace, at this step one loses the individual constants of the motion l and V^2 in the class A case, but the combination $l'''^{-1}(n^{ab} v_a{}''' v_b{}''') = l^{-1}(n^{ab} v'_a v'_b)$ remains a constant for all Bianchi types. However, this too is lost upon passing to the scale invariant variables (except for the stiff perfect fluid case)

$$
\begin{aligned}
l'''' &= e^{-\zeta \operatorname{Tr} \mathbf{F}} l''' = n u^\perp, \qquad \underset{\sim}{l}'''' = e^{2\gamma^{-1}\zeta} l'''' = \underset{\sim}{n} u^\perp, \\
v_a{}'''' &= v_b{}''''(e^{-\zeta \mathbf{F}})^b{}_a, \qquad \underset{\sim}{v}_a{}'''' = e^{-(2\gamma^{-1}-1)\zeta} v_a{}'''', \\
(\ln \underset{\sim}{l}'''')^\circ &= 2\underset{\sim}{N} a \underset{\sim}{v}_3{}''''(\underset{\sim}{v}^\perp)^{-1} - \overset{\circ}{\lambda} \operatorname{Tr} \mathbf{E} - \overset{\circ}{\zeta} \operatorname{Tr} \mathbf{F}, \\
(\underset{\sim}{v}_a{}'''')^\circ &= \underset{\sim}{v}_f{}''''(\underset{\sim}{N}(\underset{\sim}{v}^\perp)^{-1} C^f{}_{ga} \underset{\sim}{v}''''^g - \kappa_\alpha{}^f{}_a \overset{\circ}{\tilde{\Gamma}}{}^\alpha - [F^f{}_a + \delta^f{}_a(2\gamma^{-1} 1)] \overset{\circ}{\zeta}).
\end{aligned}
\tag{6.21}
$$

Similarly, although V^2 vanishes for Bianchi types II and V, the combination $l'''^{-s}(v_1{}'''^2 + v_2{}'''^2) = l^{-s}(v_1{}'^2 + v_2{}'^2)$ with $s = \frac{1}{2}$ and $s = 1$, respectively, is a constant of the motion, but this is also lost upon passing to the scale invariant variables (except for the type V stiff perfect fluid case).

When the supermomentum constraints are not degenerate, one can use them to eliminate $\mathring{\tilde{\Gamma}}^a$ and then use the super-Hamiltonian constraint to solve for either $\mathring{\lambda}$ or $\mathring{\zeta}$, leaving the β^- degree of freedom and the additional variable $\mathring{\zeta}$ or $\mathring{\lambda}$, respectively (governed by a first order equation of motion), as the only remaining gravitational variables, together with the scale invariant fluid variables $\underset{\sim}{l}''''$, $\underset{\sim}{v}_a''''$, and $\underset{\sim}{p}$. The variables ζ must be found by quadrature to express the field components in the comoving ADM frame $\{e_\alpha'''\}$, unless one makes the lapse choice $N = N_{(1)}$ which makes ζ the time variable. This choice is better suited to the Hamiltonian form of the gravitational equations of motion. When the supermomentum constraints are degenerate, one has additional first order equations of motion for the undetermined automorphism velocities or momenta.

This approach represents the most economical one from the point of view of the explicit parameter dependence of the Einstein and fluid equations, but considerably complicates these equations by the addition of more velocity dependent terms. Furthermore, one can no longer take advantage of the constants of the motion, including those additional constants which occur in the symmetric or other special cases, while the variational approach cannot directly produce all of the gravitational equations of motion. It is thus somewhat a matter of taste which approach one decides is simpler. The diagonal parametrization (6.12) together with the lapse choice $N = N_{(2)} = e^{\zeta}$ was discussed briefly by Siklos for the class B Bianchi types and used to study singularities in vacuum models [49].

To make these remarks more concrete, consider the explicit form of the Lagrangian and nonpotential force for all the nonsemisimple Bianchi types except type II (set $a = 0$ in the type V case to obtain type I)

$$
\begin{aligned}
\mathscr{L} &= L_N = N(\mathscr{E} - U_G - \mathscr{H}_M),\\
N\mathscr{E} &= e^{2\lambda}\underset{\sim}{N}^{-1}[-2\mathring{\lambda}(\mathring{\lambda} + 2\mathring{\zeta}) + 6\mathring{\beta}^{-2}\\
&\qquad + \tfrac{1}{2}e^{2(\lambda-\zeta)}(e^{4\sqrt{3}\beta^-}(\mathring{\tilde{\sigma}}^1)^2 + e^{-4\sqrt{3}\beta^-}(\mathring{\tilde{\sigma}}^2)^2) + \mathscr{G}_{33}(\mathring{\tilde{\sigma}}^3)^2],\\
NU_G &= e^{2\lambda}\underset{\sim}{N}[6a^2 + \mathscr{G}_{33}(1 - \delta^V{}_Z)],\\
N\mathscr{H}_M &= 2k\underset{\sim}{N}[e^{\zeta} l v^{\perp} - e^{2\zeta+2\lambda}p] = 2ke^{2\lambda}\underset{\sim}{N}(\underset{\sim}{l}''''\underset{\sim}{v}^{\perp} - \underset{\sim}{p}),\\
NQ &= 4ae^{2\lambda}[2a(d\lambda - d\zeta) + \mathscr{G}_{33}(1 - \delta^V{}_Z)\,\tilde{\sigma}^3],\\
\mathscr{G}_{33} &= \tfrac{1}{2}(n^{(1)}e^{2\sqrt{3}\beta^-} - n^{(2)}e^{-2\sqrt{3}\beta^-})^2 + \tfrac{1}{2}(1 - \delta^V{}_Z)(e^{2\sqrt{3}\beta^-} - e^{-2\sqrt{3}\beta^-})^2.
\end{aligned}
\tag{6.22}
$$

The equations of motion for the scale variables for the case $\underset{\sim}{N} = 1$ are then

$$
\begin{aligned}
0 &= \mathring{\mathring{\zeta}} + 2\mathring{\lambda}\mathring{\zeta} + 4(2a^2 + \mathscr{G}_{33}) - \tfrac{1}{2}e^{4\sqrt{3}\beta^-}(\mathring{\tilde{\Gamma}}^1)^2 - \tfrac{1}{2}e^{-4\sqrt{3}\beta^-}(\mathring{\tilde{\Gamma}}^2)^2\\
&\qquad - \tfrac{1}{4}e^{-2\lambda}(\partial/\partial\lambda - \partial/\partial\zeta)(N\mathscr{H}_M) + \tfrac{1}{2}e^{\zeta-2\lambda}\mathscr{H},\\
0 &= \mathring{\mathring{\lambda}} + 2\mathring{\lambda}^2 - 2a^2 - \tfrac{1}{4}e^{4\sqrt{3}\beta^-}(\mathring{\tilde{\Gamma}}^1)^2\\
&\qquad - \tfrac{1}{4}e^{-4\sqrt{3}\beta^-}(\mathring{\tilde{\Gamma}}^2)^2 - \tfrac{1}{4}e^{-2\lambda}\partial/\partial\zeta(N\mathscr{H}_M),\\
0 &= -\bar{\delta}\mathscr{L}/\bar{\delta}\beta^- = e^{2\lambda}(-\bar{\delta}\underset{\sim}{\mathscr{L}}/\bar{\delta}\beta^- + 2\mathring{\lambda}\underset{\sim}{p}_-).
\end{aligned}
\tag{6.23}
$$

On the other hand, when written in terms of the scale invariant momenta, these equations easily accommodate the alternative lapse choice $N = N_{(2)}$ or equivalently $N = 4(p_\zeta - p_\lambda)^{-1}$ which makes ζ the time variable for these Bianchi types. No further integration is then required to express the fields in a comoving frame.

The approaches so far described choose to solve the super-Hamiltonian and supermomentum constraints by eliminating the necessary number of gravitational degrees of freedom. However, one may take the opposite point of view by instead eliminating all of the fluid degrees of freedom. This is possible since the perfect fluid super-Hamiltonian and supermomentum completely determine the individual fluid variables. Two options reflecting this latter choice are available.

First one may proceed as above in each approach until the point where one solves the constraints for certain gravitational variables. Then one instead solves these constraints for the fluid variables, using the results to express the spatial energy-momentum tensor of the fluid entirely in terms of the gravitational variables. This spatial tensor then acts as a velocity-dependent force f which drives the gravitational Lagrangian/Hamiltonian system. The equations of motion of this system, representing the evolution equations expressed in terms of the gravitational variables alone, are then the only equations left to be solved [52].

The solution of the constraints for the fluid variables requires an explicit equation of state. For the usual one (1.3), this was done by Bogoyavlensky [52] and later independently by Moncrief [53]. Introduce the simplifying notation

$$h = -(2kg^{1/2})^{-1}\mathscr{H}_G, \qquad h_a = (2kg^{1/2})^{-1}\mathscr{H}^G_a, \qquad \Delta = \rho(u^\perp)^2. \tag{6.24}$$

Using the supermomentum constraint $h_a = \rho u^\perp u_a$, one may turn the super-Hamiltonian constraint into a quadratic equation for Δ which is easily solved for the appropriate root:

$$\begin{aligned} h &= \Delta + (\gamma - 1)\rho u^a u_a = \Delta + (\gamma^2\Delta)^{-1}(\gamma - 1)\, h^a h_a, \\ \Delta &= \tfrac{1}{2}h + \tfrac{1}{2}[h^2 - \gamma^{-2}(\gamma - 1)\, h^a h_a]^{1/2}. \end{aligned} \tag{6.25}$$

The alternative expression $h = \gamma\Delta - (\gamma - 1)\rho$ then yields ρ as a function of the gravitational variables:

$$\rho = (\gamma - 1)^{-1}(\gamma\Delta - h) = \tfrac{1}{2}(\gamma - 1)^{-1}\,[(\gamma - 2)h + [\gamma^2 h^2 - (\gamma - 1)\, h^a h_a]^{1/2}]. \tag{6.26}$$

This has a well defined limit as $\gamma \to 1$. The variable μ is then defined by the formula

$$\mu = \gamma\rho^{1-\gamma^{-1}}, \tag{6.27}$$

so that finally one obtains

$$\begin{aligned} l &= g^{1/2}\Delta^{1/2}\rho^{\gamma^{-1} - 1/2}, \qquad v_a = g^{1/2} l^{-1} h_a, \\ T^a{}_b &= (\gamma\Delta)^{-1}\,(h^a h_b - (\gamma - 1)\, h^c h_c \delta^a{}_b) + (\gamma - 1)\, h\delta^a{}_b. \end{aligned} \tag{6.28}$$

Similar formulas hold with respect to any spatial frame, as well as for scale invariant components. The components of the fluid driving force necessary for the Lagrangian/Hamiltonian equations of motion written in terms of parametrization (5.1), for example, are then computed from the formula (see the appendix)

$$\begin{aligned} f &\equiv kg^{1/2}T^{ab}dg_{ab} = kg^{1/2}(\langle \mathbf{T}', \mathbf{e}_A\rangle \, d\beta^A + \langle \mathbf{T}', \boldsymbol{\kappa}_a\rangle \, \tilde{\sigma}^a) \\ &= f_A \, d\beta^A + f_a \tilde{\sigma}^a. \end{aligned} \tag{6.29}$$

One need only choose initial data satisfying the inequality $\hbar > (\hbar^a \hbar_a)^{1/2}$ required for the positivity of the fluid energy density and the timelike character of the fluid flow vector field [53]. This can always be done, for example, by fixing all the gravitational variables except $p_0 = 2\,\mathrm{Tr}\,\boldsymbol{\pi}$ and choosing $|p_0|$ larger than a certain critical value. This is possible since $\hbar_a$ is independent of p_0 while $\hbar$ depends on p_0 only through the term $(12k)^{-1}\, e^{-3\beta^0} p_0{}^2$.

A second possibility is to use Taub's comoving frame $\{\bar{e}_\alpha\}$ whose associated lapse and shift are fixed by the fluid variables

$$N = \mu^{-2} v^{\perp}, \qquad \bar{N}_a = \mu^{-2} \bar{v}_a, \tag{6.30}$$

leaving only the freedom to change the spatial frame by time-independent automorphisms, once a given choice of structure constant tensor components is made. Conversely when the pressure is nonzero, the fluid variables are determined by the lapse and shift:

$$\mu = e^{-\phi}, \qquad \bar{v}_a = e^{-2\phi} \bar{N}_a, \qquad \phi \equiv \tfrac{1}{2}\ln(N^2 - \bar{N}^a \bar{N}_a). \tag{6.31}$$

Since $\bar{l}$ and $\bar{v}_a$ are time independent (implying the same for $\bar{\mathscr{H}}^G{}_a$), while the single independent thermodynamic variable is determined implicitly by the defining relation for $\bar{l}$, only the canonical gravitational variables (i.e., those other than the lapse and shift) have nontrivial equations of motion, subject of course to the gravitational constraints.

By using the barred versions of (6.24)–(6.28) one may express the lapse and shift entirely in terms of the canonical gravitational variables and the gravitational constraint functions, leading to Moncrief's results [53]

$$N = \mu^{-1} \rho^{-1/2} \Delta^{1/2}, \qquad \bar{N}_a = \mu^{-2} \bar{v}_a. \tag{6.32}$$

Here μ, ρ, Δ and $\bar{v}_a$ are now understood to be given by the barred versions of the above formulas. Given arbitrary $(\bar{g}_{ab}, \bar{K}_{ab})$ or $(\bar{g}_{ab}, \bar{\pi}^{ab})$ subject only to the constraint inequality $\hbar^2 > \hbar^a \hbar_a$, Eq. (6.32) defines the values of the lapse and shift or equivalently, the values of the fluid variables which correspond to the values of the gravitational constraint functions, thus solving the gravitational constraints.

The difficulties with the variational approach for the class B Bianchi types are so extensive here that it must be abandoned [30]. In the class A case, however, no problems arise. From the Hamiltonian point of view described by Moncrief [54], the value of the effective Hamiltonian density obtained by replacing the lapse and shift by expressions (6.32) is just the constant of the motion $\bar{l}$. Furthermore, the special automorphism group $SAut_{\bar{e}}(\mathfrak{g})$ acts canonically on the gravitational phase space as a symmetry group of the Hamiltonian so its canonical generators are constants of the motion. Among these are the gravitational supermomenta which generate the canonical action of the inner automorphism subgroup in the class A case. All of these constants of the motion may in principle be used to reduce the number of degrees of freedom but this problem has never been examined.

7. Conclusion

By recognizing and exploiting the kinematical role played by the spatial gauge group and the choice of time in the dynamics of a spatially homogeneous spacetime, the structure of the spatially homogeneous perfect fluid Einstein equations has been considerably clarified, leading to the most illuminating forms of these equations. Since spatial homogeneity provides a natural spacetime slicing, the only remaining freedom compatible with the symmetry consists of reparametrization of the slicing time function and rethreading of the slicing in a way which preserves the spatial homogeneity of the spatial frames used to reduce the field equations to ordinary differential equations.

Beginning with the usual "synchronous reference frame" [19] gauge choice of normal threading and proper time parametrization of the slicing, the symmetry of the resulting ordinary differential equations leads naturally to the emergence of two classes of nonsynchronous gauges which considerably simplify the field equations. These are the "diagonal gauge" described in Section 5 in which the spatial metric is diagonalized and the "minimal strain gauge" of Section 6 in which the spatial metric is not only diagonalized but put in its simplest diagonal form, each of these gauges having advantages and disadvantages in comparison with the other. For the semisimple Bianchi types these two gauges coincide with "minimal distortion gauge" which minimizes the time dependence of the spatial conformal metric; these latter types provide a faithful finite-dimensional representation of many aspects of the general theory developed for spatially compact spacetimes.

In all cases the gravitational constraints take a very simple form, allowing certain degrees of freedom to be entirely eliminated, and leading to reduced equations of motion for the unconstrained degrees of freedom. By exploiting the 1-parameter group of scale transformations of the unit of length, an even smaller set of coupled variables may be considered, at the expense of losing the utility of the constants of the motion and somewhat complexifying the dependence of the equations of motion on the gravitational momenta. This scale group as well as the elimination of the super-Hamiltonian constraint suggest certain preferred reparametrizations of the slicing time function.

The fluid equations of motion may also be understood in terms of the spatial gauge freedom, due to the existence of a preferred reference frame introduced by Taub in which these equations are trivial and the natural fluid variables are constants. However, this comoving fluid frame leaves the graviational equations of motion in a rather complex form. On the other hand the fluid equations of motion become nontrivial and increasingly more complicated in the sequence of frames which increasingly simplify the gravitational equations of motion. By considering the successive transformations from the comoving fluid frame to each of the latter frames (synchronous gauge, diagonal gauge and minimal strain gauge), the combinations of the natural fluid variables which remain constraints of the motion in each frame are easily obtained. Apart from these latter details the techniques described here are not limited to perfect fluid sources nor even Einsein's theory of general relativity but can be applied to any theory involving spatially homogeneous or even spatially self-similar spacetimes [4, 55, 56].

Appendix

Although an explicit parametrization of the matrix $\mathbf{S} \in \hat{G} \subset \mathrm{SAut}_e(\mathfrak{g})$ appearing in parametrization (5.1) is not required, it is necessary to introduce the right invariant frame $\{\tilde{E}_a\}$ on $\hat{G}$ with dual frame $\{\tilde{\sigma}^a\}$ associated with the basis $\{\boldsymbol{\kappa}_a\}$ of the matrix Lie algebra $\hat{\mathfrak{g}}$ of $\hat{G}$. The dual 1-forms $\tilde{\sigma}^a$ are defined by $d\mathbf{S}\,\mathbf{S}^{-1} = \boldsymbol{\kappa}_a \tilde{\sigma}^a$; if $[\boldsymbol{\kappa}_a, \boldsymbol{\kappa}_b] = \hat{C}^c{}_{ab}\boldsymbol{\kappa}_c$ defines $\hat{C}^c{}_{ab}$, then one has

$$[\tilde{E}_a, \tilde{E}_b] = -\hat{C}^c{}_{ab}\tilde{E}_c, \qquad d\tilde{\sigma}^a = \tfrac{1}{2}\hat{C}^a{}_{bc}\tilde{\sigma}^b \wedge \tilde{\sigma}^c. \tag{A.1}$$

The right invariant frame induces functions $\dot{\tilde{\sigma}}^a$ on $T\hat{G}$ and P_a on $T^*\hat{G}$ whose values at a given tangent vector or tangent covector, respectively, are just its components with respect to this frame. In local coordinates $\{\theta^i, \dot{\theta}^i\}$ on $T\hat{G}$ and $\{\theta^i, p_i\}$ on $T^*\hat{G}$ lifted from local coordinates $\{\theta^i\}$ on $\hat{G}$ (and corresponding to taking coordinate components of tangent vectors and covectors), one has

$$\begin{aligned} &\tilde{E}_a = \tilde{E}^i{}_a \partial/\partial\theta^i, \qquad \tilde{\sigma}^a = \tilde{\sigma}^a{}_i d\theta^i, \qquad \tilde{\sigma}^a{}_i \tilde{E}^i{}_b = \delta^a{}_b, \\ &P_a = \tilde{E}^i{}_a p_i, \qquad \dot{\tilde{\sigma}}^a = \tilde{\sigma}^a{}_i \dot{\theta}^i. \end{aligned} \tag{A.2}$$

In this notation one has the important formula $\dot{\mathbf{S}}\mathbf{S}^{-1} = \boldsymbol{\kappa}_a \dot{\tilde{\sigma}}^a$ used in the text to define the "automorphism velocities."

As long as functions on $T\hat{G}$ and $T * \hat{G}$ are expressed in terms of the variables $\{\mathbf{S}, \dot{\tilde{\sigma}}^a\}$ and $\{\mathbf{S}, P_a\}$, respectively, a generalized Lagrange derivative may be introduced for functions on $T\hat{G}$ by

$$\delta f/\delta\dot{\tilde{\sigma}}^a = -(\partial f/\partial\dot{\tilde{\sigma}}^a)^{\cdot} + \hat{C}^c{}_{ab}\dot{\tilde{\sigma}}^b \partial f/\partial\dot{\tilde{\sigma}}^c + \tilde{E}_a f \tag{A.3}$$

which is defined by the local coordinate computation

$$\delta f/\delta\dot{\tilde{\sigma}}^a \equiv \tilde{E}^i{}_a \delta f/\delta\theta^i = \tilde{E}^i{}_a[-(\partial f/\partial\dot{\theta}^i)^{\cdot} + \partial f/\partial\theta^i], \tag{A.4}$$

while Poisson brackets of functions on $T^*\hat{G}$ follow from the basic brackets

$$\{P_a, P_b\} = \hat{C}^c{}_{ab} P_c, \qquad \{\mathbf{S}, P_a\} = \boldsymbol{\kappa}_a \mathbf{S}, \tag{A.5}$$

results which are also easily obtained by local coordinate computations. Note the useful relation

$$\tilde{E}_a \mathbf{S} = d\mathbf{S}(\tilde{E}_a) = \boldsymbol{\kappa}_a \mathbf{S}. \tag{A.6}$$

As is often done in physics, the same symbols have been used for distinct but obviously related functions and vector fields on G, TG and T^*G to avoid complicated but more mathematically precise notation. In this spirit all of the above formulas may be carried over to $\mathscr{M}_D \times G$ and its tangent and cotangent bundles, which here are not distinguished notationally from $\mathscr{M}$ and its tangent and cotangent bundles. In particular the components of the nonpotential force Q (actually its pullback to $\mathscr{M}_D \times G$ via the parametrization map (5.1)) relevant to the equations of motion for $\dot{\tilde{\sigma}}^a$ or P_a are

$$-\delta L/\delta \dot{\tilde{\sigma}}^a = Q_a = Q(\tilde{E}_a) = \dot{P}_a - \{P_a, H\}, \tag{A.7}$$

where L and H represent the total Lagrangian and Hamiltonian, respectively, and a unit lapse function is assumed. Similarly in the context of the larger spaces which include the fluid variables, one needs the following formulas to obtain the fluid terms in the gravitational equations of motion:

$$\tilde{E}_a v'_b = -v'_c \kappa_a{}^c{}_b = -\{P_a, v'_b\}. \tag{A.8}$$

Comparison of (2.19) and (2.20) with (5.17) shows that the DeWitt metric on $\mathscr{M}' \sim \mathscr{M}_D \times \hat{G}$ may be written

$$\mathscr{G} = 4e^{3\beta^0}(6\eta_{AB}\, d\beta^A \otimes d\beta^B + \mathscr{G}_{ab} \tilde{\sigma}^a \otimes \tilde{\sigma}^b). \tag{A.9}$$

Since η_{AB} and $\mathscr{G}_{ab}$ are diagonal, the frame $\{\partial/\partial\beta^A, \tilde{E}_a\}$ is an orthogonal frame on $\mathscr{M}$ which is invariant under the action of the group $\hat{G}$. This explains the simplifications which occur in the Lagrangian/Hamiltonian equations of motion.

To make parametrization (5.1) or (5.1) with (6.12) less abstract, it is helpful to explicitly parametrize $\mathbf{S}$ and evaluate the matrix multiplication. The following parametrization of $\mathbf{S}$ is valid for the Bianchi types of the third category of Table I and is obtained from (3.11) by setting the structure constant component a to zero:

$$\mathbf{S} = e^{\theta^1 \mathbf{e}^{31} + \theta^2 \mathbf{e}^{32}} e^{\theta^3 k 0_3} = \begin{bmatrix} c_3 & -n^{(1)} s_3 & \theta^1 \\ n^{(2)} s_3 & c_3 & \theta^2 \\ 0 & 0 & 1 \end{bmatrix}. \tag{A.10}$$

This differs from (5.2) and (5.12) only in that θ^1 and θ^2 are interchanged and altered in sign. Using the diagonal parametrization (6.12) for this category, namely,

$$e^{2\boldsymbol{\beta}} = \mathrm{diag}(e^{2(\lambda+\hat{\beta}^-)}, e^{2(\lambda-\hat{\beta}^-)}, e^{2\zeta}), \tag{A.11}$$

where $\hat{\beta}^- \equiv \sqrt{3}\,\beta^-$, one then finds

$$\begin{aligned}
g_{11} &= e^{2\lambda}[e^{2\hat{\beta}^-}c_3{}^2 + e^{-2\hat{\beta}^-}(n^{(2)}s_3)^2],\\
g_{22} &= e^{2\lambda}[e^{-2\hat{\beta}^-}c_3{}^2 + e^{2\hat{\beta}^-}(n^{(1)}s_3)^2],\\
g_{12} &= -e^{2\lambda}[n^{(1)}e^{2\hat{\beta}^-} - n^{(2)}e^{-2\hat{\beta}^-}]\,c_3 s_3,\\
g_{13} &= e^{2\lambda}[e^{2\hat{\beta}^-}\theta^1 c_3 + e^{-2\hat{\beta}^-}\theta^2 n^{(2)} s_3],\\
g_{23} &= e^{2\lambda}[e^{-2\hat{\beta}^-}\theta^2 c_3 - e^{2\hat{\beta}^-}\theta^1 n^{(1)} s_3],\\
g_{33} &= e^{2\zeta} + e^{2\lambda}[e^{2\hat{\beta}^-}(\theta^1)^2 + e^{-2\hat{\beta}^-}(\theta^2)^2].
\end{aligned} \tag{A.12}$$

These can be simplified for Bianchi types VI_0, VII_0, VI_h and VII_h by using double angle formulas and introducing hyperbolic functions of $2\hat{\beta}^-$; for example, one obtains the following expression for g_{11} for Bianchi types VI_0 and VI_h:

$$g_{11} = e^{2\lambda}[\sinh 2\hat{\beta}^- + \cosh 2\hat{\beta}^- \cosh 2\theta^3]. \tag{A.13}$$

Note added in proof. We may easily circumvent the failure of either choice (5.12) or (5.13) to specify a basis $\{\boldsymbol{\kappa}_a\}$ of the offdiagonal Lie algebra $\hat{\mathfrak{g}}$ (of the 3-dimensional matrix group $\hat{G} \subset S\ \mathrm{Aut}_e(\mathfrak{g})$ for a canonical basis e of $\mathfrak{g}$) which has the same character for all class A and class B types. Consider the space $\mathscr{C}_D$ of all standard diagonal form structure constant tensor components, a space parametrized by the 4-tuple $(n^{(1)}, n^{(2)}, n^{(3)}, a)$ subject to the constraint $an^{(3)} = 0$. Introduce the Lie algebra basis valued function $\{\boldsymbol{\kappa}_a\}$ on $\mathscr{C}_D$ by

$$\boldsymbol{\kappa}_a = e^{-\alpha^a}\mathbf{k}^0_a, \qquad \mathbf{k}^0_a = -n^{(b)}\mathbf{e}^c{}_b + n^{(c)}\mathbf{e}^b{}_c,$$
$$e^{\alpha^a} = 2^{-1/2}\langle \mathbf{k}^0_a, \mathbf{k}^{0T}_a\rangle^{1/2} = 2^{-1/2}(n^{(b)2} + n^{(c)2})^{1/2}$$

and satisfying

$$[\boldsymbol{\kappa}_a, \boldsymbol{\kappa}_b] = \hat{C}^c{}_{ab}\boldsymbol{\kappa}_c, \qquad \hat{C}^c{}_{ab} = \varepsilon_{abd}\hat{n}^{cd}$$
$$\hat{\mathbf{n}} = \mathrm{diag}(\hat{n}^{(1)}, \hat{n}^{(2)}, \hat{n}^{(3)}), \qquad \hat{n}^{(a)} = n^{(a)}e^{\alpha^a - \alpha^b - \alpha^c},$$

where it is clear from the context when (a, b, c) is to be interpreted as a cyclic permutation of (1, 2, 3). This is well defined everywhere on $\mathscr{C}_D$ except for those points corresponding to Bianchi types I, II, and V (precisely those points where rank $\mathbf{n} < 2$ and the scale matrix $e^{\boldsymbol{\alpha}} = \mathrm{diag}(e^{\alpha^1}, e^{\alpha^2}, e^{\alpha^3})$ is singular) where it has finite direction dependent limits. We may therefore interpret $\{\boldsymbol{\kappa}_a\}$ as a multivalued function at these points, whose values are given by all possible limits. For the Bianchi types I, II, and V, any value may be picked to describe the dynamics in terms of a diagonal gauge decomposition of the gravitational and source variables. For the semisimple Bianchi types at the canonical points of $\mathscr{C}_D$, the new basis agrees with (5.12), but there are slight differences for the remaining canonical points. Such a continuous choice of basis for all points of $\mathscr{C}_D$ enables us to develop a more unified picture of the dynamics, as more fully described in [55].

References

1. R. T. Jantzen, *Commun. Math. Phys.* **64** (1979), 211–232.
2. R. T. Jantzen, *Ann. Inst. Henri Poincaré* **33** (1980), 121–146.
3. R. T. Jantzen, *Nuovo Cimento B* **55** (1980), 161–172.
4. R. T. Jantzen, *Ann. Phys. (N. Y.)* **127** (1980), 302–329.
5. R. T. Jantzen, *in* "Essays in Relativity" (F. J. Tipler, Ed.), Chap. 8, Academic Press, New York, 1980.
6. R. T. Jantzen, *J. Math. Phys.* **23** (1982), 1137–1146.
7. C. W. Misner, J. A. Wheeler, and K. S. Thorne, "Gravitation," Freeman, San Francisco, 1973.
8. G. F. R. Ellis, *J. Math. Phys.* **8** (1967), 1171–1194.
9. G. F. R. Ellis and J. M. Stewart, *J. Math. Phys.* **9** (1968), 1072–1082.
10. G. F. R. Ellis and M. A. H. MacCallum, *Commun. Math. Phys.* **12** (1969), 108–141; **19** (1970), 31–64; M. A. H. MacCallum, *Commun. Math. Phys.* **20** (1972), 57–84.
11. A. R. King and G. F. R. Ellis, *Commun. Math. Phys.* **31** (1972), 209–242.
12. A. R. King and G. F. R. Ellis, *Commun. Math. Phys.* **38** (1974), 119–156.
13. S. T. C. Siklos, *Commun. Math. Phys.* **58** (1978), 255–272.
14. C. W. Misner, *Phys. Rev.* **186** (1969), 1319–1327.
15. M. P. Ryan, Jr., *J. Math. Phys.* **19** (1969), 1724–1728.
16. M. P. Ryan, Jr., *Ann. Phys.* (N. Y.) **65** (1971), 506–537; **68** (1971), 541–555; **70** (1972), 301–322.
17. M. P. Ryan, Jr., "Hamiltonian Cosmology," Springer–Verlag, Berlin, 1972.
18. M. P. Ryan, Jr., *J. Math. Phys.* **15** (1974), 812–815.
19. M. P. Ryan, Jr. and L. C. Shepley, "Homogeneous Relativistic Cosmologies," Princeton Univ. Press, Princeton, 1975.
20. C. W. Misner, *in* "Magic without Magic" (J. R. Klauder, Ed.), Freeman, San Francisco, 1972.
21. J. W. York, Jr., *in* "Sources of Gravitational Radiation" (L. Smarr, Ed.), Cambridge Univ. Press, Cambridge, 1979.
22. A. E. Fischer and J. E. Marsden, *J. Math. Phys.* **13** (1972), 546–568.
23. A. E. Fischer and J. E. Marsden, *in* "Isolated Gravitating Systems in General Relativity" (J. Ehlers, Ed.) North Holland, New York, 1979; *in* "General Relativity: An Einstein Centenary Survey" (S. Hawking and W. Israel, Eds.), Cambridge Univ. Press, Cambridge, 1979.
24. Y. Choquet-Bruhat and J. W. York, Jr., *in* "General Relativity and Gravitation" (A. Held, Ed.), Vol. 1, Plenum Press, New York, 1980.
25. J. Isenberg and J. Nester, *in* "General Relativity and Gravitation" (A. Held, Ed.), Vol. 1, Plenum Press, New York, 1980.
26. L. Smarr and J. W. York, Jr., *Phys. Rev. D* **17** (1978), 2529–2551; *D* **17** (1978), 1945–1956.
27. V. Moncrief, *J. Math. Phys.* **16** (1975), 493–499, 1556–1560; **17** (1976), 1893–1902.
28. A. E. Fischer, *in* "Relativity" (M. Carmeli, S. I. Fickler, and L. Witten, Eds.), Plenum Press, New York, 1970.
29. C. B. Collins and S. W. Hawking, *Ap. J.* **180** (1973), 317–334.
30. A. H. Taub and M. A. H. MacCallum, *Commun. Math. Phys.* **25** (1972), 173–189.
31. V. N. Lukash, *Nuov. Cimento B* **35** (1976), 268–292.
32. L. P. Grischuk, A. G. Doroshkevich, and V. N. Lukash, *Sov. Phys. JETP (Eng. Transl.)* **34** (1972), 1–7.
33. K. Gödel, *Proc. Int. Congr. Math.* **I** (1950), 175–181.
34. I. Ozsváth, *J. Math. Phys.* **11** (1970), 2860–2870; **12** (1971), 1078–1082.

35. R. T. JANTZEN, Spatially homogeneous cosmology: Background and dynamics, unpublished report, University of California, Berkeley, 1977.
36. A. H. TAUB, *Ann. Math.* **53** (1951), 472–490.
37. J. A. ISENBERG, *Ann. Phys.* (*N. Y.*) **129** (1980), 223–248.
38. F. W. WARNER, "Foundations of Differentiable Manifolds and Lie Groups," p. 119, Scott, Foresman, Glenview, Ill., 1971.
39. F. B. ESTABROOK, H. D. WAHLQUIST, AND C. G. BEHR, *J. Math. Phys.* **9** (1968), 497–504.
40. B. S. DEWITT, *Phys. Rev.* **160** (1967), 1113–1148.
41. A. H. TAUB, Proceedings of the 1967 Colloque on "Fluids et champ gravitationel en relativité générale," No. 170, pp. 57–72, Centre National de la Recherche Scientifique, Paris, 1969.
42. D. KRAMER, H. STEPHANI, E. HERLT, AND M. MACCALLUM, "Exact Solutions of Einstein's Field Equations" (E. Schmutzer, Ed.), Cambridge Univ. Press, Cambridge, 1980.
43. S. T. C. SIKLOS, *J. Phys. A* **14** (1981), 395–409.
44. B. G. SCHMIDT, *Commun. Math. Phys.* **15** (1969), 329–336.
45. L. BIANCHI, *Mem. Soc. It. della Sc.* (*Dei XL*) **11** (1897), 267–352; *in* "Opere", Vol. 9, Edizione Cremonese, Rome, 1958.
46. R. ABRAHAM AND J. E. MARSDEN, "Foundations of Mechanics" (Revised 2nd ed.), Benjamin/Cummings, Reading, Mass., 1980.
47. J. W. YORK, JR., *Phys. Rev. Lett.* **28** (1972), 1082–1085.
48. This degree of freedom represents the single timelike direction of the Lorentz signature DeWitt metric which determines all of the "kinetic" (extrinsic curvature) terms in the Einstein equations as well as the cosmological volume scale function, while Tr **K** does not appear in the supermomentum constraint since the natural slicing by homogeneous hypersurfaces is a "constant Tr **K**" slicing.
49. S. T. C. SIKLOS, *Phys. Lett. A* **76** (1980), 19–21; *Gen. Rel. Grav.* **13** (1981), 433–441.
50. S. P. NOVIKOV, *Sov. Phys. JETP* (*Eng. Transl.*) **35** (1972), 1031–1042.
51. O. I. BOGOYAVLENSKY AND S. P. NOVIKOV, *Sov. Phys. JETP* (*Eng. Transl.*) **37** (1974), 747–755; *in* "Proceedings, I. G. Petrovsky Seminar," pp. 7–44, 1973; *Sel. Math. Sov.* **2** (1982) [English transl.]
52. SEE REF. [2]. However, for the class B types, the second line of Eq. (18) of Ref. [2] (Ran Kill)$^{\perp} \subseteq$ ker δ fails to hold since the symmetric tensor field W whose mixed component matrix with respect to the frame e' is $\mathbf{e}^3{}_3$ is orthogonal to the gauge directions Ran Kill but its divergence $(\delta W)^{\prime}_b = -\mathrm{Tr}\, \delta_b \mathbf{e}^3{}_3 = -2a_b$ does not vanish. This means that the true minimal strain shifts do not even satisfy the minimal strain equation here.
53. O. I. BOGOYAVLENSKY, *Sov. Phys. JETP* (*Eng. Transl.*) **43** (1976), 187–193.
54. V. MONCRIEF, *Phys. Rev. D* **16** (1977), 1702–1705.
55. R. T. JANTZEN, *in* "Proceedings of the 1982 Enrico Fermi Summer School on Gamov Cosmology" (R. Ruffini and F. Melchiorri, Eds.), North Holland, New York, in press.

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