ANNALS OF PHYSICS 127, 302–329 (1980)

Variation of Parameters in Cosmology

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Received August 23, 1979; revised December 1979

Parameters which appear in the solutions of the dynamical equations of spatially homogeneous cosmology or in the dynamical equations themselves are subject to algebraic relations imposed by the constraint equations, i.e., are confined to a constraint hypersurface in parameter space. Values of these parameters off the constraint hypersurface often correspond to solutions which have an additional stiff perfect fluid source that may or may not be flowing orthogonally to the spatially homogeneous foliation or to a related inhomogeneous but spatially self-similar solution or to a combination of the two. These possibilities are studied and explicitly illustrated, leading to a uniform derivation of most of the known exact anisotropic spatially homogeneous or spatially self-similar solutions as well as some new ones.

1. INTRODUCTION

Since their introduction by Taub nearly three decades ago [1], spatially homogeneous (SH) space–times have been studied for a variety of reasons. Because the natural spatial slices of these space–times are the orbits of a three-dimensional isometry group or "group of motions" which acts simply transitively on each slice, the Einstein equations may be reduced from partial differential equations to ordinary differential equations. Although these equations are still rather complicated, they are much more easily handled and provide an elegant finite-dimensional setting for the investigation of many of the theoretical ideas that have been developed by relativists in an attempt to better understand the structure of the Einstein equations. However, aside from these mathematical considerations, there has been much physical interest in this special class of space–times since they are the simplest generalizations of the spatially homogeneous and isotropic cosmological models. For example, they permit the study of various mechanisms which might explain the high degree of large-scale isotropy actually observed in our universe, among these mechanisms being the semiclassical quantum-mechanical effect of particle creation by the gravitational field.

Spatially self-similar (SSS) space–times, first studied by Eardley [2], are conformally related to spatially homogeneous space–times by a very special class of conformal factors. Although these conformal factors make the spatially homogeneous space-times spatially inhomogeneous, the original homogeneity group of motions remains

0003-4916/80/080302–28$05.00/0

as a group of homothetic motions of the new space–times and again the Einstein equations may be reduced to ordinary differential equations. These equations for a given spatially self-similar space–time are very closely related to the Einstein equations for a certain spatially homogeneous space–time and in fact are a continuous deformation of those equations involving essentially a single deformation parameter which determines the conformal factor. The zero value of this parameter corresponds to the spatially homogeneous case.

The key to understanding the relationship between the solutions of the equations for zero and nonzero values of this parameter involves the decomposition of the Einstein equations into a set of evolution equations for the spatial metric or "dynamical equations" subject to four constraints [3]. [The choice of the lapse function and shift vector field here is essentially arbitrary subject to the restrictions necessary for compatibility of the global reference frame with the symmetry of the space–time [4]. Without such restrictions one would not obtain ordinary differential equations.] Although the constraint equations are well defined, the set of evolution equations is only defined modulo linear combinations of the constraint functions. This is important if one first solves the evolution equations and then imposes the constraints upon the solutions, a procedure which is possible because of the well-known compatibility of the constraint equations with the evolution equations. When these are all ordinary differential equations, the constraint functions are in most cases first integrals of the evolution equations and can therefore depend only on the constants which appear in the evolution equations and the constants of integration which arise in the integration of the evolution equations, as well as the deformation parameter if it does not appear explicitly there. One may therefore think of the constraints as functions on a "parameter space." This parameter space, it turns out, may be enlarged by the addition of one or several parameters which characterize a special class of stiff perfect fluids.

If one begins at a point of the constraint solution subspace in the parameter space associated with a given class of spatially homogeneous vacuum or electromagnetic space–times, and one moves off the constraint subspace in certain directions, the new points will correspond either to the addition to the original source of these space–times of a stiff perfect fluid flowing orthogonally to the natural slicing ["stuffing"] and which in some cases can be made to flow with a component of the 4-velocity parallel to that slicing ["tilting"] or they will correspond to conformally scaling the space–time, resulting in a spatially self-similar spacetime ["stretching"], or to some combination of these operations. Of course from the point of view of the enlarged parameter space for spatially self-similar space–times with an electromagnetic field and stiff perfect fluid as sources, one remains in the constraint subspace. However, occasionally the stretching operation can be applied when the original constraints or their stuffed and tilted generalizations have no solutions, leading to spatially self-similar solutions of the full Einstein equations with no spatially homogeneous limit.

The close relationship between stiff perfect fluid solutions and vacuum solutions for SH space–times was noticed by Ellis and McCallum [5] and Barrow [6], while Ftaclas and Cohen made a similar observation for electromagnetic rather than vacuum space–times during the process of stuffing a type V SH electromagnetic solution [7]. Wainwright *et al.* [8] have developed a systematic algorithm for producing stiff perfect fluid solutions from vacuum solutions for space–times with a two-parameter Abelian group of isometries acting orthogonally transitively on two-dimensional space-like orbits [9]. This includes the "symmetric case" SH space–times [4] with a nonsemisimple homogeneity isometry group and their stretched SSS analogs. A number of examples of this technique were presented in [8]. In each case the algorithm was applied to a SSS solution which with one exception was a stretched version of a SH vacuum solution, although the stretching algorithm was not discussed, and a stuffed and tilted version resulted. The aim of the present paper is to clarify the relationship between SH and SSS vacuum and electromagnetic solutions and those with an additional uncharged stiff perfect fluid.

In Section 2, SH and SSS space–times are reviewed and the stuffing operation is described. In the following section, the stretching operation is discussed, as well as the Hamiltonian formulation for SSS dynamics. In Section 4, the vacuum non-semisimple case for SH space–times and its SSS generalization are studied and the stretching operation is extended to a solution generating technique, accompanied by stuffing and tilting. A special electromagnetic solution of Cohen and Ftaclas is also treated. Finally, Section 5 shows how the semisimple electromagnetic Taub–Nut solutions arise from the vacuum solutions and how a slightly different stuffing operation applies.

2. SH and SSS Space–Times

The notation and conventions established in [4] for SH space–times will be observed here. For a SH space–time $(M, {}^4\mathrm{g})$, $M = R \times G$ is the product manifold of the real line and a three-dimensional simply connected Lie group G, while ${}^4\mathrm{g}$ is of the form

$$ {}^4\mathrm{g} = -N(t)^2\, dt \otimes dt + g_{ab}(t)(\omega^a + N^a_t\, dt) \otimes (\omega^b + N^b_t\, dt). \tag{2.1} $$

Here the lapse function N is a function on R and the matrix $\mathbf{g}_t = g_{ab}(t)\, \mathbf{e}^b{}_a$ of 3-metric components is a parametrized curve in $\mathscr{M} \subset GL(3, R)$, the submanifold of matrices of components of inner products on R^3. $\{e^b{}_a\}$ is the natural basis of $\mathrm{gl}(3, R)$ as in [4]. $\{\omega^a\}$ is a basis of the vector space $\mathfrak{g}^*$ of left invariant 1-forms on G and is dual to a basis $e = \{e_a\}$ of the Lie algebra $\mathfrak{g}$ of left invariant vector fields on G. The shift vector field $\vec{N}_t = N^a_t e_a$ is a parametrized curve in the Lie algebra

$$ \mathfrak{X}(\mathfrak{g}) = \mathfrak{g} \oplus \mathrm{aut}(\mathfrak{g}) = \tilde{\mathfrak{g}} \oplus \mathrm{aut}(\mathfrak{g}) \subset \mathfrak{X}(G) \tag{2.2} $$

which generates the action on G of the semidirect product group

$$ \mathscr{D}(\mathfrak{g}) = R(G) \times_s \mathrm{Aut}(G) = L(G) \times_s \mathrm{Aut}(G) \subset \mathscr{D}(G) \tag{2.3} $$

of translations and automorphisms of G into itself. $\tilde{\mathfrak{g}}$ is the Lie algebra of right invariant vector fields on G and is isomorphic to the Lie algebra of generators of the left action of G on $(M, {}^4\mathrm{g})$ as an isometry group. $\{e_0 = \partial/\partial t, e_a\}$ is a comoving ADM frame on M adapted to the SH foliation [3] with dual frame $\{\omega^0 = dt, \omega^a\}$. This means that e is tangent to the foliation and invariant under dragging along by e_0, which implies the following condition:

$$C^\alpha{}_{\beta\gamma} = \omega^\alpha([e_\beta, e_\gamma]) = \delta^\alpha{}_a \delta^b{}_\beta \delta^c{}_\gamma C^a{}_{bc}\,. \tag{2.4}$$

Greek and latin indices take values in the sets $\{0, 1, 2, 3\}$ and $\{1, 2, 3\}$, respectively. The vector field $e_\perp = N^{-1}(e_0 - \vec{N})$ is the unit normal to the SH foliation. Let $\omega^\perp = N\,dt$.

Let $C^a{}_{bc} = \omega^a([e_b, e_c])$ be the components of the structure constant tensor of $\mathfrak{g}$ with respect to the basis e. The following auxiliary quantities and relations are useful:

$$\begin{gathered} C^a{}_{bc} = \tfrac{1}{2}\epsilon_{bcd} n^{ad} + a_f \delta^{fa}_{bc} = \epsilon_{bcd} C^{ad}, \\ C^{ad} = \tfrac{1}{2} C^a{}_{bc} \epsilon^{bcd}, \qquad C^{(ab)} = n^{ab}, \qquad C^{[ab]} = a_c \epsilon^{cab}, \\ 0 = a_f n^{fa} = a_f C^{fa} = a_f C^f{}_{bc}\,, \\ \mathbf{m} = g^{-1/2} n^a{}_b \mathbf{e}^b{}_a\,, \qquad \mathbf{A} = a_f \eta^{fa}{}_b \mathbf{e}^b{}_a\,, \qquad \eta^{abc} = g^{-1/2}\epsilon^{abc}, \\ \mathbf{k}_a = \mathrm{ad}_e(e_a) = C^b{}_{ac} \mathbf{e}^c{}_b\,. \end{gathered} \tag{2.5}$$

Finally, let $\mathrm{ad}_e(\xi) = \mathrm{ad}_e(\xi)^a{}_b\, \mathbf{e}^b{}_a$ be the matrix with respect to the basis e of the derivation $\mathrm{ad}(\xi) \in \mathrm{der}(\mathfrak{g})$ for $\xi \in \mathfrak{X}(\mathfrak{g})$ and let $\mathrm{ad}_e(\xi)^{\#}$ be its symmetrization with respect to $\mathbf{g}$:

$$\begin{gathered} \pounds_\xi e_a = \mathrm{ad}_e(\xi)^b{}_a\, e_b\,, \\ \mathrm{ad}_e(\xi)^{\#a}{}_b = \mathrm{ad}_e(\xi)_{(cb)}\, g^{ca}. \end{gathered} \tag{2.6}$$

Similarly let $\mathbf{K}_a = \mathbf{k}_a{}^{\#}$, i.e., $K^a{}_{bc} = C_{(b}{}^a{}_{c)}$.

Suppose $\vec{N}_t = 0$ and $\mathbf{g}_t = \mathbf{S}_t^{-1T} \bar{\mathbf{g}}_t S_t^{-1}$, where $\mathbf{S}_t \in \mathrm{Aut}_e(\mathfrak{g}) \simeq \mathrm{Aut}(G)$ is a parametrized curve in the matrix representation of the automorphism group of the Lie algebra $\mathfrak{g}$ with respect to the basis e. As discussed in [4], one can introduce a new comoving ADM frame

$$\begin{aligned} \bar{e}_0 &= \frac{\partial}{\partial \bar{t}} = e_0 + \vec{\bar{N}}, & \bar{e}_a &= S^{-1b}{}_a e_b\,, \\ \bar{\omega}^0 &= d\bar{t} = dt, & \bar{\omega}^a &= S^a{}_b \omega^b - \bar{N}^a\, dt \end{aligned} \tag{2.7}$$

such that the metric has the form (2.1) with all barred quantities appearing. The new shift vector field $\bar{N} = \vec{\bar{N}}{}^a_t \bar{e}_a$ is a parametrized curve in $\mathfrak{X}(\mathfrak{g})$ satisfying

$$\dot{\mathbf{S}}_t \mathbf{S}_t^{-1} = \mathrm{ad}_{\bar{e}}(\vec{\bar{N}}_t). \tag{2.8}$$

This result will be used in this paper only for the case where $\mathbf{S}_t$ is confined to a one-dimensional subgroup of $\mathrm{Aut}_e(\mathfrak{g})$ generated by an element $\mathbf{E} = \mathrm{ad}_e(\xi)$ for some $\xi \in \mathfrak{X}(\mathfrak{g})$. In this case simpler relations hold:

$$\mathbf{S}_t = \exp \theta\mathbf{E}, \qquad \dot{\mathbf{S}}_t\mathbf{S}_t^{-1} = \dot{\theta}\mathbf{E}. \tag{2.9}$$

The dot stands for the time derivative.

A SSS space–time $(M, {}^4\mathrm{g}')$ arises from a SH space–time $(M, {}^4\mathrm{g})$ by a conformal scaling by a conformal factor $e^{2\psi}$ whose logarithmic spatial derivative lies in $\mathfrak{g}^*$:

$${}^4\mathrm{g}' = e^{2\psi}\,{}^4\mathrm{g}, \qquad d\psi = b_a\omega^a. \tag{2.10}$$

If we assume that $\pounds_{\vec{N}}\, d\psi = 0$ to sidestep tricky shift considerations, then $d\psi$ is a time-independent exact element of $\mathfrak{g}^*$ and therefore satisfies

$$0 = d^2\psi = -\tfrac{1}{2}b_aC^a{}_{bc}\omega^b \wedge \omega^c \qquad \text{or} \qquad b_aC^a{}_{bc} = 0. \tag{2.11}$$

The existence of such an element in $\mathfrak{g}^*$ implies that G is not semisimple, so the semisimple Bianchi types VIII and IX do not allow SSS extensions. G acts as a group of homothetic motions of the space–time $(M, {}^4\mathrm{g}')$. The generating Lie algebra of homothetic Killing vector fields for this action is isomorphic to $\tilde{\mathfrak{g}}$. Eardley has devised a classification scheme for SSS metrics in terms of the equivalence of triples $(n^{ab}, a_b\,; b_b)$ under the natural action of the general linear group [2]. Those for which b_a is nonzero and $\mathfrak{g}$ is of class A or B are called class C and D, respectively. When b_a vanishes, one returns to the SH case. The covector a_b is zero and nonzero respectively for class A and class B Lie groups. Except for the single type ${}^*_1$III in Eardley's classification, a_b and b_b are always linearly dependent and $(M, {}^4\mathrm{g}')$ has a two-dimensional Abelian subgroup of G as an isometry group. For the exceptional type, this isometry group is no longer Abelian and a_b and b_b are linearly independent.

Consider the case where the shift is zero and G is not semisimple. Let e be a canonical basis of $\mathfrak{g}$ as defined in [4] so that $\mathbf{n} = \mathrm{diag}(n^{(1)}, n^{(2)}, n^{(3)})$ is diagonal with $n^{(3)} = 0$ and $a_b = a\delta^3{}_b$. For Bianchi type II one must choose $n^{(a)} = \delta^a{}_1$ rather than $n^{(a)} = \delta^a{}_3$ as in [3] so that $n^{(3)}$ vanish. If $\{x^a\}$ are canonical coordinates of the first or second kind on G with respect to the basis e and if $b_c = b\delta^3{}_c$, then one may take $\psi = bx^3$.

For the Bianchi type VI$_h$ Lie algebras where $n^{(1)} = n^{(2)} = q = 1$ in a canonical basis, it is convenient to introduce a noncanonical basis $\bar{e}_a = A^{-1b}{}_a e_b$ of $\mathfrak{g}$ related to a canonical basis e by a $\pi/4$ rotation about e_3:

$$\mathbf{A} = 2^{-1/2}\begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2^{1/2} \end{pmatrix}. \tag{2.12}$$

For these Bianchi types, $\mathbf{n} = q(\mathbf{e}^1{}_1 - \mathbf{e}^2{}_2)$ and $\bar{\mathbf{n}} = q(\mathbf{e}^2{}_1 + \mathbf{e}^1{}_2)$, while $a_b = a\delta^3{}_b = \bar{a}_b$ and $h = -a^2q^{-2}$. Type III $=$ VI$_{-1}$ is the only class B Lie algebra which allows a solution of $b_c C^c{}_{ab} = 0$ with b_c linearly independent of a_c. The exceptional case $^*_1$III occurs when one chooses the solution $\bar{b}_a = b\delta^2{}_a$ rather than $\bar{b}_a = b\delta^3{}_a = b_a$.

ψ can be uniquely defined by requiring that it vanish at the identity of G when the shift is zero. This fixes the scale of $e^{2\psi}$ and removes the ambiguity between the definition of the individual scales of $e^{2\psi}$ and the SH metric ^{4}g to which ^{4}g$'$ is conformally related. Otherwise, one could add an arbitrary function $\psi_0(t)$ to ψ and therefore change the conformally related SH metric ^{4}g by the time-dependent scaling $e^{-2\psi_0(t)}$. This also allows one to repeat the discussion of a change of frame from zero shift to nonzero shift given in the SH case, simply by working on the SH space–time to which the SSS one is conformally related and treating ψ as a space–time scalar field which is unaffected by a change of frame or shift vector field.

Each physical geometric object field Φ has a dimension q determined by its scaling properties under constant scale transformations of the unit of length. A SSS field Φ' of dimension q is always related to a SH field Φ by a scaling appropriate for its dimension:

$$\Phi' = e^{q\psi}\Phi. \tag{2.13}$$

Since the covariant metric has dimension 2, the dimension depends on the positioning of indices. The covariant Ricci and Einstein tensors as well as the energy–momentum tensor of sources have $q = 0$. For a perfect fluid, the pressure p and the energy density ρ have $q = -2$, while the velocity vector field u, the baryon number density n, and chemical potential $\mu = (\rho + p)/n$ have $q = -1$. The notation of Ref. [3] for perfect fluids and electromagnetic fields is observed here. Three-dimensional geometric object fields also have dimensions. For example, the lapse and 3-metric have dimensions 1 and 2, while the shift vector field has dimension 0. For an electromagnetic field, the contravariant electric and magnetic field densities $\mathscr{E}$ and $\mathscr{B}$ have $q = 1$. Note also that the new normal vector field and its dual 1-form are given by

$$e_\perp{}' = e^{-\psi}e_\perp\,, \qquad \omega^{\perp\prime} = e^{\psi}\omega^\perp. \tag{2.14}$$

The conformal transformation properties of the connection and various curvature tensors are evaluated in Appendix A using standard formulas which may be found in the work by Schouten [10].

The Einstein field equations for a general space–time $(M, {}^4\mathrm{g})$ may be written in many different forms. If one is using a Hamiltonian approach, one attempts to solve the following system of equations:

$$\begin{aligned} NM_{cd}\,g^{ca}g^{db} &= Ng^{1/2}({}^4G_{cd} - kT_{cd})\,g^{ca}g^{db} \\ &= \dot{\pi}^{ab} - NS(\pi, \pi)^{ab} - N(F^{ab} + kg^{1/2}T_{cd}\,g^{ca}g^{db}) = 0, \end{aligned} \tag{2.15a}$$

$$\begin{aligned} M^\perp{}_\perp &= g^{1/2}(G^\perp{}_\perp - kT^\perp{}_\perp) \\ &= \tfrac{1}{2}g^{-1/2}(\mathrm{Tr}\,\pi^2 - \tfrac{1}{2}\,\mathrm{Tr}^2\,\pi) - \tfrac{1}{2}g^{1/2}\,{}^3R - kg^{1/2}T^\perp{}_\perp = 0, \end{aligned} \tag{2.15b}$$

$$M^\perp{}_a = g^{1/2}(G^\perp{}_a - kT^\perp{}_a) = -\pi^b{}_{a|b} - kg^{1/2}T^\perp{}_a = 0. \tag{2.15c}$$

The geometrical force in the dynamical Hamiltonian equations (2.15a) is

$$NF^{ab} = -Ng^{1/2}\,{}^3G^{ab} - (N^{|ab} - g^{ab}N^{|c}{}_{|c}) + (\pounds_{\vec{N}}\pi)^{ab}. \tag{2.16}$$

Equations (2.15b) and (2.15c) are respectively called the super-Hamiltonian and supermomentum constraints. $S(\pi, \pi)$ is the usual quadratic expression in π^{ab} called the spray of the DeWitt metric on the space of 3-metrics. The slash indicates covariant differentiation with respect to the 3-metric which is used to manipulate indices on all three-dimensional quantities.

In certain cases, however, it is more convenient to replace the dynamical equations (2.15a) by an alternative form:

$$P_{ab} = {}^4R_{ab} - kT^{\mathrm{TR}}{}_{ab} = 0 \tag{2.15d}$$

where $T^{\mathrm{TR}\alpha}{}_\beta = T^\alpha{}_\beta - \frac{1}{2}\delta^\alpha{}_\beta T^\gamma{}_\gamma$ is the trace reversal of the energy–momentum tensor of the source. Consider the zero shift case for a SSS space–time. If (2.15d) is satisfied, then

$$\mathbf{M}' = M'^a{}_b e^b{}_a = -M'^\perp{}_\perp I. \tag{2.17}$$

Provided that the energy-momentum tensor is divergence-free, the tensor with components $g'^{-1/2}M'^\alpha{}_\beta$ is also divergence-free. Evaluating Eq. (A8) for a SSS metric and source satisfying (2.17) leads to the result:

$$\begin{aligned}
&(e^{-2\psi}M'^\perp{}_a)^\cdot = 0,\\
&(e^{-\psi}M'^\perp{}_\perp)^\cdot + (\ln g^{1/2})^\cdot\, e^{-\psi}M'^\perp{}_\perp + 2(a^c - b^c)\, e^{-2\psi}M'^\perp{}_c \\
&\qquad = g^{-1/2}(g^{1/2}e^{-\psi}M'^\perp{}_\perp)^\cdot + 2(a^c - b^c)\, e^{-2\psi}M'^\perp{}_c = 0.
\end{aligned} \tag{2.18}$$

Since $e^{-2\psi}M'^\perp{}_a$ are constants, they can only depend on the parameters appearing in the ordinary differential equations (2.15d) and the constants of integration which appear in the integration of those equations. If $a^c - b^c = 0$ or the algebraic constraint $e^{-2\psi}M'^\perp{}_a = 0$ is satisfied, the same statement holds for $g^{1/2}e^{-\psi}M'^\perp{}_\perp$.

A SSS stiff perfect fluid flowing orthogonally to the natural foliation has a 4-velocity u' whose SH counterpart u has components $u^\perp = 1$ and $u_a = 0$. Equations (B8) show that its energy–momentum tensor has the properties

$$T'^{\mathrm{TF}}{}_{ab} = 0 = T'^\perp{}_a\,, \tag{2.19}$$

so Eqs. (2.15d) and (2.15c) are unchanged by the addition of such a source. Since $ge^{2\psi}T'^\perp{}_\perp = gT^\perp{}_\perp = -l^2$ is constant for this fluid as shown in Appendix B, solutions of (2.15d) and (2.15c) but not of (2.15b) can be interpreted as having an additional stiff perfect fluid source present, provided that $g^{1/2}e^{-\psi}M'^\perp{}_\perp$ is not a nonnegative algebraic expression in the parameters and constants of which it is composed. In this case one can define the single quantity l characterizing the fluid by

$$g^{1/2}e^{-\psi}M'^\perp{}_\perp + l^2 = 0. \tag{2.20}$$

The addition of such a fluid to a given solution of the Einstein equations will be referred to as stuffing that solution. In Section 4 it will be seen that a large class of stuffed solutions can also be tilted so that u_a is nonzero.

3. Streching Algorithm and Hamiltonian Formulation

Suppose one starts with a pair $(G, \mathfrak{g})$ and a basis e of $\mathfrak{g}$ in which the structure constant tensor has components (n^{ab}, a_b) and one considers another pair $(\bar{G}, \bar{\mathfrak{g}})$ with basis $\bar{e}$ and structure constant tensor components $(n^{ab}, \bar{a}_b)$ where $\bar{a}_b = a_b + b_b$ and $b_c C^c{}_{ab} = 0$. This is possible only if a_b and b_b are linearly dependent since otherwise the Jacobi identities would not be satisfied by the barred structure constant tensor:

$$0 = \bar{a}_f \bar{C}^{fa} = a_f b_g \epsilon^{gfa}. \tag{3.1}$$

This in turn implies that $b_f \bar{C}^{fa} = 0$, so one may perform the conformal scaling determined by b_c on the SH metrics associated with the group $\bar{G}$, yielding SSS metrics characterized by the triple $(n^{ab}, \bar{a}_b\,;\, b_b)$. These metrics are dynamically more closely related to the SH metrics associated with the original group G than the actual group $\bar{G}$ of homothetic motions. The group $\bar{G}$ is said to be obtained from G by stretching. This stretching operation for the group was discovered while putting the metrics of [8] into the standard SSS form (2.10). Only the exceptional type *III metrics cannot be obtained in this way and will therefore be excluded from the present discussion. In certain cases [Bianchi types V and IV], G and $\bar{G}$ are isomorphic and may be identified, so that one is simply using two different bases e and $\bar{e}$ of $\mathfrak{g}$.

Since a_b and b_b must be linearly dependent, one may introduce a proportionality factor:

$$b_c = \nu a_c\,, \qquad a_c = \nu^{-1} b_c\,, \qquad \bar{a}_c = (1 + \nu)\, a_c\,. \tag{3.2}$$

Evaluating the zero shift formulas (A6) using the barred structure constant tensor components, one finds

$$\begin{aligned} e^{2\psi}\, {}^4\bar{R}'^a{}_b &= {}^4R^a{}_b - 2\nu a^a a_b = {}^4R^a{}_b - 2\nu^{-1} b^a b_b\,, \\ e^{2\psi}\, {}^4\bar{G}'^\perp{}_\perp &= {}^4G^{0\perp}{}_\perp + (3 + 2\nu)\, a_f a^f = {}^4G^{0\perp}{}_\perp + \nu^{-1}(3\nu^{-1} + 2)\, b_f b^f, \\ e^{\psi}\, {}^4\bar{G}'^\perp{}_a &= \mathrm{Tr}\, \mathbf{k}^0_a \mathbf{K} + (1 + \nu)\, a_a\, \mathrm{Tr}\, \mathbf{K} - (3 + \nu)\, a_f K^f{}_a \\ &= \mathrm{Tr}\, \mathbf{k}^0_a \mathbf{K} + (1 + \nu^{-1})\, b_a\, \mathrm{Tr}\, \mathbf{K} - (3\nu^{-1} + 1)\, b_f K^f{}_a\,. \end{aligned} \tag{3.3}$$

The superscript 0 on $\mathbf{k}_a{}^0$ and ${}^4G^{0\perp}{}_\perp$ refers to the expressions one obtains by setting the structure constant tensor components a_b to zero. The formulas which are written in terms of b_a and ν^{-1} remain valid when $\nu^{-1} = 0$, the case in which one starts with a class A metric with $a_b = 0$. The resulting metrics are called class D_0 metrics and satisfy $\bar{a}_b = b_b$. When ν^{-1} does not equal 0 or -1, one obtains the class D metrics with $b_c = f\bar{a}_c$ and $f = (1 + \nu^{-1})^{-1}$. When $\nu^{-1} = -1$, $\bar{a}_b$ vanishes and one obtains the class C metrics. When $\nu = 0$, one returns to the SH case.

The utility of the stretching operation is nicely exhibited in the Hamiltonian formulation of the Einstein equations. The shift vector field is assumed to vanish for simplicity. As in Ref. [4], introduce the SSS scalar curvature potential U_{SSS} and **its SH analog U_{SH}, both potential fields on $\mathscr{M}$:**

$$U_{\text{SSS}} = -g^{1/2}e^{2\psi}\,{}^3R' = U_{\text{SH}} + 2g^{1/2}b^c(b_c - 4a_c),$$
$$U_{\text{SH}} = -g^{1/2}\,{}^3R, \tag{3.4}$$

and the gravitational force field 1-form on $\mathscr{M}$:

$$NF_{\text{SSS}} = NF^{ab}_{\text{SSS}}\,dg_{ab} = -g^{1/2}e^{3\psi}(N'{}^3G'{}^{ab} + N'{}^{\|ab} - g'{}^{ab}N'{}^{\|c}{}_{\|c})\,dg_{ab}\,. \tag{3.5}$$

Here $\|$ indicates covariant differentiation with respect to the scaled 3-metric. The SSS extrinsic curvature and gravitational canonical momentum π are given by

$$K'_{ab} = -(2N')^{-1}\,(g'_{ab})^{\cdot} = e^{\psi}K_{ab}\,,$$
$$\pi'^{ab} = -g'^{1/2}(K'^{ab} - g'^{ab}\,K'^c{}_c) = \pi^{ab}. \tag{3.6}$$

Comparison of (2.15a) and the last formula of (A6), together with the scale invariance of π, shows that

$$F^{ab}_{\text{SSS}} = F^{ab}_{\text{SH}} - g^{1/2}(2b_cK^{cab} + 2b^ab^b + g^{ab}(b_c - 4a_c)\,b^c). \tag{3.7}$$

Using the SH results of [4], it is easy to evaluate the nonpotential component of the force field F_{SSS}:

$$F_{\text{SSS}} = -dU_{\text{SSS}} + Q_{\text{SSS}}\,,$$
$$Q_{\text{SSS}} = Q^{ab}_{\text{SSS}}\,dg_{ab} = 2g^{1/2}((a^c - b^c)\,K_c{}^{ab} - 2(a^a - b^a)(a^b - b^b))\,dg_{ab}\,. \tag{3.8}$$

Thus, the vacuum zero shift dynamical Hamiltonian equations may be derived from the Hamiltonian:

$$H = 2Ng^{1/2}e^{2\psi}G'^{\perp}{}_{\perp} = N(g^{-1/2}(\text{Tr}\,\boldsymbol{\pi}^2 - \tfrac{1}{2}\,\text{Tr}^2\,\boldsymbol{\pi}) + U_{\text{SSS}}),$$
$$\boldsymbol{\pi} = \pi^a{}_b\mathbf{e}^b{}_a\,, \tag{3.9}$$

provided one includes the nonpotential force in the momentum equation:

$$\dot{\pi}^{ab} = \{\pi^{ab}, H\} + Q^{ab}_{\text{SSS}}\,. \tag{3.10}$$

In the nonvacuum SSS case, one must add to the Hamiltonian the source potential

$$U_{\text{source}} = -2kNg^{1/2}e^{2\psi}T'^{\perp}{}_{\perp} = -2kNg^{1/2}T^{\perp}{}_{\perp} \tag{3.11}$$

expressed in terms of the proper choice of SH source variables so that as a function on $\mathscr{M}$ for fixed values of these variables it satisfies

$$-dU_{\text{source}} = kg^{1/2}T^{ab}\, dg_{ab}\,. \tag{3.12}$$

As discussed in [11], the variables $\mathscr{E}^a$ and $\mathscr{B}^a$ are appropriate for an electromagnetic field, while $\{l, v_a, n\}$ are appropriate for a perfect fluid. In the latter case, the variables μ, p, and ρ are considered to be functions of n determined by the equation of state.

Until this point, the discussion of the Hamiltonian approach has applied to all SSS metrics. Now consider only those for which the stretching operation applies and evaluate the above formulas using the barred structure constant tensor components. For example,

$$\begin{aligned}\bar{Q}^{ab}_{\text{SSS}} &= Q^{ab}_{\text{SH}} + 2g^{1/2}(g^{ab}a_c b^c - a^{(a}b^{b)}) = Q^{ab}_{\text{SH}} + \Delta Q^{ab},\\ \bar{U}_{\text{SSS}} &= U^0_{\text{SH}} + 2g^{1/2}(3a_c + 2b_c)\, a^c,\end{aligned} \tag{3.13}$$

where the superscript 0 has the same significance as above. Evaluating the matrix of components $\mathbf{Q}_{\text{SSS}} = Q_{\text{SSS}}{}^a{}_b \mathbf{e}^b{}_a$ at a diagonal metric matrix $\mathbf{g} = e^{2\boldsymbol{\beta}}$ as in [4] for $C^a{}_{bc}$ in standard diagonal form with $a_c = a\delta^3{}_c$ and $b_c = b\delta^3{}_c$, one finds in the notation of that reference

$$\begin{aligned}\Delta\mathbf{Q} &= 2abg^{1/2}g^{33}\mathbf{I}^{(3)},\\ \mathbf{Q}_{\text{SSS}} &= 2ae^{\beta^0+4\beta^+}(\mathbf{K}^0_3 + a\mathbf{e}_+ + b\mathbf{I}^{(3)}).\end{aligned} \tag{3.14}$$

These formulas make it clear that dynamically, ${}^4\bar{\mathrm{g}}'$ is much more closely related to ${}^4\mathrm{g}$ than to ${}^4\bar{\mathrm{g}}$. In the following section this relationship will lead to a solution generating mechanism for a large class of SSS metrics.

4. The Nonsemisimple Symmetric Case

In this section the SSS symmetric case space–times with a SSS stiff perfect fluid source are considered. A basis e of $\mathfrak{g}$ is assumed having the property that $\mathbf{n} = \operatorname{diag}(n^{(1)}, n^{(2)}, 0)$ and $a_c = a\delta^3{}_c$, while $b_c = b\delta^3{}_c$, therefore excluding the exceptional type *III space–times. The shift $\vec{N}$ is chosen to vanish and the curve $\mathbf{g}_t$ to lie in $\mathscr{M}_{S(3)}$, the submanifold of $\mathscr{M}$ whose points $\mathbf{g}$ satisfy $g_{13} = g_{23} = 0$. This defines the symmetric case. The following choice of variables is convenient [4, 11]:

$$\begin{aligned}&\mathbf{g} = \mathbf{S}^T\mathbf{g}_D\mathbf{S}, \qquad \mathbf{g}_D = \operatorname{diag}(g_{D11}, g_{D22}, g_{D33}) \in \mathscr{M}_D,\\ &\mathbf{S} = \exp\theta\mathbf{k}^0_3 \in \operatorname{Aut}_e(\mathfrak{g}), \qquad \mathbf{k}^0_3 = -n^{(1)}\mathbf{e}^2{}_1 + n^{(2)}\mathbf{e}^1{}_2 \in \operatorname{der}_e(\mathfrak{g}),\\ &\det\mathbf{S} = 1, \qquad \det\mathbf{g} = \det\mathbf{g}_D\,.\end{aligned} \tag{4.1}$$

When $\mathbf{n} = 0$, then $\mathbf{S} = \mathbf{I}$ and only the diagonal case is considered, i.e., the curve $\mathbf{g}_t$ lies entirely in the diagonal submanifold $\mathscr{M}_D$ of $\mathscr{M}$. The spatial components of the circulation 1-form of the fluid are assumed to satisfy $v_a = v_3\delta^3{}_a$, where v_3 is a constant of the motion as shown in Appendix B. The lapse choice $N = (g_{33})^{1/2}$ is made since it decouples $g_{33} = g_{D33}$ from the 2-metric variables $\{g_{11}, g_{22}, g_{12}\}$ and linearizes the equation of motion for that variable, as will be seen below. The discussion of Ref. [4] shows that no loss of generality is involved by considering only the diagonal rather than the symmetric case when $\mathbf{n}$ vanishes.

Let $\{e_{Da} = e_b S^{-1b}{}_a\}$ be the frame in which the matrix of metric components is $\mathbf{g}_D$ and let the subscript D indicate components with respect to this frame. The frame e_D is more convenient for calculations and corresponds to a choice of shift $\vec{N}$ such that

$$\mathrm{ad}_{e_D}(\vec{N}) = \dot{\theta}\mathbf{k}_3^0 . \tag{4.2}$$

The unique such shift in $\mathfrak{aut}(G)$ is $\vec{N} = \dot{\theta}(e_3^0 - \tilde{e}_3^0)$, where the zero superscript indicates the expression one obtains by setting the structure constant tensor component a to zero in the coordinate formulas given in [4]. Since $\mathbf{S} \in \mathrm{Aut}_e(\mathfrak{g})$, the only changes one needs to make in curvature formulas expressed in the new frame arise from transforming time derivatives. These changes are easily evaluated by computations like the following in which the formula $\dot{S}\mathbf{S}^{-1} = \dot{\theta}\mathbf{k}_3^0$ is used:

$$\begin{aligned}(\dot{\mathbf{g}})_D &= \mathbf{S}^{-1T}(\mathbf{S}^T\mathbf{g}_D\mathbf{S})^{\cdot}\,\mathbf{S}^{-1} = (\mathbf{g}_D)^{\cdot} + \dot{\theta}(\mathbf{g}_D\mathbf{k}_3^0 + \mathbf{k}_3^{0T}\mathbf{g}_D),\\ (\dot{\mathbf{K}})_D &= \mathbf{S}(\mathbf{S}^{-1}\mathbf{K}_D\mathbf{S})^{\cdot}\,\mathbf{S}^{-1} = (\mathbf{K}_D)^{\cdot} - \dot{\theta}[\mathbf{k}_3^0, \mathbf{K}_D].\end{aligned} \tag{4.3}$$

For example, the extrinsic curvature components in the new frame are

$$2N\mathbf{K}_D = -\mathbf{g}_D{}^{-1}\dot{\mathbf{g}}_D - \dot{\theta}(\mathbf{k}_3^0 + \mathbf{g}_D{}^{-1}\mathbf{k}_3^{0T}\mathbf{g}_D). \tag{4.4}$$

The first term in this formula is simply $-(\ln \mathbf{g}_D)^{\cdot}$.

Evaluation of the formulas of Appendix A with these changes yields the following expressions for the Einstein equations (2.15d), (2.15c), and (2.15b) in the case $a \neq 0$:

$$\begin{aligned}2g_{33}e^{2\psi}\bar{\mathbf{P}}'{}_D &= x^{-1}(x(\ln \mathbf{g}_D)^{\cdot})^{\cdot} + (-x^{-1}(x\Delta^2\dot{\theta})^{\cdot} + 2a\Delta^2)(y^{-1}\mathbf{e}^2{}_1 + y\mathbf{e}^1{}_2)\\ &\quad + (1 - \dot{\theta}^2)\,\Delta_2\,\mathrm{diag}(1, -1, 0) - (\Delta^2 + 4a^2\nu + v_3{}^2)\,\mathbf{e}^3{}_3 - 4a^2 l = 0,\\ e^{-2\psi}\overline{M}'{}^{\perp}{}_3 &= ax(\ln w)^{\cdot} - \tfrac{1}{2}x\Delta^2\dot{\theta} - klv_3 = 0,\\ 4g^{1/2}e^{-\psi}\overline{M}'{}^{\perp}{}_{\perp} &= -2\dot{x}x(\ln w)^{\cdot} + x^2(\ln y)^{\cdot 2} - (3 + 2\nu)(\dot{x}^2 - 4a^2x^2)\\ &\quad + x^2\Delta^2(1 + \dot{\theta}^2) + 4kl^2 + kv_3{}^2x^2 = 0.\end{aligned} \tag{4.5}$$

The definitions of the symbols appearing here are

$$\begin{aligned}\Delta &= n^{(1)}y - n^{(2)}y^{-1}, & \Delta_2 &= (n^{(1)}y)^2 - (n^{(2)}y^{-1})^2,\\ x &= (g_{D11}\,g_{D22})^{1/2}, & y &= (g_{D11}/g_{D22})^{1/2},\\ w &= g_{33}/x^{1+\nu}, & \mathbf{P} &= P^a{}_b\mathbf{e}^b{}_a .\end{aligned} \tag{4.6}$$

To set $a = 0$ in Eqs. (4.5), one must first let $a = \nu^{-1}b$ and then set $\nu^{-1} = 0$, as well as abandon the ν-dependent variable w:

$$\begin{aligned} 2g_{33}e^{2\psi}\bar{\mathbf{P}}'_D &= x^{-1}(x(\ln \mathbf{g}_D)^{\cdot})^{\cdot} - x^{-1}(x\Delta^2\dot{\theta})^{\cdot}(y^{-1}\mathbf{e}^2{}_1 + y\mathbf{e}^1{}_2) \\ &\quad + (1 - \dot{\theta}^2)\,\Delta_2\,\mathrm{diag}(1, -1, 0) - (\Delta^2 + v_3{}^2)\,\mathbf{e}^3{}_3 = 0, \\ e^{-2\psi}\bar{M}'^{\perp}{}_3 &= -b\dot{x} - \tfrac{1}{2}x\Delta^2\dot{\theta} - klv_3 = 0, \\ 4g^{1/2}e^{-\psi}\bar{M}'^{\perp}{}_{\perp} &= -2\dot{x}x(\ln g_{33})^{\cdot} - \dot{x}^2 + x^2(\ln y)^{\cdot 2} \\ &\quad + x^2\Delta^2(1 + \dot{\theta}^2) + 4kl^2 + kv_3{}^2x^2 = 0. \end{aligned} \tag{4.7}$$

According to Appendix B, the fluid variable l has the following equation of motion:

$$\dot{l} = av_3x. \tag{4.8}$$

When $a = 0$ or $v_3 = 0$, it is a constant of the motion.

Since $\mathbf{I}^{(3)} = \mathbf{e}^1{}_1 + \mathbf{e}^2{}_2 \in \mathrm{der}_e(\mathfrak{g})$, one can choose a new frame having identical properties in which x is scaled by any positive constant one pleases, so the scale of x is unimportant and will be fixed arbitrarily. The equation of motion for x is

$$g_{33}e^{2\psi}(\bar{P}'^1{}_1 + \bar{P}'^2{}_2) = x^{-1}(x(\ln x)^{\cdot})^{\cdot} - 4a^2 = x^{-1}(\ddot{x} - 4a^2x) = 0. \tag{4.9}$$

If $a = 0$, then by a choice of the origin of time and disregarding the scale, one can choose either $x = t$ or $x = 1$. However, the expression for $\bar{M}'^{\perp}{}_{\perp}$ is positive definite when $\dot{x} = 0$ and the super-Hamiltonian constraint cannot be satisfied, so $x = 1$ is not possible. If $a \neq 0$, then this equation has the first integral

$$\dot{x}^2 - 4a^2x^2 = 4a^2\epsilon, \tag{4.10}$$

where ϵ can take the values $\{1, 0, -1\}$, again disregarding the scale of x. One may choose the following solution:

$$\begin{aligned} x_\epsilon &= \tfrac{1}{2}(e^{2at} - \epsilon e^{-2at}), \\ 2a\int x_\epsilon\, dt &= (2a)^{-1}\,\dot{x}_\epsilon = \tfrac{1}{2}(e^{2at} + \epsilon e^{-2at}), \\ x_1 = \sinh 2at, \qquad & x_0 = \tfrac{1}{2}e^{2at}, \qquad x_{-1} = \cosh 2at, \end{aligned} \tag{4.11}$$

x_{-1} is its own time-reversed solution, while x_1 and x_0 have distinct time-reversed solutions which will be ignored here since they may be obtained easily by time-reversal considerations. The variable x is just the square root of the determinant of the 2-metric components and is designated by the symbol a by Siklos in his study of the SH perfect fluid symmetric case space–times [12].

The equation for l is easily integrated when $a \neq 0$:

$$l = av_3\int x\, dt + l_0 = av_3\dot{x}/(4a^2) + l_0\,. \tag{4.12}$$

Since

$$\rho = g^{-1}(l^2 - \tfrac{1}{4}v_3{}^2x^2) = \tfrac{1}{4}\epsilon v_3{}^2 + l_0(l_0 + \tfrac{1}{2}a^{-1}v_3\dot{x}), \tag{4.13}$$

the conditions $l_0 = \epsilon = 0$ imply that ρ and hence μ and v_3 vanish. When $|\epsilon| = 1$, v_3 must not be negative or ρ will eventually become negative.

By introducing the variable $\tilde{w} = g_{33}/x^{1+\nu+\frac{1}{4}a^{-2}kv_3{}^2}$ when $a \neq 0$, considerable simplification occurs:

$$\begin{aligned} e^{-2\psi}\overline{M}'^{\perp}{}_3 &= ax(\ln \tilde{w})^{\cdot} - \tfrac{1}{2}x\,\Delta^2\dot{\theta} - kl_0v_3 = 0, \\ 4g^{1/2}e^{-\psi}\overline{M}'^{\perp}{}_{\perp} &= -2a^{-1}\dot{x}(e^{-2\psi}\overline{M}'^{\perp}{}_3) + 2x\,\Delta^2(\tfrac{1}{2}(1+\dot{\theta}^2)\,x - \dot{\theta}\dot{x}/(2a)) \\ &\quad + x^2(\ln y)^{\cdot 2} - 4a^2\epsilon(3+2\nu) - \epsilon kv_3{}^2 + 4kl_0{}^2 = 0 \\ g_{33}e^{2\psi}(2\overline{P}'^3{}_3 - (4a^2\nu + v_3{}^2)(\overline{P}'_D{}^1{}_1 + \overline{P}'_D{}^2{}_2)) &= x^{-1}(x(\ln \tilde{w})^{\cdot})^{\cdot} - \Delta^2 = 0. \end{aligned} \tag{4.14}$$

Note, for example, that $\tilde{w}$ has the same equation of motion as w in the unstretched case and as $w_0 = g_{33}/x$ in the untilted, unstretched case, while the supermomentum is the same function of $\tilde{w}$ as of w except that the function l is replaced by the constant l_0. Moreover, $\tilde{w}$ (or equivalently g_{33}) does not appear in the equations of motion of the remaining variables nor does it enter into the expression for the super-Hamiltonian on the solution space of the supermomentum constraint. The remaining variables $\{x, y, \theta\}$ determine the 2-metric variables $\{g_{11}, g_{22}, g_{12}\}$ and their equations of motion are identical with the unstretched vacuum case. The only difference is that the super-Hamiltonian differs from that case by the constant term $-(4a^2\nu + \frac{1}{4}kv_3{}^2)\epsilon + 2kl_0{}^2$. When $\epsilon \neq 0$, any solution of the equations of motion for y and θ may be interpreted as corresponding to a solution of the full Einstein equations for some values of the parameters ν, v_3, and l_0. When $\epsilon = 0$, only those solutions for which the vacuum SH super-Hamiltonian is zero or negative may be interpreted in such a way.

By writing $\tilde{w} = \tilde{w}_0\tilde{w}_p$ with $\tilde{w}_p$ satisfying the condition

$$x(\ln w_p)^{\cdot} = kl_0v_3 = 2a\gamma, \tag{4.15}$$

$\tilde{w}_0$ will satisfy the same equations as $\tilde{w}$ in the untilted or vacuum SH case:

$$\begin{aligned} e^{-2\psi}\overline{M}'^{\perp}{}_3 &= ax(\ln \tilde{w}_0)^{\cdot} - \tfrac{1}{2}x\,\Delta^2\dot{\theta} = 0, \\ x^{-1}(x(\ln \tilde{w}_0)^{\cdot})^{\cdot} &- \Delta^2 = 0. \end{aligned} \tag{4.16}$$

The condition (4.15) can be integrated explicitly for each value of ϵ using integral tables. Let $\tilde{w}_p = f(\gamma, \epsilon)$:

$$\begin{aligned} f(\gamma, 1) &= (\tanh at)^{\gamma}, \qquad f(\gamma, 0) = \exp(-2\gamma\exp(-2at)), \\ f(\gamma, -1) &= \exp(\gamma \tan^{-1}(\sinh 2at)). \end{aligned} \tag{4.17}$$

In this way any solution of the vacuum SH case leads to a solution of the stuffed and tilted SSS case, provided that the modified super-Hamiltonian constraint on the constants of integration arising from the solution of the equations of motion for y and θ can be satisfied.

When $a = 0$, the equation of motion for θ takes the form

$$(x\Delta^2\dot{\theta})^{\cdot} = 0, \qquad x\Delta^2\dot{\theta} = \alpha. \tag{4.18}$$

The first integral of this equation enables one to eliminate $\dot{\theta}$ from the super-Hamiltonian constraint and from the equation of motion for y:

$$x^{-1}(x(\ln y^{\cdot})^{\cdot} + (1 - \alpha^2 x^{-2}\Delta^{-4})\,\Delta_2 = 0. \tag{4.19}$$

Once y is found, θ can be found by integrating the first integral equation, while g_{33} has the equation of motion:

$$x^{-1}(x(\ln g_{33})^{\cdot})^{\cdot} - (\Delta^2 + kv_3{}^2) = 0. \tag{4.20}$$

By writing $g_{33} = g_{33}^0 g_{33}^p$ with g_{33}^p satisfying

$$x^{-1}(x(\ln g_{33}^p)^{\cdot})^{\cdot} - kv_3{}^2 = 0, \tag{4.21}$$

then g_{33}^0 will satisfy the same equation as g_{33} in the untilted or vacuum SH case:

$$x^{-1}(x(\ln g_{33}^0)^{\cdot})^{\cdot} - \Delta^2 = 0. \tag{4.22}$$

Since $x = t$, Eq. (4.21) is easily integrated and has the following particular solution:

$$g_{33}^p = \exp(\tfrac{1}{4}kv_3{}^2t^2). \tag{4.23}$$

The actual constraints on the constants of integration arising from the SH vacuum equations of motion satisfied by $\{y, \theta, g_{33}^0\}$ differ only by constants from the vacuum SH constraints on these constants:

$$\begin{aligned} e^{-2\psi}\overline{M}'^{\perp}{}_3 &= -\tfrac{1}{2}\alpha - b - klv_3\,, \\ 4g^{1/2}e^{-\psi}\overline{M}'^{\perp}{}_{\perp} &= -2x(\ln g_{33}^0)^{\cdot} + x^2(\ln y)^{\cdot 2} - 1 + x^2\Delta^2 + \alpha^2\Delta^{-2} + 4kl^2. \end{aligned} \tag{4.24}$$

Solutions of the vacuum dynamical equations but not of the constraints are therefore easily reinterpreted in terms of stuffed, tilted, and stretched solutions.

One may attempt the following ansatz to obtain a class of special solutions of the remaining equations. It will be called the Lukash ansatz since Lukash used it with success in the vacuum type VII_h SH case [13] One seeks solutions for which $\dot{\theta} = \kappa = \pm 1$, so that $\theta = \kappa t + \theta_0$ where θ_0 is an irrelevant parameter since it leads to a constant automorphism matrix acting on the metric matrix. In order for this to make sense, $\mathbf{k}_3^0$ must not vanish so assume $n^{(1)} \neq 0$. The off-diagonal components of the dynamical equations then imply the condition

$$(\ln x\Delta^2)^{\cdot} = 2a\kappa, \qquad x\Delta^2 = \beta\exp(2\kappa at), \qquad \beta \geqslant 0. \tag{4.25}$$

However, the equation of motion for y then becomes

$$x^{-1}(x(\ln y)^{\cdot})^{\cdot} = 0, \qquad x(\ln y)^{\cdot} = \tilde{\zeta} = 2a\zeta. \tag{4.26}$$

When $a = 0$, this has the solution $y = y_0 t^{\tilde{\zeta}}$, while if $a \neq 0$, the solution is $y = y_0 f(\zeta, \epsilon)$.

For the ansatz to work there must exist values of β and of $\tilde{\zeta}$ or ζ such that the second equation of (4.25) can be satisfied by x and y. For $a \neq 0$ and $\epsilon = 1$, only the group types VI_h and VII_h for which $n^{(1)}n^{(2)} \neq 0$ are compatible with the ansatz. Assuming $| n^{(1)} | = | n^{(2)} | = 1$, one must pick $\zeta = -\frac{1}{2}$, $\beta = 2$, and $\kappa = -n^{(1)}n^{(2)}$. For $\epsilon = 0$, $x\Delta^2 = \frac{1}{2}e^{2at}\Delta^2$, so one must choose $\kappa = 1$, $\beta = \frac{1}{2}\Delta^2$, and $\zeta = 0$ so that $y = y_0$ and Δ^2 are constants. (One must choose $\kappa = -1$ for the time-reversed solution.) This yields solutions for the types VI_h, VII_h, and IV. The case $\epsilon = -1$ is incompatible with the ansatz.

If $a = 0$ where $\alpha = \beta\kappa$, then

$$x\Delta^2 = t(n^{(1)}y_0 t^{\tilde{\zeta}} - n^{(2)}y_0^{-1}t^{-\tilde{\zeta}})^2 = \beta \tag{4.27}$$

requires $n^{(2)} = 0$, $\zeta = -\frac{1}{2}$, and $\beta = (n^{(1)}y_0)^2$, leading only to stretched type II solutions. The remaining dynamical equation is

$$(x(\ln \overset{0}{g}_{33})^{\cdot})^{\cdot} = \beta, \qquad x(\ln \overset{0}{g}_{33})^{\cdot} = \beta t + \delta, \qquad \overset{0}{g}_{33} = \sigma t^{\delta} \exp(\tfrac{1}{2}\beta t). \tag{4.28}$$

The constraints then become

$$e^{-2\psi}\overline{M}'^{\perp}{}_3 = -\tfrac{1}{2}\beta\kappa - b - klv_3 = 0, \qquad 2g^{1/2}e^{-\psi}\overline{M}'^{\perp}{}_{\perp} = -\delta - \tfrac{3}{8} + 2kl^2 = 0. \tag{4.29}$$

Since $\mathbf{I}^{(3)}$ and $\mathbf{e}^1{}_1 - \mathbf{e}^2{}_2 + 2\mathbf{e}^3{}_3$ belong to $\mathrm{der}_e(\mathfrak{g})$ in this case, the scales of both x and yg_{33} are unimportant. One may assume $y_0 = 1$, leaving the meaningful part of the scale in σ. The sign of κ is irrelevant here because of the discrete automorphism with matrix diag$(1, -1, 1)$ which changes the sign of g_{12}, b, and v_3 and hence has the effect of changing the sign of κ since $g_{12} = -n^{(1)}\theta$. This solution is referred to in example 1(c) of [8].

For $a \neq 0$ the remaining equations are

$$\begin{aligned} (x(\ln \tilde{w}_0)^{\cdot})^{\cdot} &= \beta \exp(2\kappa at), \\ x(\ln \tilde{w}_0)^{\cdot} &= 2a\gamma + (2a)^{-1}\kappa\beta \exp(2\kappa at), \\ 0 = e^{-2\psi}\overline{M}'^{\perp}{}_3 &= 2a^2\gamma, \\ 0 = g^{1/2}e^{-\psi}\overline{M}'^{\perp}{}_{\perp} &= a^2(\zeta^2 - \epsilon(3 + 2\nu + \tfrac{1}{4}a^{-2}kv_3^2) + kl_0^2/a^2) + 2\beta Q(\kappa, \epsilon), \\ Q(\kappa, \epsilon) &= (x - \tfrac{1}{2}a^{-1}\kappa\dot{x}) \exp(2\kappa at) = \tfrac{1}{2}(1 - \kappa) - \tfrac{1}{2}\epsilon(1 + \kappa), \\ Q(1, \epsilon) &= -\epsilon, \qquad Q(-1, \epsilon) = 1. \end{aligned} \tag{4.30}$$

Letting $\tilde{w}_0 = \sigma f(\gamma, \epsilon)\, \tilde{w}_{0p}$, then $\tilde{w}_{0p}$ satisfies

$$x(\ln w_{0p})^{\cdot} = (2a)^{-1} \kappa\beta \exp(2\kappa a t). \tag{4.31}$$

For $\epsilon = 0$ and $\kappa = 1$, this can just be integrated and it has the particular solution $\tilde{w}_{0p} = x^{\frac{1}{2}a^{-2}\beta}$. For $\epsilon = 1$, notice the following trick which enables one to integrate the equations:

$$\begin{gathered}(\ln \tilde{w}_{0p} x^{\beta Q/a^2})^{\cdot} = 2(ax)^{-1}\beta(\kappa \exp(2\kappa a t) + \tfrac{1}{2}a^{-1}Q\dot{x}) = -\tfrac{1}{2}Q\kappa\beta a^{-1}, \\ \tilde{w}_{0p} = (x \exp(2\kappa a t))^{\frac{1}{4}a^{-2}\beta Q}.\end{gathered} \tag{4.32}$$

For the $\epsilon = 1$ SH vacuum case, $g^{1/2}M^{\perp}{}_{\perp} = -11a^2 + 4Q$, so one must take $\kappa = -1$ and only the single type VII_h with $h = a^2 = 4/11$ admits a solution of the super-Hamiltonian constraint. This is Lukash's outgoing wave solution [13], stuffed by Barrow [3], and stretched and tilted by Wainwright *et al.* [8]. For $\epsilon = 0$, $g^{1/2}e^{-\psi}\overline{M}'^{\perp}{}_{\perp} = 2\beta Q + kl_0{}^2$ which can vanish only if $l_0 = Q = 0$ requiring $\kappa = 1$. In the SH type IV case, assuming a canonical basis e so that $n^{(1)} = 1$, setting $y_0 = 1 = \beta$ yields the solution found by Harbey and Tsoubelis [14]. The SH vacuum type VII_h solution is Lukash's ingoing wave solution [13].

Assuming a canonical basis e of $\mathfrak{g}$, the quantity Δ_2 vanishes only if $y = 1$ for the type VII_h, VII_0, VI_h and VI_0 cases where $|\, n^{(1)} | = |\, n^{(2)} | = 1$. Equation (4.19) then shows that $y = 1$ is a solution of the equation of motion for y. However, in the type VII_h and VII_0 cases, the variable θ becomes irrelevant and $\mathbf{g}_t \in \mathscr{M}_{T(3)}$, the Taublike case, since

$$\mathbf{g} = \mathbf{S}^T\mathbf{g}_D\mathbf{S} = \mathbf{g}_D, \qquad \mathbf{g}_D \in \mathscr{M}_{T(3)}. \tag{4.33}$$

This is equivalent to the Taublike case for types V and I respectively [11]. These can be obtained from the type VI_h and VI_0 $y = 1$ cases by Lie group contraction so only the latter cases will be examined here.

Define $\bar{y} = \exp(-2q\theta)$ and let $q = n^{(1)} = -n^{(2)}$ so that $\Delta = 2q$ and consider the noncanonical frame discussed near Eq. (2.12). Since

$$\mathbf{k}_3^0 = -q(\mathbf{e}^2{}_1 + \mathbf{e}^1{}_2), \qquad \mathbf{A}\mathbf{k}_3^0\mathbf{A}^{-1} = -q(\mathbf{e}^1{}_1 - \mathbf{e}^2{}_2), \tag{4.34}$$

and $\mathbf{k}_0^3$ commutes with $\mathbf{g}_D$ when $y = 1$, the new matrix of components

$$\bar{\mathbf{g}} = \mathbf{A}^{-1T}\mathbf{g}\mathbf{A}^{-1} = \mathrm{diag}(\bar{g}_{11D}, \bar{g}_{22D}, \bar{g}_{33D}) \tag{4.35}$$

is diagonal and $\bar{y}^2 = \bar{g}_{11D}/\bar{g}_{22D}$. The noncanonical frame will therefore be used in this $y = 1$ case but for simplicity drop the barred notation.

The case $a \neq 0$ will be considered first. Let $\lambda = qa^{-1}$. The Einstein equations in the noncanonical frame are easily obtained from (4.5):

$$
\begin{aligned}
0 &= 2g_{33}e^{2\psi}\bar{\mathbf{P}}' \\
&= x^{-1}(x(\ln \mathbf{g}_D)^{\cdot})^{\cdot} - 4a^2 \operatorname{diag}(1+\lambda, 1-\lambda, 1+\lambda^2+\nu+\tfrac{1}{4}a^{-2}kv_3{}^2), \\
0 &= e^{-2\psi}\bar{M}'^{\perp}{}_3 = ax(\ln u)^{\cdot} - kl_0 v_3\,, \\
0 &= g^{1/2}e^{-\psi}\bar{M}'^{\perp}{}_{\perp} \qquad\qquad (4.36)\\
&= -(2a)^{-1}\, e^{-2\psi}\bar{M}'^{\perp}{}_3 + \tfrac{1}{4}x^2(\ln z)^{\cdot 2} - a^2\epsilon(3+\lambda^2+2\nu+\tfrac{1}{4}a^{-2}kv_3{}^2) + l_0{}^2, \\
z &= yx^{-\lambda}, \qquad u = g_{33}\,y^{-\lambda}x^{-(1+\nu+\frac{1}{4}a^{-2}kv_3^2)}, \\
\mathbf{g} &= \operatorname{diag}(x^{1+\lambda}z,\ x^{1-\lambda}z,\ ux^{1+\lambda^2+\nu+\frac{1}{4}a^{-2}kv_3^2}z_\lambda).
\end{aligned}
$$

The equations of motion and their solution for the new variables are

$$
\begin{aligned}
x^{-1}(x(\ln z)^{\cdot})^{\cdot} &= 0 = x^{-1}(x(\ln u)^{\cdot})^{\cdot}, \\
x(\ln z)^{\cdot} &= 2a\gamma, \qquad x(\ln u)^{\cdot} = 2a\zeta, \qquad (4.37)\\
z &= z_0 f(\gamma, \epsilon), \qquad u = u_0 f(\zeta, \epsilon).
\end{aligned}
$$

The constraints then take the form

$$
\begin{aligned}
e^{-2\psi}\bar{M}'^{\perp}{}_3 &= 2a^2\zeta - kl_0v_3 = 0, \\
g^{1/2}e^{-\psi}\bar{M}'^{\perp}{}_{\perp} &= -(2a)^{-1}\, e^{-2\psi}\bar{M}'^{\perp}{}_3 \qquad (4.38)\\
&\quad + a^2(\gamma^2 - \epsilon(3+\lambda^2+2\nu+\tfrac{1}{4}a^{-2}kv_3{}^2)) + kl_0{}^2 = 0.
\end{aligned}
$$

Now $\epsilon = -1$ solutions are possible for sufficiently negative ν. The $\epsilon = 1$ SH solutions were given by Ellis and MacCallum [5] and these were stretched, stuffed, and tilted by Wainwright *et al.* [8]. The $\epsilon = 0$ solutions require $\gamma = 0 = l_0$ and will not accept a fluid; these have a Weyl conformal curvature tensor of type N, while the other solutions are of type I or when $q = 0$ of type D (Bianchi types V and I). One can set $z_0 = 1$ since the scale of y is unimportant because of the automorphism generator $\mathbf{e}^1{}_1 - \mathbf{e}^2{}_2 \in \operatorname{der}_e(\mathfrak{g})$.

Still using the noncanonical frame for type VI_0 where $a = 0$ and $x = t$ one has

$$
\begin{aligned}
0 &= 2g_{33}e^{2\psi}\bar{\mathbf{P}}' = x^{-1}(x(\ln \mathbf{g}_D)^{\cdot})^{\cdot} - (4q^2 + kv_3{}^2)\,\mathbf{e}^3{}_3\,, \\
0 &= e^{-2\psi}\bar{M}'^{\perp}{}_3 = -qx(\ln y)^{\cdot} - klv_3 - b, \qquad (4.39)\\
0 &= 4g^{1/2}e^{-\psi}\bar{M}'^{\perp}{}_{\perp} = -2x(\ln g_{33})^{\cdot} + x^2(\ln y)^{\cdot 2} - 1 + (4q^2 + kv_3{}^2)\,x^2 + 4kl_0{}^2.
\end{aligned}
$$

These imply the following equations which are easily integrated:

$$
\begin{aligned}
&(x(\ln y)^{\cdot})^{\cdot} = 0, \qquad x(\ln y)^{\cdot} = \bar{\zeta}, \qquad y/y_0 = t^{\bar{\zeta}} \\
&(x(\ln g_{33})^{\cdot})^{\cdot} - (4q^2 + v_3{}^2)\,x = 0, \qquad x(\ln g_{33})^{\cdot} = \tfrac{1}{2}(4q^2 + v_3{}^2)\,x^2, \\
&g_{33}/g^0_{33} = t^{\nu}\exp((q^2 + \tfrac{1}{4}v_3{}^2)\,t^2), \qquad (4.40)\\
&e^{-2\psi}\bar{M}'^{\perp}{}_3 = -q\bar{\zeta} - klv_3 - b = 0, \\
&2g^{1/2}e^{-\psi}\bar{M}'^{\perp}{}_{\perp} = -\gamma + \tfrac{1}{2}(\bar{\zeta}^2 - 1) + 2kl^2 = 0.
\end{aligned}
$$

Again one can set $y_0 = 1$. These solutions were also given by Wainwright *et al.* [8]. Setting $q = 0$ yields the diagonal type I limit.

Next consider the diagonal type II case which occurs when $\theta = \dot{\theta} = 0$. This is also treated in [8]. Recall that $\mathbf{n} = \mathrm{diag}(n^{(1)}, 0, 0)$, The Einstein equations with $x = t$ take the form

$$\begin{aligned} 0 &= 2g_{33}e^{2\psi}\bar{\mathbf{P}}' = x^{-1}(x(\ln \mathbf{g}_D)^{\cdot})^{\cdot} + (n^{(1)}y)^2 \,\mathrm{diag}(1, -1, -1) - kv_3{}^2\mathbf{e}^3{}_3\,, \\ 0 &= 4g^{1/2}e^{-\psi}\overline{M}'^{\perp}{}_{\perp} = -2x(\ln g_{33})^{\cdot} + x^2(\ln y)^{\cdot 2} - 1 + x^2(n^{(1)}y)^2 + 4kl^2 + kv_3{}^2x^2, \\ 0 &= e^{-2\psi}\overline{M}'^{\perp}{}_3 = -b - klv_3\,. \end{aligned} \tag{4.41}$$

The equations of motion for y and g_{33} are

$$(x(\ln y)^{\cdot})^{\cdot} + (n^{(1)}y)^2\, x = 0 = (x(\ln g_{33})^{\cdot})^{\cdot} - (n^{(1)}y)^2\, x - kv_3{}^2x. \tag{4.42}$$

Adding these one obtains the equation of motion for $v = yg_{33}$:

$$\begin{aligned} (x(\ln v)^{\cdot})^{\cdot} - kv_3{}^2x = 0, \qquad x(\ln v)^{\cdot} = \gamma + \tfrac{1}{2}kv_3{}^2t^2, \\ v/v_0 = t^{\gamma} \exp(\tfrac{1}{4}kv_3{}^2t^2). \end{aligned} \tag{4.43}$$

The equation of motion for g_{33} is then

$$(t(\ln(g_{33} \exp(-\tfrac{1}{4}kv_3{}^2t^2)))^{\cdot})^{\cdot} - (n^{(1)})^2(g_{33} \exp(-\tfrac{1}{4}kv_3{}^2t^2))^{-2}t^{2\gamma+1} = 0 \tag{4.44}$$

and has the solution

$$\begin{aligned} g_{33} \exp(-\tfrac{1}{4}kv_3{}^2t^2) = \overset{0}{g}_{33}t^{\delta(\frac{1}{2}\delta-1)+\zeta}(1 + \tfrac{1}{4}(n^{(1)}/\delta)^2\, t^{2\delta}), \\ \delta^2 = 2(\gamma + 1). \end{aligned} \tag{4.45}$$

The constraints then become

$$\begin{aligned} e^{-2\psi}\overline{M}'^{\perp}{}_3 &= -b - klv_3 = 0, \\ g^{1/2}e^{-\psi}\overline{M}'^{\perp}{}_{\perp} &= -\zeta + 2l^2 = 0. \end{aligned} \tag{4.46}$$

As discussed for the type II Lukash ansatz case, the variable v can be scaled by an automorphism, so one can assume $v_0 = 1$.

Finally, stretched L.R.S. type V space–times containing both a stiff perfect fluid and a source-free electromagnetic field will be considered. For this case, $\mathbf{n} = 0 = \theta$ and $y = 1$ so that $\mathbf{g} = \mathrm{diag}(x, x, g_{33})$. The Maxwell equations for this case may be evaluated using (A10):

$$\begin{aligned} (2 + v)\, a\mathscr{E}^3 &= (2 + v)\, a\mathscr{B}^3 = 0, \\ \dot{\mathscr{E}}^1 - a\mathscr{B}^2 &= 0 = \dot{\mathscr{B}}^1 + a\mathscr{E}^2, \\ \dot{\mathscr{E}}^2 + a\mathscr{B}^1 &= 0 = \dot{\mathscr{B}}^2 - a\mathscr{E}^1, \\ \ddot{\mathscr{E}}^A - a^2\mathscr{E}^A &= 0 = \ddot{\mathscr{B}}^A - a^2\mathscr{B}^A, \qquad A = 1, 2. \end{aligned} \tag{4.47}$$

Setting $\mathscr{E}^3 = \mathscr{B}^3 = 0$, the remaining equations have simple exponential solutions. The equations $T^1{}_1 + T^2{}_2 = 0 = T^{12}$, which must be satisfied for this electromagnetic field to be a source for the stretched L.R.S. type V space–time, require that the exponential solutions of these equations be of the form

$$\begin{aligned}\mathscr{B}^1 &= 2\pi^{1/2} e \cos\phi \exp(\kappa a t) = -\kappa \mathscr{E}^2,\\ \mathscr{B}^2 &= 2\pi^{1/2} e \sin\phi \exp(\kappa a t) = \kappa \mathscr{E}^1,\end{aligned} \tag{4.48}$$

where $\kappa^2 = 1$ and e and ϕ are constants. Since $\mathscr{E}^a \mathscr{B}_a = 0$, this is a null field. The nonzero components of its corresponding SH energy–momentum tensor are

$$\begin{aligned}gT^3{}_3 &= xe^2 \exp(2\kappa a t) = -gT^\perp{}_\perp,\\ g^{1/2} T^\perp{}_3 &= \kappa e^2 \exp(2\kappa a t).\end{aligned} \tag{4.49}$$

Consider the Einstein equations (4.5) for this case with this SSS electromagnetic field as an additional source. The variable x has the same solutions as before and much of the remaining discussion there continues to hold. The relevant equations which change are

$$\begin{aligned} x^{-1}(x(\ln \tilde{w}_0)^\cdot)^\cdot - 2ke^2 x^{-1} \exp(2\kappa a t) &= 0,\\ e^{-2\psi} \overline{M}'^\perp{}_3 &= ax(\ln \tilde{w}_0)^\cdot - k\kappa e^2 \exp(2\kappa a t) = 0,\\ g^{1/2} e^{-\psi} \overline{M}'^\perp{}_\perp &= -\tfrac{1}{2} a^{-1} \dot{x} e^{-2\psi} \overline{M}'^\perp{}_3 + ke^2 \exp(2\kappa a t)(x - \tfrac{1}{2} a^{-1} \kappa \dot{x})\\ &\quad - \epsilon a^2 (3 + 2\nu + \tfrac{1}{4} a^{-2} v_3{}^2) + k l_0{}^2\\ &= -\tfrac{1}{2} a^{-1} \dot{x} e^{-2\psi} \overline{M}'^\perp{}_3 + ke^2 Q - \epsilon a^2 (3 + 2\nu + \tfrac{1}{4} a^{-2} k v_3{}^2) + k l_0{}^2 = 0. \end{aligned} \tag{4.50}$$

From these one obtains

$$\begin{aligned} x(\ln \tilde{w}_0)^\cdot &= 2a\zeta + k\kappa e^2 a^{-1} \exp(2\kappa a t),\\ \tilde{w}_0 = \tilde{w}_{0p} f(\zeta, \epsilon), &\qquad e^{-2\psi} \overline{M}'^\perp{}_3 = 2a\zeta = 0. \end{aligned} \tag{4.51}$$

The particular solution $\tilde{w}_{0p}$ of the equation

$$ax(\ln \tilde{w}_{0p})^\cdot - k\kappa e^2 \exp(2\kappa a t) = 0 \tag{4.52}$$

maybe found for each ϵ in a way analogous to the $\epsilon = 1$ case treated in Eqs. (4.31) and (4.32). The results are

$$\begin{aligned} \epsilon \neq 0, &\qquad \tilde{w}_{0p} = (x \exp(2\kappa a t))^{\frac{1}{4}\epsilon Q k e^2 a^{-2}},\\ \epsilon = 0,\ \kappa = 1, &\qquad \tilde{w}_{0p} = x^{k e^2 a^{-2}},\\ \epsilon = 0,\ \kappa = -1, &\qquad \tilde{w}_{0p} = \exp(\tfrac{1}{8} a^{-2} k e^2 x^{-2}). \end{aligned} \tag{4.53}$$

Notice that the super-Hamiltonian constraint forces $e = 0 = l_0$ when $\epsilon = 0$ and $\kappa = -1$. The untilted SH solutions were found by Cohen and Ftaclas [7]. Stretched type I L.R.S. solutions may be obtained by setting $a = \nu^{-1}b$ and letting $\nu^{-1} = 0$ and by using the new $a = 0$ equation for $g_{33} = g^0_{33}\, g^p_{33}$ or g^0_{33} itself; recall $x = t$ when $a = 0$:

$$\begin{aligned} x(\ln g^0_{33})^{\cdot} - 2e^2 &= 0, \qquad g^0_{33} = t^{\gamma} \exp(2ke^2 t), \\ e^{-2\psi}\overline{M}'^{\perp}{}_3 &= -b - klv_3 - \kappa k e^2 = 0, \\ 2g^{1/2}e^{-\psi}\overline{M}'^{\perp}{}_{\perp} &= -\gamma - \tfrac{1}{2} + 2kl^2 = 0. \end{aligned} \tag{4.54}$$

The idea of variation of parameters appears in another context in a paper by Dunn and Tupper [15] where a very special class of Taub-like type VI_0 SH solutions are studied. For $\mathbf{n} = q\,\mathrm{diag}(1, -1, 0)$ and $\mathbf{g} = \mathrm{diag}(x, x, g_{33}) = \exp(\mathrm{diag}(\alpha\beta t, \alpha\beta t, 2\alpha t))$, with the lapse choice $N^2 = g_{33}$, one finds

$$\begin{aligned} 2g_{33}{}^4\mathbf{R} &= \mathrm{diag}(\alpha^2\beta^2, \alpha^2\beta^2, 2\alpha^2\beta - 4q^2), \\ 4g_{33}G^{\perp}{}_{\perp} &= -\alpha^2\beta(4 + \beta) + 4q^2, \qquad G^{\perp}{}_a = 0. \end{aligned} \tag{4.55}$$

Dunn and Tupper have examined what combinations of electromagnetic and perfect fluid sources are compatible with these expressions.

5. The Taub-Like Semisimple Case

When the group G is semisimple, only SH space–times are possible and the situation is rather different. Let $\mathbf{n} = \mathrm{diag}(1, 1, n^{(3)})$ with $|n^{(3)}| = 1$; Bianchi type IX has $n^{(3)} = 1$, while type VIII has $n^{(3)} = -1$. Since $n^{(3)} \neq 0$, the lapse choice $N^2 = g_{33}$ no longer results in a decoupling of the remaining variables from g_{33} nor can one tilt a stuffed solution as above.

The stuffing operation itself depends on what set of dynamical equations are integrated. Since only the Taub-like case admits exact solutions [1, 16, 17], the metric matrix will be assumed to be of the form $\mathbf{g} = \mathrm{diag}(A, A, B)$ and the shift will be taken to vanish. Following the discussion of Bonanos [18], the lapse will be chosen to have the form $N^2 = A^m B^n$. For certain choices of the pair (m, n), certain combinations of the vacuum field equations are exactly integrable and the super-Hamiltonian places a single algebraic constraint on the constants of integration which appear. The multiple of the super-Hamiltonian which is a constant of the motion for a given choice may be determined from Eq. (A8) as in Eq. (2.18). Some choices are compatible with the introduction of an electromagnetic field or an untilted stiff perfect fluid as a source.

Using the formulas of Appendix A, one finds the following curvature expressions:

$$\begin{aligned} {}^4\mathbf{R} &= \mathrm{diag}({}^4R^1{}_1, {}^4R^1{}_1, {}^4R^3{}_3), \qquad {}^4G^{\perp}{}_a = 0, \\ 2N^2\, {}^4R^1{}_1 &= (\ln A)^{\cdot\cdot} - (\ln A)^{\cdot}\,(\ln NA^{-1}B^{-1/2})^{\cdot} + N^2A^{-2}(2n^{(3)}A - B), \\ 2N^2\, {}^4R^3{}_3 &= (\ln B)^{\cdot\cdot} - (\ln B)^{\cdot}(\ln NA^{-1}B^{-1/2})^{\cdot} + N^2A^{-2}B, \\ 2N^2\, {}^4G^{\perp}{}_{\perp} &= -(\ln A)^{\cdot}\,(\ln B)^{\cdot} - \tfrac{1}{2}(\ln A)^{\cdot 2} + N^2A^{-2}(-2n^{(3)}A + \tfrac{1}{2}B). \end{aligned} \tag{5.1}$$

The following combinations of these expressions for special values of (m, n) are useful:

$$m = 2: \quad 2N^2R^3{}_3 = (\ln B)^{\cdot} + \tfrac{1}{2}(1 - n)(\ln B)^{\cdot} + B^{1+n}, \tag{5.2a}$$

$$n = -1: \quad 2N^2({}^4R^1{}_1 + {}^4G^{\perp}{}_{\perp}) = (\ln A)^{\cdot\cdot} + \tfrac{1}{2}(1 - m)(\ln A)^{\cdot 2} - \tfrac{1}{2}A^{m-2}, \tag{5.2b}$$

$$m = n + 1: \quad 2N^2({}^4R^1{}_1 + {}^4R^3{}_3) = (\ln AB)^{\cdot\cdot} + \tfrac{1}{2}(1 - n)(\ln AB)^{\cdot 2} + 2n^{(3)}(AB)^n, \tag{5.2c}$$

$$n = 1: \quad 4N^2\,{}^4G^{\perp}{}_{\perp} = -(\ln AB)^{\cdot 2} + (\ln B)^{\cdot 2} + A^{m-2}(B^2 - 4n^{(3)}AB). \tag{5.2d}$$

For the choice $(m, n) = (2, 1)$ made by Taub [1], the vacuum equation ${}^4R^3{}_3 = 0$ has the first integral $(\ln B)^{\cdot 2} + B^2 = \gamma^2$. This can be directly integrated yielding the solution $B = \gamma \operatorname{sech} \gamma t$, neglecting the integration constant corresponding to the origin of time. Then (5.2d) has the form

$$4g\,{}^4G^{\perp}{}_{\perp} = -(\ln AB)^{\cdot 2} + \gamma^2 - 4n^{(3)}AB. \tag{5.3}$$

By letting $z = (4AB)^{-1}$ and solving the equation

$$4g\,{}^4G^{\perp}{}_{\perp} = -z^{-2}(\dot{z} - \gamma^2 z^2 + n^{(3)}) = -4kl^2,$$

one obtains the result

$$z = \tfrac{1}{2}\delta^{-1}(e^{\delta(t-t_0)} + n^{(3)}e^{-\delta(t-t_0)}), \qquad \delta = (\gamma^2 + 4kl^2)^{1/2}. \tag{5.4}$$

Thus, solutions of the vacuum equations in which $g\,{}^4G^{\perp}{}_{\perp}$ is taken to be a negative constant $-kl^2$ are interpretable as having an untilted stiff perfect fluid source characterized by the value of l. This solution was given by Barrow [6] for the type IX case.

For the choice $(m, n) = (2, -1)$ made by Brill [17], the relevant vacuum equations and their solutions are

$$\begin{aligned} 2N^2\,{}^4R^3{}_3 &= B^{-1}(\ddot{B} + B) = 0, \\ 2N^2({}^4R^1{}_1 + {}^4G^{\perp}{}_{\perp}) &= -2w^{-1}(\ddot{w} + \tfrac{1}{4}w) = 0, \qquad w = A^{-1/2}, \\ B &= B_0 \cos t, \qquad w = A_0^{-1/2} \cos \tfrac{1}{2}(t - t_0), \\ A\,{}^4G^{\perp}{}_{\perp} &= \dot{B}\dot{w}w - B\dot{w}^2 + \tfrac{1}{4}Bw^2 - n^{(3)} = \tfrac{1}{4}(B_0/A_0)\cos t_0 - n^{(3)} = 0. \end{aligned} \tag{5.5}$$

For the choice $(m, n) = (0, -1)$ made by Misner [16], the relevant vacuum equations and their solutions are

$$\begin{aligned} 2N^2({}^4R^1{}_1 + {}^4G^{\perp}{}_{\perp}) &= \tfrac{1}{2}A^{-2}(2A\ddot{A} - \dot{A}^2 - 1) = 0, \\ 2N^2({}^4R^1{}_1 + {}^4R^3{}_3) &= (AB)^{-1}((AB)^{\cdot\cdot} + 2n^{(3)}) = 0, \\ A = (2l_N)^{-2}t^2 + l_N{}^2, &\qquad AB = -n^{(3)}(t^2 - 2ml_Nt + 4l_N{}^2\gamma), \\ A^2\,{}^4G^{\perp}{}_{\perp} &= -2(\gamma + l_N{}^2) = 0. \end{aligned} \tag{5.6}$$

For solutions of the Misner dynamical equations, one can show that ${}^4\mathbf{G} = G^{\perp}{}_{\perp}\,\mathrm{diag}(-1, -1, 1)$. Using this result and Eq. (A8) for the divergence-free Einstein tensor, a computation similar to (2.18) shows that $(A^2\,{}^4G^{\perp}{}_{\perp})^{\cdot} = 0$. For solutions of the Brill dynamical equations, ${}^4\mathbf{G} = G^{\perp}{}_{\perp}\,\mathrm{diag}(0, 0, 1)$ and $(A\,{}^4G^{\perp}{}_{\perp})^{\cdot} = 0$. This explains which multiples of the super-Hamiltonian were chosen to be evaluated above.

When adding an untilted stiff perfect fluid, the Misner and Brill equations for AB and B, respectively, remain unchanged but the equations for A pick up a term involving AB and B, respectively, therefore changing the character of the solutions. On the other hand, these equations easily accommodate the introduction of an L.R.S. electromagnetic field:

$$\begin{gathered}
\mathscr{E}^a = \mathscr{E}^3\delta^a{}_3\,, \qquad \mathscr{B}^a = \mathscr{B}^3\delta^a{}_3\,, \\
U = -\tfrac{1}{2}C^c{}_{ad}C^d{}_{bc}(\mathscr{E}^a\mathscr{E}^b + \mathscr{B}^a\mathscr{B}^b) = (\mathscr{E}^3)^2 + (\mathscr{B}^3)^2 = 8\pi k^{-1}e^2, \\
-T^{\perp}{}_{\perp} = -T^3{}_3 = T^1{}_1 = T^2{}_2 = k^{-1}e^2A^2.
\end{gathered} \tag{5.7}$$

The constant of the motion U for any SH electromagnetic field here allows one to evaluate the energy–momentum tensor without explicit expressions for the electric and magnetic field densities.

The Misner dynamical equations are unchanged by the addition of an electromagnetic source which therefore can only introduce the electromagnetic parameter e into the super-Hamiltonian constraint on the constants of integration:

$$A^2({}^4G^{\perp}{}_{\perp} - kT^{\perp}{}_{\perp}) = -2(\gamma + l_N{}^2 - n^{(3)}e^2) = 0. \tag{5.8}$$

The Brill equation for B does change but only in a trivial way, again corresponding to the introduction of the electromagnetic parameter into the super-Hamiltonian constraint:

$$\begin{aligned}
2N^2({}^4R^3{}_3 - kT^3{}_3) &= B^{-1}(\ddot{B} + B + 2e^2) = 0, \\
B &= B_0 \cos t - 2e^2, \\
A({}^4G^{\perp}{}_{\perp} - kT^{\perp}{}_{\perp}) &= \dot{B}\dot{w}w - B\dot{w}^2 + \tfrac{1}{4}Bw^2 - n^{(3)} + e^2w^2 \\
&= \tfrac{1}{4}(B_0/A_0)\cos t_0 - n^{(3)} + \tfrac{1}{2}e^2/A_0 = 0.
\end{aligned} \tag{5.9}$$

The Brill equation for w is unchanged.

The Taub equations allow the introduction of both an untilted stiff perfect fluid and an electromagnetic field. Although both equations change with the introduction of an electromagnetic field, they remain integrable:

$$\begin{aligned}
&2N^2({}^4R^3{}_3 - kT^3{}_3) = (\ln B)^{\cdot\cdot} + B^2 + 2e^2B = 0, \\
&(\ln B)^{\cdot 2} + B^2 + 8e^2B = \gamma^2, \\
&4N^2({}^4G^{\perp}{}_{\perp} - kT^{\perp}{}_{\perp}) = -z^2(\dot{z}^2 - \gamma^2z^2 + n^{(3)}) = 0.
\end{aligned} \tag{5.10}$$

The equation for z remains the same in terms of the first integral of the new equation for B, while the equation for B itself can only be integrated implicitly:

$$t - t_0 = \int dBB^{-1}(\gamma^2 - 4e^2B - B^2)^{-1/2}$$
$$= -\gamma^{-1}\ln(B^{-1}((\gamma^2 - 4e^2B - B^2)^{1/2} + \gamma) - e^2/2\gamma). \qquad (5.11)$$

A perfect fluid can be incorporated into this solution merely by replacing γ by $\delta = (\gamma^2 + 4kl^2)^{1/2}$ in the equation for B.

6. Conclusion

By exploiting the relationship of the constraint equations of general relativity to a given choice of evolution equations and by choosing geometrical variables adapted to the symmetry of both these sets of equations, this paper has been able to relate several classes of cosmological models in a simple way and to systematize a large number of existing exact solutions as well as produce some new ones. The mathematical machinery introduced here to accomplish this end is much more efficient than brute force calculations centered on a single class of cosmological models of the same symmetry type. Moreover, the adaption of the geometrical variables to the symmetry of the differential equations not only leads to the most natural choice of variables for the problem but also automatically relates different symmetry types to each other rather than treating them as isolated cases. This is important since the way in which solutions differ or are similar is perhaps more relevant than any particular exact solution in the sense that one is usually interested in general properties of classes of solutions.

Analogous techniques are also useful in studying the general case where exact solutions are no longer possible. There it is especially important to understand as much as possible about the structure of the equations before attempting numerical or qualitative studies. Particular exact solutions then prove useful as checks on these methods.

Appendix A: SSS Space–Times and the Stretching of SH Metrics

The following formulas for the conformal scaling of a metric with components $g_{\alpha\beta}$ by a conformal factor $\sigma = e^{2\psi}$ in n dimensions may be found in Schouten's book [10]. Although written for coordinate components, they remain valid for frame components. The semicolon and ∇ refer to the covariant derivative associated with the unscaled metric which is used to manipulate indices on unprimed quantities.

$$
\begin{aligned}
g'_{\alpha\beta} &= e^{2\psi} g_{\alpha\beta} \,, \\
S_\gamma = (\ln \sigma)_{;\gamma} = 2\psi_{;\gamma} \,, \qquad S_{\alpha\beta} &= 2S_{\beta;\alpha} - S_\alpha S_\beta + \tfrac{1}{2} g_{\alpha\beta} S^\gamma{}_\gamma \,, \\
\Gamma'^\alpha{}_{\beta\gamma} &= \Gamma^\alpha{}_{\beta\gamma} + \tfrac{1}{2}(\delta^\alpha{}_\beta S_\gamma + \delta^\alpha{}_\gamma S_\beta - S^\alpha g_{\beta\gamma}), \\
R'_{\alpha\beta} &= R_{\alpha\beta} - \tfrac{1}{4}(n-2)\, S_{\alpha\beta} - \tfrac{1}{4} g_{\alpha\beta} S^\gamma{}_\gamma \,, \\
G'_{\alpha\beta} &= G_{\alpha\beta} - \tfrac{1}{4}(n-2)(S_{\alpha\beta} - g_{\alpha\beta} S^\gamma{}_\gamma), \\
e^{2\psi} R' &= R - \tfrac{1}{2}(n-1)\, S^\gamma{}_\gamma \,.
\end{aligned}
\tag{A1}
$$

The conventions of Ref. [3] are followed with the exception of the connection components:

$$
\begin{aligned}
\Gamma^\alpha{}_{\beta\gamma} &= \omega^\alpha(\nabla_{e_\beta} e_\gamma) \\
&= \tfrac{1}{2} g^{\alpha\delta}(g_{\beta\delta,\gamma} + g_{\gamma\delta,\beta} - g_{\beta\gamma,\delta}) + \tfrac{1}{2} C^\alpha{}_{\beta\gamma} + C_{(\beta}{}^\alpha{}_{\gamma)} \,.
\end{aligned}
\tag{A2}
$$

$\{e_\alpha\}$ and $\{\omega^\alpha\}$ are the frame and dual frame, respectively, and $C^\alpha{}_{\beta\gamma} = \omega^\alpha([e_\beta\,, e_\gamma])$ are the structure functions for the frame. The comma indicates ordinary differentiation of a function by one of the frame vectors.

Formulas (A1) with $S_\gamma = 2\delta^c{}_\gamma b_c$ and $n = 4$ will be used to evaluate the connection and curvature components for the SSS metric (2.10) in terms of those quantities for the SH metric (2.1) which can themselves be evaluated using standard formulas like (A2) and recalling (2.4) and the fact that SH components depend only on t. The shift vector field will be assumed to vanish throughout this appendix. One finds that the only nonzero connection components are

$$
\begin{aligned}
\Gamma^0{}_{00} &= (\ln N)^{\cdot}, & 2NK_{ab} &= \dot g_{ab} \,, \\
\Gamma^0{}_{ab} &= -N^{-1} K_{ab} \,, & \Gamma^a{}_{0b} &= \Gamma^a{}_{b0} = -NK^a{}_b \,, \\
\Gamma^a{}_{bc} &= \tfrac{1}{2} C^a{}_{bc} + K^a{}_{bc} \,.
\end{aligned}
\tag{A3}
$$

The components in the first two lines are unchanged by the conformal scaling; the components which change are

$$
\begin{aligned}
\Gamma'^0{}_{a0} &= b_a = \Gamma'^0{}_{0a} \,, \qquad \Gamma'^a{}_{00} = N^2 b^a, \\
\Gamma'^a{}_{bc} &= \Gamma^a{}_{bc} + \delta^a{}_b b_c + \delta^a{}_c b_b - b^a g_{bc} \,.
\end{aligned}
\tag{A4}
$$

The following curvature formulas are useful:

$$
\begin{aligned}
{}^4G^\perp{}_a &= \mathrm{Tr}\, \mathbf{k}_a \mathbf{K} - 2a_f K^f{}_a \,, \\
2\, {}^4G^\perp{}_\perp &= \mathrm{Tr}\, \mathbf{K}^2 - \mathrm{Tr}^2\, \mathbf{K} - {}^3R, \\
{}^4\mathbf{R} &= -N^{-1} g^{-1/2} (g^{1/2} \mathbf{K})^{\cdot} + {}^3\mathbf{R}, \\
{}^3\mathbf{R} &= 2\mathbf{m}^2 - \mathbf{m}\, \mathrm{Tr}\, \mathbf{m} - \boldsymbol{l}(\mathrm{Tr}\, \mathbf{m}^2 - \tfrac{1}{2} \mathrm{Tr}^2\, \mathbf{m}) + [\mathbf{m}, \mathbf{A}] - 2a_f a^f \boldsymbol{l}, \\
{}^3R &= -(\mathrm{Tr}\, \mathbf{m}^2 - \tfrac{1}{2} \mathrm{Tr}^2\, \mathbf{m}) - 6a_f a^f, \\
\mathbf{K} &= K^a{}_b \mathbf{e}^b{}_a \,, \qquad {}^4\mathbf{R} = {}^4R^a{}_b \mathbf{e}^b{}_a \,, \qquad {}^3\mathbf{R} = {}^3R^a{}_b \mathbf{e}^b{}_a \,.
\end{aligned}
\tag{A5}
$$

Evaluating the remaining formulas (A1), one obtains the following results:

$$\begin{aligned} e^{2\psi}\, {}^4G'^{\perp}{}_{\perp} &= {}^4G^{\perp}{}_{\perp} + b^c(b_c - 4a_c), \\ e^{2\psi}\, {}^4G'^{\perp}{}_a &= {}^4G^{\perp}{}_a + 2b_c K^c{}_a\,, \\ e^{2\psi}\, {}^4R'^a{}_b &= {}^4R^a{}_b + 2b^a b_b + 2\delta^a{}_b(a_c - b_c)\, b^c + 2b_c K^{ca}{}_b\,, \\ e^{2\psi}\, {}^4G'^a{}_b &= {}^4G^a{}_b + 2b^a b_b + 2b_c K^{ca}{}_b + \delta^a{}_b(b_c - 4a_c)\, b^c. \end{aligned} \tag{A6}$$

On the other hand, using $n = 3$ and $S_c = 2b_c$ one finds

$$e^{2\psi}\, {}^3R' = {}^3R + 2b^c(4a_c - b_c). \tag{A7}$$

The following divergence formulas are very useful. Let $T'_{\alpha\beta} = T_{\alpha\beta}$ be the components of a symmetric tensor of zero dimension and $F'^{\alpha\beta} = e^{-3\psi}F^{\alpha\beta}$ the components of an electromagnetic field tensor, with $\mathscr{E}^a = g^{1/2}F^{\perp a}$ and $\mathscr{B}^a = g^{1/2}F^{*\perp a} = \frac{1}{2}\epsilon^{abc}F_{bc}$. When used with primed quantities, the semicolon and $\perp$ symbol refer to the SSS metric.

$$\begin{aligned} e^{2\psi}T'^{\beta}{}_{a;\beta} &= -N^{-1}g^{-1/2}(g^{1/2}T^{\perp}{}_a)^{\cdot} + \operatorname{Tr} \boldsymbol{\delta}_a \mathbf{T} - b_a \operatorname{Tr} \mathbf{T} + 2b_d T^d{}_a - b_a T^{\perp}{}_{\perp}\,, \\ e^{\psi}T'^{\beta}{}_{\perp;\beta} &= N^{-1}g^{-1/2}(g^{1/2}T^{\perp}{}_{\perp})^{\cdot} + \operatorname{Tr} \mathbf{KT} + 2(a^c - b^c)\, T^{\perp}{}_c\,, \\ \boldsymbol{\delta}_a &= \mathbf{k}_a - a_c \delta^b{}_a \mathbf{e}^c{}_b\,, \qquad \operatorname{Tr} \boldsymbol{\delta}_a = 0, \\ \mathbf{T} &= T^a{}_b \mathbf{e}^b{}_a\,, \end{aligned} \tag{A8}$$

$$\begin{aligned} Ng^{1/2}e^{3\psi}F'^{a\beta}{}_{;\beta} &= -\dot{\mathscr{E}}^a + Ng^{-1/2}\mathscr{B}_d(n^{da} + (a_c - b_c)\,\epsilon^{cda}), \\ Ng^{1/2}e^{3\psi}F^{*\prime a\beta}{}_{;\beta} &= -\dot{\mathscr{B}}^a - Ng^{-1/2}\mathscr{E}_d(n^{da} + (a_c - b_c)\,\epsilon^{cda}), \\ g^{1/2}e^{2\psi}F'^{\perp\beta}{}_{,\beta} &= (b_c - 2a_c)\,\mathscr{E}^c, \\ g^{1/2}e^{2\psi}F^{*\prime\perp\beta}{}_{;\beta} &= (b_c - 2a_c)\,\mathscr{B}^c. \end{aligned} \tag{A9}$$

The latter equations, upon stretching the Lie group G to $\bar{G}$ as in Section 3, take the form

$$\begin{aligned} Ng^{1/2}e^{3\psi}\bar{F}'^{a\beta}{}_{;\beta} &= -\dot{\mathscr{E}}^a + Ng^{-1/2}\mathscr{B}_d C^{da}, \\ Ng^{1/2}e^{3\psi}\bar{F}^{*\prime a\beta}{}_{;\beta} &= -\dot{\mathscr{B}}^a - Ng^{-1/2}\mathscr{E}_d C^{da}, \\ g^{1/2}e^{2\psi}\bar{F}'^{\perp\beta}{}_{;\beta} &= -(2a_c + b_c)\,\mathscr{E}^c, \\ g^{1/2}e^{2\psi}\bar{F}^{*\prime\perp\beta}{}_{;\beta} &= -(2a_c + b_c)\,\mathscr{B}^c. \end{aligned} \tag{A10}$$

Notice that the dynamical equations do not involve b_a and hence are the same as the unstretched SH case. The constraint equations are also in general equivalent to that case, adding further support for the utility of the group stretching operation. The energy–momentum tensor of the electromagnetic field will be required in the text. It has the following components in the SH case:

$$\begin{aligned} -4\pi T^{\alpha}{}_{\beta} &= F^{\alpha}{}_{\gamma} F^{\gamma}{}_{\beta} + \tfrac{1}{4}\delta^{\alpha}{}_{\beta} F^{\gamma\delta} F_{\gamma\delta}\,, \\ 4\pi g T^{ab} &= -(\mathscr{E}^a \mathscr{E}^b + \mathscr{B}^a \mathscr{B}^b) + \tfrac{1}{2} g^{ab}(\mathscr{E}_c \mathscr{E}^c + \mathscr{B}_c \mathscr{B}^c), \\ -4\pi g^{1/2} T^{\perp}{}_{\perp} &= \tfrac{1}{2} g^{-1/2}(\mathscr{E}_c \mathscr{E}^c + \mathscr{B}_c \mathscr{B}^c), \\ 4\pi g^{1/2} T^{\perp}{}_{a} &= \epsilon_{abc} \mathscr{E}^b \mathscr{B}^c. \end{aligned} \tag{A11}$$

APPENDIX B: Stiff Perfect Fluids

For a perfect fluid with energy density ρ, pressure p, and 4-velocity vector field u, the components of the energy–momentum tensor are

$$T^{\alpha}{}_{\beta} = (\rho + p)\, u^{\alpha} u_{\beta} + p\delta^{\alpha}{}_{\beta}\,. \tag{B1}$$

Assuming an equation of state $p = p(\rho)$ exists, implying an isoentropic flow, and following Misner, Thorne, and Wheeler [8], introduce the baryon number density n and chemical potential $\mu = (\rho + p)/n$. These satisfy

$$d \ln n = (\rho + p)^{-1}\, d\rho, \qquad d \ln \mu = (\rho + p)^{-1}\, dp. \tag{B2}$$

For an equation of state $p = (\gamma - 1)\rho$, one finds

$$\rho = \rho_0 (n/n_0)^{\gamma}, \qquad \mu = \gamma(\rho_0/n_0)(n/n_0)^{\gamma-1}. \tag{B3}$$

For simplicity, set $\rho_0 = n_0 = 1$. A stiff fluid has $\gamma = 2$, in which case one has

$$\rho = p = n^2, \qquad \mu = 2n. \tag{B4}$$

Following Taub [19], introduce the circulation 1-form v by $v_{\alpha} = \mu u_{\alpha}$, the natural fluid ADM generator $\vec{t}$ by $t^{\alpha} = \mu^{-1} u^{\alpha}$, and the quantity $l = n({}^4 g)^{1/2}\, u^0 = n g^{1/2} u^{\perp}$ which is a scalar density on each hypersurface in a slicing of the spacetime. For a stiff fluid, $l = \frac{1}{2} g^{1/2} v^{\perp}$. The dimensions of l, v, and $\vec{t}$ are 2, 0, and 0, respectively.

The conservation equations $T^{\beta}{}_{\alpha;\beta} = 0$ are easily thrown into the following form:

$$(n u^{\beta})_{;\beta} = 0, \qquad v_{\alpha;\beta} t^{\beta} = -(\ln \mu)_{;\alpha}\,. \tag{B5}$$

The latter equation is equivalent to

$$\pounds_{\vec{t}}\, v = 0. \tag{B6}$$

Thus, in a slicing of the space–time generated from an initial space-like slice by dragging along by the vector field $\vec{t}$, the components of v in a comoving ADM frame $\{e_0 = \vec{t}, e_a\}$ adapted to the slicing [see Section 2] are time-independent. The same is true of l since in such a frame $u^a = 0$ and $(nu^\beta)_{;\beta} = N^{-1}g^{-1/2}\dot{l}$. Note that $v_0 = v(\hat{t}) = -1$ in such a frame, so only the restriction of the 1-form v to elements of the slicing carries any information.

For a stiff perfect fluid, $p = \rho$ and the energy–momentum tensor and its trace reversal have components

$$T^\alpha{}_\beta = (2u^\alpha u_\beta + \delta^\alpha{}_\beta)\,\rho,$$
$$T^{\mathrm{TR}\alpha}{}_\beta = 2u^\alpha u_\beta \rho. \tag{B7}$$

Assuming zero shift vector field one has

$$-gT^\perp{}_\perp = \rho g(u^\perp)^2 + \rho g(u^\perp)^2 - 1) = l^2 + \tfrac{1}{4}gg^{ab}v_a v_b\,,$$
$$g^{1/2}T^\perp{}_a = lv_a\,, \tag{B8}$$
$$T^{\mathrm{TR}a}{}_b = \tfrac{1}{2}v^a v_b\,.$$

For a SSS space–time with a SSS stiff perfect fluid source, any fluid quantity Y of dimension d that has been introduced above satisfies

$$Y' = e^{q\psi}Y, \tag{B9}$$

where Y' is the SSS quantity and Y the corresponding SH quantity. The equations of motion for l and v_a following from the SSS conservation equations assuming zero shift vector field are

$$e^{2\psi}(n'u'^\beta)_{;\beta} = \dot{l} - 2Nl(a_c - b_c)\,v^c/v^\perp = 0,$$
$$e^{2\psi}(v'_{a;\beta}v'^\beta - \tfrac{1}{2}(\mu'^2)_{;a}) = v^\perp\dot{v}_a - v_c C^c{}_{ab}v^b = 0. \tag{B10}$$

For a stiff perfect fluid, the first of these may be rewritten as

$$\dot{l} = Ng^{1/2}(a_c - b_c)\,v^c. \tag{B11}$$

Note that when $v_a = 0$, l is a constant and $-gT^\perp{}_\perp = l^2$. Even when $v_a \neq 0$, l is a constant if $a_a - b_a = 0$.

Applying the Lie group stretching operation of Section 3, one finds the following equations of motion when written in terms of the new structure constant tensor components:

$$\dot{l} = Ng^{1/2}a_c v^c,$$
$$v^\perp\dot{v}_a = v_c\bar{C}^c{}_{ab}v^b, \tag{B12}$$
$$\bar{C}^a{}_{bc} = \epsilon_{bcd}n^{ad} + (a_f + b_f)\,\delta^{fa}_{bc}\,.$$

For a symmetric case perfect fluid source, the circulation 1-form satisfies $\dot{v}_a = 0$ which requires v_a to be an eigenvector of $\bar{C}^{ab}$ [11]:

$$v_d\bar{C}^{da} = \lambda v^a. \tag{B13}$$

When $\bar{a}_c = a_c + b_c = 0$, v_a must be an eigenvector of n^{ab}, while if $\bar{a}_c \neq 0$, v_c and $\bar{a}_c$ must be linearly dependent. For the symmetric case space–times of Section 4 where $\mathbf{n} = \text{diag}(n^{(1)}, n^{(2)}, 0)$, $v_a = v_3\delta^3{}_a$ is time-independent. Since $n^{(3)} = 0$, the rotation of such a fluid vanishes [11].

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