MAY 1989 LECTURES

Corso di Fisica Teorica (relatività generale)

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APPRECIATING THE ARTICLE:

General Relativity and Kinetic Theory

by Jürgen Ehlers

Proceedings "Enrico Fermi" School XLVII, 1969, Varenna: General Relativity and Cosmology, edited by RK Sachs, Academic Press, New York, 1971, pp 1-70.

$\boxed{\text{PART 1}}$ MATHEMATICAL PRELIMINARIES (PHASE SPACE GEOMETRY)

PRELIMINARIES

We need some tensor algebra notation. Summary:

n-dimensional vector space V (elements: "vectors")

with basis $\{e_\alpha\}_{\alpha=1,\dots,n}$

dual space V^* of real-valued linear functions (elements: "covectors" "1-forms")

dual basis $\{\omega^\alpha\}$ defined by $\underbrace{\omega^\alpha(e_\beta)}_{} = \delta^\alpha{}_\beta$

evaluation of 1-form on vector

components: $X = X^\alpha e_\alpha$, $X^\alpha = \underbrace{X(\omega^\alpha)}_{} \equiv \omega^\alpha(X)$

evaluation of vector on 1-form using identification $(V^*)^* \cong V$

$$\sigma = \sigma_\alpha \omega^\alpha, \quad \sigma_\alpha = \sigma(e_\alpha)$$

$\binom{p}{q}$-tensor over V = real-valued linear function of p 1-form arguments and q vector arguments

$$T(\underbrace{\sigma_{(1)},\dots,\sigma_{(p)}}_{p \text{ 1-forms}}, \underbrace{X_{(1)},\dots,X_{(q)}}_{q \text{ vectors}}) \in \mathbb{R}$$

basis: $\{e_{\alpha_1}\otimes\cdots\otimes e_{\alpha_p}\otimes\omega^{\beta_1}\otimes\cdots\otimes\omega^{\beta_q}\}$

components: $T = T^{\alpha_1\cdots\alpha_p}{}_{\beta_1\cdots\beta_q}\, e_{\alpha_1}\otimes\cdots\otimes e_{\alpha_p}\otimes\omega^{\beta_1}\otimes\cdots\otimes\omega^{\beta_q}$

$$T^{\alpha_1\cdots\alpha_p}{}_{\beta_1\cdots\beta_q} = T(\omega^{\alpha_1},\dots,\omega^{\alpha_p}, e_{\beta_1},\dots,e_{\beta_q})$$

where the tensor product of two tensors is defined by

$$\underbrace{\underset{\binom{p_1}{q_1}}{T} \otimes \underset{\binom{p_2}{q_2}}{S}}_{\binom{p_1+p_2}{q_1+q_2}}\,(\sigma_{(1)},\dots,\sigma_{(p_1)},\sigma_{(p_1+1)},\dots,\sigma_{(p_1+p_2)}, X_{(1)},\dots,X_{(q_1)}, X_{(q_1+1)},\dots,X_{(q_1+q_2)})$$

$$= T(\sigma_{(1)},\dots,\sigma_{(p_1)}, X_{(1)},\dots,X_{(q_1)})\, S(\sigma_{(p_1+1)},\dots,\sigma_{(p_1+p_2)}, X_{(q_1+1)},\dots,X_{(q_1+q_2)})$$

just multiplies the values of the factor tensors

SYMMETRY AND ANTISYMMETRY

Define $\delta^{\alpha_1\cdots\alpha_p}_{\beta_1\cdots\beta_p} = \begin{cases} \operatorname{sgn}\sigma, & \text{if } (\beta_1,\dots,\beta_p) = (\sigma(\alpha_1),\dots,\sigma(\alpha_p)) \text{ is a permutation } \sigma \text{ of the } p\text{-tuplet } (\alpha_i) \\ 0 & \text{otherwise} \end{cases}$

and $|\delta|^{\alpha_1\cdots\alpha_p}_{\beta_1\cdots\beta_p} = |\delta^{\alpha_1\cdots\alpha_p}_{\beta_1\cdots\beta_p}|$.

Then $\frac{1}{p!}\delta^{\alpha_1\cdots\alpha_p}_{\beta_1\cdots\beta_p}$ and $\frac{1}{p!}|\delta|^{\alpha_1\cdots\alpha_p}_{\beta_1\cdots\beta_p}$ project out the totally antisymmetric and totally symmetric parts of a $\binom{0}{p}$-tensor or a $\binom{p}{0}$-tensor; for example, if T is a $\binom{0}{p}$-tensor:

$$\mathrm{ALT}(T)_{\alpha_1\cdots\alpha_p} = \frac{1}{p!}\delta^{\beta_1\cdots\beta_p}_{\alpha_1\cdots\alpha_p} T_{\beta_1\cdots\beta_p} \equiv T_{[\alpha_1\cdots\alpha_p]}.$$

$$\mathrm{SYM}(T)_{\alpha_1\cdots\alpha_p} = \frac{1}{p!}|\delta|^{\beta_1\cdots\beta_p}_{\alpha_1\cdots\alpha_p} T_{\beta_1\cdots\beta_p} \equiv T_{(\alpha_1\cdots\alpha_p)}.$$

(NOTE $\mathrm{ALT}(\mathrm{ALT}(T)) = \mathrm{ALT}(T)$, $\mathrm{SYM}(\mathrm{SYM}(T)) = \mathrm{SYM}\,T$)

Antisymmetric tensors prove to be more useful than symmetric tensors for certain purposes. An antisymmetric tensor

$$\mathrm{ALT}(T) = T \leftrightarrow T_{\alpha_1\cdots\alpha_p} = \frac{1}{p!}\delta^{\beta_1\cdots\beta_p}_{\alpha_1\cdots\alpha_p} T_{\beta_1\cdots\beta_p} = T_{[\alpha_1\cdots\alpha_p]}$$

is called a p-form:

$$T = T_{\alpha_1\cdots\alpha_p}\,\omega^{\alpha_1}\otimes\cdots\otimes\omega^{\alpha_p} = \frac{1}{p!}\delta^{\beta_1\cdots\beta_p}_{\alpha_1\cdots\alpha_p} T_{\beta_1\cdots\beta_p}\,\omega^{\alpha_1}\otimes\cdots\otimes\omega^{\alpha_p}$$

$$\equiv \frac{1}{p!} T_{\beta_1\cdots\beta_p}\,\omega^{\beta_1}\wedge\cdots\wedge\omega^{\beta_p}$$

where the wedge product ~~of~~ p 1-forms is just the antisymmetrized tensor product (times $p!$)

Note $\omega^{\beta_1\cdots\beta_p} = \underbrace{\delta^{\beta_1\cdots\beta_p}_{\alpha_1\cdots\alpha_p}}_{\frac{1}{p!}\delta^{\gamma_1\cdots\gamma_p}_{\alpha_1\cdots\alpha_p}\delta^{\beta_1\cdots\beta_p}_{\gamma_1\cdots\gamma_p}} \omega^{\alpha_1}\otimes\cdots\otimes\omega^{\alpha_p} = \frac{1}{p!}\delta^{\beta_1\cdots\beta_p}_{\gamma_1\cdots\gamma_p}\,\omega^{\gamma_1}\wedge\cdots\wedge\omega^{\gamma_p}$

is totally antisymmetric in its indices, so only $\underline{1}$ of its $p!$ permutations is linearly independent for a given set of unordered indices. Let $|\beta_1\cdots\beta_p|$ be the ordered index sets:

$$\beta_1 < \cdots < \beta_p.$$

Then $\{\omega^{|\alpha_1} \wedge \ldots \wedge \omega^{\alpha_p|}\}$ is a basis of the space of antisymmetric ~~p of~~ $\binom{0}{p}$-tensors (p-forms) and

$$T = \frac{1}{p!} T_{\alpha_1 \ldots \alpha_p} \omega^{\alpha_1} \wedge \ldots \wedge \omega^{\alpha_p} = T_{\alpha_1 \ldots \alpha_p} \omega^{|\alpha_1} \wedge \ldots \wedge \omega^{\alpha_p|}$$

(sum over ordered sets of indices only)

All these definitions hold also for antisymmetric $\binom{p}{0}$-tensors, called p-vectors.

FACT. Suppose $\{X_{(1)}, \ldots, X_{(p)}\}$ is a linearly independent set, there determining a p-dimensional linearly subspace of V, for which it is a basis. Then any other basis of the subspace wedged together, yields a multiple of $X_{(1)} \wedge \ldots \wedge X_{(p)}$. If $\{X_{(1)}, \ldots, X_{(p)}\}$ is an orthonormal basis of the subspace with respect to an inner product $g = g_{\alpha\beta}\, \omega^\alpha \otimes \omega^\beta$ on V, then the absolute value of the multiple is the volume of the p-parallelopiped formed from the p other basis vectors, and the sign is the relative orientation of the two bases.

MANIFOLDS

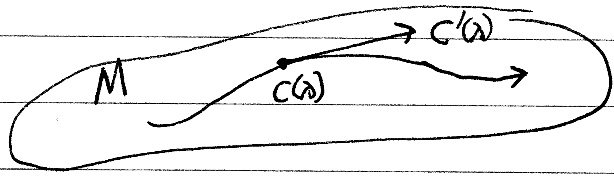
Given an n-dimensional manifold M with local coordinates $\{x^\alpha\}_{\alpha=1,\dots,n}$, partial derivatives $\partial_\alpha = \frac{\partial}{\partial x^\alpha}$, coordinate differentials dx^α.

Given a parametrized curve c through a point $p \in M$:

$$c(0)=p\,,\quad c^\alpha(\lambda) = x^\alpha(c(p))\,,\quad c'^\alpha(\lambda) = \frac{d}{d\lambda}\, c^\alpha(\lambda)\,,$$

the chain rule enables us to identify the tangent vector to the curve with a 1st order partial differential operator on functions f over M:

$$\frac{d}{d\lambda} f(c(\lambda)) = \frac{\partial f}{\partial x^\alpha}\Big|_{c(\lambda)} c'^\alpha(\lambda) = \underbrace{c'^\alpha(\lambda)\frac{\partial}{\partial x^\alpha}\Big|_{c(\lambda)}}\; f$$



$\equiv c'(\lambda)$, the tangent vector to c at $c(\lambda)$

The tangent space TM_p to M at p is the n-dimensional vector space of all possible tangent vectors at p to curves through p:

$$TM_p = \left\{ X^\alpha \frac{\partial}{\partial x^\alpha}\Big|_p \;\middle|\; (X^\alpha) \in \mathbb{R}^n \right\},$$

for which $\{\partial_\alpha|_p\}$ provides a basis.

Holding f fixed in $\quad Xf = X^\alpha \partial_\alpha|_p f \quad = df|_p(X)$
yields a real-valued linear function on TM_p
which we call the differential of f at p.

Thus the dual space $(TM_p)^* \equiv T^*M_p$ is the space of differentials at p of all possible (differentiable) functions on M

$$T^*M_p = \{ \sigma_\alpha\, dx^\alpha|_p \mid (\sigma_\alpha) \in \mathbb{R}^n \}$$

for which $\{dx^\alpha|_p\}$ provides a basis.

One can take an arbitrary basis $\quad e_\alpha = e^\beta{}_\alpha\, \partial_\beta$
and dual basis $\quad \omega^\alpha = \omega^\alpha{}_\beta\, dx^\beta\,, \quad (\omega^\alpha{}_\beta) = (e^\beta{}_\alpha)^{-1}$
of the tangents spaces on M, called a frame & dual frame.

Now we can carry over the discussion of a general n-dimensional vector space V to each tangent space on M, yielding the algebra of tensor fields on M. The algebra of antisymmetric tensor fields is called the exterior algebra of M.

A p-form can be integrated on a p-dimensional subspace of M (submanifold with possible edges and corners).
In sloppy notation, supposed $x^\alpha = C^\alpha(u^1, \ldots, u^p)$ is a parametrized surface for certain ranges of the parameters u^i.
Then $\quad dx^\alpha = \frac{\partial C^\alpha}{\partial u^\beta} du^\beta \quad$ is a 1-form on $\mathbb{R}^p$

and

$$\int_C \sigma = \int_C \sigma_{|\alpha_1\ldots\alpha_p|}\, dx^{\alpha_1}\wedge\ldots\wedge dx^{\alpha_p} \qquad \text{(now substitute)}$$

$$= \int \sigma_{|\alpha_1\ldots\alpha_p|}(C(u))\, \frac{\partial C^{[\alpha_1}}{\partial u^{i_1}}\cdots\frac{\partial C^{\alpha_p]}}{\partial u^{i_p}}\, du^{i_1}\wedge\ldots\wedge du^{i_p}$$

$$= \int \sigma_{|\alpha_1\ldots\alpha_p|}(C(u))\, \frac{\partial C^{[\alpha_1}}{\partial u^{1}}\cdots\frac{\partial C^{\alpha_p]}}{\partial u^{n}}\, du^{1}\wedge\ldots\wedge du^{p}$$

$$\equiv \int\cdots\int \sigma_{[\alpha_1\ldots\alpha_p]}(C(u))\, \frac{\partial C^{[\alpha_1}}{\partial u^{1}}\cdots\frac{\partial C^{\alpha_p]}}{\partial u^{n}}\, du^{1}\ldots du^{p}$$

(iterated integral over parameter ranges in usual sense of calculus.)

For a metric $g = g_{\alpha\beta}\,\omega^\alpha\otimes\omega^\beta$ on M, one has the volume element n-form

$$\eta = \frac{1}{n!}\eta_{\alpha_1\ldots\alpha_n}\,\omega^{\alpha_1}\wedge\ldots\wedge\omega^{\alpha_n} = \eta_{1\ldots n}\,\omega^1\wedge\ldots\wedge\omega^n$$

$$= g^{1/2}\,\omega^1\wedge\ldots\wedge\omega^n$$

$$g^{1/2} \longleftarrow |\det(g_{\alpha\beta})|^{1/2}$$

which determines volumes of regions of M.

Note on ORIENTATIONS

The space of nonzero n-forms ~~on a manifold~~ at a point on a manifold has two disjoint 1-dimensional components. If one can pick one of these at each point in a continuous manner, M is called orientable and this choice provides M with an orientation.
On E^3, $dx^1 \wedge dx^2 \wedge dx^3$ for any righthanded cartesian coordinate system provides ordinary Euclidean space with its standard orientation.

A parametrized surface has an orientation corresponding to the orientation of $du^1 \wedge \ldots \wedge du^p$ in the parameter space. Equivalently, the p-vector field $\left(\frac{\partial c^\alpha}{\partial u^1}\frac{\partial}{\partial x^\alpha}\right) \wedge \ldots \wedge \left(\frac{\partial c^\alpha}{\partial u^p}\frac{\partial}{\partial x^\alpha}\right)$ on the surface determines this orientation.

For a (nonnull) hypersurface, a choice of normal induces an orientation on the hypersurface used in Stoke's theorem, in such a way that if it is parametrized then

$$\left(\frac{\partial c^\alpha}{\partial u^1}\frac{\partial}{\partial x^\alpha}\right) \wedge \ldots \wedge \left(\frac{\partial c^\alpha}{\partial u^{n-1}}\frac{\partial}{\partial x^\alpha}\right) \wedge n = \begin{pmatrix}\text{positive}\\ \text{function}\end{pmatrix} \frac{\partial}{\partial x^1} \wedge \ldots \wedge \frac{\partial}{\partial x^n}$$

must hold if the orientation is the same as the one corresponding to $du^1 \wedge \ldots \wedge du^{n-1}$, provided $dx^1 \wedge \ldots \wedge dx^n$ has the orientation of M.

For an unparametrized p-surface, one must assign an orientation before one can integrate p-forms.

Read about this elsewhere.

INDUCED VOLUME ELEMENTS

For a hypersurface Σ with a normalizable normal vector field, one has a unit normal vector field defined on Σ

$$n = n^\alpha e_\alpha, \qquad |n\cdot n| = |n^\alpha n_\alpha| = 1.$$

The volume element induced on the hypersurface is obtained by contracting the volume element on M with the unit normal,

$$\eta_\Sigma = \underset{\uparrow}{n} \lrcorner\, \eta = \frac{1}{(n-1)!} n^{\alpha_1}\eta_{\alpha_1\cdots\alpha_n}\, \omega^{\alpha_1}\wedge\cdots\wedge\omega^{\alpha_n}$$

evaluate first vector argument of η on n

CALCULATION:

$$n \lrcorner\, \eta = (n^\alpha e_\alpha) \lrcorner \left(\frac{1}{n!}\eta_{\alpha_1\cdots\alpha_n}\delta^{\alpha_1\cdots\alpha_n}{}_{\beta_1\cdots\beta_n}\,\omega^{\beta_1}\otimes\cdots\otimes\omega^{\beta_n}\right)$$

$$= \frac{1}{n!}\, n^\alpha\, \underbrace{\eta_{\alpha_1\cdots\alpha_n}\underbrace{\delta^{\alpha_1\cdots\alpha_n}{}_{\beta_1\cdots\beta_n}\,\delta^{\beta_1}{}_\alpha}_{\delta^{\alpha_1\cdots\alpha_n}{}_{\alpha\cdots\beta_n}}}_{} \,\omega^{\beta_2}\otimes\cdots\otimes\omega^{\beta_n}$$

$$e_\alpha \lrcorner\, \omega^{\beta_1} = \omega^{\beta_1}(e_\alpha) = \delta^{\beta_1}{}_\alpha$$

$$n^\alpha \eta_{\alpha\cdots\beta_n} = \frac{1}{(n-1)!}\,\delta^{\alpha_2\cdots\alpha_n}{}_{\beta_2\cdots\beta_n}\, n^\alpha\eta_{\alpha\alpha_2\cdots\alpha_n}$$

$$= \frac{1}{(n-1)!}\, n^\alpha \eta_{\alpha\alpha_2\cdots\alpha_n}\,\omega^{\alpha_2}\wedge\cdots\wedge\omega^{\alpha_n}$$

A null hypersurface has a nonnormalizable normal $\quad n^\alpha n_\alpha = 0$

and the corresponding $(n-1)$-form $\quad \eta = n \lrcorner\, \eta \quad$ (defined up to a scalar)

is degenerate in the sense that its restriction to the hypersurface (which contains the normal as a tangent vector) vanishes.

Neither the induced volume element nor the rescaled differential solid angle extend to null pseudospheres but a different rescaling does. The problem is that the single null direction in the null pseudosphere has zero length, so a rescaling by the ~~"radius"~~ inverse "radius" σ of the induced volume element as that radius $\sigma \to 0$ leads to a nondegenerate volume element.

To see this let $\{y^\alpha\}$ be arbitrary cartesian coordinates, with $g = g_{\alpha\beta}\, dy^\alpha \otimes dy^\beta$.

For a given nonnull pseudosphere $g_{\alpha\beta} y^\alpha y^\beta = \epsilon\sigma^2$, $\sigma > 0$, $|\epsilon| = 1$ one can solve this quadratic equation in any one of the coordinates for the two roots, as functions of the remaining coordinates. One can then represent ~~a~~ piece of a pseudosphere as a graph. Suppose we solve for the last coordinate.

Let $i = 1, \ldots, n-1$:

$$y^n = y^n(y^1, \ldots, y^{n-1}), \quad y^n > 0$$

$$g_{\alpha\beta} y^\alpha y^\beta = \epsilon\sigma^2 \xrightarrow{d} g_{\alpha\beta} y^\alpha dy^\beta = 0 \quad \text{(on surface)}$$

$$g_{\alpha\beta} y^\alpha dy^\beta = \underbrace{y_\beta\, dy^\beta}_{\text{covariant normal}}$$

$$dy^n = -\frac{y_i\, dy^i}{y_n}$$

$\to$ contravariant normal: $y^\alpha \dfrac{\partial}{\partial y^\alpha}$

normalized: $n = \sigma^{-1} y^\alpha \dfrac{\partial}{\partial y^\alpha}$

Induced volume element:

$$\boxed{\text{Define } \delta^{1\cdots n}_{\alpha_1\cdots\alpha_n} \equiv \epsilon_{\alpha_1\cdots\alpha_n}}$$

$$n \lrcorner \eta = \frac{1}{(n-1)!} g^{1/2} \left(\frac{y^\alpha}{\sigma}\right) \epsilon_{\alpha\alpha_2\cdots\alpha_n}\, dy^{\alpha_2} \wedge \cdots \wedge dy^{\alpha_n}$$

omit this index; omit this from product

$$= \frac{g^{1/2}}{\sigma} \Big[y^n \epsilon_{n1\cdots n-1}\, dy^1 \wedge \cdots \wedge dy^{n-1} + y^i \epsilon_{i1\cdots\hat{i}\cdots n}\, dy^1 \wedge \cdots \wedge \widehat{dy^i} \wedge \cdots \wedge \underbrace{dy^n}_{-\frac{y_i dy^i}{y_n}} \Big]$$

$$= \frac{g^{1/2}}{\sigma} \underbrace{\Big[y^n + \frac{y^i y_i}{y_n} \Big]}_{\frac{\epsilon\sigma^2}{y_n}} \epsilon_{n1\cdots n-1}\, dy^1 \wedge \cdots \wedge dy^{n-1}$$

$$= \frac{\sigma g^{1/2}}{\epsilon y_n}\, dy^1 \wedge \cdots \wedge dy^{n-1} \underbrace{\epsilon_{n1\cdots n-1}}_{(-1)^{n-1}}$$

$\leftarrow$ orientation factor check books

Now I have to admit that I am working from memory and have forgotten about the convention for inducing an orientation on a hypersurface. It probably should be that

$$(-1)^{n-1}\, n \lrcorner \eta = \eta \llcorner n$$

↑ evaluate η on last vector argument

is the induced volume element, in which case

$$\eta \llcorner n = \sigma \left(\frac{g^{1/2}}{\epsilon y_n}\, dy^1 \wedge \ldots \wedge dy^{n-1} \right)$$

↑ explicit factor of radius.

$$\sigma^{-1}(\eta \llcorner n) = \frac{g^{1/2}}{\epsilon y_n}\, dy^1 \wedge \ldots \wedge dy^{n-1}$$

~~Suppose we consider the case~~ We can now take the limit as $\sigma \to 0$ and obtain a volume element on the limiting null pseudosphere in the pseudo-Euclidean case. Call it Π_0, following EHLERS.

For the 4-dimensional Lorentz case, another convention $1,2,3,4 \to 0,1,2,3$ screws up the sign of the orientation anyway.

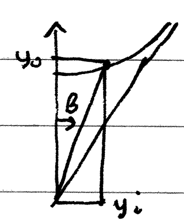
Solving for y_0 leads to (assume inertial coordinates)

$$\Pi_\sigma = \frac{g^{1/2}}{|y_0|}\, dy^1 \wedge dy^2 \wedge dy^3 \qquad (\cancel{\text{since } \epsilon = -1}$$

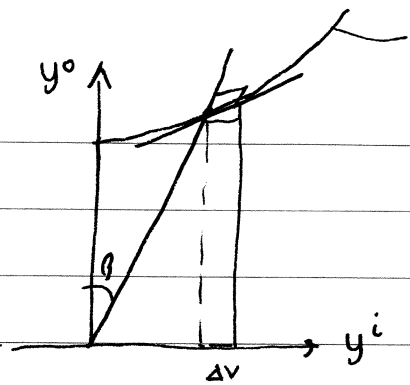
on the future timelike hyperboloid ($\epsilon = -1$, $y^0 > 0$) of radius σ, with limit Π_0 on the future ~~timelike~~ null cone. ~~[Note $g^{1/2} = 1$]~~.

[Interesting Question: If one takes such a limit in different systems of inertial coordinates, does one obtain a unique Π_0 or a 1-parameter family?]

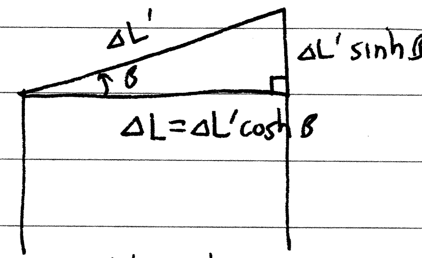
For inertial coordinates, $g^{1/2} = 1$ and $|y_0| = y^0 = \sqrt{\sigma^2 + y^i y_i} = \sigma\gamma = \sigma \cosh \beta$



and this factor is just the Lorentz contraction factor needed to convert the spatial volume, times σ in the nonnull case (the extra factor σ^{-1} introduced above)



$y_\alpha y^\alpha = \sigma^2$ forward timelike pseudosphere



Lorentzian Geometry

$$\Delta L' = \frac{\Delta L}{\cosh\beta} \rightarrow \Delta V' = \frac{\Delta V}{\cosh\beta}$$

(length contraction relative to radial observer)

For a point on the future timelike pseudosphere

$$(y^\alpha) = (y^0, y^i) = \sigma(\gamma, \gamma v^i) = (E, p^i)$$

we can interpret σ as the mass m of ~~an inert~~ a particle and the vector (y^α) as its 4-momentum, with E then interpretable as the energy and p^i as the spatial momentum, and v^i as the spatial velocity.

We then have a volume element on the mass shell for each mass $m \geq 0$ if we interpret this copy of Minkowski space as ~~the~~ a tangent space to Minkowski space.

TANGENT / COTANGENT BUNDLES, VELOCITY / MOMENTUM PHASE SPACE

Given a manifold M, introduce the space of all tangent vectors to all points of M

$$TM = \{ X_p \mid X_p \in TM_p, \ p \in M \} \quad = \text{ tangent bundle}$$

and the same for tangent 1-forms

$$T^*M = \{ \sigma_p \mid \sigma_p \in T^*M_p, \ p \in M \} \quad - \text{ cotangent bundle}$$

Given coordinates $\{x^\alpha\}$ on M, one can introduce induced coordinates on the two bundles:

on TM: $\{q^\alpha, \dot{q}^\alpha\}$

$$q^\alpha(X_p) = x^\alpha(p) \qquad \text{coordinates of } p$$

$$\dot{q}^\alpha(X_p) = dx^\alpha|_p (X_p) = X^\alpha|_p \qquad \text{coord components of } X_p$$

on T^*M: $\{q^\alpha, p_\alpha\}$

$$q^\alpha(\sigma_p) = x^\alpha(p) \qquad \text{coordinates of } p$$

$$p_\alpha(\sigma_p) = \sigma_p(\partial_\alpha|_p) = \sigma_\alpha|_p \qquad \text{coord components of } \sigma_p$$

[Often one is sloppy and uses the same kernel symbol x in place of q]

When M is the configuration space of a classical mechanical system TM and T^*M are called the velocity and momentum phase spaces respectively.

EXAMPLE. M has a metric $g = g_{\alpha\beta} dx^\alpha \otimes dx^\beta$. Then

$$T = \tfrac{1}{2} g_{\alpha\beta} \dot{q}^\alpha \dot{q}^\beta \qquad (\text{interpret naturally } g_{\alpha\beta} \text{ as functions on } TM)$$

is the kinetic energy function and the usual Lagrange equations with $L=T$ give the equations for affinely parametrized geodesics on M

$$\frac{d}{dt}(g_{\alpha\beta}\dot{q}^\beta) - \frac{\partial}{\partial q^\gamma}(g_{\alpha\beta})\,\dot{q}^\alpha\dot{q}^\beta = 0$$

$$\ddot{q}^\alpha + \Gamma^\alpha{}_{\beta\gamma}\,\dot{q}^\beta\dot{q}^\gamma = 0$$

[Exercise: derive these equations, identifying $\frac{d}{dt}$ with "$\cdot$"; t is an affine parameter]

The Legendre transformation

$$p_\alpha = \frac{\partial L}{\partial \dot{q}^\alpha} = g_{\alpha\beta}\,\dot{q}^\beta$$

corresponds to lowering indices with the metric, leading to the Hamiltonian kinetic energy function on momentum phase space:

$$T = \tfrac{1}{2} g^{\alpha\beta} p_\alpha p_\beta = H$$

The Hamiltonian equations follow from this Hamiltonian.

The energy is conserved $\quad \frac{dH}{dt} = \frac{\partial H}{\partial t} = 0 \rightarrow H = E$

and when nonzero, one can normalize the affine paramèter to correspond to arc length.

USEFUL FACT: If ξ is a Killing vector field of the metric $\xi_{(\alpha;\beta)}=0$, then the function

$P_\xi = \xi^\alpha p_\alpha$ on T^*M corresponding to

$\tilde{P}_\xi = \xi_\alpha \dot{q}^\alpha$ on TM, is a constant of the motion, representing the inner product of the killing vector with the velocity

EXERCISE: Show $\{P_\xi, H\} = 0$ (Poisson bracket).

The equations of motion on velocity phase space are first order equations in the dependent variables $q^\alpha, \dot{q}^\alpha$ and the ~~dedependent~~ independent variable t:

$$\frac{d}{dt} q^\alpha = \dot{q}^\alpha$$

total covariant derivative

$$\frac{d}{dt} \dot{q}^\alpha = -\Gamma^\alpha{}_{\beta\gamma} \dot{q}^\beta \dot{q}^\gamma \qquad \left(\leftrightarrow \frac{D\dot{q}^\alpha}{dt} = 0 \right)$$

and represent the integral curves of the vector field

$\mathbb{L}$ for Louiville

$$\mathbb{L} = \dot{q}^\alpha \frac{\partial}{\partial q^\alpha} - \Gamma^\alpha{}_{\beta\gamma} \dot{q}^\beta \dot{q}^\gamma \frac{\partial}{\partial \dot{q}^\alpha}$$

on the tangent bundle (velocity phase space)

[Recall on M a vector field $X = X^\alpha \partial_\alpha$ has integral curves $C(t)$ such that $\frac{dc^\alpha(t)}{dt} = X^\alpha(C(t))$, where $c^\alpha(t) = x^\alpha(C(t))$, hence the above expression for the components of $\mathbb{L}$.]

In the case of a 4-dimensional Lorentzian metric, we must use another symbol like λ for the affine parameter to avoid confusion with a time variable, and we can throw in an electromagnetic field $F = \frac{1}{2} F_{\alpha\beta} dx^\alpha \wedge dx^\beta$ and obtain the equations of motion for a particle of charge e as a vector field on velocity phase space:

$$\mathbb{L} = \dot{q}^\alpha \frac{\partial}{\partial q^\alpha} + \left(eF^\alpha{}_\beta \dot{q}^\beta - \Gamma^\alpha{}_{\beta\gamma} \dot{q}^\beta \dot{q}^\gamma \right) \frac{\partial}{\partial \dot{q}^\alpha}$$

corresponding to

$$\frac{d}{d\lambda} q^\alpha = \dot{q}^\alpha$$

$$\frac{d}{d\lambda} \dot{q}^\alpha = -\Gamma^\alpha{}_{\beta\gamma} \dot{q}^\beta \dot{q}^\gamma + eF^\alpha{}_\beta \dot{q}^\beta$$

$$\Big[\leftrightarrow \underbrace{\frac{D\dot{q}^\alpha}{d\lambda} = eF^\alpha{}_\beta \dot{q}^\beta}_{\text{Lorentz force law}} \Big]$$

Since the velocity must be timelike or null the conserved energy must be ~~negative~~ non positive, and defines the mass

$$E = \frac{1}{2} g_{\alpha\beta} \dot{q}^\alpha \dot{q}^\beta = -m^2, \quad m \geq 0$$

When $m \neq 0$, then $m\lambda = \tau$ is the proper time measured by the charged particle and $\dfrac{d\dot{q}^\alpha}{d\tau} = \dfrac{e}{m} F^\alpha{}_\beta \dot{q}^\beta$.

Now only the forward light cone and its interior are relevant to the motion of physical particles, so one considers a reduced phase space, the tangent subbundle of future timelike or null tangent vectors to spacetime. Each mass hyperboloid, including the degenerate case of the lightcone, has an invariant volume element derived above should we need to integrate in the phase space. The ~~full volum~~ total volume element on the 7-dimensional ~~spa~~ mass m phase space (subspace of TM with $g_{\alpha\beta}\dot{q}^\alpha \dot{q}^\beta = -m^2$, $\dot{q}^\alpha$ future pointing) can be taken as

$$\Omega_m = \eta \wedge \omega_m$$

$$= \left(g^{1/2}\, dq^0 \wedge dq^1 \wedge dq^2 \wedge dq^3\right)\left(\frac{g^{1/2}}{|\dot{q}_0|}\, d\dot{q}^1 \wedge d\dot{q}^2 \wedge d\dot{q}^3\right)$$

$$= \frac{|g|}{|\dot{q}_0|}\, dq^0 \wedge dq^1 \wedge dq^2 \wedge dq^3 \wedge d\dot{q}^1 \wedge d\dot{q}^2 \wedge d\dot{q}^3$$

The constancy of the ~~energy~~ kinetic energy $T = \frac{1}{2} g_{\alpha\beta}\dot{q}^\alpha \dot{q}^\beta \ (= -\frac{1}{2}m^2)$ for the charged particle equations of motion is equivalent to the condition

$$\underbrace{\mathbb{L}\,(\underbrace{g_{\alpha\beta}\dot{q}^\alpha \dot{q}^\beta}_{\text{function}})}_{\text{derivative of function along "flow vector" } \mathbb{L} \text{ on velocity phase space.}} = 0 \quad \text{on velocity phase space}$$

This means $\mathbb{L}$ is tangent to the mass m velocity phase space, defining a vector field $\mathbb{L}_m$ on each such space.

[NOTE: EHLERS uses the notation p^α for $\dot{q}^\alpha$.]

EXTERIOR DERIVATIVE AND STOKES THEOREM

The differential of a function (or 0-form)

$$df = \partial_\alpha f \, dx^\alpha$$

may be generalized to the EXTERIOR DERIVATIVE OF A P-FORM

$$F = \frac{1}{p!} F_{\alpha_1 \dots \alpha_p} dx^{\alpha_1} \wedge \dots \wedge dx^{\alpha_p}$$

$$dF = \frac{1}{p!} dF_{\alpha_1 \dots \alpha_p} \wedge dx^{\alpha_1} \wedge \dots \wedge dx^{\alpha_p} \qquad \text{(Definition)}$$

$$= \qquad \text{(calculation)}$$

$$\frac{1}{p!} \partial_\beta F_{\alpha_1 \dots \alpha_p} dx^\beta \wedge dx^{\alpha_1} \wedge \dots \wedge dx^{\alpha_p}$$

$$= \frac{1}{p!} \partial_{[\beta} F_{\alpha_1 \dots \alpha_p]} dx^\beta \wedge dx^{\alpha_1} \wedge \dots \wedge dx^{\alpha_p}$$

$$= \frac{1}{(p+1)!} (dF)_{\beta \alpha_1 \dots \alpha_p} dx^\beta \wedge dx^{\alpha_1} \wedge \dots \wedge dx^{\alpha_p}$$

so $$(dF)_{\beta \alpha_1 \dots \alpha_p} = \frac{(p+1)!}{p!} \partial_{[\beta} F_{\alpha_1 \dots \alpha_p]} = (p+1) \partial_{[\beta} F_{\alpha_1 \dots \alpha_p]}$$

so the exterior derivative is just the antisymmetrized derivative of the coordinate components, apart from a numerical counting factor.

It is clearly a linear operation, and since partial derivatives commute, $d^2 F \equiv 0$, while one has the wedge product rule

$$d(\underset{\substack{\uparrow \\ p\text{-form}}}{F} \wedge \underset{\substack{\uparrow \\ q \text{ form}}}{G}) = dF \wedge G + (-1)^p F \wedge dG$$

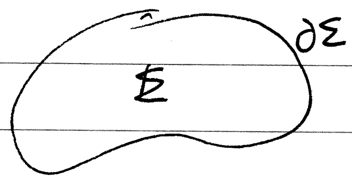
to respect the antisymmetry of the wedge product.

$$\downarrow$$

$$\left[\underset{p}{F} \wedge \underset{q}{G} = (-1)^{pq} G \wedge F \right]$$

why is this true?

STOKE'S THEOREM: Read about it elsewhere. (Schutz, for example).

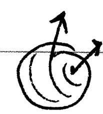


Basically if Σ is a p-dimensional submanifold with boundary $\partial\Sigma$, Σ with an orientation compatible with the "outward orientation" on $\partial\Sigma$ (more discussion than I care to give here and conveniently sidestepped by Ehlers)

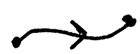
then for a $(p-1)$-form F

$$\int_{\partial\Sigma} F = \int_{\Sigma} dF \, . \qquad \text{STOKE'S THM}$$

This includes for $n=3$:

$p=3$ Gauss's theorem 

$p=2$ Stoke's theorem 

$p=1$ Fundamental thm of calculus 

which are usually rewritten in terms of the Euclidean metric on E^3 using normal vector fields. However, NO METRIC is required for Stoke's Thm. It is just convenient to rewrite it in terms of a metric when one is handy.

<u>EXAMPLE</u> $p=n$: (define $dx^{\alpha_1\cdots\alpha_p} \equiv dx^{\alpha_1}\wedge\cdots\wedge dx^{\alpha_p}$)

$$F = \frac{1}{(n-1)!} F_{\alpha_2\cdots\alpha_n} dx^{\alpha_2\cdots\alpha_n} = \frac{1}{(n-1)!} F^{\alpha_1}\eta_{\alpha_1\cdots\alpha_n} dx^{\alpha_2\cdots\alpha_n} \equiv F^{\alpha_1}\sigma_{\alpha_1}$$

$$dF = \frac{1}{(n-1)!} (F^{\alpha_1} g^{1/2})_{,\beta}\, \epsilon_{\alpha_1\cdots\alpha_n} dx^{\beta\alpha_2\cdots\alpha_n} \quad \text{but } \beta \text{ must } = \alpha_1$$

$$= \frac{1}{(n-1)!} \underbrace{(F^{\alpha_1} g^{1/2})_{,\alpha_1} g^{-1/2}}_{F^{\alpha_1}{}_{;\alpha_1}} \left(\frac{\eta_{\alpha\alpha_2\cdots\alpha_n} dx^{\alpha\alpha_2\cdots\alpha_n}}{n}\right) = F^{\alpha}{}_{;\alpha}\,\eta$$

compensate for extra sum over α

$$\int_{\partial R} F = \int_{\partial R} F^{\alpha}\sigma_{\alpha} = \int_R dF = \int_R F^{\alpha}{}_{;\alpha}\,\eta$$

"dS_α" vector differential of surface area

another story see Schutz.

divergence of vector field $F^\alpha\partial_\alpha$ dual to $(n-1)$ form F

MAKING CONTACT WITH THE NONRELATIVISTIC PHASE SPACES

The ~~sec~~ 7-dim mass m velocity phase space corresponds to the direct product of the time coordinate axis with the nonrelativistic 6-dim velocity phase space with coordinates $\{q^i, \dot{q}^i\}$ $i=1,2,3$.

To recover the analog of velocity phase space at a "moment of time", one must consider the velocity phase space over a ~~hyp~~ spacelike hypersurface in spacetime, which is itself a hypersurface in the 7-dim mass m velocity phase space.

We then need a measure on such hypersurfaces which is invariant under the phase flow (1-parameter group of diffeomorphisms of the vector field $\mathbb{L}_m$). Any 6-form will provide a measure on a hypersurface.

$$\boxed{\underset{p\text{-form}}{F} \wedge \underset{q\text{-form}}{G} = (-1)^{pq}\, G \wedge F}$$

The choice

$$\omega_m \equiv \mathbb{L}_m \lrcorner\, \Omega_m \qquad \text{definition}$$

$$\underset{(\text{calculation})}{=} \left(\dot{q}^\alpha \frac{\partial}{\partial q^\alpha} + (eF^\alpha{}_\beta \dot{q}^\beta - \Gamma^\alpha{}_{\beta\gamma}\dot{q}^\beta \dot{q}^\gamma)\frac{\partial}{\partial \dot{q}^\alpha}\right) \lrcorner\, (\overset{4\text{-form}}{\eta} \wedge \overset{3\text{ form}}{\Pi_m})$$

$$= \underbrace{\left(\dot{q}^\alpha \frac{\partial}{\partial q^\alpha} \lrcorner\, \eta\right)}_{\frac{1}{3!}\dot{q}^\alpha \eta_{\alpha\beta\gamma\delta}\, dx^\beta \wedge dx^\gamma \wedge dx^\delta \;\equiv\; \dot{q}^\alpha \sigma_\alpha} \wedge \Pi_m + \Big[\underbrace{(eF^\alpha{}_\beta \dot{q}^\beta - \Gamma^\alpha{}_{\beta\gamma}\dot{q}^\beta \dot{q}^\gamma)}_{Q^\alpha} \underbrace{\frac{\partial}{\partial \dot{q}^\alpha} \lrcorner\, \Pi_m}_{\frac{1}{3!}\frac{\eta_{0\delta\beta\gamma}\, d\dot{q}^\delta \wedge d\dot{q}^\beta \wedge d\dot{q}^\gamma}{|\dot{q}|}}\Big] \wedge \eta\, \underbrace{(-1)^{3\cdot 4}}_{1}$$

σ_α: "dS_α" hypersurface volume element

$$\frac{1}{2!}\frac{Q^\alpha}{|\dot{q}|}\eta_{0\alpha\beta\gamma}\, d\dot{q}^\beta \wedge d\dot{q}^\gamma$$

$$= \dot{q}^\alpha \sigma_\alpha \wedge \Pi_m + \frac{1}{2|P_0|}\eta_{0\alpha\beta\gamma}(F^\alpha{}_\delta \dot{q}^\delta - \Gamma^\alpha{}_{\delta\rho}\dot{q}^\delta \dot{q}^\rho)\, d\dot{q}^\beta \wedge d\dot{q}^\gamma \wedge \eta$$

assigns to a hypersurface Σ a volume element which corresponds to the projection of its tangent space orthogonal to $\mathbb{L}_m$ and which is invariant under the phase flow since $d\omega_m = 0$.

INVARIANCE UNDER PHASE FLOW:

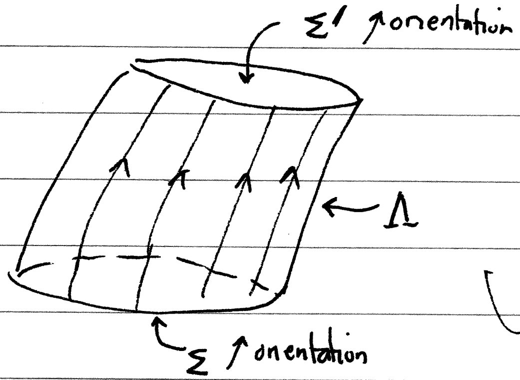
In terms of Lie derivatives

$$\mathcal{L}_{\mathbb{L}_m}\Omega_m = \mathbb{L}_m \lrcorner \underbrace{d\Omega_m}_{\substack{=0 \text{ since}\\ \text{already maximum}\\ \text{degree}}} + \underbrace{d(\underbrace{\mathbb{L}_m\lrcorner\Omega_m}_{\omega_m})}_{=0} = 0 \quad \text{so}$$

$$\mathcal{L}_{\mathbb{L}_m}\omega_m = \mathcal{L}_{\mathbb{L}_m}(\mathbb{L}_m\lrcorner\Omega_m) \underset{\substack{\uparrow\\ \text{product rule}\\ \text{for Lie derivatives}}}{=} (\underbrace{\mathcal{L}_{\mathbb{L}_m}\mathbb{L}_m}_{\substack{=0\\ \text{since } = [\mathbb{L}_m,\mathbb{L}_m]}})\lrcorner\Omega_m + \mathbb{L}_m\lrcorner\underbrace{\mathcal{L}_{\mathbb{L}_m}\Omega_m}_{=0}$$

the condition $d\omega_m = 0$ shows that both Ω_m and ω_m are invariant under the flow along the vector field $\mathbb{L}_m$.

If you are unfamiliar with Lie derivatives, then Stoke's thm shows the invariance of a finite volume:



Apply Stoke's thm to a closed tube of phase orbits in mass m phase space

$$\partial T = \Sigma \cup \Sigma' \cup \Lambda$$

~~$\partial T = \emptyset$ (no boundary)~~

$$\cancel{0 = \int_{\partial T}\omega_m = \int_T d\omega_m = \int_{\Sigma'}\omega_m}$$

$$0 = \int_T \underbrace{d\omega_m}_{=0} = \int_{\partial T}\omega_m = \int_{\Sigma'}\omega_m - \int_\Sigma \omega_m + \underbrace{\int_\Lambda \omega_m}_{=0}$$

$\omega_m|_\Lambda = 0$

so $\int_\Sigma \omega_m = \int_{\Sigma'}\omega_m$ = constant of phase flow

It remains to show that $d\omega_m = 0$.

Derivation of $d\omega_m = 0$

Following Ehlers, do the calculation at a point where

$$\left.\begin{array}{l} g_{\alpha\beta} = \eta_{\alpha\beta}, \quad \partial_\gamma g_{\alpha\beta} = 0 \to \Gamma^\alpha{}_{\beta\gamma} = 0 \\ \qquad\qquad\qquad \to d\eta_{\alpha\beta\gamma\delta} = 0 \to d\sigma_\alpha = 0 \end{array}\right\} \text{on spacetime}$$

$\downarrow$

$$-\dot q_0 = \dot q^0 = (1 + \delta_{ij}\dot q^i \dot q^j)^{1/2} \to \frac{\partial}{\partial q^\alpha}(\dot q^0) = 0 \quad \Big\} \text{ on phasespace (mass } m)$$

$$\omega_m = \underbrace{\dot q^\alpha \sigma_\alpha \wedge \Pi_m}_{} - \underbrace{\frac{1}{2\dot q^0}\eta_{ijk0}(F^i{}_\alpha \dot q^\alpha - \Gamma^i{}_{\mu\nu}\dot q^\mu \dot q^\nu)\, d\dot q^j \wedge d\dot q^k \wedge \eta}_{}$$

only $\frac{\partial}{\partial q^\alpha}$ derivatives of the coefficients can enter here because $\Pi_m \sim d\dot q^1 \wedge d\dot q^2 \wedge d\dot q^3$

but

"$d^{(q)}\dot q^\alpha$" $= \delta^\alpha_0\, d^{(q)}\dot q^0 = 0$

$d^{(q)}\sigma_\alpha = 0$

$d^{(q)}\Pi_m = \; = 0$

so this term has zero exterior derivative at the special point & hence zero exterior derivative everywhere.

only $\frac{\partial}{\partial \dot q^i}$ derivatives of these coefficients can enter (use the notation $d^{(\dot q)}$ for the restricted differential)

$\downarrow$ exterior derivative ($d^{(\dot q)}$)

$$\frac{1}{2(\dot q^0)^2}\, d\dot q^0 \wedge \eta_{ijk0} F^i{}_\alpha \dot q^\alpha\, d\dot q^j \wedge d\dot q^k \wedge \eta$$

$$-\frac{1}{2\dot q^0}\eta_{ijk0}\, F^i{}_\alpha d\dot q^\alpha \wedge d\dot q^j \wedge d\dot q^k \wedge \eta$$

$$F^i{}_0\, d\dot q^0 + \underbrace{F^i{}_\ell\, d\dot q^\ell}_{\substack{\shortparallel \\ 0 \text{ antisymmetry} \\ \text{(inertial coordinates)}}}$$

$$\frac{1}{2(\dot q^0)^2}\left(-\frac{\dot q_\ell\, d\dot q^\ell}{\dot q^0} \wedge d\dot q^j \wedge d\dot q^k\right) \wedge \eta\; \eta_{ijk0} F^i{}_\alpha \dot q^\alpha$$

$\downarrow$

$$\underbrace{F^i{}_j \dot q_i \dot q^j}_{\substack{0 \text{ by} \\ \text{antisym}}} + \underbrace{F^i{}_0 \dot q_i \dot q^0}_{\text{cancel}}$$

$\boxed{\text{qed}}$

SPECIAL CASE: Hypersurface in spacetime lifted to phase space by choosing (smoothly) a subspace of each mass hyperboloid at each point $\rightarrow$ hypersurface Σ in mass m phase space

Restricting η to this hypersurface gives zero since it is a 4-form being restricted to a 3-dimensional hypersurface in spacetime so the restriction of the second term in ω_m to Σ is zero.

$$\omega_m \big|_\Sigma = \left(\dot{q}^\alpha \sigma_\alpha \wedge \pi_m \right)\Big|_\Sigma + 0 .$$

~~Choose inertial coordinates at~~

Suppose the hypersurface in spacetime is spacelike with adapted coordinates so that it is $x^0 = \text{const}$. Choose inertial coordinates at a point on this hypersurface.

Then
$$\omega_m \big|_\Sigma = \dot{q}^0 \sigma_0 \wedge \pi_m = \left(\dot{q}^0\right)\left(dq^1 \wedge dq^2 \wedge dq^3\right)\left(\frac{1}{\dot{q}^0}\, d\dot{q}^1 \wedge d\dot{q}^2 \wedge d\dot{q}^3\right)$$
$$= \text{"}\, d^3q \wedge d^3\dot{q} \,\text{"}$$

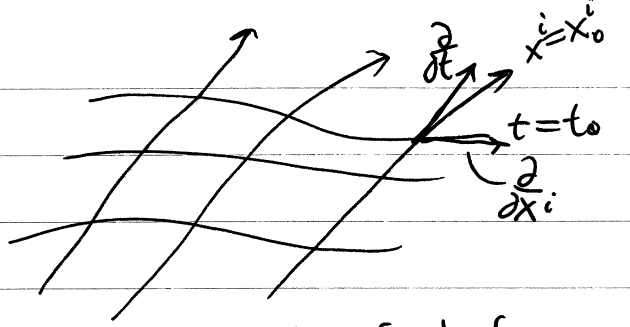
which is just the product of the volume element on the spacelike hypersurface in spacetime with the usual volume element of the spacelike momentum variables

Thus ω_m appropriately generalizes the 3-dimensional phase space volume to the coordinate invariant mass m velocity phase space formulation.

This covers the mathematical tools of EHLERS 1–32.
Now we must turn to the description of a gas of particles.

$\boxed{\text{END OF PART 1}}$

AFTERTHOUGHT: MASS RESCALING



spacetime, sliced by family of spacelike hypersurfaces

$$\{X^\alpha\} = \{X^0, X^i\} = \{t, x^i\}$$

no proper time function exists unless slicing geodesically parallel.

orthogonal decomposition of metric:

$$g = -N dt \otimes N dt \quad \leftarrow \text{time metric}$$

$$+ \; g_{ij} (dx^i + N^i dt) \otimes (dx^j + N^j dt)$$

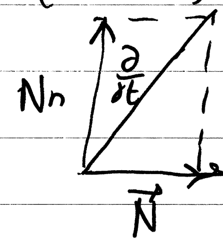
spatial metric

$$\frac{\partial}{\partial t} = N n + \underbrace{\vec{N}}_{N^i \frac{\partial}{\partial x^i}}$$

lapse function

unit normal $n \cdot n = -1$

shift vector field



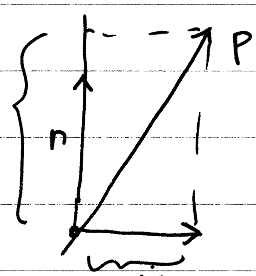
$$N dt = \text{"}d\tau\text{"}$$

not true differential; just notation like "dV" or "dS_α"

(~~differential of function τ only along individual time lines~~) proper time separation of hypersurfaces

Tangent subbundle over hypersurface $t = t_0$ is phase space at "moment of time $t = t_0$", n defines a preferred axis in each tangent space to decompose quantities

$$E = p^\perp$$



$p^i \frac{\partial}{\partial x^i}$ spatial momentum.

$E = \gamma m$ if $m \neq 0$.

$$\boxed{\text{USE EHLERS NOTATION} \quad \dot{q}^\alpha \to p^\alpha}$$

$$\Omega_m = \frac{\overbrace{d^4x}^{{}^4g^{1/2} dx^{0123}} \wedge d^3p}{E} = \frac{\overbrace{\text{"}d\tau\text{"}}^{N dt} \wedge \overbrace{\text{"}d^3x\text{"}}^{{}^3g^{1/2} dx^{123}} \wedge \overbrace{\text{"}d^3p\text{"}}^{{}^3g^{1/2} dp^{123}}}{E}$$

volume element on 7-d mass m phase space.

solutions of equations of motion:

$$\underbrace{\frac{\partial}{\partial \lambda}}_{\text{tangent in phase space}}\Big|_{q(\lambda)\,p(\lambda)} = \underbrace{\mathbb{L}_m}_{\text{Louiville vector field in phase space.}}\Big|_{q(\lambda)\,p(\lambda)}$$

if $m \neq 0$

$$m \, d\lambda = d\tau_p$$

"proper time parametrization for trajectory"

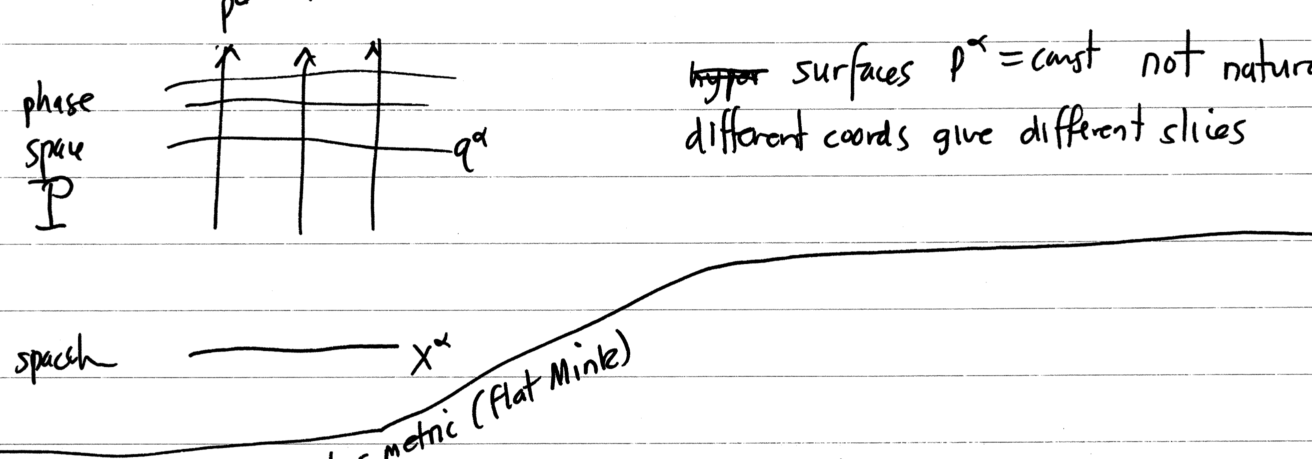
$$\omega_m = \mathbb{L}_m \lrcorner \, \Omega_m = \left(\frac{1}{m}\mathbb{L}_m\right) \lrcorner \, (m \Omega_m) \quad \text{if } m \neq 0$$

but $d\tau = \gamma \, d\tau_p$ (time dilation)

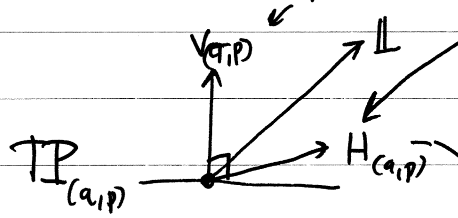
$$\text{"}\frac{\partial}{\partial \tau_p}\text{"} \lrcorner \left(\frac{d\tau \; {}^3g^{1/2} dx^{123} \; {}^3g^{1/2} dp^{123}}{\gamma} \right) \qquad \frac{E}{m} = \gamma$$

$$= \text{"}dV_x \wedge dV_p\text{"} \quad \text{usual phase space volume differential.}$$

metric on phase space forall $m \geq 0$? fiber bundle.



~~hyper~~ surfaces $p^\alpha = \text{const}$ not natural
different coords give different slices



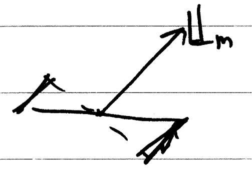
must pick out horizontal subspaces — connection does that — See JAN NOTES.

isomorphic to spacetime tangent space — use spacetime metric.

get metric on phase space.

$$\mathbb{L} = \underbrace{p^\alpha \frac{\partial}{\partial q^\alpha}}_{\text{translation along coords } q^\alpha} + \underbrace{(eF^\alpha{}_\beta p^\beta - \Gamma^\alpha{}_{\beta\gamma} p^\beta p^\gamma)}_{\text{deflection in fiber direction.}} \frac{\partial}{\partial p^\alpha}$$

$\omega_m = \mathbb{L}_m \lrcorner \Omega_m$ = nonintegrable surface area volume element
assigns surface area only to projection "orthogonal" (?) to $\mathbb{L}_m$ (natural projection of forms & vectors — not metric?)



leftover: $df \wedge \underbrace{(X \lrcorner n)} = (Xf)\, n$:

$$f_{,\beta}\, dx^\beta \wedge \frac{1}{(n-1)!} X^\alpha n_{\alpha\alpha_2\ldots\alpha_n} dx^{\alpha_2\ldots\alpha_n} = f_{,\beta} X^\alpha \frac{1}{(n-1)!} \underbrace{n_{\alpha\alpha_2\ldots\alpha_n}}_{\delta(1\ldots n)} dx^{\overbrace{\beta\alpha_2\ldots\alpha_n}^{\delta(1\ldots n)}}$$

same

$$X^\beta f_{,\beta} \frac{1}{(n-1)!} \frac{1}{n} n_{\alpha\alpha_2\ldots\alpha_n} dx^{\alpha\alpha_2\ldots\alpha_n}$$

$$= (Xf)\, n.$$

PART 2: RELATIVISTIC GAS

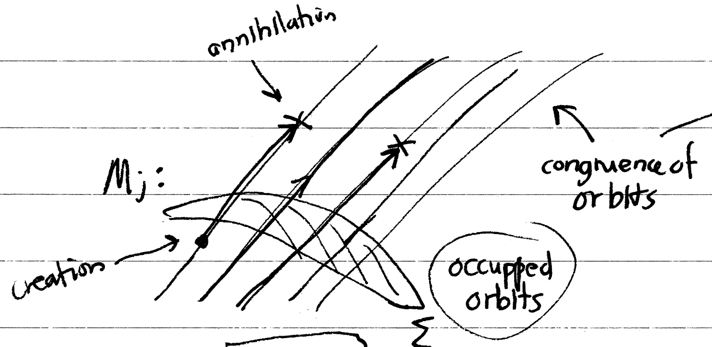
MULTISPECIE GAS

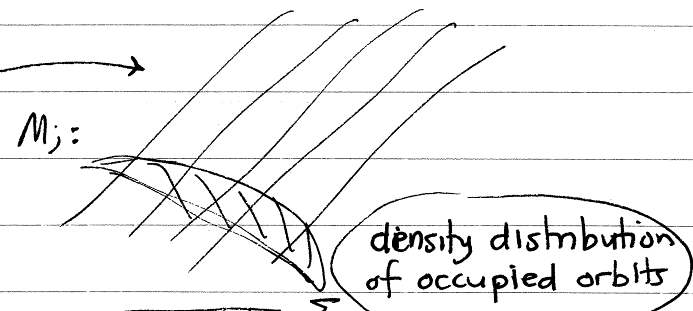
EHLERS NOTATION: $\mathcal{M}_m =$ mass m phase space (7-dim)

Now add subscript j to label particle species. and shorten notation

$$\mathcal{M}_{m_j}, \ \mathbb{L}_{m_j} \longrightarrow \mathcal{M}_j, \ \mathbb{L}_j \quad \text{etc.}$$

For each particle: $\underset{\text{mass}}{m_j}, \ \underset{\text{charge}}{e_j}, \ \underset{\text{spin}}{s_j}, \ \ldots$ eepton # or spectral type (stars)





microstate

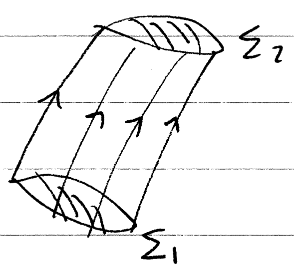
macrostate

calculated from some ensemble of microstates

$N_j[\Sigma] =$ # occupied orbits (states) intersecting Σ

$\overline{N}_j[\Sigma] =$ flux through Σ of fictitious fluid flowing in $\mathcal{M}_j$ with velocity $\mathbb{L}_j$ (λ-"time").
(Ehlers uses v instead of λ)

$$= \int_\Sigma \underbrace{f_j}_{\text{species } j \text{ distribution function}} \omega_j$$



flow tube D with ends Σ_1, Σ_2

$N_j[\partial D] = N_j[\Sigma_2] - N_j[\Sigma_1]$

$=$ $\underline{\text{NET}}$ number of collisions in D

$= \underbrace{\overset{\text{NET}}{\#}\text{ creations}}_{\text{extra occupied orbits remaining at } \Sigma_2} - \underbrace{\overset{\text{NET}}{\#}\text{ annihilations}}_{\text{extra occupied orbits on } \Sigma_1 \text{ not on } \Sigma_2}$

EHLERS omits the word $\underline{\text{net}}$:

a creation then annihilation $\underline{\text{within}}$ D has no effect on this number.

$$\overline{N}_j[\partial D] = \int_{\partial D} f_j \omega_j$$

$=$ average number of net collisions in D

NOW EHLERS DROPS OVERBAR NOTATION USED TO DISTINGUISH MICRO/MACRO STATE.

For $\Sigma \sim$ (spacetime hypersurface G, subspace $K_x \subset$ ~~each~~ species j mass hyperboloid)
this becomes

$$N_j[\Sigma] = \int_G \sigma_\alpha \left\{ \int_{K_x} f_j \, p^\alpha \pi_j \right\}$$

"dS_α"

defines preferred axis (normal n to G)
in tangent space, picks out $n_\alpha p^\alpha = {}^{n}p_\perp$"
which cancels $p_\perp$ in denom of π_j to yield "$\int f_0 \, \underline{dp^{123}}$"
in appropriate coordinates

If we have a region $D \subset M_j$, the average (net) number of collisions is:

$$N_j[\partial D] = \int_{\partial D} f_j \omega_j = \int_D \underbrace{d(f_j \omega_j)}$$

since $d\omega_j = 0$

$$df_j \wedge \omega_j = df_j \wedge (\mathbb{L}_j \lrcorner \Omega_j) = (\mathbb{L}_j f_j)\Omega_j$$

$$= \int_D \mathbb{L}_j(f_j)\, \Omega_j$$

where $\mathbb{L}_j(f_j) = p^\alpha \dfrac{\partial f_j}{\partial x^\alpha} + (e_j F^\alpha{}_\beta p^\beta - \Gamma^\alpha{}_{\beta\gamma} p^\beta p^\gamma) \dfrac{\partial f_j}{\partial p^\alpha} = \dfrac{\partial f_j}{\partial q^\alpha}\dfrac{dq^\alpha}{d\lambda} + \dfrac{\partial f_j}{\partial p_\alpha}\dfrac{dp^\alpha}{d\lambda}$

$= \dfrac{df_j}{d\lambda}$ derivative of f_j along phase orbit (soln of eq's of motion)

= density of collisions in phase space (7-dim)
wrt Ω_j measure.

So for a region G in spacetime and $D = \{(x,p) \mid x \in G,\ p \in K_x \subset P_{m_j}(x)\}$

$$N_j[\partial D] = \int_G n \left\{ \int_{K_x} \mathbb{L}_j(f_j)\, \pi_j \right\}$$ = avg net # collisions in which a particle enters the phase space region

---

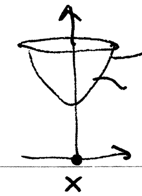
$$0 = \mathbb{L}_j(f_j) = p^\alpha \text{"}\frac{\nabla f_j}{\partial x^\alpha}\text{"} + e_j F^\alpha{}_\beta p^\beta \frac{\partial f_j}{\partial p^\alpha} = \frac{df_j}{d\lambda}$$ *LOUIVILLE EQ*

collisionfree system or detailed balancing between creations & annihilations of particles of type j with 4-momentum p at x.

$$\left\{ \begin{array}{l} \text{volume element} \\ \Pi_j = \Pi_{m_j} \sim \text{"}\frac{dp^{123}}{|p_0|}\text{"} \end{array} \right.$$

AVERAGING OVER MOMENTUM VARIABLES

TM_x



$P_{m_j}: \; p^\alpha p_\alpha = -m_j^2$

P_j

- $$N_j^\alpha(x) = \int_{P_j(x)} f_j(x,p)\, p^\alpha \Pi_j$$ particle 4-current density (species j)

$$J_j^\alpha = e_j N_j^\alpha \,, \quad J^\alpha = \sum_j J_j^\alpha$$ electric current densities.

Inertial frame: $$N_j^0(x) = \int_{P_j(x)} f_j \, p^0 \frac{dp^{123}}{|p_0|} = \int_{P_j(x)} f_j \, dp^{123} = n_j$$ number density of j particles in LRS of inertial frame

Local Rest space.

$$N_j^a(x) = \int f_j \underbrace{\frac{p^a}{p^0}}_{v^a} dp^{123} = n_j \underbrace{\langle v_j^a \rangle_j}_{\text{average 3-velocity in LRS}}$$

where $$\boxed{\langle \text{function}(a,p) \rangle_j \equiv \frac{\int f_j \, \text{function}(a,p)\, \Pi_j}{\int f_j \Pi_j}}$$

weighted average with weight function f_j

- $$T_j^{\alpha\beta}(x) = \int_{P_j(x)} p^\alpha p^\beta f_j \Pi_j \qquad T^{\alpha\beta} = \sum_j T_j^{\alpha\beta}$$

partial / total momentum flux densities.

inertial frame:

$$T_j^{00} = \mu_j = n_j \langle E_j \rangle$$ energy density ($p^0 \equiv E$)

$$T_j^{0a} = n_j \langle p^a \rangle_j$$ momentum density

$$T_j^{ab} = n_j \langle p^a v^b \rangle_j$$ kinetic pressure tensor

$$p_j = \tfrac{1}{3} T_j{}^a{}_a = \tfrac{1}{3} n_j \langle p^a v_a \rangle_j$$ mean kinetic pressure

$$\rho_j = m_j n_j$$ rest mass density

property of defintion of $T^{\alpha\beta}$:

$T_{\alpha\beta}V^\alpha V^\beta > 0$ for all V^α st $V^\alpha V_\alpha \leq 0$. (timelike, null)

$$\to \quad T^{\alpha\beta} = \begin{cases} \mu u^\alpha u^\beta + p^{\alpha\beta}, & u^\alpha \text{ future directed timelike eigenvector}, \ p^{\alpha\beta}u_\beta = 0 \\ \text{or } k^a k^b, & \text{if. monochromatic photon stream. (no dispersion — all same wave 4-vector)} \end{cases}$$

u^α = dynamical mean 4-velocity since in inertial frame aligned with this 4-vector, the momentum density vanishes:

$$\left(\begin{array}{l} T_0{}^\beta = u^\alpha T_\alpha{}^\beta = -\mu u^\beta + 0 = -\mu u^\beta \\ T^{0\beta} = \mu u^\beta \to T^{0a} = 0 \quad \text{(inertial frame)} \end{array} \right.$$

$$u^\alpha_{kin} = \frac{N_j^\alpha}{|N_j^\beta N_j^\beta|^{1/2}}$$ = kinematical mean velocity $\to$ in adapted inertial frame $\langle V_j \rangle = 0$

mean ~~pose~~ 3-velocity vanishes.

A "monochromatic" particle stream (no dispersion — all have same 4-velocity) has $T^{\alpha\beta} = \mu u^\alpha u^\beta$ (DUST).

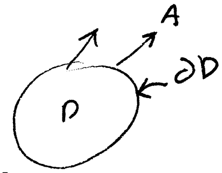
For mixed species, other possibilities: u

use baryon number instead of particle number:

$$B^\alpha = \sum_j b_j N_j^\alpha = \rho_b u_b^\alpha, \quad \rho_b = [B^\alpha B_\alpha]^{1/2}$$

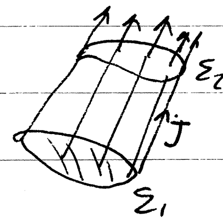
CONSERVATION LAWS

Gauss's Thm:



$$\int_D \mathbb{X}^\alpha{}_{;\alpha}\, \eta = \int_{\partial D} \mathbb{X}^\alpha \sigma_\alpha$$

= flux of $\mathbb{X}$ out of D



$\int_\Sigma J^\alpha \sigma_\alpha$ = charge enclosed in Σ

= cross-section independent for a fluxtube of J^α. iff $J^\alpha{}_{;\alpha} = 0$

(conservation of electric charge)

Similarly conservation of baryon number: $B^\alpha{}_{;\alpha} = (\rho_b u_b^\alpha)_{;\alpha} = 0$

Reversing the discussion, $N_j{}^\alpha{}_{;\alpha}$ = spacetime production density for j-particles. (created by inelastic collisions) per spacetime volume.

For the usual "cylindrical region" $\hat{D} = \{(x,p) \mid x \in D,\ (x,p) \in M_j\}$

$$\boxed{\int_{\partial\hat{D}} f_j \omega_j} \overset{\rightarrow}{=} \int_{\hat{D}} \mathbb{L}_j(f_j)\, \Omega_j = \int_{x\in D} \eta \left\{ \int_{P_j(x)} \mathbb{L}_j(f_j)\, \pi_j \right\}$$

$$\downarrow \qquad \parallel$$

$$\int_{x\in\partial D} \sigma_\alpha \left\{ \int_{P_j(x)} p^\alpha f_j \pi_j \right\} \equiv \int_{\partial D} \sigma_\alpha N_j^\alpha \underset{\text{Gauss's THM}}{=} \int_D N_j{}^\alpha{}_{;\alpha}\, \eta$$

for arbitrary D

so

balance equation for j-particles.

$$\boxed{N_j{}^\alpha{}_{;\alpha} = \int_{P_j} \mathbb{L}_j(f_j)\, \pi_j}$$

Take $\sum_j e_j \times$

$$\boxed{J^\alpha{}_{;\alpha} = \sum_j \int_{P_j} e_j \mathbb{L}_j(f_j)\, \pi_j}$$

balance equation for electric charge.

To calculate $T^{\alpha\beta}{}_{;\beta}$:

1) Recall $\quad d(f_j \omega_j) = \mathbb{L}_j(f_j)\Omega_j \quad$ for a function f_j on phase space

2) Stokes Thm $\quad \int_{\partial\hat{D}} f_j \omega_j = \int_{D} \mathbb{L}_j(f_j)\Omega_j$

3) special choice of $f_j = V_\beta p^\beta f_j$, with $V_{a;b} = 0$ at $x_0 \in D$, V_α 1-form on D

4) insert in 2) :

$$\int_{\partial\hat{D}} f_j \omega_j = \int_{\hat{D}} \mathbb{L}_j(f_j)\Omega_j = \int_D \eta \left\{ \int_{P_j} \mathbb{L}_j(f_j)\pi_j \right\} \equiv \int \eta \left\{ \int_{P_j} \mathbb{L}_j(V_\beta p^\beta f_j)\pi_j \right\}$$

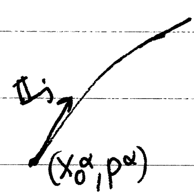
$$\| $$

$$\int_{\partial D} \sigma_\alpha \left\{ \int p^\alpha f_j \pi_j \right\} = \int_{\partial D} \sigma_\alpha \left\{ \int p^\alpha p^\beta f_j \pi_j \right\} V_\beta \equiv \int_{\partial D} \sigma_\alpha T^{\alpha\beta} V_\beta$$

$$\| $$

$$\int_D (T^{\alpha\beta} V_\beta)_{;\alpha}\, \eta$$

so: $\displaystyle (V_\alpha T^{\alpha\beta})_{;\beta} = \int_{P_j} \underbrace{\mathbb{L}_j(V_\alpha p^\alpha f_j)}\pi_j$



along particle orbit at x_0 : $\quad \underbrace{\frac{D}{d\lambda}(V_\alpha p^\alpha f_j)}_{\substack{\text{absolute covariant}\\ \text{derivative along}\\ \text{orbit at } x_0}} = V_\alpha \Big(p^\alpha \underbrace{\frac{df_j}{d\lambda}}_{\mathbb{L}_j(f_j)} + f_j \underbrace{\frac{Dp^\alpha}{d\lambda}}_{e_j F^\alpha{}_\beta p^\beta} \Big)$

So at x_0: $\quad T^{\alpha\beta}{}_{;\alpha} V_\beta = V_\alpha(\quad) \longrightarrow$ V_α arbitrary at $x_0 \rightarrow$

$$\boxed{T_j^{\alpha\beta}{}_{;\beta} = \underbrace{F^\alpha{}_\beta \underbrace{\left(\int_{P_j} e_j p^\beta f_j \pi_j\right)}_{J_j^\beta}}_{\substack{\text{electromagnetic}\\ \text{4-force density}}} + \underbrace{\int_{P_j} p^\alpha \mathbb{L}_j(f_j)\pi_j}_{\substack{\text{collisional}\\ \text{4-force density}}}}$$

4-momentum balance for j-particles.

FIELD EQUATIONS So far no use of Field equations.

case 1) $g_{\alpha\beta}$, $F_{\alpha\beta}$ given fields, apply above to test gas.

case 2) determine them by ~~average~~ using averaged gas quantities as source:

$$G^{\alpha\beta} + \Lambda g^{\alpha\beta} = T^{\alpha\beta} = T_K^{\alpha\beta} + T_{EM}^{\alpha\beta} \quad , \quad T_{EM}^{\alpha\beta} = F^{\alpha}{}_{\gamma}F^{\gamma\beta} - \tfrac{1}{4}g^{\alpha\beta}F_{\gamma\delta}F^{\gamma\delta}$$

$$F_{[\alpha\beta,\gamma]} = 0 \qquad F^{\alpha\beta}{}_{;\beta} = J^{\beta}$$

consequence: $\underbrace{T^{\alpha\beta}{}_{;\beta}}_{\downarrow} = 0 = J^{\alpha}{}_{;\alpha}$

$$T_K{}^{\alpha\beta}{}_{;\beta} = -T_{EM}{}^{\alpha\beta}{}_{;\beta} = \cdots = F^{\alpha\beta}J_{\beta}$$

$$0 = J^{\alpha}{}_{;\alpha} = \boxed{\sum_j \int_{P_j} e_j \mathbb{L}_j(f_j)\pi_j}$$

$$0 = T_K{}^{\alpha\beta}{}_{;\beta} = F^{\alpha}{}_{\beta}J^{\beta} + \sum_j \int_{P_j} p^{\alpha}\mathbb{L}_j(f_j)\pi_j \rightarrow \boxed{0 = \sum_j \int_{P_j} p^{\alpha}\mathbb{L}_j(f_j)\pi_j}$$

conservation of charge & 4-momentum in collisions

conditions on time-evolution of f_j by Einstein Maxwell eq's.

satisfied automatically if Liouville equation holds:

$\mathbb{L}_j(f_j) = 0 \rightarrow$ 1) collisionfree gas

2) detailed balancing of creations and annihilations.

Vlasov-Landau approx. of plasma physics, stellar dynamics $\rangle$ $\boxed{\text{collisionfree charged selfgravitating gas}}$

PART 3

BOLTZMANN'S COLLISION INTEGRAL (no fun)

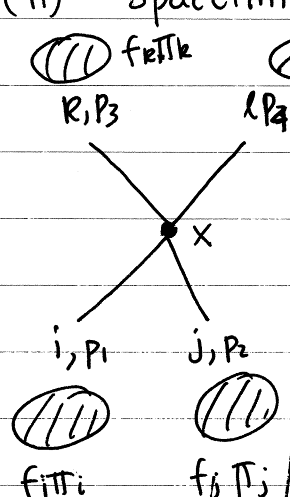
(i) Rescale mass-hyperboloid volume element (recall Ehlers uses $\hbar = 1$, $h = 2\pi\hbar \to 2\pi$)

$$f_j \pi_j \to f_j \pi_j \quad \text{but} \quad f_j \to \frac{r_j}{(2\pi)^3} f_j \;,\; \pi_j \to \frac{(2\pi)^3}{r_j}$$

so that it corresponds to classical limit of density of quantum states in phase space (r_j = spin degeneracy).

~~(ii) Abbreviate~~

(ii) Spacetime density of collisions (2 in, 2 out case)



I cannot give plausible motivation for these factors

"increase density by 1" bose

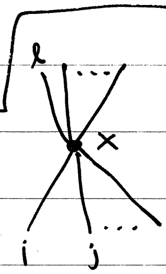
$$\underbrace{(f_i \pi_i)(f_j \pi_j)}_{\text{total \# in}} \quad \underbrace{W(ip_1, jp_2 ; kp_3, \ell p_4)}_{\substack{\text{1 pair in, 1 pair out} \\ \text{probability}}} \underbrace{\begin{bmatrix} (f_k+1)(f_\ell+1) \\ (1-f_k)(1-f_\ell) \end{bmatrix} \pi_k \pi_\ell}_{\text{total \# out}}$$

~~Lorentz-invariant crosssection~~

$\sim \delta(p_1+p_2-p_3-p_4)\, \sigma(p_i, p_2, p_3, p_4)$

"$f_k = 1$, prob $= 0$ to enter
$f_k = 0$, prob $= 1$ to enter" fermion

In general



, integrate :

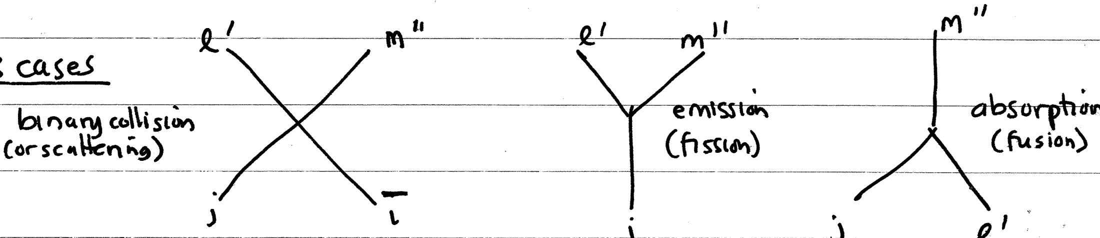
Collision density at x : $\int_{K_1} \pi_i \int_{K_2} \pi_j \cdots \int_{K_3} \pi_\ell \cdots \{ f_i(x,p_1) f_j(x,p_2) \cdots (1 \pm f_\ell(x,p_3)) \cdots \}$
$W(ip_1, jp_2, \ldots ; \ell p_3, \ldots)$

for specified momentum ranges

for unspecified mom ranges

$$= \Big(\int_{P_i} \pi_i f_i\Big) \Big(\int_{P_j} \pi_j f_j\Big) \cdots \Big(\int_{P_\ell} \pi_\ell f_\ell\Big) \cdots \{ \underbrace{(f_\ell^{-1} \pm 1)}_{\equiv \hat{f}_\ell} \cdots \} W(ip_1, jp_2, \ldots ; \ell p_3, \ldots)$$

Consider 3 cases



bar or subscript notation to differentiate momentum variables :

$m'' \sim (m, p'')$ abbreviation

Also use abbreviation $\sum_{i=1}^{N} \int_{P_\ell} \Pi_\ell' f_\ell(x, p') \to \int_{\ell'}$

Sumover all possible such collisions,

~~and get~~ ~~remove density of incident j particles~~; and get:

Collision density at (x,p) for particles of type j =

$$\underbrace{f_j}_{\text{not integrated over}} \Big[\tfrac{1}{2} \int_{\bar{\imath}} \int_{\ell'} \int_{m''} (\hat{f}_j \hat{f}_{\bar{\imath}} - \hat{f}_{\ell'} \hat{f}_{m'}) W_{j\bar{\imath};\ell' m''}$$

$$+ \int_{\ell'} \int_{m''} (\hat{f}_j \hat{f}_{\ell'} - \hat{f}_{m'}) V_{j\ell';m''} + \tfrac{1}{2} \int_{\ell'} \int_{m''} \int_{j} (\hat{f}_j - \hat{f}_{\ell'} \hat{f}_{m'}) V_{\ell' m'; j} \Big]$$

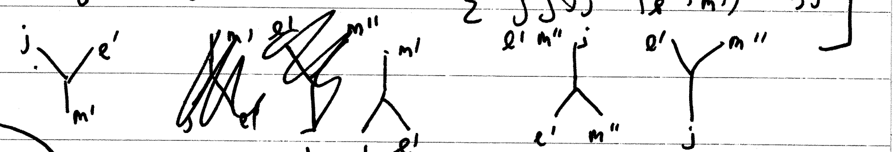
symmetry $= W_{\ell' m''; j\bar{\imath}}$

$= V_{m''; j\ell'}$

etc

missing $\frac{1}{2}$

here not necessary. Why?



only final $\hat{f}$ factors appear

factors of $\frac{1}{2}$ to avoid overcounting

Thus if $\mathbb{L}_j f_j$ represents the collision density on phase space

$$\boxed{f_j^{-1} \mathbb{L}_j f_j = \tfrac{1}{2} \int\!\!\int\!\!\int (\hat{f}_j \hat{f}_{\bar{\imath}} - \hat{f}_{\ell'} \hat{f}_{m'}) W_{j\bar{\imath};\ell' m''} + \int\!\!\int (\;) + \tfrac{1}{2} \int\!\!\int (\;)}$$

GENERALIZED BOLTZMANN EQ.

compatibility with derived above $\quad \int_j e_j f_j^{-1} \mathbb{L}_j(f_j) = 0 = \int p^\alpha f_j f_j^{-1} \mathbb{L}_j(f_j)$ (conservation of electric charge & ~~number density~~ 4-momentum)

if $W, V \sim$ delta function in charge differences as well as in change in momentum. (cross-section conditions)

CONNECTION WITH CROSS-SECTIONS

Observer with 4-velocity u:

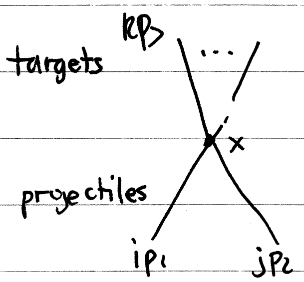
$$f_R \hat{f}_R (\ldots) \ldots$$

$$dp^{123}(k_1) \} = f_i(-u\cdot p_1)\Pi_i(k_1)$$

$$f_i(x,p_1)\, E_1 \Pi_i(k_1) \quad \times \quad f_j(x,p_2)\, E_2 \Pi_j(k_2)\, |V_2 - V_1|$$

spacetime collision density $=$ (density of target particles (in phasespace)) (relative flux density of projectile particles) dQ^u

differential cross-section for collisions of a given type
$(i p_1, j p_2 \longrightarrow k p_3, \ldots)$



$$= \left(\int_{k_1} \Pi_i f_i\right)\left(\int_{k_2} \Pi_j f_j\right)\left(\int_{k_3} \Pi_k f_k\right)\cdots\{\hat{f}_k \cdots\} W(ip_1, jp_2; kp_3, \ldots)$$

$$|V_2 - V_1| = \text{relative 3-velocity} = \left|\frac{p_2}{-u\cdot p_2} - \frac{p_1}{-u\cdot p_1}\right| = \frac{|(u\cdot p_1)p_2 - (u\cdot p_2)p_1|}{(u\cdot p_1)(u\cdot p_2)}$$

$$\frac{p}{-u\cdot p} = (1, V), \quad \Delta p = (0, \Delta V)$$

$$f_k \hat{f}_k(x p_3)\,\cancel{\ldots}$$

so $\cancel{f_i f_j}$ $f_i(x,p_1)\, f_j(x,p_2)\, \Pi_i(k_1)\, \Pi_j(k_2)\, |(u\cdot p_1)p_2 - (u\cdot p_2)p_1|\, dQ^u =$

$$f_i(\,,\,)\, f_j(\;)\quad \Pi_i(k_1)\, \Pi_j(k_2)\quad f_k \Pi_k(k_3)\, \hat{f}_k\, W(ip_1, jp_2; kp_3 \ldots)$$

so $dQ^u\, |(u\cdot p_1)p_2 - (u\cdot p_2)p_1| = \underbrace{W(ip_1, jp_2; kp_3, \ldots)\, \Pi_k \wedge \ldots}_{\text{Lorentz invariant}}$

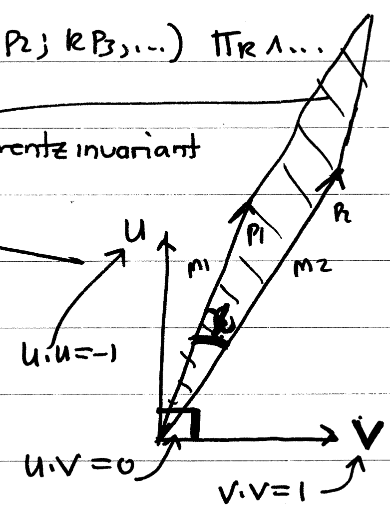
this has same value for all observers in plane of p_1 & p_2

~~$u = A p_1 + B p_2$~~

~~$= (u\cdot p_2)p_1 - (u\cdot p_1)p_2$~~

$\therefore$ this must have same value for all such observers, including p_1, p_2 rest frames and center of mass frame

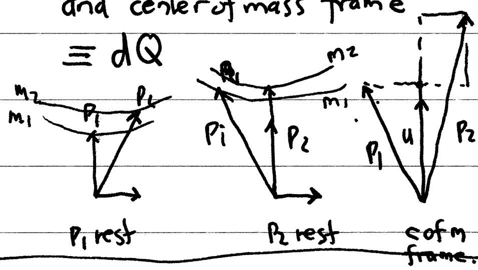
$\equiv dQ$



$|p_1 \wedge p_2|$ = area of parallelopiped in hyperbolic geometry:
$= m_1 m_2 \sinh\chi$

$p_1 \wedge p_2\,(u, v) = \text{area}$

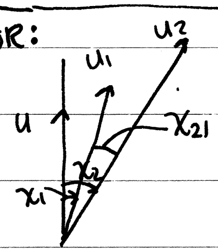
$\underbrace{(p_1 \wedge p_2)(u, \bullet)}_{\text{vector}} = \text{area } V$



relative angle χ is a boost invariant

$$|u\cdot(p_1 \wedge p_2)| = \text{area}$$
$$= m_1 m_2 \sinh\chi = \text{ind of } u$$
$$= \sqrt{(p_1\cdot p_2)^2 - m_1^2 m_2^2}$$

OR:



$p_1 = m_1 u_1,\ p_2 = m_2 u_2$

$-p_1\cdot p_2 = m_1 m_2 \cosh\chi_{21}$

$-p_1\cdot u = m_1 \cosh\chi_1$

$-p_2\cdot u = m_2 \cosh\chi_2$

$$V_u^{rel} = |\tanh\chi_2 - \tanh\chi_1| = \left|\frac{\sinh\chi_2}{\cosh\chi_2} - \frac{\sinh\chi_1}{\cosh\chi_1}\right| = \left|\frac{\sinh\chi_{21}}{\cosh\chi_1\cosh\chi_2}\right|$$

$$V_u^{rel}\cosh\chi_1\cosh\chi_2 = |\sinh\chi_{21}|$$

$$m_1 m_2 V_u^{rel}\cosh\chi_1\cosh\chi_2 = m_1 m_2 \underbrace{|\sinh\chi_{21}|}_{\text{Lorentz invariant}} = |(u\cdot p_1)p_2 - (u\cdot p_2)p_1|$$

ENTROPY See: P.T. Landsberg, *Proc. Phys. Soc. London* 74, 486 (1959)

(i) $S = -k\sum_i P_i \ln P_i$, $\sum P_i = 1$ · from information theory (equil or non equil)

(ii) For system characterized by occupation numbers:

$$P_i \to P\{n_j\} = P\{n_1, n_2, \dots\} = \text{prob of } n_j \text{ particles in } j\text{th Quantum state}$$

(iii) If $P_j(n_j)$ = prob of n_j particles in j-th Q-state is independent of other prob's:

$$P\{n_j\} = \prod_j P_j(n_j)$$

$$\therefore S = -k\sum_{\{n_j\}} \left[\prod_j P_j(n_j)\right] \ln\left[\prod_k P_k(n_k)\right] = -k\sum_{\{n_j\}} \prod_k \left[P_k(n_k)^{P_k(n_k)}\right]$$

$$= -k\sum_{\{n_j\}} \sum P_j(n_j)\ln P_j(n_j) = -k\sum_{q=0}^{?}\sum_{n_j=0}^{q} P_j(n_j)\ln P_j(n_k)$$

q = max # particles in a given state

EQUILIBRIUM: $\delta S = 0$ subject to constraints $\sum \bar n_j = N$, $\sum \epsilon_j \bar n_j = U$

where $\bar n_j$ = average occupation # of jth state: (N total number, U total energy)

$$\delta(S + \alpha N + \beta U) = 0 \qquad \alpha, \beta \text{ Lagrange multipliers.}$$

$$\to \sum_j \delta(S^{(j)} + \alpha N^{(j)} + \beta U^{(j)}) = 0 \quad \text{where } S^{(j)} = -k P_j(n_j)\ln P_j(n_j),\ N^{(j)} = \bar n_j,\ U^{(j)} = \epsilon_j \bar n_j$$

(i) FERMIONS: $n_j = 0$ or 1; $P(0) = P_0$, $P(1) = P_1 \to \bar n = 0P_0 + 1P_1 = P_1$, $P_0 = 1-\bar n$

so $P_j(1) = \bar n_j$, $P_j(0) = 1-\bar n_j$

and $\boxed{S = -k\sum_j\left[(1-\bar n_j)\ln(1-\bar n_j) + \bar n_j\ln\bar n_j\right]} \equiv \sum_j S^{(j)}$

$$0 = \frac{d}{d\bar n_j}\left[S^{(j)} + \alpha N^{(j)} + \beta U^{(j)}\right] = -k\Big[\underbrace{(-1)\ln(1-\bar n_j) + \frac{(1-\bar n_j)}{1-\bar n_j}(-1) + \ln\bar n_j + 1}_{\ln\bar n_j - \ln(1-\bar n_j) = \ln\frac{\bar n_j}{1-\bar n_j} = \ln(\bar n_j^{-1}-1)^{-1}}\Big] + \alpha + \beta\epsilon_j$$

$$\to (\bar n_j^{-1}-1)^{-1} = e^{\alpha+\beta\epsilon_j} \qquad \boxed{\bar n_j = \left[e^{-(\alpha+\beta\epsilon_j)} + 1\right]^{-1}}$$

(ii) BOSONS: Assume prob (additional particle in state) is independent of # already in state:

$P_j(N_j) = c\, q_j^{N_j}$ $\quad 0 \le q_j \le 1$, where q_j is fixed for a given Q-state j.

normalization: $1 = c\sum_{N_j=0}^{\infty} q_j^{N_j} = \frac{c}{1-q_j}$ geometric series $\to c = 1-q_j \to P_j(N_j) = (1-q_j)q_j^{N_j}$

average: $\bar n_j = c\sum_{N_j} N_j q_j^{N_j} = c q_j \frac{\partial}{\partial q_j}\sum q_j^{N_j} = c q_j\frac{\partial}{\partial q_j}\left[\frac{1}{1-q_j}\right] = \frac{c q_j}{(1-q_j)^2} = \frac{q_j}{1-q_j} = \frac{1}{q_j^{-1}-1}$;

$$S = -k\sum_j\Big\{\sum_{N_j=0}^{\infty}\underbrace{(1-q_j)}_{c}q_j^{N_j}\log(1-q_j)q_j^{N_j}\Big\}$$

$\log(1-q_j) + N_j\log q_j$

$\left[-\log(\bar n_j+1) + N_j\log\frac{\bar n_j}{\bar n_j+1}\right]$

$q_j^{-1}-1 = \frac{1}{\bar n_j}$, $q_j^{-1} = \frac{1}{\bar n_j}+1 = \frac{\bar n_j+1}{\bar n_j}$

$q_j = \frac{\bar n_j}{\bar n_j+1}$, $q_j - 1 = -\frac{1}{\bar n_j+1}$

use $\sum c q_j^{N_j} = 1$, $\sum c N_j q_j^{N_j} = \bar n_j$

$$= -k\sum_j\left\{-\log(\bar n_j+1) + \bar n_j\log\frac{n_j}{n_j+1}\right\} = \boxed{-k\sum_j\left\{n_j\log n_j - (1+n_j)\log(1+n_j)\right\} = S}$$

Thus we get the formula:

$$S = -k \sum_j \left[\bar{n}_j \ln \bar{n}_j \mp (1 \pm \bar{n}_j) \ln (1 \pm \bar{n}_j) \right]$$

$$\bar{n}_j = \frac{1}{\left(e^{-(\alpha + \beta \epsilon_j)} \mp 1\right)}$$

<u>sign</u>
upper: bosons
lower: fermions

(boson case follows by similar calculation)

The entropy result is quoted by Ehlers in § 4.12 on page 49. (He uses $k=1$)

Now with classical correspondence $\bar{n}_\lambda \rightarrow f_j(x, p_\lambda)$

$\uparrow$ quantum state with momentum p_λ in a cell ~~of size~~ $K_\lambda \subset P_j$ of size: $\underbrace{\omega_j(G \times K_\lambda)}_{=\sigma_\alpha(G)\, p^\alpha \pi_j(K_\lambda)} = 1$

so define

$$\boxed{S_j^\alpha = -\int_{P_j(x)} \{ f_j \ln f_j \mp (1 \pm f_j) \ln (1 \pm f_j) \} p^\alpha \pi_j}$$

$\boxed{\text{entropy flux density of } j\text{-species}}$

so $S_j^0 = -U_\alpha S_j^\alpha = -\int_{P_j(x)} \{ f_j \ln f \dots \} dp^{123}$, etc. (inertial frame at x)

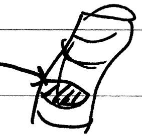
[in analogy with particle ~~number density~~ flux 4-current density.

$$N_j^\alpha = \int_{P_j(x)} f_j\, p^\alpha \pi_j$$

(see discussion above)]

$S^\alpha = \sum_j S_j^\alpha$ total entropy flux density.

$S[\Sigma] = \int_\Sigma S^\alpha \sigma_\alpha$ total entropy flux through surface Σ.

Σ  for world tube of gaseous body $\longrightarrow$ total entropy of body at "time" Σ (spacelike hypers.)

CLASSICAL LIMIT $f_j \rightarrow 0$, $\ln(1 \pm f_j) \rightarrow \pm f_j$

$$S_j^\alpha(x) \rightarrow -\int (f_j \ln f_j - f_j)\, p^\alpha \pi_j = N_j^\alpha - \int p^\alpha f_j \ln f_j\, \pi_j$$

(classical Boltzmann expression ?)

ENTROPY PRODUCTION DENSITY

Recall if $X^\alpha = \int_{P_j} F_j \, p^\alpha \pi_j$, then $X^\alpha{}_{;\alpha} = \int_{P_j} \mathbb{L}_j(F_j)\pi_j$

Apply to S^α:

$$S^\alpha_{j;\alpha} = -\int \underbrace{\mathbb{L}_j\{ f_j \ln f_j \mp (1 \pm f_j)\ln(1\pm f_j)\}}_{\underbrace{\frac{\partial}{\partial f_j}\{\ \}}_{\ln f_j + 1 \mp (\pm \ln(1\pm f_j) \pm 1)} \mathbb{L}_j f_j} \pi_j = \int (\ln \hat{f}_j)\, \mathbb{L}_j f_j \, \pi_j$$

$$= \int f_j \pi_j \cdot f_j^{-1} \mathbb{L}_j(f_j) \ln \hat{f}_j$$

$$\doteq -\left[\ln f_j + 1 \cancel{\ \mp\ } - \ln(1\pm f_j) \cancel{-1}\right] = \ln\left(\frac{1\pm f_j}{f_j}\right) = \ln \hat{f}_j$$

$$S^\alpha{}_{;\alpha} = \underbrace{\left(\sum_j \int f_j \pi_j\right)}_{\int_j} f_j^{-1} \ln \hat{f}_j \, \underbrace{\mathbb{L}_j(f_j)}_{\iiint_{\bar{\imath}\, l'm''} (\hat{f}_j \hat{f}_{\bar{\imath}} - \hat{f}_{l'} \hat{f}_{m''}) W_{j\bar{\imath};l'm''} + \iint() + \frac{1}{2}\iint()}$$

$$= \frac{1}{2} \iiiint_{j\bar{\imath}\, l'm''} \ln \hat{f}_j \, (\hat{f}_j \hat{f}_{\bar{\imath}} - \hat{f}_l \hat{f}_m) \, W_{j\bar{\imath};l'm''} + \cdots$$

$\frac{1}{2}(\ln \hat{f}_j + \ln \hat{f}_{\bar{\imath}}) \hat{f}_j \hat{f}_i$

$= \frac{1}{2} \ln(\hat{f}_j \hat{f}_{\bar{\imath}}) \hat{f}_j \hat{f}_i$

all indices summed & integrated over — use symmetry of W

$\frac{1}{2}(\ln \hat{f}_j + \ln \hat{f}_{\bar{\imath}}) \hat{f}_l \hat{f}_m$

$\frac{1}{2} \ln(\hat{f}_j \hat{f}_i) \hat{f}_l \hat{f}_m$

$$\frac{1}{2} \ln(\hat{f}_j \hat{f}_i)(\hat{f}_j \hat{f}_i - \hat{f}_l \hat{f}_m)$$

$$\doteq \frac{1}{4}\left[\ln(\hat{f}_j \hat{f}_i) - \ln(\hat{f}_l \hat{f}_m)\right](\hat{f}_j \hat{f}_i - \hat{f}_l \hat{f}_m) = \frac{1}{4}(\ln(\hat{f}_j \hat{f}_i))\hat{f}_j \hat{f}_i + \frac{1}{4} \ln(\hat{f}_l \hat{f}_m) \hat{f}_l \hat{f}_m$$

$$- \frac{1}{4} \ln \hat{f}_j \hat{f}_i \, \hat{f}_l \hat{f}_m - \frac{1}{4} \ln \hat{f}_l \hat{f}_m \, \hat{f}_i \hat{f}_j$$

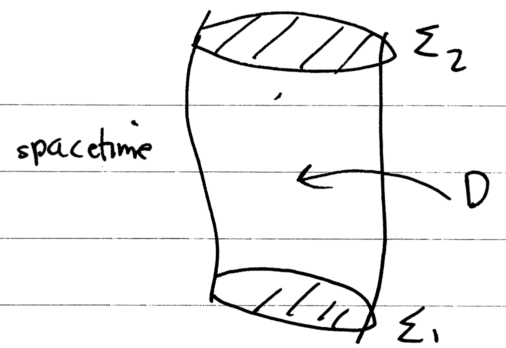
$$= \frac{1}{2} \ln(\hat{f}_j \hat{f}_i)(\hat{f}_j \hat{f}_i - \hat{f}_l \hat{f}_m) \checkmark$$

$$\boxed{S^\alpha{}_{;\alpha} = \frac{1}{8} \iiiint \ln\left(\frac{\hat{f}_j \hat{f}_{\bar{\imath}}}{\hat{f}_{l'} \hat{f}_{m'}}\right)(\hat{f}_j \hat{f}_i - \hat{f}_l \hat{f}_{m'}) W_{j\bar{\jmath}l'm''} + \cdots} \geq 0$$

$(a-b)\ln\frac{a}{b} = (a-b)(\ln a - \ln b) \geq 0$ for any $a, b > 0$

$$\hat{f}_j = f_j^{-1} \pm 1 = \frac{1 \pm f_j}{f_j} \} \geq 0$$

BOLTZMANN'S H-THEOREM



worldtube of gaseous body (no gas outside tube)

Stokes Thm:

$$\underbrace{\int_{\Sigma_2} S^\alpha \sigma_\alpha - \int_{\Sigma_1} S^\alpha \sigma_\alpha}_{\int_{\partial D} S^\alpha \sigma_\alpha} = \int_D S^\alpha{}_{;\alpha} \eta \geq 0$$

total entropy of body never decreases in time.

PART 4

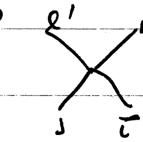
A gas described by $(g_{\alpha\beta}, F_{\alpha\beta}; f_j)$ on spacetime and its associated phase spaces is in a $\boxed{\text{stationary state}}$ if $(g_{\alpha\beta}, F_{\alpha\beta})$ are stationary and f_j ~~are~~ invariant under the induced transformations of the phase spaces. Such a gas has vanishing entropy production $S^\alpha{}_{;\alpha} = 0$ and constant total entropy $\int_\Sigma S^\alpha \sigma_\alpha$ = constant if Σ is a slice of the world tube of the gas.

What about the converse? If a gas has $S^\alpha{}_{;\alpha}=0$, does this imply stationarity? The integrand consists of terms of the form

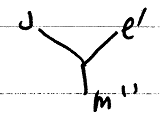
$(a-b)(\ln a - \ln b)$

which vanish iff $a=b$ or equivalently $\ln a = \ln b$; so

$$\log \hat{f}(j) + \ln \hat{f}(i) - \ln \hat{f}(l') - \ln \hat{f}(m'') = 0$$



$$\ln \hat{f}(j) + \log \hat{f}(l') - \ln \hat{f}(m'') = 0$$



must hold for all possible collisions, which means $\ln \hat{f}_j$ must be an ADDITIVE COLLISION INVARIANT.

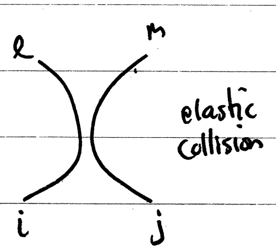
From the Boltzmann equation one then has $\mathbb{L}_j(f_j) = 0$ for all $p \in P_j(x)$ as the condition for ZERO ENTROPY PRODUCTION.

(DETAILED BALANCING: # particles thrown into state p at x = # particles thrown out)

DEFINE THIS TO BE AN EQUILIBRIUM STATE.

(i) functional dependence of f_j on momentum variables:

The only additive collision invariants are:



$$h(p) = -\beta_\alpha p^\alpha + \alpha_j$$

β_α: same for i, j, l, m

α_j: $\alpha_j = \alpha_m$, $\alpha_i = \alpha_l$. (ie depends only on particle type)

NO TIME TO SIMPLIFY PROOF. LEFT AS EXERCISE.

$\ln \hat{f}_j = -\beta_\alpha p^\alpha - \alpha_j \quad \longrightarrow \quad f_j(x,p) = \dfrac{1}{e^{-\alpha_j(x) - \underbrace{\beta_\alpha(x) p^\alpha}_{\beta(x) E}} \mp 1} \geq 0$

$\parallel$

$f_j^{-1} \pm 1$

bosons:

$e^{-\alpha - \beta \cdot p} > 1$

$\alpha + \beta \cdot p < 0$

$\alpha_j \leq -\beta_\alpha p^\alpha \quad \forall p$

minimum value: $-\beta_\alpha(m_j p^\alpha) = \beta m_j$

For f_j to vanish at infinity ($E_j >> m_j$) on P_j :

β^α = timelike future directed vector.

$= \underbrace{\beta u^\alpha}$, $\quad u^\alpha u_\alpha = -1$

mean 4-velocity.

$-\beta_\alpha p^\alpha = -\beta u_\alpha p^\alpha = \beta E$ energy relative to mean 4-vel

additive collision invariant only if $\quad \alpha_j + \alpha_i = \alpha_e + \alpha_m \quad$ elastic coll

$\alpha_j + \alpha_e = \alpha_m \quad$ absorption.

thermal energy of particle.

Note f_j is a function only of

$E = \sqrt{m^2 + p_i p^i}$ (orthonormal frame adapted to mean velocity)

and so is $\boxed{\text{isotropic}}$

So the defining integrals for the particle current density & energy-momentum tensor reduce to:

$N_j^\alpha = n_j u^\alpha$

$T_j^{\alpha\beta} = (\mu_j + p_j) u^\alpha u^\beta + p_j g^{\alpha\beta}$

n_j = number density, μ_j = energy density $\Big\}$ wrt u^α

p_j = pressure

and kinematical & dynamical mean velocity of each component $= u^\alpha$.

Similarly the entropy flux density reduces to.

$S_j^\alpha = s_j u^\alpha \qquad s_j$ = entropy density of j species.

REST OF SECTION PRETTY STRAIGHTFORWARD.

(ii) functional dependence on position variables:

Since f_j is a function of $\alpha_j + \beta_\alpha p^\alpha$, $\quad \mathbb{L}_j(f_j) = 0 \rightarrow \mathbb{L}_j(\alpha_j + \beta_\alpha p^\alpha) = 0$

which means that along each particle trajectory $\quad \alpha_j + \beta_\alpha p^\alpha$ is a constant.

This condition is

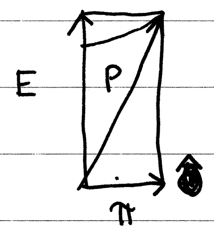
$$0 = \mathbb{L}_j(\alpha_j + \beta_\alpha p^\alpha) = (\alpha_{,\alpha} + e\beta_\beta F^\beta{}_\alpha)p^\alpha + \beta_{\alpha;\beta}\, p^\alpha p^\beta$$

exercise!! (do it)

where $p_\alpha p^\alpha = -m^2$. Decompose:

$$p^\alpha = E u^\alpha + \pi e^\alpha$$

$\vec{e}$: unit vector = normalized spatial momentum orthogonal to u



plane of $\vec{u}, \vec{e}$

$$E^2 - \pi^2 = m^2$$

Also: $g_{\alpha\beta} = -u_\alpha u_\beta + h_{\alpha\beta} \leftarrow$ spatial metric wrt u^α: $h_{\alpha\beta} \equiv u_\alpha u_\beta + g_{\alpha\beta}$

And Introduce: $\quad \mathbb{A}_\alpha \equiv \alpha_{,\alpha} + e\beta_\beta F^\beta{}_\alpha$

Result: $\quad \underbrace{\mathbb{A}_\alpha(Eu^\alpha + \pi e^\alpha) + \beta_{(\alpha;\beta)}(Eu^\alpha + \pi e^\alpha)(Eu^\beta + \pi e^\beta)}$ $= 0$

of form: $\quad X + Y_\alpha e^\alpha + Z_{\alpha\beta} e^\alpha e^\beta = 0$

which for fixed x, E, π is a function of the unit vector $\vec{e}$, ie a function on S^2 (quadratic) which may be decomposed into spherical harmonic components for $\ell = 0, 1, 2$ (since only quadratic). Each component must vanish separately:

$$\ell = 0: \quad 0 = X + \tfrac{1}{3} Z_{\alpha\beta} g^{\alpha\beta} = E^2 \beta_{\alpha;\beta} u^\alpha u^\beta + \mathbb{A}_\beta u^\beta E + \tfrac{1}{3}\underbrace{(E^2 - m^2)}_{\pi^2} h^{\alpha\beta}\beta_{\alpha;\beta}$$

$$\ell = 1: \quad 0 = Y^\gamma = [\mathbb{A}_\alpha + 2Eu^\beta \beta_{(\beta;\alpha)}]\underbrace{h^{\alpha\gamma}}_{\substack{\text{spatial metric} \\ \text{wrt } u^\alpha}}$$

$$\ell = 2: \quad 0 = Z_{\alpha\beta} - \tfrac{1}{3}(\mathrm{Tr}\, Z)h_{\alpha\beta} = h^{\alpha\gamma}h^{\beta\delta}\beta_{\gamma;\delta} - \tfrac{1}{3}h^{\alpha\beta}h^{\gamma\delta}\beta_{(\gamma;\delta)}$$

So $\quad m\, h^{\alpha\beta}\beta_{\alpha;\beta} = 0 \qquad \mathbb{A}_\alpha u^\alpha = 0 \qquad \beta_{\alpha;\beta} u^\alpha u^\beta = -\tfrac{1}{3} h^{\alpha\beta} \beta_{\alpha;\beta}$

$\mathbb{A}_\alpha h^{\alpha\beta} = 0 \qquad u^\alpha \beta_{(\alpha;\beta)} h^{\beta\gamma} = 0 \qquad$ [since true for <u>all</u> E]

$$\mathbb{A}_\alpha = \alpha_{,\alpha} + e\beta_\beta F^\beta{}_\alpha = 0 \quad \rightarrow \quad (d\alpha)_\alpha = -eT^{-1}u_\beta F^\beta{}_\alpha \equiv \frac{e}{T} E_\alpha$$

$\hookrightarrow \vec{E} = \frac{T}{e} d\alpha = \quad$ ($-F_{0\alpha}$ in orthonormal frame)

and

$$\beta_{(\alpha;\beta)} = \beta_{(\gamma;\delta)} \delta^{\gamma}_{\alpha} \delta^{\delta}_{\beta} = \beta_{(\gamma;\delta)} (h^{\gamma}{}_{\alpha} - u^{\gamma} u_{\alpha})(h^{\delta}{}_{\beta} - u^{\delta} u_{\beta})$$

$$= \beta_{(\gamma;\delta)} h^{\gamma}{}_{\alpha} h^{\delta}{}_{\beta} + \underbrace{\beta_{\gamma;\delta} u^{\gamma} u^{\delta}}_{-\frac{1}{3} h^{\gamma\delta} \beta_{\gamma;\delta}} \underbrace{u_{\alpha} u_{\beta}}_{-g_{\alpha\beta} + h_{\alpha\beta}} - 2 \underbrace{\beta_{(\gamma;\delta)} u^{\gamma} h^{\delta}{}_{(\beta} u_{\alpha)}}_{=0}$$

$$= \underbrace{(\beta_{(\gamma;\delta)} - \tfrac{1}{3} h_{\gamma\delta} h^{\mu\nu} \beta_{\mu;\nu}) h^{\gamma}{}_{\alpha} h^{\delta}{}_{\beta}}_{=0} + \underbrace{\tfrac{1}{3} h^{\gamma\delta} \beta_{\gamma;\delta}}_{=0 \text{ if } m \neq 0} g_{\alpha\beta}$$

$$= \lambda g_{\alpha\beta}, \quad \lambda = 0 \text{ if } m > 0$$

Result of (i), (ii):

A gas is in equilibrium iff

a) $f_j(x, p) = [\exp(-\alpha_j(x) - \beta_\alpha(x) p^\alpha) \mp 1]^{-1}$

b) $\beta^\alpha \doteq \frac{u^\alpha}{T}$ is a $\begin{pmatrix}\text{Killing} \\ \text{conformal Killing}\end{pmatrix}$ vector field if $\begin{Bmatrix} m>0 \\ m \geq 0 \end{Bmatrix}$

c) $d\alpha = \frac{e\vec{E}}{T}$

If any component of the gas has positive mass, the region must have a timelike KVF and so must be stationary.

~~The temperature squared gives the~~ $-T^{-2}$ is just the norm of the Killing vector field β^α which is related to a 3-dimensional gravitational potential by:

Let ξ be a dimensionless KVF $\beta^\alpha = \frac{1}{T_0} \xi^\alpha$

Define $\mathcal{U} \equiv \frac{1}{2} \ln(-\xi^\alpha \xi_\alpha)$ or $-\xi^\alpha \xi_\alpha = e^{2\mathcal{U}}$

So $\beta = \frac{1}{T} = \frac{1}{T_0} e^{\mathcal{U}} \rightarrow \boxed{e^{\mathcal{U}} = \frac{T_0}{T}}$

and $\alpha = \beta \tilde{\mu} = \frac{\tilde{\mu}}{T} \rightarrow$ chemical pot depends on grav potential like T when α constant:

(1) If $F_{\alpha\beta} = 0$

(2) If $\vec{E} = 0$

If $F_{\alpha\beta}$ is also invariant (which must be the case except for null solutions)
then $F = dA$ has an invariant 4-potential A_α
(dont confuse with Ehler's previous use of symbol A_α!)

So A_α = $\cancel{0}$ $\alpha_{,\alpha} + e \underbrace{\beta_\beta F^\beta{}_\alpha}_{\beta^\beta(A_{\alpha,\beta} - A_{\beta,\alpha})} = 0 \longrightarrow$ use coord inv. notation

$$0 = d\alpha + e \underbrace{\beta}_{\frac{1}{T_0}\xi} \lrcorner\, dA = d\alpha + \frac{e}{T_0}\Big[\underbrace{\mathcal{L}_\xi A}_{0 \text{ by invariante}} - d(\xi \lrcorner A)\Big]$$

$$= d(\underbrace{\alpha - e\beta^\alpha A_\alpha}_{\gamma = \text{constant}})$$

$$-\alpha - \beta^\alpha p_\alpha = -\gamma - e\beta^\alpha A_\alpha - \beta^\alpha p_\alpha = -\gamma - \beta^\alpha(p_\alpha + eA_\alpha)$$

so: $\boxed{f_0(x,p) = \left[\exp\left(-\gamma - \beta_\alpha(p^\alpha + eA^\alpha)\right) \mp 1\right]^{-1}}$

Then Ehlers discusses the nonrelativistic limit.
Straight forward.

Also See Ehlers, Geren, Sachs JMP 9, 1344 (1968)
Isotropic solutionals of the Einstein Liouville Equations.