Let $e = (e_\alpha)$, $\theta = (\theta^\alpha)$ be a frame with dual basis: $\begin{cases} \theta^\alpha(e_\beta) = \delta^\alpha_\beta \\ [e_\alpha, e_\beta] = C^\gamma_{\alpha\beta} e_\gamma \\ d\theta^\alpha = -\frac{1}{2} C^\alpha_{\beta\gamma}\theta^\beta \wedge \theta^\gamma \end{cases}$

Let: $\nabla_{e_\alpha} e_\beta = \Gamma^\gamma_{\alpha\beta} e_\gamma$

Define "CONNECTION ONE-FORMS with respect to this frame: $\omega^\beta{}_\alpha = \Gamma^\beta{}_{\gamma\alpha}\theta^\gamma$

so that $\nabla_X e_\alpha = \omega^\beta{}_\alpha(X) e_\beta$

TORSION TENSOR: $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] \equiv \Theta^\alpha(X,Y) e_\alpha$

TORSION TWO-FORMS: $T(e_\alpha, e_\beta) \equiv T^\gamma{}_{\alpha\beta} e_\gamma \qquad \Theta^\alpha \equiv \frac{1}{2} T^\alpha{}_{\beta\gamma}\theta^\beta \wedge \theta^\gamma$

$$\boxed{\text{note: } T^\gamma{}_{\alpha\beta} = 2\Gamma^\gamma_{[\alpha\beta]} - C^\gamma_{\alpha\beta}}$$

CURVATURE TENSOR / CURVATURE FORMS: $R(X,Y)Z \equiv \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]} Z \qquad R(e_\alpha, e_\beta)e_\gamma \equiv R^\delta{}_{\gamma\alpha\beta} e_\delta$

$R(X,Y)e_\beta \equiv \Omega^\alpha{}_\beta(X,Y) e_\alpha \qquad \Omega^\alpha{}_\beta = \frac{1}{2} R^\alpha{}_{\beta\gamma\delta}\theta^\gamma \wedge \theta^\delta$

COMPUTATION:

$$T(X,Y) = \Theta^\alpha(X,Y) e_\alpha = \nabla_X Y - \nabla_Y X - [X,Y] = \underbrace{\nabla_X(Y^\alpha e_\alpha) - \nabla_Y(X^\alpha e_\alpha)}_{X(Y^\alpha) - Y(X^\alpha) + Y^\alpha \nabla_X e_\alpha - X^\alpha \nabla_Y e_\alpha} - [X,Y]$$

$$= \{X\theta^\alpha(Y) - Y\theta^\alpha(X) - \theta^\alpha([X,Y])\} e_\alpha + \underbrace{\{Y^\alpha \omega^\beta{}_\alpha(X) - X^\alpha \omega^\beta{}_\alpha(Y)\}}_{\{\omega^\beta{}_\alpha(X)\theta^\alpha(Y) - \omega^\beta{}_\alpha(Y)\theta^\alpha(X)\}} e_\beta +$$

(Recall $d\omega(X,Y) = X\cdot\omega(Y) - Y\omega(X) - \omega([X,Y])$)

$$= \{d\theta^\alpha(X,Y) + \omega^\alpha{}_\beta \wedge \theta^\beta(X,Y)\} e_\alpha$$

RESULT. $\boxed{\Theta^\alpha = d\theta^\alpha + \omega^\alpha{}_\beta \wedge \theta^\beta \leftrightarrow \Theta = d\theta + \omega\wedge\theta}$ CARTAN'S FIRST STRUCTURAL FORMULA.

COMPUTATION:

$$R(X,Y)e_\beta = \Omega^\alpha{}_\beta(X,Y) e_\alpha = \nabla_X\nabla_Y e_\beta - \nabla_Y\nabla_X e_\beta - \nabla_{[X,Y]} e_\beta$$

$$\nabla_X(\nabla_Y e_\beta) = \nabla_X\{\omega^\alpha{}_\beta(Y) e_\alpha\} = X\omega^\alpha{}_\beta(Y) e_\alpha + \omega^\alpha{}_\beta(Y)\omega^\gamma{}_\alpha(X) e_\gamma$$

$$\nabla_{[X,Y]} e_\beta = \omega^\alpha{}_\beta([X,Y]) e_\alpha$$

$$\Omega^\alpha{}_\beta(X,Y) e_\alpha = \{X\omega^\alpha{}_\beta(Y) - Y\omega^\alpha{}_\beta(X) - \omega^\alpha{}_\beta([X,Y]) + \omega^\alpha{}_\gamma(X)\omega^\gamma{}_\beta(Y) - \omega^\alpha{}_\gamma(Y)\omega^\gamma{}_\beta(X)\} e_\alpha$$

$$= \{d\omega^\alpha{}_\beta(X,Y) + \omega^\alpha{}_\gamma \wedge \omega^\gamma{}_\beta(X,Y)\} e_\alpha$$

RESULT $\boxed{\Omega^\alpha{}_\beta = d\omega^\alpha{}_\beta + \omega^\alpha{}_\gamma \wedge \omega^\gamma{}_\beta \leftrightarrow \Omega = d\omega + \omega\wedge\omega}$ CARTAN'S SECOND STRUCTURAL FORMULA.

Note. $d\omega^\alpha{}_\beta(e_\gamma, e_\delta) = \partial_\gamma \Gamma^\alpha_{\delta\beta} - \partial_\delta \Gamma^\alpha_{\gamma\beta} - C^\mu_{\gamma\delta}\Gamma^\alpha_{\mu\beta}. \qquad \partial_\gamma \equiv e_\gamma$

$\omega^\alpha{}_\mu \wedge \omega^\mu{}_\beta(e_\gamma, e_\delta) = \Gamma^\alpha{}_{\gamma\mu}\Gamma^\mu{}_{\delta\beta} - \Gamma^\alpha{}_{\delta\mu}\Gamma^\mu{}_{\gamma\beta}$

$$\boxed{R^\alpha{}_{\beta\gamma\delta} = 2\partial_{[\gamma}\Gamma^\alpha_{\delta]\beta} + 2\Gamma^\alpha_{[\gamma|\mu}\Gamma^\mu_{\delta]\beta} - \Gamma^\alpha_{\mu\beta} C^\mu_{\gamma\delta}}$$

Ricci Identities:
$$\begin{cases} 2\,\nabla_{[\alpha}\nabla_{\beta]}X^{\gamma} = X^{\delta}R^{\gamma}{}_{\delta\alpha\beta} - T^{\delta}{}_{\alpha\beta}\nabla_{\delta}X^{\gamma} \\ 2\,\nabla_{[\alpha}\nabla_{\beta]}\omega_{\gamma} = -\omega_{\delta}R^{\delta}{}_{\gamma\alpha\beta} - T^{\delta}{}_{\alpha\beta}\nabla_{\delta}\omega_{\gamma} \end{cases}$$

Note:
$$\left\{ \begin{array}{l} [\nabla_{\alpha}\nabla_{\beta}X^{\gamma}] \equiv \\ (\nabla\nabla X)^{\gamma}{}_{\beta\alpha} \end{array} \right\} \equiv X^{\gamma}{}_{;\beta\alpha}$$

COMPUTATION:

$$\nabla X = (\nabla_{\beta}X^{\gamma})\, e_{\gamma}\otimes\theta^{\beta}$$
$$= \nabla_{\beta}X\otimes\theta^{\beta}$$

$$\nabla\nabla X = (\nabla_{\alpha}\nabla_{\beta}X^{\gamma})\, e_{\gamma}\otimes\theta^{\beta}\otimes\theta^{\alpha}$$
$$= \underbrace{\nabla_{\alpha}(\nabla_{\beta}X)}_{\nabla_{\alpha}\nabla_{\beta}X}\otimes\theta^{\beta}\otimes\theta^{\alpha} + (\nabla_{\beta}X)\otimes\underbrace{\nabla_{\alpha}\theta^{\beta}}_{-\Gamma_{\alpha}{}^{\beta}{}_{\gamma}\theta^{\gamma}}\otimes\theta^{\alpha} = \{\nabla_{\alpha}\nabla_{\beta}X - (\nabla_{\gamma}X)\Gamma^{\gamma}{}_{\alpha\beta}\}\otimes\theta^{\beta}\otimes\theta^{\alpha}$$

pour notation

$$\nabla\wedge\nabla X \equiv 2\,\mathrm{ALT}(\nabla\nabla X) = (\nabla_{[\alpha}\nabla_{\beta]}X^{\gamma})\, e_{\gamma}\otimes\theta^{\beta}\wedge\theta^{\alpha} \equiv [\nabla_{[\alpha}\nabla_{\beta]}X]\otimes\theta^{\beta}\wedge\theta^{\alpha}$$
$$= \{\nabla_{[\alpha}\nabla_{\beta]}X - \Gamma^{\gamma}_{[\alpha\beta]}\nabla_{\gamma}X\}\otimes\theta^{\beta}\wedge\theta^{\alpha}$$
$$= \Big[\tfrac{1}{2}\{2\nabla_{[\alpha}\nabla_{\beta]} - \nabla_{[\alpha,\beta]}\}X - \tfrac{1}{2}\{\underbrace{2\Gamma^{\gamma}_{[\alpha\beta]} - C^{\gamma}_{\alpha\beta}}_{T^{\gamma}_{\alpha\beta}}\}\nabla_{\gamma}X\Big]\otimes\theta^{\beta}\wedge\theta^{\alpha}$$
$$= \tfrac{1}{2}(X^{\delta}R^{\gamma}{}_{\delta\alpha\beta} - T^{\gamma}{}_{\alpha\beta}\nabla_{\gamma}X)\otimes\theta^{\beta}\wedge\theta^{\alpha} \qquad \text{QED}$$

SYMBOLICALLY:
$$\boxed{\begin{aligned} \nabla\wedge\nabla X &= \Omega X - (\nabla X)\Theta \\ \nabla\wedge\nabla\sigma &= -\sigma\Omega - (\nabla\sigma)\Theta \end{aligned}}$$

Note $\nabla\wedge\nabla f = 0 - \nabla f\,\Theta$
Similar formulas exist for higher rank tensors.

Bianchi Identities

$$\Theta^{\alpha} = d\theta^{\alpha} + \omega^{\alpha}{}_{\beta}\wedge\theta^{\beta}$$
$$\Omega^{\alpha}{}_{\beta} = d\omega^{\alpha}{}_{\beta} + \omega^{\alpha}{}_{\gamma}\wedge\omega^{\gamma}{}_{\beta}$$

COMP. $d\Theta^{\alpha} = d\omega^{\alpha}{}_{\beta}\wedge\theta^{\beta} - \omega^{\alpha}{}_{\beta}\wedge d\theta^{\beta} = (\Omega - \omega\wedge\omega)\wedge\theta - \omega\wedge(\Theta - \omega\wedge\theta)$
$$= \Omega\wedge\theta - \underline{\omega\wedge\omega\wedge\theta} - \omega\wedge\Theta + \underline{\omega\wedge\omega\wedge\theta}$$

RESULT $\boxed{d\Theta^{\alpha} + \omega^{\alpha}{}_{\beta}\wedge\Theta^{\beta} = \Omega^{\alpha}{}_{\beta}\wedge\theta^{\beta} \leftrightarrow d\Theta + \omega\wedge\Theta = \Omega\wedge\theta}$ BIANCHI'S FIRST IDENTITY

COMP. $d\Omega = d\omega\wedge\omega - \omega\wedge d\omega = (\Omega - \omega\wedge\omega)\wedge\omega - \omega\wedge(\Omega - \omega\wedge\omega)$
$$= \Omega\wedge\omega - \omega\wedge\Omega + 0$$

RESULT $\boxed{d\Omega^{\alpha}{}_{\beta} + \omega^{\alpha}{}_{\gamma}\wedge\Omega^{\gamma}{}_{\beta} - \Omega^{\alpha}{}_{\gamma}\wedge\omega^{\gamma}{}_{\beta} = 0 \leftrightarrow d\Omega + \omega\wedge\Omega - \Omega\wedge\omega = 0}$ BIANCHI'S SECOND IDENTITY

Note. For $\Theta^{\alpha} \equiv 0$, $0 = \Omega^{\alpha}{}_{\beta}\wedge\theta^{\beta} = \frac{1}{2}R^{\alpha}{}_{\beta\gamma\delta}\,\theta^{\beta}\wedge\theta^{\gamma}\wedge\theta^{\delta} = \frac{1}{2}R^{\alpha}{}_{[\beta\gamma\delta]}\,\theta^{\beta}\wedge\theta^{\gamma}\wedge\theta^{\delta}$
$$\rightarrow \quad R^{\alpha}{}_{[\beta\gamma\delta]} = 0.$$

The Cartan Structural Equations are equivalent to the definition of the torsion and curvature tensors, but stated in the language of forms. The Bianchi identities are a trivial consequence following immediately from the Cartan Structural Equations.

TENSOR VALUED FORMS

In the third chapter of the book of Alvarez is it said that all indices are not created equal. Different indices perform different functions. Examples.

(1) $\nabla_{e_\alpha} e_\beta = \Gamma^\gamma{}_{\alpha\beta} e_\gamma$ defines the connection coefficients, but

$\nabla_X e_\beta = \Gamma^\gamma{}_{\alpha\beta} X^\alpha e_\gamma = \Gamma^\gamma{}_{\alpha\beta}\theta^\alpha(X) e_\gamma$ shows that the index α has a natural function as a one-form index, so we define the connection one-forms: $\omega^\alpha{}_\beta \equiv \Gamma^\alpha{}_{\gamma\beta}\theta^\gamma$

so that $\nabla_X e_\beta = \omega^\alpha{}_\beta(X) e_\alpha$. This equation shows that $\omega^\alpha{}_\beta(X)$ acts as a linear transformation of the frame vectors, so we call $\omega^\alpha{}_\beta$ a "$gl(n\mathbb{R})$-valued one-form," since ~~evaluating it on a vector~~ leads to a linear transformation of the tangent space.

Then the covariant differential of e_β is: $\nabla e_\beta = e_\alpha \otimes \omega^\alpha{}_\beta$,

So the covariant differential of a general vector field is: $\nabla X = e_\alpha \otimes \omega^\alpha{}_\beta X^\beta = (e_\alpha \otimes \theta^\beta \otimes \omega^\alpha{}_\beta)(X)$

ie we are lead to define: $\omega = e_\alpha \otimes \theta^\beta \otimes \omega^\alpha{}_\beta$ $\qquad - dX^\alpha e_\alpha$

(2) Because $T^\gamma{}_{\alpha\beta} = T^\gamma{}_{[\alpha\beta]}$: $T = T^\gamma{}_{\alpha\beta} e_\gamma \otimes \theta^\alpha \otimes \theta^\beta = e_\gamma \otimes \frac{1}{2} T^\gamma{}_{\alpha\beta}\theta^\alpha \wedge \theta^\beta \equiv e_\gamma \otimes \Theta^\gamma$

ie the antisymmetric pair of indices leads us naturally to defining a set of n two-forms Θ^γ or a vector-valued twoform: $\Theta \equiv e_\gamma \otimes \Theta^\gamma \equiv T$.

(3) Because $R^\gamma{}_{\delta\alpha\beta} = R^\gamma{}_{\delta[\alpha\beta]}$: $R = R^\gamma{}_{\delta\alpha\beta} e_\gamma \otimes \theta^\delta \otimes \theta^\alpha \otimes \theta^\beta = e_\gamma \otimes \theta^\delta \otimes \frac{1}{2} R^\gamma{}_{\delta\alpha\beta}\theta^\alpha \wedge \theta^\beta$

$R \equiv e_\gamma \otimes \theta^\delta \otimes \Omega^\gamma{}_\delta \equiv \Omega$

ie the antisymmetric pair of indices leads us naturally to defining a set of two-forms $\Omega^\alpha{}_\beta$ or a $gl(n\mathbb{R})$-valued two form or $\binom{1}{1}$ tensor-valued form $\Omega \equiv R$.

So the torsion and curvature tensors are most naturally interpreted as operators on 2-surface elements; namely if X, Y are 2 vectors $R(X,Y)$ is that linear transformation of the tangent space which assigns to Z minus its first order variation under parallel transport around the "parallelogram". [actually a pentangle since need $[X,Y]$ to "close parallelogram"]. Similarly T assigns a vector $T(X,Y)$ to the area element spanned by X, Y which represents the failure of the "parallel transport parallelogram" ~~to close~~ determined by X, Y to close.

In general an $\binom{r}{s}$ tensor-valued p-form is simply an $\binom{r+p}{s}$ tensor which is alternating in p of its covariant indices (in fact more indices may be alternating but we could choose to single out p of them): $S = S^{\alpha_1 \ldots \alpha_s}{}_{\beta_1 \ldots \beta_r \gamma_1 \ldots \gamma_p} e_{\alpha_1} \otimes \ldots \otimes \theta^{\beta_1} \ldots \otimes \theta^{\gamma_1} \otimes \ldots \otimes \theta^{\gamma_p}$

$$= e_{(\alpha)} \otimes \theta^{(\beta)} \otimes \left\{ \frac{1}{p!} S^{(\alpha)}{}_{(\beta)\gamma_1 \ldots \gamma_p} \theta^{\gamma_1} \wedge \ldots \wedge \theta^{\gamma_p} \right\}$$

$$\equiv e_{(\alpha)} \otimes \theta^{(\beta)} \otimes S^{(\alpha)}{}_{(\beta)}$$

Thus $S^{(\alpha)}{}_{(\beta)}$ is a set of p-forms or an $\binom{r}{s}$ tensor valued p-form S.

<u>Alternative interpretation.</u> Let $\epsilon_\alpha, \epsilon^\beta$ be the standard basis and dual basis of $\mathbb{R}^n$ ie ϵ_α is a column vector $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}^\alpha$ and ϵ^β is a row vector $(0 \ldots \overset{\beta}{1} \ldots 0)$ and $\epsilon^\beta{}_\alpha \equiv \epsilon_\alpha \epsilon^\beta$ is their matrix product with a 1 in the αth row, β^{th} column, zero elsewhere, ie $\{\epsilon^\beta{}_\alpha\}$ is the

canonical basis of $gl(n\mathbb{R})$. It turns out that we shall be primarily be concerned with vector, covector- and $\binom{1}{1}$ tensor valued forms. In the expression for such a tensor-valued p form replace e_α, θ^β, $e_\alpha \otimes \theta^\beta$ by $\epsilon_\alpha, \epsilon^\beta$, $\epsilon_\alpha \epsilon^\beta \equiv \epsilon^\beta{}_\alpha$ in the tensor-valued factor:

$$e = (e_1, \dots, e_n) = e_\alpha \epsilon^\alpha \qquad \Theta = \epsilon_\alpha \Theta^\alpha$$

$$\theta = \begin{pmatrix} \theta^1 \\ \vdots \\ \theta^n \end{pmatrix} = \epsilon_\alpha \theta^\alpha \qquad X = \epsilon_\alpha X^\alpha$$

$$\omega = \| \omega^\alpha{}_\beta \| = \omega^\alpha{}_\beta \epsilon^\beta{}_\alpha$$

$$\Omega = \| \Omega^\alpha{}_\beta \| = \Omega^\alpha{}_\beta \epsilon^\beta{}_\alpha$$

X is a vector-valued zero form, e a covector-valued zero form.

Note a $\binom{1}{1}$ tensor-valued 1-form but a "connection-valued" 1-form, because $\Gamma^\alpha{}_{\beta\gamma}$ doesn't transform like a tensor under change of basis. Similarly we could define other geometric object valued forms.

Using this notation and the conventions of matrix multiplication we can write matrix equations to replace certain equations involving such forms. Examples:

$\nabla e = e\omega$ for $\nabla e_\alpha = e_\beta \omega^\beta{}_\alpha \equiv e_\beta \otimes \omega^\beta{}_\alpha$

$\omega \wedge \theta$ for $\omega^\alpha{}_\beta \wedge \theta^\beta$

$e \Omega X$ for $e_\alpha \Omega^\alpha{}_\beta X^\beta \equiv e_\alpha \otimes \Omega^\alpha{}_\beta X^\beta$

Note occasional suppression of $\otimes$.

Multiplication (matrix) can occur only if adjacent indices are in the proper position: $\alpha \quad \alpha$

Thus: $\theta(e) = I$ for $\theta^\alpha(e_\beta) = \delta^\alpha{}_\beta$

In other words we are going to be sloppy.

It is convenient to have both interpretations in mind when writing tensor-valued forms, ie Ω simultaneously stands for the Riemann tensor $e_\alpha \otimes \theta^\beta \otimes \Omega^\beta{}_\alpha$ and the matrix of one forms $\| \Omega^\beta{}_\alpha \|$, although depending on the context one interpretation may be explicit.

~~Note that if~~

Note that if S is an $\binom{r}{s}$-tensor valued zero form, $\nabla S = \nabla_\alpha S \otimes \theta^\alpha$ is an $\binom{r}{s}$ tensor-valued one form. Also $e\theta = e_\alpha \otimes \theta^\alpha$ is just what we called Θ in the former interpretation, and it is just the identity operator on each tangent space with frame components:

$$\delta^\alpha{}_\beta \qquad \left(\Theta = e_\alpha \otimes \theta^\alpha = \delta^\alpha{}_\beta e_\alpha \otimes \theta^\beta = \text{Cartan's } d\theta \right).$$

CHANGE OF BASIS.

If $\boxed{e' = e \cdot a}$ then $\boxed{\theta' = a^{-1} \cdot \theta}$ so that $\theta'(e') = I$. ie $\begin{cases} e'_\alpha = e_\beta a^\beta{}_\alpha \\ \theta'^\alpha = a^{-1\alpha}{}_\beta \theta^\beta \end{cases}$

$$\nabla e'_\beta = (\nabla_{e'} e'_\beta) \otimes \theta'^\alpha = \nabla_{e_\gamma} (e_\delta a^\delta{}_\beta)\, a^\gamma{}_\alpha \otimes \theta'^\alpha = \{ (\nabla_{e_\gamma} e_\delta) a^\delta{}_\beta + e_\delta (\nabla_{e_\gamma} a^\delta{}_\beta) \} \otimes \theta^\gamma$$

$$= e_\mu \Gamma^\mu{}_{\gamma\delta} a^\delta{}_\beta \otimes \theta^\gamma + e_\delta \otimes \underbrace{(\partial_\gamma a^\delta{}_\beta)}_{da^\delta{}_\beta} \theta^\gamma = e'_\nu \otimes a^{-1\nu}{}_\mu \{ \omega^\mu{}_\delta a^\delta{}_\beta + da^\mu{}_\beta \} \equiv e_\nu{}' \otimes \omega^\nu{}_\mu{}'$$

$$\boxed{\omega' = a^{-1}(\omega a + da)}$$

The torsion and curvature tensors "transform as tensors" under the frame change:

$$\Theta'^\alpha = a^{-1\alpha}{}_\beta \Theta^\beta \quad \leftrightarrow \quad \boxed{\Theta' = a^{-1} \Theta}$$

$$\Omega'^\alpha{}_\beta = a^{-1\alpha}{}_\gamma \Omega^\gamma{}_\delta a^\delta{}_\beta \quad \leftrightarrow \quad \boxed{\Omega' = a^{-1} \Omega a} = \mathrm{ad}(a^{-1})(\Omega) \quad \leftarrow \left\{ \begin{array}{l} \Omega \text{ often called a "2-form} \\ \text{of type ad."} \end{array} \right\}$$

COVARIANT EXTERIOR DERIVATIVE

We would like to extend the exterior derivative d defined for ordinary forms to a covariant exterior derivative D defined for tensor-valued forms.

On the one extreme we have ordinary forms for which: $D\beta \equiv d\beta$

On the other we have tensor-valued zero-forms: $DQ \equiv \nabla Q$ $\quad (Df \equiv \nabla f \equiv df$ for functions $f)$

Remark. If Q is a tensor-valued zero-form and β a form we can write

$Q \otimes \beta = Q \wedge \beta$ analogous to writing $f\beta = f \wedge \beta$ for functions f (zero-forms)

We extend D to tensor-valued p-forms S by: $D(S \wedge \beta) = DS \wedge \beta + (-1)^p S \wedge d\beta$

where β is an ordinary form. Note that for S a scalar-valued p-form, ie ordinary form, this is just the usual rule for forms. Thus:

$$S = \underbrace{e_{(\alpha)} \otimes \theta^{(\beta)}}_{\substack{\text{tensor-valued} \\ \text{zero-form}}} \otimes \underbrace{S^{(\alpha)}{}_{(\beta)}}_{\substack{\text{ordinary} \\ \text{2-forms}}} = (e_{(\alpha)} \otimes \theta^{(\beta)}) \wedge S^{(\alpha)}{}_{(\beta)}$$

$$DS = \underbrace{D(e_{(\alpha)} \otimes \theta^{(\beta)})}_{\nabla(e_{(\alpha)} \otimes \theta^{(\beta)})} \wedge S^{(\alpha)}{}_{(\beta)} + (e_{(\alpha)} \otimes \theta^{(\beta)}) \wedge \underbrace{DS^{(\alpha)}{}_{(\beta)}}_{dS^{(\alpha)}{}_{(\beta)}}$$

Fact: $\nabla e_\alpha = e_\beta \otimes \omega^\beta{}_\alpha$, $\nabla \theta^\alpha = -\omega^\alpha{}_\beta \otimes \theta^\beta$

Thus almost by inspection:

$$DS = [e_{(\alpha)} \otimes \theta^{(\beta)}] \Big\{ dS^{(\alpha)}{}_{(\beta)} + \sum_{i=1}^{s} \omega^{\alpha_i}{}_\gamma \wedge S^{\alpha_1 \cdots \gamma \cdots \alpha_s}{}_{(\beta)} \underbrace{- \sum_{i=1}^{r} \omega^\gamma{}_{\beta_i} S^{(\alpha)}{}_{\beta_1 \cdots \gamma \cdots \beta_r}}_{-(-1)^p \sum_{i=1}^{r} S^{(\alpha)}{}_{\beta_1 \cdots \gamma \cdots \beta_r} \wedge \omega^\gamma{}_{\beta_i}} \Big\}$$

Thus the covariant exterior derivative of an $\binom{r}{s}$ tensor-valued p-form is an $\binom{r}{s}$ tensor valued $(p+1)$-form, and we will lapse into writing $DS^{(\alpha)}{}_{(\beta)}$ for $(DS)^{(\alpha)}{}_{(\beta)}$ exactly as we write $\nabla_\alpha X^\beta$ for $(\nabla X)^\beta{}_\alpha$.

$$\boxed{\begin{array}{l}\text{or define} \\ \underbrace{D\{S^{(\alpha)}{}_{(\beta)}\}}_{\substack{\text{collection} \\ \text{of p-forms}}} = \underbrace{\{DS\}^{(\alpha)}{}_{(\beta)}}_{\substack{\text{components of} \\ \text{tensor-valued p-form}}}\end{array}}$$

Θ, Θ are vector-valued 1-forms and 2-forms respectively so:

$$D\Theta = d\Theta + \omega \wedge \Theta \quad \leftrightarrow \quad D\theta^\alpha = d\theta^\alpha + \omega^\alpha{}_\beta \wedge \theta^\beta$$

$$D\Theta = d\Theta + \omega \wedge \Theta \quad \leftrightarrow \quad D\Theta^\alpha = d\Theta^\alpha + \omega^\alpha{}_\beta \wedge \Theta^\beta$$

Ω is a $\binom{1}{1}$ tensor valued 2-form so:

$$D\Omega = d\Omega + \omega \wedge \Omega - \Omega \wedge \omega \quad \leftrightarrow \quad D\Omega^\alpha{}_\beta = d\Omega^\alpha{}_\beta + \omega^\alpha{}_\gamma \wedge \Omega^\gamma{}_\beta - \Omega^\alpha{}_\gamma \wedge \omega^\gamma{}_\beta$$

ω is a $\binom{1}{1}$ valued one-form, but not a $\binom{1}{1}$ tensor valued form so we really shouldn't use these formulas, but even if we tried: $D\omega = d\omega + \omega \wedge \omega - \omega \wedge \omega = d\omega$.

Thus $De = \Theta\omega$ defines the connection forms, and recalling the Cartan Structural equations and Bianchi Identities, they can be written:

$$\left.\begin{array}{l}\Theta = D\theta \\ \Omega = d\omega + \omega \wedge \omega\end{array}\right\} \text{Cartan S.E's} \qquad \left.\begin{array}{l}D\Theta = \Omega \wedge \theta \\ D\Omega = 0\end{array}\right\} \text{Bianchi I's}$$

Note that D enjoys many of the properties of d. For instance if $f: M \to \bar{M}$ and $S, \bar{S}$ are tensor-valued forms on $M, \bar{M}$ respectively such that $S = f^* \bar{S}$ [here we think of $S, \bar{S}$ as the collection of p-forms $S^{(\alpha)}{}_{(\beta)}$, $\bar{S}^{(\alpha)}{}_{(\beta)}$ for which f^* makes sense individually $S^{(\alpha)}{}_{(\beta)} = f^* \bar{S}^{(\alpha)}{}_{(\beta)}$] then provided that $\omega = f^* \bar{\omega}$

we have $DS^{(\alpha)}{}_{(\beta)} = f^* \bar{D}\bar{S}^{(\alpha)}{}_{(\beta)}$, ie "$D \circ f^* = f^* \circ \bar{D}$". $\left\{ \begin{array}{l} \text{since } d \circ f^* = f^* \circ d \text{ and} \\ f^*(\beta \wedge \gamma) = f^*(\beta) \wedge f^*(\gamma). \end{array} \right.$

If g is a $\binom{2}{0}$ symmetric tensor, it is a $\binom{2}{0}$ tensor-valued zeroform so:

$$Dg = dg - \omega g - g\omega \quad \leftrightarrow \quad Dg_{\alpha\beta} = dg_{\alpha\beta} - \omega^\gamma{}_\alpha g_{\beta\gamma} - g_{\gamma\alpha}\omega^\gamma{}_\beta$$
$$\equiv dg_{\alpha\beta} - (\omega_{\alpha\beta} + \omega_{\beta\alpha})$$

If g is covariant constant $Dg_{\alpha\beta} = 0$. If $g_{\alpha\beta}$ = constants: $\omega_{\alpha\beta} + \omega_{\beta\alpha} = 0$ (ie for an orthonormal frame say.)

<u>COMPONENT FORM OF COVARIANT EXTERIOR DERIVATIVE.</u> (Schouten 5.32, 10.10).

Preliminaries: (i) $V_{[\alpha_1 \ldots \alpha_p]} = \frac{1}{p!} \delta^{\beta_1 \ldots \beta_p}_{\alpha_1 \ldots \alpha_p} V_{\beta_1 \ldots \beta_p} = \frac{1}{p(p-1)!} \left\{ \delta^{\beta_1}_{\alpha_1} \delta^{\beta_2 \ldots \beta_p}_{\alpha_2 \ldots \alpha_p} - \delta^{\beta_2}_{\alpha_2} \delta^{\beta_1 \beta_3 \ldots \beta_p}_{\alpha_1 \ldots \alpha_p} + \delta^{\beta_1}_{\alpha_3} \delta^{\beta_2 \ldots \beta_p}_{\alpha_1 \ldots \alpha_p} - \ldots \right\}$

$$= \frac{1}{p} \sum_{s=1}^{p} (-1)^{s-1} V_{\alpha_s [\alpha_1 \ldots \hat{\alpha}_s \ldots \alpha_p]}$$

(ii) $$d\omega(X_1 \ldots X_{p+1}) \equiv \sum_{j=1}^{p+1} (-1)^{j+1} X_j \omega(X_1 \ldots \hat{X}_j \ldots X_{p+1}) + \frac{1}{2} \sum_{i,j} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots X_{p+1})$$

so $$(d\omega)_{\alpha_1 \ldots \alpha_{p+1}} = \underbrace{\sum_{j=1}^{p+1} (-1)^{j-1} \partial_j \omega_{\alpha_1 \ldots \hat{\alpha}_j \ldots \alpha_{p+1}}}_{(p+1)\ \partial_{[\alpha_1} \omega_{\alpha_2 \ldots \alpha_{p+1}]}} + \frac{1}{2} \sum (-1)^i \underbrace{\left[\underbrace{\sum (-1)^j C^k{}_{\alpha_i \alpha_j} \omega_{k \alpha_1 \ldots \hat{\alpha}_i \ldots \hat{\alpha}_j \ldots \alpha_{p+1}}}_{p\, C^k{}_{\alpha_i [\alpha_j} \omega_{|k| \alpha_1 \ldots \hat{\alpha}_i \ldots \alpha_{p+1}]}} \right]}_{\frac{1}{2} p(p+1)\ C^k{}_{[\alpha_2 \alpha_1} \omega_{|k| \alpha_3 \ldots \alpha_{p+1}]}}$$

$$(d\omega)_{\alpha_1 \ldots \alpha_{p+1}} = (p+1) \left\{ \partial_{[\alpha_1} \omega_{\alpha_2 \ldots \alpha_{p+1}]} - \frac{p}{2} C^k{}_{[\alpha_1 \alpha_2} \omega_{|k| \alpha_3 \ldots \alpha_{p+1}]} \right\} \quad \text{where } \partial_\alpha \equiv e_\alpha,\ [e_\alpha, e_\beta] = C^\gamma{}_{\alpha\beta} e_\gamma.$$

Now compute; given that S is an $\binom{r}{s}$-tensor valued p-form:

$$\nabla_\mu S^{\alpha_1 \ldots \alpha_s}{}_{\beta_1 \ldots \beta_r \gamma_1 \ldots \gamma_p} = \partial_\mu S^{(\alpha)}{}_{(\beta)(\gamma)} + \Gamma^{\alpha_1}_{\mu\nu} S^{\nu \alpha_2 \ldots \alpha_s}{}_{(\beta)(\gamma)} + \ldots - \Gamma^\nu_{\mu\beta_1} S^{(\alpha)}{}_{\nu \beta_2 \ldots \beta_r (\gamma)} - \ldots$$
$$\underbrace{- \Gamma^\nu_{\mu\gamma_1} S^{(\alpha)}{}_{(\beta)\nu\gamma_2 \ldots \gamma_p} - \ldots - \Gamma^\nu_{\mu\gamma_p} S^{(\alpha)}{}_{(\beta)\gamma_1 \ldots \gamma_{p-1}\nu}}$$
$$- \sum_{s=1}^{p} (-1)^{s+1} \Gamma^\nu_{\mu\gamma_s} S^{(\alpha)}{}_{(\beta)\nu\gamma_1 \ldots \hat{\gamma}_s \ldots \gamma_p} \underset{\text{by (i)}}{=} -p\, \Gamma^\nu_{[\mu\gamma_1} S^{(\alpha)}{}_{|(\beta)\nu|\gamma_2 \ldots \gamma_p]}$$

$$\nabla_{[\mu} S^{(\alpha)}{}_{|(\beta)|\gamma_1 \ldots \gamma_p]} = \partial_{[\mu} S^{(\alpha)}{}_{(\beta)\gamma_1 \ldots \gamma_p]} + \Gamma^{\alpha_1}_{[\mu|\nu|} S^{\nu\alpha_2 \ldots \alpha_s}{}_{(\beta)\gamma_1 \ldots \gamma_p]} + \ldots - \Gamma^\nu_{[\mu|\beta_1|} S^{(\alpha)}{}_{|\nu\beta_2 \ldots \beta_r|\gamma_1 \ldots \gamma_p]}$$
$$- p\, \Gamma^\nu_{[\mu\gamma_1} S^{(\alpha)}{}_{|(\beta)|\nu|\gamma_2 \ldots \gamma_p]}$$

Now $\Gamma^\nu_{[\mu\gamma_1]} = \frac{1}{2} C^\nu{}_{\mu\gamma_1} + \frac{1}{2} T^\nu{}_{\mu\gamma_1}$. The $C^\nu{}_{\mu\gamma_1}$ term plus the $\partial_{[\mu}$ - term yield dS. The other terms ~~added to~~ in the first line added to this term is just DS, apart from a common factor $(p+1)$

$$\boxed{\{DS^{(\alpha)}{}_{(\beta)}\}_{\mu\gamma_1 \ldots \gamma_p} \equiv (p+1) \left\{ \nabla_{[\mu} S^{(\alpha)}{}_{|(\beta)|\gamma_1 \ldots \gamma_p]} + \frac{p}{2} T^\nu{}_{[\mu\gamma_1} S^{(\alpha)}{}_{|(\beta)|\nu|\gamma_2 \ldots \gamma_p]} \right\}}$$

Compare $[dB]_{\mu\gamma_1 \ldots \gamma_p} = (p+1) \left\{ \partial_{[\mu} B_{\gamma_1 \ldots \gamma_p]} - \ldots \right\}$. So the Bianchi Identities are:

(I) $\nabla_{[\mu} T^\gamma{}_{\alpha\beta]} + T^\nu{}_{[\mu\alpha} T^\gamma{}_{|\nu|\beta]} = R^\gamma{}_{[\mu\alpha\beta]}$

(II) $\nabla_{[\mu} R^\alpha{}_{|(\beta)|\gamma\delta]} + T^\nu{}_{[\mu\gamma} R^\alpha{}_{|(\beta)|\delta]\nu} = 0$

Thus in a torsionless space D is just the properly normalized antisymmetric covariant derivative and can thus be defined as $DS(X_1 \ldots X_{p+1}) = \sum_{j=1}^{p+1} (-1)^{j+1} (\nabla_{X_j} S)(X_1 \ldots \hat{X}_j \ldots X_{p+1}) + \cancel{\sum_{i,j} (-1)^{i+j} S([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots X_{p+1})}$

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THE EXTERIOR DERIVATIVE EXPOSED

First an innocent computation:

$$\nabla_{[\alpha_1}\omega_{\alpha_2\ldots\alpha_{p+1}]} = \partial_{[\alpha_1}\omega_{\alpha_2\ldots\alpha_{p+1}]} - p\underbrace{\Gamma^{\beta}_{[\alpha_1\alpha_2}}_{\frac{1}{2}(T^{\beta}_{\alpha_1\alpha_2}+C^{\beta}_{\alpha_1\alpha_2})}\omega_{|\beta|\alpha_3\ldots\alpha_{p+1}]}$$

$$= \underbrace{\partial_{[\alpha_1}\omega_{\alpha_2\ldots\alpha_{p+1}]} - \tfrac{1}{2}pC^{\beta}_{[\alpha_1\alpha_2}\omega_{|\beta|\alpha_3\ldots\alpha_{p+1}]}}_{\frac{1}{p+1}(d\omega)_{\alpha_1\ldots\alpha_{p+1}}} - \frac{p}{2}T^{\beta}_{[\alpha_1\alpha_2}\omega_{|\beta|\alpha_3\ldots\alpha_{p+1}]}$$

(I) $$(d\omega)_{\alpha_1\ldots\alpha_{p+1}} = (p+1)\nabla_{[\alpha_1}\omega_{\alpha_2\ldots\alpha_{p+1}]} + \frac{p(p+1)}{2}T^{\beta}_{[\alpha_1\alpha_2}\omega_{|\beta|\alpha_3\ldots\alpha_{p+1}]}$$

(II) or: $$d\omega(X_1\ldots X_{p+1}) = \underbrace{\sum_{j=1}^{p+1}(-1)^{j+1}\underbrace{(\nabla_{X_j}\omega)(X_1\ldots\hat{X}_j\ldots X_{p+1})}_{\nabla_{X_j}\{\omega(X_1\ldots\hat{X}_j\ldots X_{p+1})\}-\sum_{i=1}^{p+1}\omega(X_1\ldots\nabla_{X_j}X_i\ldots\hat{X}_j\ldots X_{p+1})}}_{\sum_{j=1}^{p+1}(-1)^{j+1}\nabla_{X_j}\{\omega(X_1\ldots\hat{X}_j\ldots X_{p+1})\}-\frac{1}{2}\sum_{ij}(-1)^{i+j}\omega(\nabla_{X_j}X_i-\nabla_{X_i}X_j,\,X_1\ldots\hat{X}_i\ldots\hat{X}_j\ldots X_{p+1})} - \frac{1}{2}\sum_{ij}(-1)^{i+j}\omega(T(X_iX_j),X_1,\ldots\hat{X}_i\ldots\hat{X}_j\ldots X_{p+1})$$

But $T(X_iX_j) - \nabla_{X_i}X_j + \nabla_{X_j}X_i = -[X_iX_j]$, so

(III) $$d\omega(X_1\ldots X_{p+1}) = \sum_{j=1}^{p+1}(-1)^{j+1}\nabla_{X_j}\{\omega(X_1\ldots\hat{X}_j\ldots X_{p+1})\} + \frac{1}{2}\sum_{ij}(-1)^{i+j}\omega([X_iX_j],X_1\ldots\hat{X}_i\ldots\hat{X}_j\ldots X_{p+1})$$

Thus we have an exactly equivalent formula for d but expressed in terms of the covariant derivative. But notice that this formula as a differential operator can act not only on ordinary forms but on any tensor-valued form. [Note that we could have written down (III) immediately from the definition of $d\omega(X_1\ldots X_{p+1})$ since $\nabla_{X_j}\{\omega(X_1\ldots\hat{X}_j\ldots X_{p+1})\} \equiv X_j\{\omega(X_1\ldots\hat{X}_j\ldots X_{p+1})\}$; but the alternate expressions (I), (II) are also useful which explains our choice of action (?)] So we call it big D, the exterior covariant derivative, an operator which takes $\binom{r}{s}$ tensor valued p-forms to $\binom{r}{s}$ tensor-valued (p+1)-forms.

For a $\binom{0}{0}$ tensor-valued p-form ω ~~clearly~~ by definition $D\omega = d\omega$.

For an $\binom{r}{s}$ tensor-valued 0-form S clearly: $DS = \nabla S$

Thus D is the natural interpolation between d and ∇.

(See reverse for easy derivation of the Ricci Identities)

Note D also annihilates n-forms.

$$\{DS^{(\alpha)}{}_{(\beta)}\}_{\gamma\delta_1\dots\delta_p} = (p+1)\{\nabla_{[\gamma}S^{(\alpha)}{}_{|(\beta)|\delta_1\dots\delta_p]} + \tfrac{p}{2}T^{\delta}{}_{[\gamma\delta_1}S^{(\alpha)}{}_{|(\beta)|\delta_2\dots\delta_p]}\}$$

$$\left.\begin{aligned} Df &= df \\ D^2f &= d^2f = 0 \end{aligned}\right\} \quad 0 = 2\{\nabla_{[\gamma}\nabla_{\delta]}f + \tfrac{1}{2}T^{\delta}{}_{[\gamma\delta]}\nabla_\delta f\} \qquad \boxed{\nabla_{[\alpha}\nabla_{\beta]}f = -T^{\gamma}{}_{\alpha\beta}\nabla_\gamma f}$$

$$DX^\alpha = dX^\alpha + \omega^\alpha{}_\beta X^\beta \qquad D^2X^\alpha = \cancel{d^2X^\alpha} + d\omega^\alpha{}_\beta X^\beta - \omega^\alpha{}_\beta\wedge dX^\beta + \omega^\alpha{}_\beta\wedge(dX^\beta + \omega^\beta{}_\gamma X^\gamma)$$

$$D^2X^\alpha = \Omega^\alpha{}_\beta X^\beta$$

$$D^2X = \Omega\llcorner X$$

$$2\{\nabla_{[\gamma}\nabla_{\delta_1]}X^\alpha + \tfrac{1}{2}T^{\delta}{}_{[\gamma\delta_1]}\nabla_\delta X^\alpha\} = X^\beta R^\alpha{}_{\beta\gamma\delta_1}$$

$$\boxed{2\nabla_{[\alpha}\nabla_{\beta]}X^\gamma = X^\delta R^\gamma{}_{\delta\alpha\beta} - T^{\delta}{}_{\alpha\beta}\nabla_\delta X^\gamma}$$

Suppose $T^{\alpha\beta}$ is a symmetric tensor on a V_4. Define the vector valued one form: $T = e_\alpha T^\alpha{}_\beta\theta^\beta$

Since $D\eta\dots \equiv 0$ for $\Theta = 0$, $\quad {}^*T = e_\alpha T^{\alpha\beta}\eta_\beta$

$$D^*T^\alpha = DT^{\alpha\beta}\wedge\eta_\beta = T^{\alpha\beta}{}_{;\gamma}\underbrace{\theta^\gamma\wedge\eta_\beta}_{\delta^\gamma_\beta\eta} = T^{\alpha\beta}{}_{;\beta}\eta$$

Both the energy momentum and Einstein tensors are of this type.

Let F be a 2-form: $F = \frac{1}{2}F_{\alpha\beta}\theta^{\alpha\beta}$, $\quad {}^*F = \frac{1}{2}F_{\alpha\beta}\eta^{\alpha\beta} = \frac{1}{2}F^{\alpha\beta}\eta_{\alpha\beta}$

$$d^*F = D^*F = \tfrac{1}{2}DF^{\alpha\beta}\wedge\eta_{\alpha\beta} = \tfrac{1}{2}F^{\alpha\beta}{}_{;\gamma}\underbrace{\theta^\gamma\wedge\eta_{\alpha\beta}}_{(\delta^\gamma_\beta\eta_\alpha - \delta^\gamma_\alpha\eta_\beta)} = F^{\alpha\gamma}{}_{;\gamma}\eta_\alpha$$

Define $j = j_\alpha\theta^\alpha$: $\quad d^*F = {}^*j \leftrightarrow F^{\alpha\beta}{}_{;\beta} = j^\alpha$.

$dF = 0 \Leftrightarrow F_{[\alpha\beta;\gamma]} = 0$ can alternately be written $0 = \cancel{d^*F = DF = \tfrac{1}{2}DF_{\alpha\beta}\wedge\theta^{\alpha\beta} = \tfrac{1}{2}DF^{\alpha\beta}\wedge\theta_{\alpha\beta} = \tfrac{1}{2}F^{\alpha\beta}{}_{;\gamma}}$?

Define the vector valued one form: $\quad F = e_\alpha F^\alpha{}_\beta\theta^\beta$

$$DF = e_\alpha\{DF^\alpha{}_\beta\}\theta^\beta = e_\alpha g^{\alpha\gamma}(DF)_{\gamma\beta}\theta^\beta$$

$$0 = D^2g_{\alpha\beta} = -g_{\gamma\beta}\Omega^\gamma{}_\alpha - g_{\alpha\gamma}\Omega^\gamma{}_\beta = -(\Omega_{\beta\alpha} + \Omega_{\alpha\beta}) \rightarrow \quad g^{\alpha\beta}\Omega_{\alpha\beta} = \Omega^\alpha{}_\alpha = 0; \qquad \boxed{\mathrm{TR}\,\Omega = 0 \Leftarrow D^2g = 0}$$

$$0 = Dg_{\alpha\beta} = dg_{\alpha\beta} + \omega_{\alpha\beta} + \omega_{\beta\alpha} \qquad \boxed{\mathrm{TR}\,\omega = 0 \Leftarrow Dg = 0 = dg}$$

Ricci Identities

$$DS^{\alpha_1\dots}{}_{\beta_1\dots} = dS^\alpha{}_{\beta_1\dots} + \omega^\alpha{}_\gamma\wedge S^{\gamma\dots}{}_{\beta_1\dots} - (-1)^p S^{\alpha\dots}{}_\gamma\wedge\omega^\gamma{}_{\beta\dots}$$

$$DS^{\gamma\dots}{}_{\beta_1\dots} = dS^{\gamma\dots}{}_{\beta_1\dots} + \omega^\gamma{}_\delta\wedge S^{\delta\dots}{}_{\beta_1\dots} + \dots - (-1)^p S^{\gamma\dots}{}_{\delta\dots}\wedge\omega^\delta{}_{\beta_1} - \dots$$

$$DS^{\alpha_1\dots}{}_{\gamma\dots} = dS^{\alpha_1\dots}{}_{\gamma\dots} + \omega^{\alpha_1}{}_\delta\wedge S^{\delta\dots}{}_{\gamma\dots} + \dots - (-1)^p S^{\alpha_1\dots}{}_{\delta\dots}\wedge\omega^\delta{}_\gamma$$

$$\begin{aligned} D(DS^{\alpha_1\dots}{}_{\beta_1\dots}) = dDS^{\alpha_1\dots}{}_{\beta_1\dots} &= -\omega^{\alpha_1}{}_\gamma\wedge dS^{\gamma\dots}{}_{\beta_1\dots} - (-1)^p dS^{\alpha_1\dots}{}_\gamma\wedge\omega^\gamma{}_{\beta_1} + d\omega^{\alpha_1}{}_\gamma\wedge S^{\gamma\dots}{}_{\beta_1\dots} - (-1)^{2p}S^{\alpha_1\dots}{}_\gamma\wedge d\omega^\gamma{}_{\beta_1} + \\ &+ \omega^\alpha{}_\gamma\wedge DS^\gamma{}_\beta + \dots \quad + \omega^{\alpha_1}{}_\gamma\wedge dS^{\gamma\dots}{}_{\beta_1\dots} + \omega^{\alpha_1}{}_\gamma\wedge\omega^\gamma{}_\delta\wedge S^{\delta\dots}{}_{\beta_1\dots} - (-1)^p\omega^{\alpha_1}{}_\gamma\wedge S^\gamma{}_{\delta\dots}\wedge\omega^\delta{}_{\beta_1} + \dots \\ &- (-1)^{p+1}DS^{\alpha_1\dots}{}_\gamma\wedge\omega^\gamma{}_{\beta_1} \quad + (-1)^p\{dS^{\alpha_1\dots}{}_\gamma\wedge\omega^\gamma{}_{\beta_1} + \omega^{\alpha_1}{}_\delta\wedge S^{\delta\dots}{}_\gamma\wedge\omega^\delta{}_{\beta_1}\} - (-1)^p S^{\alpha_1\dots}{}_\gamma\wedge\omega^\delta{}_\gamma\wedge\omega^\gamma{}_{\beta_1} + \dots \end{aligned}$$

$$\boxed{D^2S^{\alpha_1\dots}{}_{\beta_1\dots} = \Omega^{\alpha_1}{}_\gamma\wedge S^{\gamma\dots}{}_{\beta_1\dots} + \dots - S^{\alpha_1\dots}{}_{\gamma\dots}\wedge\Omega^\gamma{}_{\beta_1} - \dots}$$

Linear Connection expressed in terms of $T^{\alpha}{}_{\beta\gamma}$ and an auxiliary symmetric tensor field $g_{\alpha\beta}$ of rank n and $\nabla_\gamma g_{\beta\gamma}$

$\boxed{\text{Schouten III § 3,4,5}}$ Let $g^{\alpha\beta}$ be the inverse. Raise, lower indices with $g_{\alpha\beta}, g^{\alpha\beta}$.

Define $\nabla_\alpha g^{\beta\gamma} \equiv Q_\alpha{}^{\beta\gamma}$. Then $g^{-1}g = 1$ $\quad (\nabla_\alpha g^{-1})g = -g^{-1}\nabla_\alpha g$ $\quad \nabla_\alpha g = -g\nabla_\alpha g^{-1} g$ $\quad \nabla_\alpha g_{\beta\gamma} = -Q_{\alpha\beta\gamma}$

Define $\Gamma_{\alpha\beta\gamma} = g_{\alpha\delta}\Gamma^{\delta}{}_{\beta\gamma}$. Then $-Q_{\alpha\beta\gamma} = \partial_\alpha g_{\beta\gamma} - g_{\delta\gamma}\Gamma^{\delta}{}_{\alpha\beta} - g_{\beta\delta}\Gamma^{\delta}{}_{\alpha\gamma} = \partial_\alpha g_{\beta\gamma} - \Gamma_{\gamma\alpha\beta} - \Gamma_{\beta\alpha\gamma}$.

$$\partial_\gamma g_{\alpha\beta} = \Gamma_{\beta\gamma\alpha} + \Gamma_{\alpha\gamma\beta} - Q_{\gamma\alpha\beta}$$
$$\partial_\beta g_{\gamma\alpha} = \Gamma_{\alpha\beta\gamma} + \Gamma_{\gamma\beta\alpha} - Q_{\beta\gamma\alpha}$$
$$-\partial_\alpha g_{\beta\gamma} = -\Gamma_{\gamma\alpha\beta} - \Gamma_{\beta\alpha\gamma} + Q_{\alpha\beta\gamma}$$

Define $\{\gamma\alpha\beta\} \equiv \gamma\alpha\beta + \alpha\beta\gamma + \beta\gamma\alpha$.

$\{\gamma,\alpha\beta\} \equiv \frac{1}{2}\partial_{\{\alpha}g_{\gamma\beta\}}$, $\quad \left\{{\gamma \atop \alpha\beta}\right\} = g^{\gamma\delta}\{\delta,\alpha\beta\}$.

$$\partial_{\{\gamma}g_{\alpha\beta\}} = \underbrace{2\Gamma_{\alpha(\gamma\beta)}} + \left[2\Gamma_{\alpha[\gamma\beta]} - 2\Gamma_{\alpha[\gamma\beta]}\right] + 2\Gamma_{\beta[\gamma\alpha]} + 2\Gamma_{\gamma[\beta\alpha]} - Q_{\{\gamma\alpha\beta\}}$$

$$\Gamma_{\alpha\gamma\beta} = \frac{1}{2}\partial_{\{\gamma}g_{\alpha\beta\}} - \Gamma_{\{\gamma[\alpha\beta]\}} + \frac{1}{2}Q_{\{\gamma\alpha\beta\}}$$

Recall $\Gamma^{\gamma}{}_{\alpha\beta} = \frac{1}{2}(T^{\gamma}{}_{\alpha\beta} + C^{\gamma}{}_{\alpha\beta})$

$$\begin{cases} \Gamma_{\alpha\gamma\beta} = \{\alpha,\gamma\beta\} + \frac{1}{2}\left[T_{\{\gamma\alpha\beta\}} + C_{\{\gamma\alpha\beta\}} - Q_{\{\gamma\alpha\beta\}}\right] \\ \Gamma^{\alpha}{}_{\gamma\beta} = \left\{{\alpha \atop \gamma\beta}\right\} + \frac{1}{2}\left[T_{\gamma}{}^{\alpha}{}_{\beta} - T^{\alpha}{}_{\beta\gamma} + T_{\beta\gamma}{}^{\alpha} + C\cdots - Q\cdots\right] \end{cases}$$

$\Gamma_{\gamma\alpha\beta} = \underbrace{\{\gamma,\alpha\beta\}}_{\frac{1}{2}g_{\{\gamma\alpha,\beta\}}} + \frac{1}{2}\left[C_{\{\gamma\alpha\beta\}} + T_{\{\gamma\alpha\beta\}}\right]$

$V_n = \{M_n, \nabla, g, \nabla g = 0, T = 0\}$ = Riemannian geometry ($Q = T = 0$, in a coord system $\Gamma^{\alpha}{}_{\beta\gamma} = \left\{{\alpha \atop \beta\gamma}\right\}$)

$U_n = \{M_n, \nabla, g, \nabla g = 0\}$ $\qquad \nabla g = 0$ metric connection

$W_n = \{M_n, \nabla, g, \nabla_\alpha g_{\beta\gamma} = -Q_\alpha g_{\beta\gamma}\}$ Weyl's geometry $\qquad \nabla g = -Q \otimes g$ semi-metric connection.

under a conformal transformation Q trans. suitably, get same connection.

In U_n, $[\nabla, \text{raising/lowering indices}] = 0$.

Ricci Identity: $\nabla_{[\gamma}\nabla_{\delta]}g_{\alpha\beta} = -\frac{1}{2}R^{\rho}{}_{\alpha\gamma\delta}g_{\rho\beta} - \frac{1}{2}R^{\rho}{}_{\beta\gamma\delta}g_{\alpha\rho} - T^{\rho}{}_{\gamma\delta}\nabla_\rho g_{\alpha\beta}$

$$= -\frac{1}{2}(R_{\beta\alpha\gamma\delta} + R_{\alpha\beta\gamma\delta}) - (\,) = -R_{(\alpha\beta)\gamma\delta} - T^{\rho}{}_{\gamma\delta}\nabla_\rho g_{\alpha\beta}$$

$$R_{(\alpha\beta)\gamma\delta} = \nabla_{[\gamma}Q_{\delta]\alpha\beta} + T_{\gamma\delta}{}^{\rho}Q_{\rho\alpha\beta}$$

$$\boxed{R_{(\alpha\beta)\gamma\delta} = 0 \quad \text{for metric connection}}$$

If $Q_{(\gamma\delta)\alpha\beta} \equiv 0$ then $Q_{\gamma\delta\alpha\beta}$ satisfies the identity:

$$Q_{\alpha\beta\gamma\delta} \equiv Q_{\gamma\delta\alpha\beta} - \frac{3}{2}\left(Q_{[\beta\alpha\gamma]\delta} + Q_{[\beta\delta\gamma]\alpha} + Q_{[\delta\alpha\gamma]\beta} + Q_{[\alpha\beta\delta]\gamma}\right)$$
$$+ Q_{\gamma\beta(\delta\alpha)} + Q_{\alpha\gamma(\delta\beta)} + Q_{\beta\delta(\alpha\gamma)} + Q_{\delta\alpha(\beta\gamma)} + Q_{\delta\gamma(\beta\alpha)} + Q_{\alpha\beta(\delta\gamma)}$$

Let $Q_{\gamma\delta\alpha\beta} = R_{\alpha\beta\gamma\delta}$. Recall 1st Bianchi Identity: $R^{\alpha}{}_{[\beta\gamma\delta]} = \nabla_{[\beta}T^{\alpha}{}_{\gamma\delta]} + T^{\epsilon}{}_{[\beta\gamma}T^{\alpha}{}_{|\epsilon|\delta]}$

Hence for a metric connection: $\quad R_{\gamma\delta\alpha\beta} = R_{\alpha\beta\gamma\delta} + (\nabla T, T^2)_{\alpha\beta\gamma\delta}$

Thus in a U_n, if we define a Ricci Tensor it will not be symmetric, and if we define an Einstein tensor it wont have zero divergence, nor be symmetric, nor possibly well defined? see Heyl.

Sup

suppose we let $\bar{\omega}^a{}_b = \omega^a{}_b + S^a{}_b$

$$\bar{\Omega}^a{}_b = d\bar{\omega}^a{}_b + dS^a{}_b + (\omega^a{}_c + S^a{}_c)\wedge(\omega^c{}_b + S^c{}_b) = \Omega^a{}_b + dS^a{}_b + \omega^a{}_c\wedge S^c{}_b + S^a{}_c\wedge\omega^c{}_b + S^a{}_c\wedge S^c{}_b$$
$$= \Omega^a{}_b + DS^a{}_b + S^a{}_c\wedge S^c{}_b.$$

It is amusing to note that if we defined $D\omega = d\omega$ then
$\bar{\Omega}^a{}_b = \Omega^a{}_b(\omega,\omega) + \Omega^a{}_b(S,\omega)$
where $\Omega^a{}_b(S,\omega) \equiv DS + S\wedge S$, D wrt ω

Let $S^a{}_{cb} = \frac{1}{2}T_{\{cab\}}$ and one separates the U_4 connection into its Riemannian part plus a torsion contribution.

If define $D\omega = d\omega + \omega\wedge\omega$

$$d\theta^a + \omega^a{}_b \wedge \theta^c = \tfrac{1}{2} T^a{}_{bc}\theta^{bc}$$
$$-\tfrac{1}{2} C^a{}_{bc}\theta^{bc} + \Gamma^a{}_{[bc]}\theta^{bc}$$

$$\Gamma^a{}_{[bc]} = \tfrac{1}{2}(C^a{}_{bc} + T^a{}_{bc})$$

$$g_{\alpha\gamma,\beta} = \Gamma_{\gamma\beta\alpha} + \Gamma_{\alpha\beta\gamma}$$
$$-g_{\alpha\beta,\gamma} = -\Gamma_{\alpha\gamma\beta} - \Gamma_{\beta\gamma\alpha}$$
$$g_{\beta\gamma,\alpha} = \Gamma_{\beta\alpha\gamma} + \Gamma_{\gamma\alpha\beta}$$

$$g_{\{\gamma\alpha,\beta\}} = 2\Gamma_{\gamma(\alpha\beta)} + 2\Gamma_{\alpha[\beta\gamma]} - 2\Gamma_{\beta[\gamma\alpha]} - 2\Gamma_{\gamma[\alpha\beta]}$$
$$2\Gamma_{\gamma[\alpha\beta]}$$

$$2\Gamma_{\gamma\alpha\beta} = g_{\{\gamma\alpha,\beta\}} + C_{\{\gamma\alpha\beta\}} + \underbrace{T_{\{\gamma\alpha\beta\}}}_{K_{\gamma\alpha\beta}}$$

<u>Consider a coordinate system:</u>

$$\Gamma^a{}_{(bc)} = \tfrac{1}{2}\Gamma^a{}_{bc}$$

$$\nabla_X X = X^a{}_{,a}X^b + \Gamma^a{}_{(bc)}X^bX^c$$
$$X = \frac{dx^a}{dt}$$

$$0 = \underbrace{g_{\alpha\beta,\gamma} - \Gamma_{\alpha\gamma\beta} - \Gamma_{\beta\gamma\alpha}}_{0} \underbrace{- \{\alpha;\gamma\beta\} - \{\beta;\gamma\alpha\} - K_{\alpha\gamma\beta} - K_{\beta\gamma\alpha}}_{0}$$

$$K_{(\gamma|\alpha|\beta)} = 0$$

$$\cancel{T_{\gamma\alpha\beta} - T_{\alpha\beta\gamma} + T_{\beta\gamma\alpha}}$$
$$\cancel{T_{\beta\alpha\gamma} - T_{\alpha\gamma\beta} + T_{\gamma\beta\alpha}}$$

$\boxed{\text{Torsion doesn't affect geodesics. ?}}$

$$\begin{cases} \tilde{\Gamma}^\gamma{}_{\alpha\beta} = \Gamma^\gamma{}_{\alpha\beta} - T^\gamma{}_{\alpha\beta} \\ \bar{\Gamma}^\gamma{}_{\alpha\beta} = \tfrac{1}{2}(\Gamma^\gamma{}_{\alpha\beta} + \tilde{\Gamma}^\gamma{}_{\alpha\beta}) = \Gamma^\gamma{}_{\alpha\beta} - \tfrac{1}{2}T^\gamma{}_{\alpha\beta} \end{cases}$$

unique torsionless connection having same geodesics.

$$g_{\gamma\alpha,\beta} = \Gamma_{\alpha\beta\gamma} + \Gamma_{\gamma\beta\alpha}$$
$$-g_{\alpha\beta,\gamma} = -\Gamma_{\beta\gamma\alpha} - \Gamma_{\gamma\gamma\beta}$$
$$g_{\beta\gamma,\alpha} = \Gamma_{\gamma\alpha\beta} + \Gamma_{\beta\alpha\gamma}$$

$$g_{\{\gamma\alpha,\beta\}} = 2\Gamma_{\gamma(\alpha\beta)} + 2\Gamma_{\alpha[\beta\gamma]} + 2\Gamma_{\beta[\alpha\gamma]}$$
$$= 2\Gamma_{\gamma\alpha\beta} - 2\Gamma_{\gamma[\alpha\beta]} + 2\Gamma_{\alpha[\beta\gamma]} - 2\Gamma_{\beta[\gamma\alpha]}$$

$$2\Gamma_{\gamma\alpha\beta} = g_{\{\gamma\alpha,\beta\}} + 2\underbrace{\Gamma_{\{\gamma[\alpha\beta]\}}}$$
$$\cancel{C_{\{\gamma\alpha\beta\}}} + T_{\{\gamma\alpha\beta\}}$$

THE EXTERIOR COVARIANT DERIVATIVE, THE TRANSPOSED CONNECTION, AND LIE DERIVATIVES

We working a fixed frame.

Let ∇ be a connection with coninection forms $\omega^\alpha{}_\beta = \Gamma^\alpha{}_{\gamma\beta}\theta^\gamma$, torsion $\Theta^\alpha = \frac{1}{2}T^\alpha{}_{\beta\gamma}\theta^\beta\wedge\theta^\gamma$,
related by $2\Gamma^\alpha{}_{[\beta\gamma]} = T^\alpha{}_{\beta\gamma} + C^\alpha{}_{\beta\gamma}$. Define the partial contraction $T^\alpha{}_\beta = T^\alpha{}_{\beta\gamma}\theta^\gamma$, $\Theta^\alpha = \frac{1}{2}\theta^\beta\wedge T^\alpha{}_\beta$

Define the <u>transposed connection</u> $\tilde\nabla$ by $\tilde\omega^\alpha{}_\beta = \omega^\alpha{}_\beta + T^\alpha{}_\beta$.

$$\tilde\Gamma^\alpha{}_{\beta\gamma} = \Gamma^\alpha{}_{\beta\gamma} + T^\alpha{}_{\gamma\beta}, \quad \begin{cases} 2\tilde\Gamma^\alpha{}_{[\beta\gamma]} = \tilde T^\alpha{}_{\beta\gamma} + C^\alpha{}_{\beta\gamma} = -T^\alpha{}_{\beta\gamma} + C^\alpha{}_{\beta\gamma} \\ \tilde\Gamma^\alpha{}_{(\beta\gamma)} = \Gamma^\alpha{}_{(\beta\gamma)} \end{cases} \qquad \tilde T^\alpha{}_{\beta\gamma} = -T^\alpha{}_{\beta\gamma}$$

$$\tilde\Gamma^\alpha{}_{\beta\gamma} = \Gamma^\alpha{}_{(\beta\gamma)} + \tfrac{1}{2}(C^\alpha{}_{\beta\gamma} + T^\alpha{}_{\beta\gamma})$$

$$\tilde\Gamma^\alpha{}_{\beta\gamma} = \Gamma^\alpha{}_{(\beta\gamma)} + \tfrac{1}{2}(C^\alpha{}_{\beta\gamma} - T^\alpha{}_{\beta\gamma}) = \Gamma^\alpha{}_{\gamma\beta} + C^\alpha{}_{\beta\gamma}$$

Thus in a coordinate system: $\tilde\Gamma^\alpha{}_{\beta\gamma} = \Gamma^\alpha{}_{\gamma\beta}$.

<u>Examples</u> X a vector field. $\tilde D X^\alpha = dX^\alpha + \tilde\omega^\alpha{}_\beta X^\beta = DX^\alpha + T^\alpha{}_\beta X^\beta = DX^\alpha + X\lrcorner\Theta^\alpha$

φ_α covector-valued form $\tilde D\varphi_\alpha = d\varphi_\alpha - \tilde\omega^\beta{}_\alpha\wedge\varphi_\beta = D\varphi_\alpha - T^\beta{}_\alpha\wedge\varphi_\beta$

This motivates a general formula.

$\boxed{\text{The } \sigma\text{'s}}$ Let $S^{(\alpha)}{}_{(\beta)}$ be a $\binom{r}{s}$ tensor-valued p-form, ie $S = [e_{(\alpha)}\otimes\theta^{(\alpha)}]\otimes S^{(\alpha)}{}_{(\beta)}$.

$$D\varphi = [e_\alpha\otimes\theta^{(\beta)}]\otimes\Big\{dS^{(\alpha)}{}_{(\beta)} + \underbrace{\sum_{i=1}^{s}\omega^{\alpha_i}{}_\gamma\wedge S^{\alpha_1\ldots\gamma\ldots\alpha_s}{}_{(\beta)} - \sum_{i=1}^{r}\omega^\gamma{}_{\beta_i}\wedge S^{(\alpha)}{}_{\beta\ldots\gamma\ldots\beta_r}}_{\sigma^{(\alpha)}{}_{(\beta)}{}^{(\gamma)}{}_{(\delta)}\,\omega^\sigma{}_\rho\wedge S^{(\delta)}{}_{(\gamma)}}\Big\}$$

If S_A symbolizes a tensor-valued p-form then: $DS_A = dS_A + \sigma_A{}^B{}^\gamma{}_\delta\,\omega^\delta{}_\gamma\wedge S_B$

These σ's are more general than their appearance in big D but show up any time an operation obeys the Liebnitz rule. For instance $p=0$ yields the special case of ~~the covariant~~ derivative with the same σ's. Similarly the Lie Derivative will involve the same σ's but with $\omega^\alpha{}_\beta$ replaced by $+\theta^\alpha(\mathcal{L}_X e_\beta)$ (note $\omega^\alpha{}_\beta(X) = \theta^\alpha(\nabla_X e_\beta)$). Let's compute the formula:

$$\mathcal{L}_X S = [e_{(\alpha)}\otimes\theta^{(\beta)}]\otimes\mathcal{L}_X\varphi^{(\alpha)}{}_{(\beta)} + (\mathcal{L}_X[e_{(\gamma)}\otimes\theta^{(\delta)}])\otimes\varphi^{(\gamma)}{}_{(\delta)}$$

But $0 = \mathcal{L}_X\theta^\beta(e_\alpha) = (\mathcal{L}_X\theta^\beta)(e_\alpha) + \theta^\beta(\mathcal{L}_X e_\alpha) \Rightarrow (\mathcal{L}_X\theta^\beta) = -\theta^\beta(\mathcal{L}_X e_\alpha)\theta^\alpha$

$(0 = \nabla_X\theta^\beta(e_\alpha) = (\nabla_X\theta^\beta)(e_\alpha) + \theta^\beta(\nabla_X e_\alpha) \rightarrow (\nabla_X\theta^\beta) = -\theta^\beta(\nabla_X e_\alpha)\theta^\alpha)$

So by the Liebnitz rule:

$$\mathcal{L}_X(e_{(\gamma)}\otimes\theta^{(\delta)}) = \sum_{i=1}^{s}\theta^{\gamma_i}(\mathcal{L}_X e_{\gamma_i})\,e_{\alpha_1\ldots\rho\ldots\alpha_s}\otimes\theta^{(\beta)} - \sum_{i=1}^{r}\theta^{\beta_i}(\mathcal{L}_X e_\rho)\,e_{(\alpha)}\otimes\theta^{\beta_1\ldots\rho\ldots\beta_r}$$

$$= \sigma^{(\alpha)}{}_{(\beta)}{}^{(\gamma)}{}_{(\delta)}{}^\rho{}_\sigma\,\theta^\sigma(\mathcal{L}_X e_\rho)\,e_{(\alpha)}\otimes\theta^{(\beta)} \qquad \text{(same for } \nabla)$$

So we have the preliminary formula:

$$\{\mathcal{L}_X S\}_A = \mathcal{L}_X S_A + \sigma_A{}^B{}^\gamma{}_\delta\,\theta^\delta(\mathcal{L}_X e_\gamma)\,S_B$$

If $p=0$, $\mathcal{L}_X S_A = X(S_A)$ and we are done.

Recall that on forms we have the identity: $\mathcal{L}_X = X\lrcorner d + dX\lrcorner$. and $X\lrcorner(\alpha\wedge\beta) = (X\lrcorner\alpha)\wedge\beta + (-1)^p\alpha\wedge(X\lrcorner\beta)$

$$\mathcal{L}_X S_A = X\lrcorner dS_A + d(X\lrcorner S_A) = X\lrcorner(DS_A - \sigma_A{}^B{}^\beta{}_\alpha\,\omega^\alpha{}_\beta\wedge S_B) + D(X\lrcorner S_A) - \sigma_A{}^B{}^\beta{}_\alpha\,\omega^\alpha{}_\beta\wedge(X\lrcorner S_B)$$

$$= X\lrcorner DS_A + D(X\lrcorner S_A) - \sigma_A{}^B{}^\beta{}_\alpha\,(X\lrcorner\omega^\alpha{}_\beta)\,S_B \qquad (A)$$

using $X\lrcorner(\omega^\alpha{}_\beta\wedge S_B) = (X\lrcorner\omega^\alpha{}_\beta)\wedge S_B - \omega^\alpha{}_\beta\wedge(X\lrcorner S_B)$

$$[X, e_\alpha] = \nabla_X e_\alpha - \nabla_{e_\alpha}X - T(X, e_\alpha) = (X\lrcorner\omega^\beta{}_\alpha - \nabla_\alpha X^\beta - T^\beta{}_{\gamma\alpha}X^\gamma)\,e_\beta = [X\lrcorner\omega^\beta{}_\alpha - \tilde\nabla_\alpha X^\beta]\,e_\beta$$

$$\theta^\beta([X, e_\alpha]) = X\lrcorner\omega^\beta{}_\alpha - \tilde\nabla_\alpha X^\beta$$

$\boxed{\text{Note: if } [X, e_\alpha] = 0 \quad \tilde\nabla_\alpha X^\beta = X\lrcorner\omega^\beta{}_\alpha}$

Thus: $\boxed{\{\mathcal{L}_X S\}_A = X \lrcorner DS_A + D(X \lrcorner S_A) - \sigma_A{}^{B}{}_{B}{}^{\alpha}(\tilde{\nabla}_B X^\alpha) S_B}$ ~~omit 2nd term if p=0~~

If $p=0$ we omit the second term and $X \lrcorner DS_A = \nabla_X S_A$ and we have the formula:

$$\{\mathcal{L}_X S\}^{(\alpha)}{}_{(\beta)} = \nabla_X S^{(\alpha)}{}_{(\beta)} - \sigma^{(\alpha)}{}_{(\beta)}{}^{(\gamma)}{}_{(\delta)}{}^{\beta}{}_{\alpha}(\tilde{\nabla}_\beta X^\alpha) S^{(\delta)}{}_{(\gamma)}$$

Example: $(\mathcal{L}_X g)_{\alpha\beta} = X^\gamma \nabla_\gamma g_{\alpha\beta} + g_{\alpha\gamma}\tilde{\nabla}_\beta X^\gamma + g_{\gamma\beta}\tilde{\nabla}_\alpha X^\gamma$

So in a V_n: $(\mathcal{L}_X g)_{\alpha\beta} = 2\nabla_{(\alpha} X_{\beta)}$. ~~Also all the prev~~

In general in a V_n in a coord system.

$$\{\mathcal{L}_X S\}^{(\alpha)}{}_{(\beta)} = X(S^{(\alpha)}{}_{(\beta)}) - \sigma^{(\alpha)}{}_{(\beta)}{}^{(\gamma)}{}_{(\delta)}{}^{\beta}{}_{\alpha}(\partial_\beta X^\alpha) S^{(\delta)}{}_{(\gamma)}$$

Thus in a V_n (in coord syst) we can substitute covariant for ordinary derivatives in the formula for the Lie derivative.

Note for a vector field $\mathcal{L}_X Y^\alpha = X \lrcorner DY^\alpha - Y^\beta \tilde{\nabla}_\beta X^\alpha = X^\beta \nabla_\beta Y^\alpha - Y^\beta \tilde{\nabla}_\beta X^\alpha$

$\mathcal{L}_X Y = \nabla_X Y - \tilde{\nabla}_Y X$ which is just a restatement of $[X,Y] = \nabla_X Y - \nabla_Y X - T(X,Y)$

<u>Alternative trick</u> We have the preliminary formula (which we skipped above)

$$(\mathcal{L}_X S)_A = X \lrcorner DS_A + D(X \lrcorner S_A) + \sigma_A{}^{B}{}_{B}{}^{\alpha}\{\theta^\alpha([X,e_\alpha]) - X \lrcorner \omega^\beta{}_\alpha\}$$

Choose a basis with $\mathcal{L}_X e_\alpha = 0$, then $\tilde{\nabla}_\beta X^\alpha = X \lrcorner \omega^\alpha{}_\beta$ so that we get the above formula, which is then true in any frame.

Also in such a frame: $(\mathcal{L}_X \omega)^\alpha{}_\beta = \mathcal{L}_X \omega^\alpha{}_\beta = X \lrcorner d\omega^\alpha{}_\beta + d(X \lrcorner \omega^\alpha{}_\beta) = X \lrcorner (\Omega^\alpha{}_\beta - \omega^\alpha{}_\gamma \wedge \omega^\gamma{}_\beta) + d(X \lrcorner \omega^\alpha{}_\beta)$

$$= X \lrcorner \Omega^\alpha{}_\beta + d(\underbrace{X \lrcorner \omega^\alpha{}_\beta}_{\nabla_\beta X^\alpha}) - \underbrace{(X \lrcorner \omega^\alpha{}_\gamma)}_{\tilde{\nabla}_\gamma X^\alpha} \omega^\gamma{}_\beta + \underbrace{(X \lrcorner \omega^\gamma{}_\beta)}_{\tilde{\nabla}_\beta X^\gamma} \omega^\alpha{}_\gamma$$

$$[\mathcal{L}_X \omega]^\alpha{}_\beta = D(\tilde{\nabla}_\beta X^\alpha) + X \lrcorner \Omega^\alpha{}_\beta$$ Schouten 5.47.

~~A symmetry~~ The one parameter group of diffeomorphisms generated by X is a symmetry if

$\mathcal{L}_X g = 0 = \mathcal{L}_X \omega$ For a metric connection: $\begin{cases} g_{\alpha\gamma}\tilde{\nabla}_\beta X^\gamma + g_{\gamma\beta}\tilde{\nabla}_\alpha X^\gamma = 0 \\ D(\tilde{\nabla}_\beta X^\alpha) + X \lrcorner \Omega^\alpha{}_\beta = 0 \end{cases}$

In a V_n; $\nabla = \tilde{\nabla}$: $\nabla_\alpha X_\beta + \nabla_\beta X_\alpha = 0$.

And $D(\underbrace{\nabla_\beta X_\alpha}_{\text{sym}}) + X \lrcorner \underbrace{\Omega_{(\alpha\beta)}}_{\text{sym}}$. Check $D\,\nabla_{[\alpha} X_{\beta]} + X \lrcorner \Omega_{\alpha\beta} = 0$

<u>Loose end</u> We mentioned a general formula for $\tilde{D}$; it is trivially:

$$\tilde{D} S_A = DS_A + \sigma_A{}^{B}{}_{B}{}^{\alpha} T^\alpha{}_\beta \wedge S_B$$

Finally we derive a variational formula involving D; let δ be a local variation ie δ commutes with d and $\delta(\omega \wedge \eta) = \delta\omega \wedge \eta + \omega \wedge \delta\eta$. Then:

$$\delta\Omega^\alpha{}_\beta = \delta[d\omega^\alpha{}_\beta + \omega^\alpha{}_\gamma \wedge \omega^\gamma{}_\beta] = d\,\delta\omega^\alpha{}_\beta - \omega^\gamma{}_\beta \wedge \delta\omega^\alpha{}_\gamma + \omega^\alpha{}_\gamma \wedge \delta\omega^\gamma{}_\beta = D\,\delta\omega^\alpha{}_\beta$$

$\boxed{\delta\Omega^\alpha{}_\beta = D\,\delta\omega^\alpha{}_\beta}$ Note that $\delta\omega^\alpha{}_\beta$ is a tensor-valued form

Notation: $\sigma(A) \equiv \sigma^\alpha{}_\beta A^\beta{}_\alpha$ $\begin{cases} D \equiv d + \sigma(\omega)\wedge \\ \mathcal{L}_X = C_X \circ D + D \circ C_X - \sigma(\tilde{\nabla} X) \end{cases}$

The unit pseudotensor and the Hodge Star operator (on a U_4)

First: $\epsilon_{\alpha\beta\gamma\delta} \stackrel{*}{=} \epsilon^{\alpha\beta\gamma\delta}$ are the Levi-Civita indicators $\epsilon_{0123} = 1$

Define $\eta_{\alpha\beta\gamma\delta} = \sqrt{-g}\,\epsilon_{\alpha\beta\gamma\delta}$, a tensor. $g \equiv \det(g_{\alpha\beta})$.

Then $\eta^{\alpha\beta\gamma\delta} = -\frac{1}{\sqrt{-g}}\epsilon^{\alpha\beta\gamma\delta}$.

Define also
$$\begin{cases} \eta_{\alpha\beta\gamma} = \eta_{\alpha\beta\gamma\delta}\theta^\delta \\ \eta_{\alpha\beta} = \frac{1}{2!}\eta_{\alpha\beta\gamma\delta}\theta^{\gamma\delta} \\ \eta_\alpha = \frac{1}{3!}\eta_{\alpha\beta\gamma\delta}\theta^{\beta\gamma\delta} \\ \eta = \frac{1}{4!}\eta_{\alpha\beta\gamma\delta}\theta^{\alpha\beta\gamma\delta} = \eta_{0123}\theta^{0123} \end{cases}$$

$$D\eta_{\alpha\beta\gamma\delta} = 0$$
$$D\eta_{\alpha\beta\gamma} = \eta_{\alpha\beta\gamma\delta}\Theta^\delta$$
$$D\eta_{\alpha\beta} = \eta_{\alpha\beta\gamma}\wedge\Theta^\gamma$$
$$D\eta_\alpha = \eta_{\alpha\beta}\wedge\Theta^\beta$$

pf. $0 = D(\eta^{\alpha\beta\gamma\delta}\eta_{\alpha\beta\gamma\delta}) = 2\eta^{\alpha\beta\gamma\delta}D\eta_{\alpha\beta\gamma\delta} \to D\eta_{\alpha\beta\gamma\delta} = 0$ (we used $Dg = 0$)

$D\eta_{\alpha\beta\gamma} = D(\eta_{\alpha\beta\gamma\delta}\theta^\delta) = \eta_{\alpha\beta\gamma\delta}D\theta^\delta = \eta_{\alpha\beta\gamma\delta}\Theta^\delta$ using 1st Cartan eq., etc.

We define the Hodge-Star operator (for more details see Hodge-Star notes).
$$\begin{cases} *\,\theta^{\alpha_1\cdots\alpha_p} = \eta^{\alpha_1\cdots\alpha_p} \\ *\,1 = \eta \end{cases}$$
, where we raise and lower indices on $\eta_{\alpha_1\cdots\alpha_p}$ indiscriminately.

It is almost obvious that
$$\begin{cases} \theta^\sigma\wedge\eta_\alpha = \delta^\sigma_\alpha\eta \\ \theta^\sigma\wedge\eta_{\alpha\beta} = \delta^\sigma_\beta\eta_\alpha - \delta^\sigma_\alpha\eta_\beta \\ \theta^\sigma\wedge\eta_{\alpha\beta\gamma} = \delta^\sigma_\gamma\eta_{\alpha\beta} + \delta^\sigma_\beta\eta_{\gamma\alpha} + \delta^\sigma_\gamma\eta_{\beta\gamma} \\ \theta^\sigma\wedge\eta_{\alpha\beta\gamma\delta} = \delta^\sigma_\delta\eta_{\alpha\beta\gamma} - \delta^\sigma_\gamma\eta_{\alpha\beta\gamma} + \delta^\sigma_\beta\eta_{\alpha\gamma\delta} - \delta^\sigma_\alpha\eta_{\beta\gamma\delta} \end{cases}$$

$\xi^{ab}\wedge\theta^c = 2g^{c[a}\xi^{b]}$

Other useful formulas follow by raising indices.

$$\boxed{\eta_{\alpha\beta}\wedge\theta^{\gamma\delta} = \eta\,\delta^{\gamma\delta}_{\alpha\beta}}$$

A very useful result is:
$$\eta_\alpha{}^\beta\wedge\Omega^\alpha{}_\beta = \tfrac{1}{2}\eta_\alpha{}^\beta{}_{\gamma\delta}\theta^{\gamma\delta}\wedge\tfrac{1}{2}R^\alpha{}_{\beta\mu\nu}\theta^{\mu\nu} = \frac{1}{4}\underbrace{\eta_{\alpha\beta\gamma\delta}}_{\sqrt{-g}\,\epsilon_{\alpha\beta\gamma\delta}}R^{\alpha\beta}{}_{\mu\nu}\underbrace{\theta^{\gamma\delta\mu\nu}}_{\epsilon^{\gamma\delta\mu\nu}\theta^{0123}}$$
$$= \frac{1}{4}\eta\underbrace{\epsilon_{\alpha\beta\gamma\delta}\epsilon^{\gamma\delta\mu\nu}}_{2\delta^{\mu\nu}_{\alpha\beta}}R^{\alpha\beta}{}_{\mu\nu} = \eta R^{\alpha\beta}{}_{\alpha\beta} = \eta g^{\beta\gamma}R^\alpha{}_{\beta\alpha\gamma} = \eta R$$
$$\boxed{\eta R = \eta_\alpha{}^\beta\wedge\Omega^\alpha{}_\beta}$$

Let M be a spacelike hypersurface in the space-time $N = U_4$, with unit normal n, and let $\{n, e_a\} = \{e_\alpha\}_{\alpha=1,2,3}$ be a normal adapted basis wrt M. Let g be the induced metric on M.

Then Define
$$\xi_{abc} = \sqrt{g}\,\epsilon_{abc} \qquad \epsilon_{abc} \equiv \epsilon_{0abc}.$$
$$\xi_{ab} = \xi_{abc}\theta^c$$
$$\xi_a = \tfrac{1}{2}\xi_{ab}\theta^{ab}$$
$$\xi = \tfrac{1}{3!}\xi_{abc}\theta^{abc}$$

The 3-Hodge star operator:
$$*\,\theta^{a_1\cdots a_p} = \xi^{a_1\cdots a_p}$$
$$*\,1 = \xi$$
Indices raised with g.

The relation between the 3- and-4 tensor fields is: $\xi = n\lrcorner\,\eta$.

Another useful result is: $\boxed{\eta^{ab}\wedge\theta^{cd} = (g^{ac}g^{bd} - g^{ad}g^{bc})\eta}$ $\leftarrow$ $\boxed{\eta_{ab}\wedge\theta^{cd} = \delta^{cd}_{ab}\eta}$

pf. $\eta^{ab}\wedge\theta^c = \eta^b g^{ac} - \eta^a g^{bc}$, $\eta^{ab}\wedge\theta^{cd} = \eta^b\wedge\theta^d g^{ac} - \eta^a\wedge\theta^d g^{bc} = (g^{bd}g^{ac} - g^{ad}g^{bc})\eta$

This is also true with $\eta \to \xi$.

We shall also need a variation formula, using $\delta\xi = \frac{1}{2}\delta g_{fg}g^{fg}\xi$:

$$\delta\xi^a{}_b = \delta(g^{ac}\xi_{cb}) = -g^{af}g^{cg}\delta g_{fg}\xi_{cb} + \tfrac{1}{2}\delta g_{fg}g^{fg}g^{ac}\xi_{cb} = \tfrac{1}{2}(g^{ag}g^{fg} - g^{af}g^{cg} - g^{ag}g^{cf})\delta g_{fg}\xi_{cb}$$
$$\boxed{\delta\xi^a{}_b = G^{acfg}\delta g_{fg}\xi_{cb}}$$
$$G^{acfg} \equiv \tfrac{1}{2}(g^{ac}g^{fg} - g^{af}g^{cg} - g^{ag}g^{cf})$$
$$= (-g\tfrac{1}{2})\,(\text{DeWitt supermetric})$$

$$R\eta = \eta^{\nu\alpha}{}_{\beta} \wedge \Omega^{\beta}{}_{\alpha}$$

$$\delta\Omega^{\nu}{}_{\beta} = D(\delta\Omega^{\nu}{}_{\beta})$$

$$\delta\eta^{\alpha}{}_{\beta} = \tfrac{1}{2}\left[g^{\alpha\gamma}g^{\mu\nu} - 2g^{\alpha(\mu}g^{\gamma)\nu}\right]\eta_{\gamma\beta}\,\delta g_{\mu\nu}$$

$$\eta^{\nu}{}_{\beta} \wedge \theta^{\sigma\rho} = \left(g^{\nu\sigma}\delta^{\rho}_{\beta} - g^{\nu\rho}\delta^{\sigma}_{\beta}\right)\eta$$

$$\delta(R\eta) = \cancel{\delta(\eta^{\alpha}{}_{\beta}\wedge\Omega^{\beta}{}_{\alpha})}\ \delta\eta^{\alpha}{}_{\beta}\wedge\Omega^{\beta}{}_{\alpha} + \eta^{\alpha}{}_{\beta}\wedge\delta\Omega^{\beta}{}_{\alpha}$$

$$\cancel{\delta\eta^{\alpha}{}_{\beta}\wedge\Omega^{\beta}{}_{\alpha} = \tfrac{1}{2}\left[g^{\alpha\gamma}g^{\mu\nu} - 2g^{\alpha(\mu}g^{\gamma)\nu}\right]\eta_{\gamma\beta}\,\delta g_{\mu\nu}\wedge R^{\beta}{}_{\alpha\sigma\rho}\,\theta^{\sigma\rho}}$$

$$\cancel{\tfrac{1}{2}\,\eta^{\alpha}{}_{\beta}\wedge\theta^{\sigma\rho}\,R^{\beta}{}_{\alpha\sigma\rho}}$$

$$\delta\eta^{\alpha}{}_{\beta}\wedge\Omega^{\beta}{}_{\alpha} = \tfrac{1}{2}\left[g^{\alpha\gamma}g^{\mu\nu} - 2g^{\alpha(\mu}g^{\gamma)\nu}\right]\eta_{\gamma\beta}\wedge\Omega^{\beta}{}_{\alpha}\,\delta g_{\mu\nu}$$

$$= \Big[\tfrac{1}{2}\underbrace{\eta^{\alpha}{}_{\beta}\wedge\Omega^{\beta}{}_{\alpha}}_{-R\eta}\,g^{\mu\nu} - \underbrace{\eta^{(\mu}{}_{\beta}\wedge\Omega^{\beta|\nu)}}_{-R^{\mu\nu}}\Big]\delta g_{\mu\nu}$$

$$\eta^{\nu}{}_{\beta}\wedge\Omega^{\beta\mu} = \tfrac{1}{2}\,\eta^{\nu}{}_{\beta}\wedge\theta^{\sigma\rho}R^{\beta\mu}{}_{\sigma\rho} = \tfrac{1}{2}\left(g^{\nu\sigma}\delta^{\rho}_{\beta} - g^{\nu\rho}\delta^{\sigma}_{\beta}\right)\eta\,R^{\beta\mu}{}_{\sigma\rho} = R^{\beta\mu\nu}{}_{\beta} = -R^{\mu\nu}$$

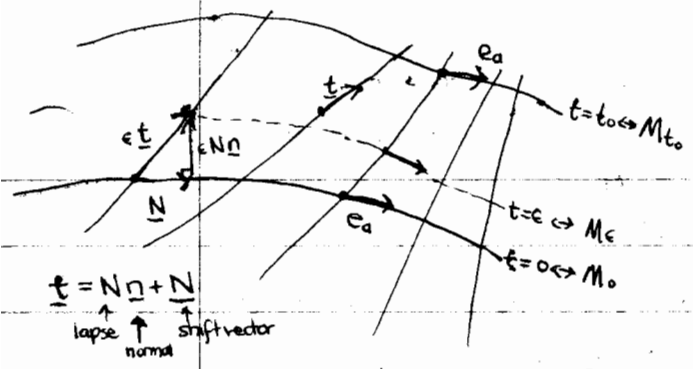
$$\delta(R\eta) = \delta(\eta_{\beta}{}^{\alpha}\wedge\Omega^{\beta}{}_{\alpha}) = \left[\tfrac{1}{2}Rg^{\mu\nu} - R^{(\mu\nu)}\right]\eta\,\delta g_{\mu\nu} + \underbrace{\underbrace{\eta_{\beta}{}^{\alpha}\wedge D(\delta\omega^{\beta}{}_{\alpha})}_{\underbrace{D(\eta_{\beta}{}^{\alpha}\wedge\delta\omega^{\beta}{}_{\alpha})}_{d(\eta_{\beta}{}^{\alpha}\wedge\delta\omega^{\beta}{}_{\alpha})} - \underbrace{D\eta_{\beta}{}^{\alpha}}_{0}\wedge\delta\omega^{\beta}{}_{\alpha}}}_{A}$$

$$\longrightarrow\ \eta_{\alpha}{}^{\beta}\wedge\delta\Omega^{\alpha}{}_{\beta} = g^{\epsilon\beta}\delta R^{\alpha}{}_{\epsilon\mu\nu}\ \underbrace{\eta_{\alpha\beta}\wedge\theta^{\mu\nu}}_{\eta_{\alpha\beta\gamma\delta}\theta^{\gamma\delta\mu\nu} = \eta\,\delta^{\mu\nu}_{\alpha\beta}} = \eta\, g^{\epsilon\beta}\delta R^{\alpha}{}_{\epsilon\alpha\beta} = \eta\, g^{\epsilon\beta}\delta R_{\epsilon\beta}$$

$$\delta\mathcal{S}^{a}{}_{b}\wedge\Omega_{a}{}^{b} = -G^{ab}\delta g_{ab}\,\mathcal{S}$$

$$\delta(\mathcal{S}^{a}{}_{b}\wedge\Omega_{a}{}^{b}) = -G^{ab}\delta g_{ab}\,\mathcal{S} + d(\mathcal{S}^{a}{}_{b}\wedge\delta\omega^{b}{}_{a})$$

ADM procedure



We start with a manifold M_0 assumed imbedded in a space-time N and an arbitrary vector field $\underline{t}$ (everywhere nonzero)

We define a one-parameter family of hypersurfaces by dragging M_0 along the integral curves of $\underline{t}$:

$$M_t \equiv \exp(t\,\underline{t})\, M_0$$

If the integral curves of $\underline{t}$ are nowhere tangent to M_0 then on some interval $[-\epsilon, \epsilon] \times M_0 \subset N$ a function t is defined, such that M_{t_0} coincides with the hypersurface $t = t_0$.

An <u>ADM frame</u> for the family $\{M_t\}$ is a frame $\{e_0, e_a\}$ such that e_a lie in TM

For such a frame, the distribution $\{e_a\}$ is hypersurface forming: $[e_a, e_b] = e_c C^c{}_{ab}$, i.e. $C^0{}_{ab} \equiv 0$.

$\{t, e_a\} \equiv \{e_\alpha\}$ is a special case of an ADM frame, which we dub the <u>adapted ADM frame</u>.

The condition that the t = constant hypersurfaces agree with the integral submanifolds of the distribution $\{e_a\}$ (given that the $t=0$ hypersurfaces is one of these submanifolds) is that

$$e^t{}_\alpha e_a = e_b R^b{}_a \quad \text{or} \quad \mathcal{L}_{\underline{t}} e_a = e_b R^b{}_a$$

In the adapted ADM frame this just says: $[e_0, e_a] = e_b C^b{}_{0a}$, i.e. $C^0{}_{0a} = 0$.

Note that t is the affine parameter on the integral curves of $\underline{t}$, zeroed on the hypersurface M_0.

If x^i are any coordinates on M_0 then (t, x^i) are ADM coordinates on some part of N (which may be M_0 itself if $\underline{t}$ merely generates a diffeomorphism of M_0 into itself), i.e. comoving coordinates wrt $\underline{t}$, zeroed by M_0, where x^i are defined to be constant on the integral curves of $\underline{t}$.

By definition the integral curve through x_i $C_{x_i}(t)$ satisfies: $\dfrac{d(x^\mu{}_0 C_{x_i})}{dt} = t^\mu$

But $x^\mu{}_0 C_{x_i}(t) = (t, x^i)$ also by def. implying $t^\mu = \delta^\mu{}_0$ $\quad \underline{t} = \frac{\partial}{\partial t}$.

In these coords, $(\mathcal{L}_{\underline{t}} \phi)_A = \frac{\partial}{\partial t} \phi_A$ for any geometric object with components ϕ_A.

By way of direct generalization, a <u>comoving adapted ADM frame</u> is an adapted ADM frame in which the e_a are dragged along the integral curves of $\underline{t} \equiv e_0$: $\mathcal{L}_{e_0} e_a = [e_0, e_a] = e_b C^b{}_{0a} = 0$

In a comoving adapted ADM frame, $C^0{}_{ab} = C^a{}_{0a} = C^b{}_{0a} = 0$, i.e. $C^\alpha{}_{\beta\gamma} = 0$ if any index is 0.

Again, $(\mathcal{L}_{\underline{t}} \phi)_A = e_0(\phi_A)$ for any geometric objects with components ϕ_A in this basis.

In any adapted ADM frame we have $\theta^0 = dt$ where θ^α is the dual frame., provided of course that dt makes sense.

Two matters require discussion. First the geometry on N induces an intrinsic geometry and an extrinsic geometry on each of the elements of the family $\{M_t\}$, which must be investigated. (3+1 split). Secondly, we have the question of dynamics; how is the extrinsic & intrinsic geometry of the members of the family to be interrelated if N is to be an acceptable space-time, i.e. a solution of some set of field equations. or an extremal of some chosen variational principle. Here the ADM method must be discussed.

Suppose for the moment we are in the V_4 theory.

Let M_0 be a space-like hypersurface imbedded in some Riemannian space-time $(N, \langle\,\rangle)$, and let M_t be the surface obtained from M_0 under the action of $e^{t\underline{t}}$ for some vector field $\underline{t}$.

The 4-metric $\bar{g} \equiv \langle , \rangle$ induces a metric g_t on each M_t and M_0 induces a metric on each M_t by dragging, namely $(\exp t\underline{t})^* g_0 \equiv \gamma_t$.

$\exp t\underline{t}$ is an isometry $(M_0, g_0) \to (M_t, \gamma_t)$.

The three geometry $\mathcal{G}_t$ of the imbedded surface M_t is the equivalence class of (M_t, g_t), equivalence defined by $(X, g) \sim (X', g') \leftrightarrow \exists$ diffeo $f: X \to X'$, st $f^* g' = g$.

The ADM equations: $\mathcal{L}_{\underline{t}} g_t = (\;)$, $\mathcal{L}_{\underline{t}}(\pi_t) = (\;)$

will give us information on (M_t, g_t), the induced structure on the surface as it moves through spacetime. The Lie derivative enters because one wants to compare (M_t, g_t) with the "static" geometry (M_t, γ_t) in the limit $t \to 0$.

$\boxed{\text{IMBEDDING IN A } U_4}$ $\equiv (N, \bar{g} \equiv \langle\,,\rangle, \bar{\nabla}, \bar{\nabla}\bar{g}=0)$ Lorentzian ~~signat~~ metric.

Let M be a space-like hypersurface imbedded in N: $f: M \to N$.

We show that M has an induced $(g, \nabla, \nabla g = 0)$ structure, and then examine the left overs ie the extrinsic geometry of M, not necessarily in this order.

[we identify X, f^*X; etc]

Induced metric on $M \equiv g \equiv f^*\bar{g}$. Thus for $X, Y \in TM$ $\quad g(X,Y) = \bar{g}(X,Y)$.

$n \equiv$ unit (time-like) normal to M: $\langle n,n\rangle = \epsilon$, $\epsilon_{+---} = 1$, $\epsilon_{-+++} = -1$ $\quad \langle n, X\rangle = 0 \quad X \in TM$

Define the extrinsic curvature tensor: $X, Y \in TM$ $\quad K(X,Y) = \langle n, \bar{\nabla}_X Y\rangle = -\langle \bar{\nabla}_X n, Y\rangle$

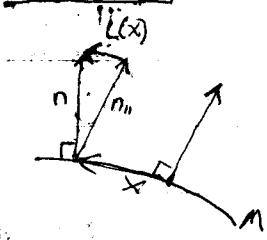
from $\langle n, Y\rangle = 0 \Rightarrow \nabla_X \langle\,,\rangle$

Recall $X, Y \in TM \to [X,Y] \in TM$.

$K(X,Y) - K(Y,X) = \langle \bar{\nabla}_X Y - \bar{\nabla}_Y X, n\rangle = \langle \bar{T}(XY) + [X,Y], n\rangle = \langle \bar{T}(XY), n\rangle$.

Note $0 = X\langle n,n\rangle = 2\langle \bar{\nabla}_X n, n\rangle \to \bar{\nabla}_X n \in TM$, so Define the Weingarten Map $L: TM \to TM$

$L(X) = -\bar{\nabla}_X n$, $\quad K(X,Y) = \langle L(X), Y\rangle$



$L(X)$ measures how the ∥ transported normal along M (along the direction X) deviates from the normal.

$\langle L(X), Y\rangle - \langle X, L(Y)\rangle = K(XY) - K(YX) = \langle \bar{T}(XY), n\rangle$

L is not self-adjoint and the projection of the torsion onto M is a measure of the non self-adjointness

In a V_4, L is self-adjoint so all the eigenvalues (principal curvatures) are real and the eigenvectors (principal directions) may be chosen orthogonal. Along a principal direction X, $\nabla_X n \propto X$.

In a U_4, L is not self-adjoint, but since $n = 3$ ($\dim M$), it has at least one real eigenvalue.

Define the induced connection ∇ by subtracting off the out-of-surface component:

$X, Y \in M$. $\quad \nabla_X Y = \bar{\nabla}_X Y - \langle n, \bar{\nabla}_X Y\rangle \frac{n}{\langle n,n\rangle} = \bar{\nabla}_X Y - K(X,Y)\frac{n}{\langle n,n\rangle}$.

{ One easily verifies that ∇ satisfies the axioms for a connection.

The induced torsion is therefore: $\quad T(XY) = \nabla_X Y - \nabla_Y X - [X,Y] \quad X, Y \in TM$.

$\bar{T}(XY) = \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X,Y] = [K(XY) - K(YX)]\frac{n}{\langle n,n\rangle} + T(XY) = \langle n, \bar{T}(X,Y)\rangle \frac{n}{\langle n,n\rangle} + T(XY)$

So the restriction of the 4-torsion operator to M is the sum of two operators, the projection of $\bar{T}$ on the normal, and the induced 3-torsion operator.

(Replacing n by any other offsurface vector, drop normal)

Let $\{e_a\}$ be a basis on M, then $\{n, e_a\} \equiv \{e_\alpha\}_{\alpha=1,123}$ is a normal basis adapted to M. We restrict ourselves to such a basis in the following. $(n \equiv e_\perp)$

$K_{ab} \equiv K(e_a, e_b)$, $\quad L(e_a) \equiv L_a{}^b e_b$, $\quad K_{ab} = \langle L(e_a), e_b\rangle = L_a{}^c g_{cb} = L_{ab}$, $\quad L_a{}^b = K_a{}^b$,

$\nabla_{e_a} e_b = \Gamma^c_{ab} e_c \qquad \Gamma_{a,bc} = \langle e_a, \bar{\nabla}_{e_b} e_c\rangle = \langle e_a, \nabla_{e_b} e_c + K(e_b e_c)\frac{n}{\langle n,n\rangle}\rangle = \langle e_a, \nabla_{e_b} e_c\rangle = \Gamma_{abc}$

For convenience define the extrinsic curvature one-forms: $K_a(X) = K(X, e_a)$, $\quad K_a = K_{ba}\theta^b$

There are several extrinsic torsion tensors we could define. Two useful ones are:

$X, Y \in TM$. $\quad \tau(X,Y) \equiv \langle X, \bar{Q}(n,Y)\rangle \qquad \tau_{ab} = \tau(e_a, e_b) = \bar{Q}_{a\perp b} \qquad \tau = \tau_{ab}\theta^a\theta^b$

$q(X) = \epsilon\langle n, \bar{Q}(n,X)\rangle \qquad q_a = q(e_a) = \bar{Q}^\perp{}_{\perp a} \qquad q = q_a\theta^a$

Note that in the normal adapted frame: $\bar{g}_{\alpha\beta} = \begin{pmatrix} \epsilon & 0\,0\,0 \\ 0 & \\ 0 & g_{ab} \\ 0 & \end{pmatrix} \quad \bar{g}^{\alpha\beta} = \begin{pmatrix} \epsilon & 0\,0\,0 \\ 0 & \\ 0 & g^{ab} \\ 0 & \end{pmatrix}$ $\qquad$ It is easily shown that $\nabla g = 0$

So we can raise and lower latin indices to latin indices.

It will be convenient to define $\chi_{ab} \equiv \underset{\bar{Q}_{(a\perp\perp b)}}{\tau_{(ab)}}$; $\rho_{ab} = \underset{\bar{Q}_{[a\perp\perp b]}}{\tau_{[ab]}}$ or $\lambda(XY) \equiv \frac{1}{2}[\langle X, \bar{Q}(nY)\rangle + \langle Y, \bar{Q}(nX)\rangle]$ $\quad X, Y \in TM$.

$\rho(XY) \equiv \frac{1}{2}[\langle X, \bar{T}(nY)\rangle - \langle X, \bar{T}(nX)\rangle]$

Let $\sim$ denote restriction of a 4-dimensional object to M: | Recall: $f^*\circ\bar{d} = d\circ f^*$

$f: M\to N \quad f^*: \Lambda^r N \to \Lambda^r M \;,\; f^*\bar{\alpha} \equiv \tilde{\alpha}$ | $f^*(\omega\wedge\eta) = f^*\omega\wedge f^*\eta \equiv \tilde{\omega}\wedge\tilde{\eta}$

$X\in TM$

$$\left.\begin{aligned}\bar{\nabla}_X e_a &= \bar{\omega}^\alpha{}_a(X)e_\alpha = \bar{\omega}^\perp{}_a(X)n + \bar{\omega}^b{}_a(X)e_b\\ \tilde{\nabla}_X e_a &= K(X\lrcorner e_a)\frac{n}{\langle n,n\rangle} + \omega^b{}_a(X)e_b\end{aligned}\right\}\text{ compare}$$

$\bar{\omega}^b{}_a(X) = \tilde{\omega}^b{}_a(X)$, $\bar{\omega}^\perp{}_a(X) = \tilde{\omega}^\perp{}_a(X)$ since $X\in TM$ so $\left\{\begin{aligned}K(X\lrcorner e_a) &= \boxed{\langle n,n\rangle\,\tilde{\omega}^\perp{}_a(X) = K_a(X)}\\ f^*\bar{\omega}^a{}_b &\equiv \boxed{\tilde{\omega}^a{}_b = \omega^a{}_b}\end{aligned}\right.$

$$f^*\bar{\Theta}^a = f^*[\bar{d}\theta^a + \bar{\omega}^a{}_\alpha\wedge\theta^\alpha] \underset{\substack{f^*\bar{\omega}^a{}_b = \tilde{\omega}^a{}_b = \omega^a{}_b\\ f^*\theta^a = \theta^a}}{\overset{f^*\theta^\perp = 0}{=\!=\!=}} d\theta^a + \omega^a{}_b\wedge\theta^b = \Theta^a$$

(a) $\boxed{\tilde{\Theta}^a = f^*\bar{\Theta}^a = \Theta^a}$ in an arbitrary adapted frame (an ADM frame).

This is just the tangential part of $\bar{T}(XY) = \langle n,\bar{T}(XY)\rangle\frac{n}{\langle n,n\rangle} + T(XY)$ $\quad X,Y\in TM$.

Toget the normal part:

$$\tilde{\Theta}^\perp = f^*\bar{\Theta}^\perp = f^*[\bar{d}\theta^\perp + \bar{\omega}^\perp{}_\alpha\wedge\theta^\alpha]$$

Now $0 = d\langle n,n\rangle = d\bar{g}_{\perp\perp}$, $0 = \bar{D}\bar{g}_{\perp\perp} = \bar{d}g_{\perp\perp} - \bar{\omega}_{\perp\perp} - \bar{\omega}_{\perp\perp} = -2\bar{\omega}_{\perp\perp}$ $\quad\boxed{\bar{\omega}^\perp{}_\perp = 0}$

$$\tilde{\Theta}^\perp = d(f^*\theta^\perp) + f^*\bar{\omega}^\perp{}_\alpha\wedge f^*\theta^\alpha = \tilde{\omega}^\perp{}_a\wedge\theta^a = \langle n,n\rangle K_a\wedge\theta^a$$

(b) $\boxed{\tilde{\Theta}^\perp = f^*\bar{\Theta}^\perp = \langle n,n\rangle K_a\wedge\theta^a}$ in any ADM frame. Note that this is just $K(XY) - K(YX) = \langle n,\bar{T}(XY)\rangle$.

(a) and (b) together reproduce $\bar{T}(XY) = \langle n,\bar{T}(XY)\rangle\frac{n}{\langle n,n\rangle} + T(XY)$.

<u>Gauss-Codazzi</u>

$$\tilde{\Omega}^a{}_b = f^*\bar{\Omega}^a{}_b = f^*[\bar{d}\bar{\omega}^a{}_b + \bar{\omega}^a{}_\alpha\wedge\bar{\omega}^\alpha{}_b] = d\tilde{\omega}^a{}_b + \tilde{\omega}^a{}_c\wedge\tilde{\omega}^c{}_b + \tilde{\omega}^a{}_\perp\wedge\tilde{\omega}^\perp{}_b$$

But $\tilde{\omega}^a{}_\perp = \epsilon K_a$: $\tilde{\omega}^a{}_\perp = \bar{g}^{a\alpha}\bar{\omega}_{\alpha\perp} = g^{ab}\bar{\omega}_{b\perp}$ $\quad 0 = \bar{D}\bar{g}_{\perp c} = \bar{d}g_{\perp a} - \bar{\omega}_{\perp a} - \bar{\omega}_{a\perp}$

so $\tilde{\omega}^a{}_\perp = -g^{ab}\tilde{\omega}_{\perp b} = -\langle n,n\rangle g^{ab}\tilde{\omega}^\perp{}_b = -g^{ab}K_b = -K^a$ $\quad\boxed{\tilde{\omega}^a{}_\perp = -K^a}$

Also $\tilde{\omega}^a{}_b = \omega^a{}_b$ so:

$$\tilde{\Omega}^a{}_b = d\omega^a{}_b + \omega^a{}_c\wedge\omega^c{}_b - \langle n,n\rangle K^a\wedge K_b = \Omega^a{}_b - \langle n,n\rangle K^a\wedge K_b$$

(a) $\boxed{\tilde{\Omega}^a{}_b = f^*\Omega^a{}_b = \Omega^a{}_b - \langle n,n\rangle K^a\wedge K_b}$ in any ADM frame.

$$f^*\bar{\Omega}^\perp{}_a = f^*[\bar{d}\bar{\omega}^\perp{}_a + \bar{\omega}^\perp{}_\alpha\wedge\bar{\omega}^\alpha{}_a] \underset{\tilde{\omega}^b{}_a = \omega^b{}_a}{\overset{\bar{\omega}^\perp{}_\perp = 0}{=\!=\!=}} d\tilde{\omega}^\perp{}_a - \omega^b{}_a\wedge\tilde{\omega}^\perp{}_b$$

(b) $\boxed{\tilde{\Omega}^\perp{}_a = d\omega^\perp{}_a - \omega^b{}_a\wedge\tilde{\omega}^\perp{}_b = \langle n,n\rangle[dK_a - \omega^a{}_b\wedge K_b] = \langle n,n\rangle DK_a}$ in any ADM frame.

The equations (a) relate the restrictions of the ~~torsion pr. tangent~~ tangential parts of the torsion and curvature operators to the intrinsic torsion, curvature and the extrinsic curvature, while the equations (b) do the same for the normal parts.

By (b) $\bar{T}^\perp{}_{ab} = \langle n,n\rangle K_{[ab]}$. Again it is convenient to define another extrinsic ~~curvatu~~ torsion tensor: $\mathcal{K}_{ab} \equiv K_{[ab]} \equiv \frac{1}{2}\bar{T}^\perp{}_{ab}\langle n,n\rangle$ $\quad \mathcal{K}(XY) = \frac{1}{2}\langle n,\bar{T}(XY)\rangle$

$\equiv \frac{1}{2}\bar{T}_{\perp ab}$

In all we have defined 3 extrinsic ~~curvatu~~ torsion tensors, $\mathcal{T}(X,Y)$, $\mathcal{Q}(X)$, $\mathcal{K}(XY)$ and $T(XY)$ we have devided into its symmetric and antisymmetric parts $\lambda(XY)$, $\rho(XY)$.

Note $\mathcal{K}$ is already antisymmetric, being the antisymmetric part of the extrinsic curvature, K, whose partial contraction in any ADM frame we have defined as $K_a = K_{ba}\theta^b$.

$[(TRK)\, n^\alpha + a^\alpha]$

$\bar{d}\,(TRK\ \bar{n}_\perp + a^a \bar{n}_a)$

$\bar{n}_\perp \quad \frac{1}{3!}\, n_{\perp abc}\, dx^{abc}$

$\bar{d} = \left(dt\frac{\partial}{\partial t} + dx^i \frac{\partial}{\partial x^i}\right) \wedge$

only

$n_a = \frac{1}{2!}\, n_{aobc}\, dx^b dx^c$

$-\frac{\partial}{\partial t}(\pi^{ij} g_{ij}) + N^i TR\pi$

$n^\alpha = \left(\frac{1}{N}, -\frac{N^a}{N}\right)$

N in $\bar{n}$ cancels

$a^a = \left(-\frac{N^a}{N}\right)_{,b}$

$\boxed{\begin{array}{l} a^\alpha n_\alpha = 0 \\ a^0 = 0 \end{array}}$

$N^a{}_{;\beta} n^\beta = n^a{}_{,\beta} n^\beta + n^\gamma \Gamma^a_{\beta\gamma} n^\beta$

$n_\mu = -N \delta^0_\mu$

$a_\mu = n_{\mu,\beta} n^\beta = n_{\mu,\beta} n^\beta - n_\gamma \Gamma^\gamma_{\beta\alpha} n^\alpha$

$n_{\mu,\beta} n^\beta - \frac{1}{N} n^0 \Gamma^0_{\beta\alpha} n^\alpha$

g^0

$\delta^0_\mu \left(\frac{1}{N}\right)_{,\beta} n^\beta$

$N n_\mu$

$a_\mu = Z_n n_\mu = n_{\mu,\beta} n^\beta + n^\gamma{}_{,\mu} n_\gamma$

$+ \frac{n_\mu}{N} N_{,\beta} n^\beta - N n^0{}_{,\mu} + \frac{N_{,\mu}}{N} = \frac{N_{,\alpha}}{N} (\delta^\alpha_\beta + n^\alpha n_\beta)$

$(\delta^\alpha_\beta + n^\alpha n_\beta)\, \delta^\beta_1 = \delta^\alpha_1$

$a_\mu = \frac{N_{,\alpha}}{N}$

$a^a = N_{,\beta} (g^{a\beta} + n^a n^\beta)$

g^{ab}

$a^a{}_{,a}$

$N^{,a}$

3-divergence

$-2d(\pi^{ij} N_j\, n)$

$$\boxed{\mathcal{L} = -\frac{\partial}{\partial t}(TR\pi) - 2\left[\pi^{ij} N_j - \frac{1}{2} N^i TR\pi + N^{,i} g^{1/2}\right]_{,i} + \pi^{ij} \partial_t g_{ij} - N\mathcal{H} - N^i \mathcal{H}_i}$$

$-2\bar{d}\,(TRK\, \bar{n}_\perp + a^a \bar{n}_a) = -2\bar{d}\,(TRK\, n^\alpha + a^\alpha)\, \bar{n}_\alpha$ 4 divergence

ADM Formulation of the U_4 Theory

Picking up at the close of the ADM Procedure discussion, we assume for the Gravitational Lagrangian: $\mathbb{K} \equiv \bar{\eta}_\alpha{}^\beta \wedge \bar{\Omega}^\alpha{}_\beta \equiv R\eta$.

A solution $\{M_t\}$ of the field equations will therefore extremize $\int_V \mathbb{K}$ on any volume V of N. (in vacuum). What we mean by a solution $\{M_t\}$ will become clearer later.

We first decompose the Lagrangian in a (3+1) split. Note $\bar{\eta}_{(\alpha\beta)} = 0 = \bar{\Omega}_{\alpha\beta}$.

We work in a normally adapted ADM basis, in which we can raise and lower Latin indices to Latin indices. We also choose the signature $(-+++)$: $\langle n, n\rangle = -1$.

$$\mathbb{K} = \bar{\eta}_{\alpha\beta}\wedge\bar{\Omega}^{\beta\alpha} = 2\bar{\eta}_{a\perp}\wedge\bar{\Omega}^{\perp a} + \bar{\eta}_{ab}\wedge\bar{\Omega}^{ba} = 2\bar{\eta}^a{}_\perp\wedge\bar{\Omega}^\perp{}_a + \bar{\eta}^a{}_b\wedge\bar{\Omega}^b{}_a$$

The nonvanishing terms of $\bar{\eta}^a{}_b = \frac{1}{2}\bar{\eta}^a{}_{b\alpha\beta}\theta^\alpha\wedge\theta^\beta$ must contain a $\perp$ so $\bar{\eta}^a{}_b = \underbrace{\bar{\eta}^a{}_{b\perp c}}_{\bar{\eta}_\perp{}^a{}_{bc} = \mathcal{J}^a{}_{bc}}\theta^\perp\wedge\theta^c$

$\bar{\eta}^a{}_b\wedge\bar{\Omega}^b{}_a = \bar{\eta}^a{}_b\wedge\bar{\Omega}^b{}_a$ since terms of the form $\bar{R}^b{}_{a\perp c}$ give zero which wedged by $\bar{\eta}^a{}_b$.

$$= \bar{\eta}^a{}_b\wedge[\Omega^b{}_a + K^b\wedge K_a] \quad \text{by Gauss-Codazzi (a)}$$

$$2\bar{\eta}^a{}_\perp\wedge\bar{\Omega}^\perp{}_a = 2\bar{\eta}^a{}_\perp\wedge(\bar{d}\bar{\omega}^\perp{}_a + \bar{\omega}^\perp{}_b\wedge\bar{\omega}^b{}_a) \quad \text{by Gauss-Codazzi (b)}$$

$$= \bar{d}(2\bar{\eta}^a{}_\perp\wedge\bar{\omega}^\perp{}_a) - 2\underbrace{\bar{d}(\bar{\eta}^a{}_\perp)}_{\bar{D}\bar{\eta}^a{}_\perp - \bar{\omega}^a{}_\alpha\wedge\bar{\eta}^\alpha{}_\perp + \bar{\omega}^a{}_\perp\wedge\bar{\eta}^\perp{}_\beta}\wedge\bar{\omega}^\perp{}_a + 2\bar{\eta}^a{}_\perp\wedge\bar{\omega}^\perp{}_b\wedge\bar{\omega}^b{}_a \quad (\text{by } \bar{d}(\alpha\wedge\beta) = \ldots)$$

$$= \bar{d}(2\bar{\eta}^a{}_\perp\wedge\bar{\omega}^\perp{}_a) - 2(\bar{D}\bar{\eta}^a{}_\perp)\wedge\bar{\omega}^\perp{}_a - 2\bar{\eta}^a{}_b\wedge\bar{\omega}^b{}_\perp\wedge\bar{\omega}^\perp{}_a \quad (\text{using } \bar{\eta}^\perp{}_\perp = 0 = \bar{\omega}^\perp{}_\perp)$$

$$= -2(\bar{D}\bar{\eta}^\perp{}_a)\wedge\bar{\omega}^\perp{}_a - 2\bar{\eta}^a{}_b\wedge K^b\wedge K_a + \bar{d}(2\bar{\eta}^a{}_\perp\wedge\bar{\omega}^\perp{}_a)$$

$$\boxed{\mathbb{K} = \bar{\eta}^a{}_b\wedge[\Omega_{ab} - K_a\wedge K_b] + 2(\bar{D}\bar{\eta}^a{}_\perp)\wedge\bar{\omega}^\perp{}_a - \bar{d}(2\bar{\eta}^a{}_\perp\wedge\bar{\omega}^\perp{}_a)}$$

In the V_4 theory the second term vanishes and the first is just the ADM Lagrangian $(K_{ab} = K_{ba})$:

$$\mathbb{K}_{ADM} = \eta^{ab}\wedge[\tfrac{1}{2}R_{abcd} - K_{ca}K_{db}]\theta^c\wedge\theta^d = (\tfrac{1}{2}(R_{abcd} - K_{ca}K_{db})g^{ac}g^{bd} - g^{ad}g^{bc})\bar{\eta}$$

$$= \{\tfrac{1}{2}(R - (-R)) - (K^2 - K_b{}^aK_a{}^b)\} = [R + K_b{}^aK_a{}^b - (K_a{}^a)^2]\bar{\eta}$$

$$(R - \mathbb{K}K^2 + \mathbb{K}^2K)\eta$$

$$\eta\,\delta^{ca}_{ab}(\tfrac{1}{2}R^{ab}{}_{cd} - K_c{}^aK_d{}^b)$$

We now examine the extra terms.

$$\bar{D}\bar{\eta}^a{}_\perp = \bar{\eta}^a{}_{\perp b}\wedge\Theta^b \qquad \Theta^b = T^b{}_{\perp d}\theta^\perp\wedge\theta^d + \tilde{\Theta}^b, \quad \tilde{\Theta}^b = 0$$

$$\bar{D}\bar{\eta}^a{}_\perp\wedge\bar{\omega}^\perp{}_a = -\bar{\eta}^a{}_{\perp b}\wedge\Theta^b\wedge\bar{\omega}^\perp{}_a + \bar{\eta}^a{}_{\perp b}\wedge\bar{T}^b{}_{\perp d}\theta^\perp\wedge\theta^d\wedge\bar{\omega}^\perp{}_a$$

Let $X\in TM^\perp$, $X = \xi n$.

$$\bar{\omega}^\perp{}_a(X) = -\langle n, \bar{\nabla}_X e_a\rangle = \xi\langle\bar{\nabla}_n n, e_a\rangle = \xi a_a = a_a\theta^\perp(X)$$

$$K_a = -\tilde{\omega}^\perp{}_a \qquad \text{so } \boxed{\bar{\omega}^\perp{}_a = a_a\theta^\perp - K_a}$$

This belongs in the Imbedding section.

Now $\bar{\eta}^a{}_{\perp b}\wedge\Theta^b\wedge K_a = 0$ since it is an element of $\Lambda^4(M)$. Hence

$$\mathbb{K} = \bar{\eta}^{ab}\wedge[\Omega_{ab} - K_a\wedge K_b] - \bar{d}(2\bar{\eta}^a{}_\perp\wedge\bar{\omega}^\perp{}_a) + 2\theta^\perp\wedge\bar{\eta}^a{}_{\perp b}\wedge[\bar{T}^b{}_{\perp d}\theta^d\wedge K_a - a_a\Theta^b]$$

But $\bar{\eta}^a{}_{\perp b} = -\theta^\perp\wedge\mathcal{J}^a{}_b$, $\bar{\eta}^{ab} = \bar{\eta}_\perp{}^a{}_{bc}\theta^\perp\wedge\theta^c = \theta^\perp\wedge\mathcal{J}^a{}_{bc}\theta^c = \theta^\perp\wedge\mathcal{J}^{ab}$

$$\boxed{\mathbb{K} = \theta^\perp\wedge\mathcal{J}^{ab}\wedge[\Omega_{ab} - K_a\wedge K_b] - 2\theta^\perp\wedge\mathcal{J}^{ab}\wedge[\bar{T}_{b\perp d}\theta^d\wedge K_a - a_a\Theta_b] - \bar{d}(2\bar{\eta}^a{}_\perp\wedge\bar{\omega}^\perp{}_a)}$$

Note: $\bar{\eta}^a{}_\perp\wedge\bar{\omega}^\perp{}_a = \bar{\eta}^a{}_\perp\wedge(a_a\theta^\perp - K_a) = -K_a\bar{\eta}^a{}_\perp + a_a\theta^\perp\wedge\bar{\eta}^a{}_\perp = -K_{ba}\theta^b\wedge\bar{\eta}^a{}_\perp + a_a\theta^\perp\wedge\bar{\eta}^a{}_\perp$

$$= -K_{ba}(\delta^b_\perp\bar{\eta}^a - g^{ba}\bar{\eta}_\perp) + a_a(\delta^\perp_\perp\bar{\eta}^a - g^{\perp a}\bar{\eta}_\perp) = K_a{}^a\bar{\eta}_\perp + a^a\bar{\eta}_a = [(\mathbb{K}K)n^\alpha + a^\alpha]\bar{\eta}_\alpha$$

only involves symmetric part

Compare with MTW 21.88.

Finally recall $\Upsilon_{ab} \equiv \bar{Q}_{a\perp b}$:

$$\boxed{\mathbb{K} = \theta^\perp \wedge \mathcal{J}^{ab} \wedge [\Omega_{ab} - K_a \wedge K_b + 2 a_a \Theta_b - 2\Upsilon_{bd}\theta^d \wedge K_a] - \bar{d}(2\bar{n}^a{}_\perp \wedge \bar{\omega}^\perp{}_a)} = \theta^\perp \wedge (\text{3-form on } M) - \bar{d}(\,).$$

We now go to an arbitrary ~~adapt~~ ADM frame $\{t, e_a\} \equiv \{e_\alpha\}_{\alpha=0123}$ with dual frame $\{\theta^0, \theta^a\}$

Let $\{e_\alpha{}'\} = \{n, e_a\}$ denote the old frame, $\theta^{\alpha\prime}$ the old dual frame.

By definition of the lapse and shift: $\boxed{\underline{t} = Nn + \underline{N}}$, $n = \frac{1}{N}t - \underline{N}/N$

Let $e_\alpha = e_\beta{}' A^\beta{}_\alpha$. Then $A^\beta{}_\alpha = \begin{pmatrix} N & N^1 & N^2 & N^3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ $\theta^{\alpha\prime} = A^\alpha{}_\beta \theta^\beta$ $\begin{cases} \theta^\perp = N\theta^0 \\ \theta'^a = N^a\theta^0 + \theta^a \end{cases}$

Also $\theta^\perp \wedge \theta'^a \wedge \theta'^b \wedge \theta'^c = N\theta^0 \wedge (N^a\theta^0 + \theta^a) \wedge (N^b\theta^0 + \theta^b) \wedge (N^c\theta^0 + \theta^c) = N\theta^0 \wedge \theta^a \wedge \theta^b \wedge \theta^c$

From now on we neglect the 4-exact form in $\mathbb{K}$ and define the ADM Lagrangian to be

$$\boxed{\mathcal{L} = N\mathcal{J}^{ab} \wedge [\Omega_{ab} - K_a \wedge K_b + 2 a_a \Theta_b - 2\Upsilon_{bd}\theta^d \wedge K_a]}$$

The 4 Lagrangian is $\theta^0 \wedge \mathcal{L} + \bar{d}(\,)$. Note $\theta^0 = dt$.

We now evaluate the acceleration a_a.

~~$n_0 = \langle n, e_0\rangle = \langle n, t\rangle = \frac{1}{N}$ $n^0 = g^{0\alpha} n_\alpha = g^{00} n_0 = \langle \theta^0, \theta^0\rangle \frac{1}{N} =$~~
~~$n_a = \langle n, e_a\rangle = 0$~~

First

$$\bar{g}_{\alpha\beta} = \left[\begin{array}{c|c} N_a N^a - N^2 & N_a \\ \hline N_a & g_{ab} \end{array}\right] \qquad \bar{g}^{\alpha\beta} = \left(\begin{array}{c|c} -\frac{1}{N^2} & N^a/N^2 \\ \hline N^a/N^2 & g^{ab} - N^a N^b/N^2 \end{array}\right) \qquad \begin{array}{l} n_\alpha = (-N, 0\,0\,0) \\ n^\alpha = (\frac{1}{N}, -\frac{N^a}{N}). \end{array}$$

$[n, e_d] = n^\alpha [e_\alpha, e_d] - e_d(n^\alpha) e_\alpha = n^\alpha C^\gamma{}_{\alpha d} e_\gamma - n^\alpha{}_{,d} e_\alpha$.

$\langle n, [n, e_d]\rangle = \langle n, n^\alpha C^\gamma{}_{\alpha d} e_\gamma - n^\alpha{}_{,d} e_\alpha\rangle = n_0 n^\alpha C^0{}_{\alpha d} - n^0{}_{,d} n_0$.

But in any ADM frame $C^0{}_{0d} = C^0{}_{ad} = 0$, $\langle n, [n, e_d]\rangle = N\left(\frac{1}{N}\right)_{,d} = -N^{-1}N_{,d}$

$a_a = \langle \bar{\nabla}_n n, e_a\rangle = -\langle n, \bar{\nabla}_n e_a\rangle = -\langle n, \bar{\nabla}_{e_a} n + [n, e_a] + \bar{\mathbb{T}}(n, e_a)\rangle$

$= -\langle n, [n, e_a] + \bar{\mathbb{T}}(n, e_a)\rangle = \boxed{\bar{T}^\perp{}_{\perp a} + N^{-1}N_{,a} \equiv a_a}$

Recall $q(x) \equiv -\langle n, \bar{\mathbb{T}}(n, x)\rangle$ $q_a \equiv -\langle n, \bar{T}(n, e_a)\rangle$ $\boxed{a_a = q_a + N^{-1}N_{,a}}$

$\mathcal{L} = N\mathcal{J}^{ab} \wedge [\Omega_{ab} - K_a \wedge K_b + 2 a_a \Theta_b - 2\Upsilon_{bd}\theta^d \wedge K_a] + 2N_{,a}\mathcal{J}^a{}_b \wedge \Theta^b$

But $N_{,a}\mathcal{J}^a{}_b \wedge \Theta^b = (e_a \lrcorner dN) \wedge \mathcal{J}^a{}_b \wedge \Theta^b = dN \wedge e_a \lrcorner (\mathcal{J}^a{}_b \wedge \Theta^b)$ $\left(dN \wedge \underset{1}{\mathcal{J}^a}{}_{\underset{2}{b}} \wedge \underset{34}{\Theta^b} = 0\right)$

$= -dN \wedge \mathcal{J}^a{}_b \wedge T^b{}_a$ (since $e_a \lrcorner \mathcal{J}^a{}_b = 0$) $= -d(N\mathcal{J}^a{}_b \wedge T^b{}_a) + Nd(\mathcal{J}^a{}_b \wedge T^b{}_a)$

$\mathcal{L} = N\mathcal{J}^{ab} \wedge [\Omega_{ab} - K_a \wedge K_b - 2\Upsilon_{bd}\theta^d \wedge K_a + 2 a_a \Theta_b] + 2Nd(\mathcal{J}^a{}_b \wedge T^b{}_a) - d(2N\mathcal{J}^a{}_b \wedge T^b{}_a)$

We now neglect the exact form $d(2N\mathcal{J}^a{}_b \wedge T^b{}_a)$.

REWRITING THE LAGRANGIAN IN FIRST ORDER FORM

We now compute the antisymmetric and symmetric parts of the extrinsic curvature. We work in a comoving ADM frame

The first we've already done: $K_{[ab]} = -\frac{1}{2}\bar{\Gamma}^{1}{}_{ab} \equiv \mathcal{K}_{ab}$

$$K_{(ab)} = \langle \bar{\nabla}_{(a} e_{b)}, n\rangle = n^{\alpha}\bar{\Gamma}_{\alpha,(ab)} = n^{0}\bar{\Gamma}_{0,ab} + n^{c}\Gamma_{c,(ab)} = N^{-1}\left(\bar{\Gamma}_{0(ab)} - N^{c}\Gamma_{c,(ab)}\right)$$

Recall $2\bar{\Gamma}_{0ab} = g_{0a,b} + g_{0b,a} - g_{ab,0} + \bar{T}_{aob} + \bar{T}_{boa} - \bar{T}_{oba} + C_{aob} + C_{boa} - C_{oba}$

$$2\bar{\Gamma}_{0(ab)} = g_{0(a,b)} + \bar{T}_{(a|0|b)} - \frac{1}{2}\dot{g}_{ab} = N_{(a,b)} + \bar{T}_{(a|0|b)} - \frac{1}{2}\dot{g}_{ab} \qquad (\cdot \equiv \partial_0 \equiv e_0)$$

$$K_{(ab)} = (2N)^{-1}\left[N_{a,b} + N_{b,a} - \dot{g}_{ab} - 2N^{c}\Gamma_{c,(ab)} + \bar{Q}_{aob} + \bar{Q}_{boa}\right] \qquad \boxed{Q \equiv T}$$

$$= (2N)^{-1}\left[N_{a,b} - N_c\Gamma^{c}{}_{ba} + N_{b,a} - N_c N^{c}{}_{ab} - \dot{g}_{ab} + \bar{Q}_{aob} + \bar{Q}_{boa}\right]$$

$$= (2N)^{-1}\left[2g_{c(a}\nabla_{b)}N^{c} - \dot{g}_{ab} + \bar{T}_{aob} + \bar{T}_{boa}\right]$$

From now on $\sim$ will only appear in the transposed connection. Recall $\tilde{\nabla}_a X^b = \nabla_a X^b + T^{b}{}_{ca}X^{c}$.

$$K_{(ab)} = (2N)^{-1}\left[2g_{c(a}\left\{\tilde{\nabla}_{b)}N^{c} - T^{c}{}_{|d|b)}N^{d}\right\} - \dot{g}_{ab} + \bar{T}_{aob} + \bar{T}_{boa}\right]$$

$$= (2N)^{-1}\left[2g_{c(a}\tilde{\nabla}_{b)}N^{c} - \dot{g}_{ab} - 2N^{c}\bar{Q}_{(a|c|b)} + 2\bar{T}_{(a|0|b)}\right]$$

Recall: $\tau_{ab} \equiv \langle e_a, \bar{T}(n, e_b)\rangle = \langle e_a, \frac{1}{N}\bar{T}(e_0, e_b) - \frac{1}{N}\bar{T}(N, e_b)\rangle = \frac{1}{N}\left(T_{aob} - N^{c}T_{acb}\right)$

$$N\tau_{ab} = Q_{(a|0|b)} - N^{c}Q_{(a|c|b)}$$

$$K_{(ab)} = (2N)^{-1}\left[2g_{c(a}\tilde{\nabla}_{b)}N^{c} - \dot{g}_{ab} + 2N\tau_{ab}\right] = \frac{1}{2N}\left[2g_{c(a}\tilde{\nabla}_{b)}N^{c} - \dot{g}_{ab}\right] + \tau_{(ab)}$$

But in our comoving ADM frame: $\mathcal{L}_t = \partial_t = \partial_0 \equiv \cdot$

$$K_{(ab)} = \frac{1}{2N}\left[g_{ca}\tilde{\nabla}_{b}N^{c} + g_{cb}\tilde{\nabla}_{a}N^{c} - (\mathcal{L}_t g)_{ab}\right] + \tau_{(ab)} = \underbrace{\boxed{\frac{1}{2N}\left[(\mathcal{L}_N g)_{ab} - (\mathcal{L}_t g)_{ab}\right] + \tau_{(ab)} = K_{(ab)}}}_{(-\frac{1}{2})(\mathcal{L}_{\dot{n}} g)_{ab}}$$

TRUE IN ANY ADM frame.

Now we write $\mathcal{L} = T - V$

$$\begin{cases} T \equiv -N\mathcal{J}^{ab}\wedge\left[K_a\wedge K_b + 2\tau_{bd}\theta^{d}\wedge K_a\right] & \leftarrow \text{contains the "velocities"} \\ V \equiv -N\mathcal{J}^{ab}\wedge\left[\Omega_{ab} + 2q_a{}^{c}\Theta_b\right] - 2N\, d\left(\mathcal{J}^{a}{}_{b}\wedge T^{b}{}_{a}\right) \end{cases}$$

To find the momentum conjugate to g_{ab} we vary T wrt $\dot{g}_{ab}$:

$$\delta_{\dot{g}}T = -N\mathcal{J}^{ab}\wedge\left[\delta_{\dot{g}}K_a\wedge K_b + K_a\wedge\delta_{\dot{g}}K_b + 2\tau_{bd}\theta^{d}\wedge\delta_{\dot{g}}K_a\right] = -2N\mathcal{J}^{ab}\wedge\left[K_a - \tau_{ad}\theta^{d}\right]\wedge\theta^{c}\,\delta_{\dot{g}}K_{cb}$$

Only $K_{(bc)}$ involves $\dot{g}_{ab}$ so from above: $\delta_{\dot{g}}K_{cb} = -\frac{1}{2N}\delta\dot{g}_{cb}$

$$\delta_{\dot{g}}T = \mathcal{J}^{ab}\wedge\left[K_a - \tau_{ad}\theta^{d}\right]\wedge\theta^{c}\,\delta\dot{g}_{bc}$$

Hence Π^{ab} (momentum conjugate to g_{ab}) $= \mathcal{J}^{c(a}\wedge\left[K_c - \tau_{cd}\theta^{d}\right]\wedge\theta^{b)} = \left(K_c - \tau_{cd}\theta^{d}\right)\wedge\theta^{(b}\wedge\mathcal{J}^{|c|a)}$

$$= \left(K_{ac} - \tau_{cd}\right)\theta^{d}\wedge\underbrace{\theta^{(b}\wedge\mathcal{J}^{|c|a)}}_{\left(g^{ba}g^{cd} - g^{c(b}g^{a)d}\right)\mathcal{J}_d}$$

$$= \left(K_{dc} - \tau_{cd}\right)\left(g^{ba}g^{cd} - g^{c(b}g^{a)d}\right)\mathcal{J}$$

$$\boxed{\Pi^{ab} = \left[(K-\tau)g^{ab} - \left(K^{(ab)} - \tau^{(ab)}\right)\right]\mathcal{J} \qquad \Pi = g_{ab}\Pi^{ab} = 2(K-\tau)\mathcal{J}}$$

Define $\sigma^{ab} = K^{(ab)} - \tau^{(ab)} = *\left(\frac{1}{2}\Pi g^{ab} - \Pi^{ab}\right)$ $\qquad \sigma = \frac{1}{2}*\Pi$ $\qquad$ Note $\Pi^{ab}\wedge\sigma_{ab} = \frac{1}{2}\Pi\wedge\Pi^{*} - \Pi^{ab}{}_{A}{}^{*}\Pi_{ab}$.

From the formula for $K^{(ab)}$: $\dot{g}_{ab} = (\mathcal{L}_N g)_{ab} - 2N\sigma_{ab}$, so σ_{ab} is just the part which contains the velocities

Note also: $\sigma^{2} = \frac{1}{4}(*\Pi)^{2}$ $\qquad \sigma^{ab}\sigma_{ab} = -\frac{1}{4}(*\Pi)^{2} + *\Pi^{ab}*\Pi_{ab}$

Define the gravitational Hamiltonian: $\mathcal{H}_G \equiv \Pi^{ab}\dot{g}_{ab} - \mathcal{L} = \Pi^{ab}\dot{g}_{ab} - T + V$ to be written in terms of the momenta.

We now work on the inversion.

(1) $\Pi^{ab}\dot{g}_{ab} = \Pi^{ab}\left[(\mathcal{L}_N g)_{ab} - 2N\sigma_{ab}\right] = 2\Pi_a{}^b\tilde{\nabla}_b N^a + 2N(\Pi^{ab}\wedge\Pi^*{}_{ab} - \frac{1}{2}\Pi\wedge{}^*\Pi)$

$-T = N(K_a\wedge K_b + 2\Upsilon_{bd}\theta^d\wedge K_a)\wedge\mathcal{J}^{ab} = N(K_{da}K_{cb} + 2\Upsilon_{bd}K_{ca})\{\theta^d\wedge\theta^c\wedge\mathcal{J}^{ab} = (g^{cb}g^{ad} - g^{ca}g^{bd})\}$

$= N(K^2 - K^{ba}K_{ab} + 2\Upsilon^{ab}K_{ab} - 2\Upsilon K)\mathcal{J}$ ∴ Now substitute $K_{(ab)} = \sigma_{ab} + \Upsilon_{(ab)}$

$= N(K^2 - K^{(ab)}K_{(ab)} + 2\Upsilon^{(ab)}K_{(ab)} - 2\Upsilon K + K^{[ab]}K_{[ab]} + 2\Upsilon^{[ab]}K_{[ab]})\mathcal{J}$

$= N((\sigma+\Upsilon)^2 - [\sigma^{ab} + \Upsilon^{(ab)}][\sigma_{ab} + \Upsilon_{(ab)}] + 2\Upsilon^{(ab)}(\sigma_{ab} + \Upsilon_{(ab)}) - 2\Upsilon(\sigma+\Upsilon) + K^{[ab]}K_{[ab]} + 2\Upsilon^{[ab]}K_{[ab]})\mathcal{J}$

$= N(\sigma^2 - \sigma^{ab}\sigma_{ab} + \Upsilon^{(ab)}\Upsilon_{(ab)} - \Upsilon^2 + K^{[ab]}K_{[ab]} + 2\Upsilon^{[ab]}K_{[ab]})\mathcal{J}$

But $\sigma^2 = \frac{1}{4}({}^*\Pi)^2$ $\sigma^{ab}\sigma_{ab} = -\frac{1}{4}({}^*\Pi)^2 + ({}^*\Pi^{ab})({}^*\Pi_{ab})$, $\lambda_{ab} \equiv \Upsilon_{(ab)}$, $\rho_{ab} \equiv \Upsilon_{[ab]}$, $\mathcal{X}_{ab} \equiv K_{(ab)}$

(2) $-T = N\left[\frac{1}{2}\Pi\wedge{}^*\Pi - \Pi^{ab}\wedge{}^*\Pi_{ab} + \lambda_{ab}\wedge{}^*\lambda^{ab} - \lambda\wedge{}^*\lambda + \mathcal{X}_{ab}\wedge{}^*\mathcal{X}^{ab} + 2\rho_{ab}\wedge{}^*\mathcal{X}^{ab}\right]$

$$\boxed{\mathcal{H}_G = 2\Pi_a{}^b\tilde{\nabla}_b N^a + V + N(\Pi^{ab}\wedge{}^*\Pi_{ab} - \tfrac{1}{2}\Pi\wedge{}^*\Pi + \lambda_{ab}\wedge{}^*\lambda^{ab} - \lambda\wedge{}^*\lambda + \mathcal{X}_{ab}\wedge{}^*\mathcal{X}^{ab} + 2\rho_{ab}\wedge{}^*\mathcal{X}^{ab})}$$

$\mathcal{H}_G = 2\Pi_a{}^b\tilde{\nabla}_b N^a + N\mathcal{H}$ defines the super-Hamiltonian $\mathcal{H}$.

$$\boxed{\mathcal{H} = [\Pi^{ab}\wedge{}^*\Pi_{ab} - \tfrac{1}{2}\Pi\wedge{}^*\Pi + \lambda^{ab}\wedge{}^*\lambda_{ab} - \lambda\wedge{}^*\lambda + \mathcal{X}_{ab}\wedge{}^*\mathcal{X}^{ab} + 2\rho_{ab}\wedge{}^*\mathcal{X}^{ab} - \mathcal{J}^{ab}\wedge(\Omega_{ab} + 2q_a\Theta_b) - 2d(\mathcal{J}^a{}_b\wedge T^b{}_a)]}$$

Also $2\Pi_a{}^b\tilde{\nabla}_b N^a = 2\Pi_a{}^b e_b\lrcorner\tilde{D}N^a = (2e_b\lrcorner\Pi_a{}^b)\wedge\tilde{D}N^a \equiv \Pi_a\wedge\tilde{D}N^a$ using $\Pi_a{}^b\wedge DN^a = 0$

$\Pi_a \equiv 2e_b\lrcorner\Pi_a{}^b$ $\mathcal{H}_G = \Pi_a\wedge\tilde{D}N^a + N\mathcal{H}$.

But $d(\Pi_a\wedge N^a) = \tilde{D}(\Pi_a\wedge N^a) = N^a D\Pi_a + \Pi_a\wedge\tilde{D}N^a$ so $2\Pi_a{}^b\tilde{\nabla}_b N^a = d(N^a\Pi_a) - N^a\tilde{D}\Pi_a$

Define the supermomentum $\mathcal{H}_a = -\tilde{D}\Pi_a$

$$\begin{cases}\mathcal{H}_G = N\mathcal{H} + N^a\mathcal{H}_a + d(N^a\Pi_a) \\ \mathcal{L} = \Pi^{ab}\dot{g}_{ab} - \mathcal{H}_G\end{cases}$$

Recall we have already neglected 2 exact forms.

We've been using a comoving ADM frame in which $\dot{g}_{ab} = \mathcal{L}_t g_{ab}$, so: $\boxed{\mathcal{L} = \Pi^{ab}(\mathcal{L}_t g)_{ab} - \mathcal{H}_G}$ in an arbitrary ADM frame.

We want to express this in terms of $\dot{g}_{ab}$, $\dot{\theta}^a \equiv \mathcal{L}_t\theta^a$

$\mathcal{L}_t g = \mathcal{L}_t(g_{ab}\theta^a\otimes\theta^b) = \dot{g}_{ab}\theta^a\otimes\theta^b + g_{ab}(\mathcal{L}_t\theta^a\otimes\theta^b + \theta^a\otimes\mathcal{L}_t\theta^b) = [\frac{\partial g_{ab}}{\partial t} + 2g_{c(a}e_{b)}\lrcorner\dot{\theta}^c]\theta^a\otimes\theta^b$

$\mathcal{L}_t\theta^a = (e_c\lrcorner\dot{\theta}^a)\theta^c$, $(\mathcal{L}_t g)_{ab} = \dot{g}_{ab} + 2g_{c(a}e_{b)}\lrcorner\dot{\theta}^c$

$\Pi^{ab}(\mathcal{L}_t g)_{ab} = \Pi^{ab}\dot{g}_{ab} + \Pi^{ab}(2g_{c(a}e_{b)}\lrcorner\dot{\theta}^c) = \Pi^{ab}\dot{g}_{ab} + 2\Pi_a{}^b\wedge e_b\lrcorner\dot{\theta}^a = \Pi^{ab}\dot{g}_{ab} + (2e_b\lrcorner\Pi_a{}^b)\wedge\dot{\theta}^a$

So Π_a the momentum conjugate to θ^a is given by. $\Pi_a = 2e_b\lrcorner\Pi_a{}^b$

SUMMARY We decomposed the 4-dim curvature scalar Lagrangian in a 3+1 manner, neglecting one exact 4-form and 2 exact 3-forms, arriving at the first order form:

$$\mathcal{L} = \Pi^{ab}\dot{g}_{ab} + \Pi_a \wedge \dot{\theta}^a - \mathcal{H}_G \quad , \quad \mathcal{H}_G = N\mathcal{H} + N^a \mathcal{H}_a \qquad (\cdot \equiv \mathcal{L}_{\underline{t}})$$

$\Pi_a = 2 e_b \lrcorner \Pi_a{}^b$ conjugate to θ^a, supermomentum $\mathcal{H}_a = -\tilde{D}\Pi_a$, super-Hamiltonian $\mathcal{H}$:

$$\mathcal{H} = \Pi^{ab} \wedge {}^*\Pi_{ab} - \tfrac{1}{2}\Pi \wedge {}^*\Pi + \lambda_{ab} \wedge {}^*\lambda^{ab} - \lambda \wedge {}^*\lambda + \mathcal{K}_{ab} \wedge {}^*\mathcal{K}^{ab} + 2\rho_{ab} {}^* \wedge \mathcal{K}^{ab} - \mathcal{T}^{ab} \wedge [\Omega_{ab} + 2 q_a \Theta_b] - 2\, d(\mathcal{T}^a{}_b \wedge \Gamma^b{}_a)$$

where $\lambda_{ab} = \overline{\mathcal{T}}_{(a\perp\perp b)}$, $\rho_{ab} = \overline{\mathcal{T}}_{[a\perp\perp b]}$, $q_a = \overline{\mathcal{T}}^{\perp}{}_{\perp a}$, $\mathcal{K}_{ab} = \tfrac{1}{2}\overline{\mathcal{T}}_{\perp(ab)}$

are various extrinsic torsion tensors.

VARIATION OF THE LAGRANGIAN

The momenta conjugate to the frame $\{\theta^a\}$ and the metric are not independent: $\Pi_a = 2 e_b \lrcorner \Pi_a{}^b$ (because we require the metric be covariant constant) so we can't simultaneously vary Π_a and $\Pi_a{}^b$ independently. We therefore assume a comoving ADM frame. The variational principle will consist of independent variations of the metric coefficients g_{ab}, the conjugate momentum Π^{ab} the lapse, the shift and the torsion fields. Before varying the Lagrangian we derive useful preliminary results.

Let σ be a 3-form: $\sigma = s\mathcal{J}$ so that ${}^*\sigma = s$,

$$\delta\sigma = \delta(s\mathcal{J}) = (\delta s)\mathcal{J} + s\,\delta\mathcal{J} = \delta({}^*\sigma)\mathcal{J} + s\,\tfrac{1}{2}\delta g_{ab} g^{ab}\,\mathcal{J}$$

$${}^*\delta\sigma = \delta({}^*\sigma) + (\delta g_{ab})\,\tfrac{1}{2} g^{ab}\,{}^*\sigma \qquad \boxed{\delta({}^*\sigma) = {}^*\delta\sigma - \tfrac{1}{2}(\delta g_{ab})\, g^{ab}\,{}^*\sigma} \qquad (1)$$

Using this and $\alpha \wedge {}^*\beta = \beta \wedge {}^*\alpha$ (α, β p-forms) we get a formula for $\delta(\sigma^A \wedge {}^*\sigma_A)$ σ_A a tensorial-3 form

$$\delta(\sigma^A \wedge {}^*\sigma_A) = (\delta\sigma^A) \wedge {}^*\sigma_A + \sigma^A \wedge \delta({}^*\sigma_A) = (\delta\sigma^A) \wedge {}^*\sigma_A + (\delta\sigma_A) \wedge {}^*\sigma^A - \tfrac{1}{2} g^{ab}\delta g_{ab}\, \sigma^A \wedge {}^*\sigma_A \qquad (2)$$

Traditionally, if σ^A is a tensor density this is just the variation of $g^{+\frac{1}{2}}\sigma^A\sigma_A$.

$$\delta\sigma_{cb} = \delta(g_{ac} g_{bd} \sigma^{bd}) = (\delta g_{ac})\sigma^c{}_b + (\delta g_{bd})\sigma^d{}_a + g_{ac} g_{bd}\,\delta\sigma^{bd} \qquad (3)$$

Recall the formula $\delta\mathcal{J}^a{}_b = G^{acfg}\,\delta g_{fg}\,\mathcal{J}_{cb}$ $\qquad G^{acfg} = \tfrac{1}{2}(g^{ac}g^{fg} - g^{af}g^{cg} - g^{ag}g^{cf})$ $\qquad (4)$

We also require $\delta(\omega_A \wedge {}^*\omega^A)$ for zeroforms ω_A; ${}^*\omega^A = \omega^A \mathcal{J}$:

$$\delta(\omega_A \wedge {}^*\omega^A) = \delta(\omega_A \omega^A \mathcal{J}) = \delta\omega_A \wedge {}^*\omega^A + (\delta\omega^A) \wedge {}^*\omega_A + \tfrac{1}{2}\delta g_{ab} g^{ab}\,\omega_A \wedge {}^*\omega^A \qquad (5)$$

Last, recall $\delta\Omega^\alpha{}_\beta = D\,\delta\omega^\alpha{}_\beta$ $\qquad (6)$.

We proceed with the variation of the Hamiltonian density $\mathcal{H}_G = N\mathcal{H} + N^a\mathcal{H}_a$

$$\delta\mathcal{H}_G = (\delta N)\mathcal{H} + (\delta N^a)\mathcal{H}_a + N\,\delta\mathcal{H} + N^a\,\delta\mathcal{H}_a$$

The terms $\mathcal{H}, \mathcal{H}_a$ are called Hamiltonian constraints, which arise from the diffeomorphism gauge group of the theory.

Now Π^{ab}, Π are 3-forms so we use (2) (3):

$$\delta(\Pi^{ab} \wedge {}^*\Pi_{ab}) = \delta\Pi^{ab} \wedge {}^*\Pi_{ab} + \{\delta\Pi_{ab} = 2\delta g_{a[a}\Pi^c{}_{b]} + g_{ac} g_{bd}\,\delta\Pi^{cd}\} \wedge {}^*\Pi^{ab} - \tfrac{1}{2} g^{ab}\delta g_{ab}\,\Pi^{cd} \wedge {}^*\Pi_{cd}$$

$$= (\delta\Pi^{ab}) \wedge 2\,{}^*\Pi_{ab} + (\delta g_{ab})\,[2\Pi_a{}^{(a} \wedge {}^*\Pi^{b)c} - \tfrac{1}{2} g^{ab}\,\Pi^{cd} \wedge {}^*\Pi_{cd}]$$

$$\delta(\Pi \wedge {}^*\Pi) = 2\,\delta\Pi \wedge {}^*\Pi - \tfrac{1}{2}\delta g_{ab} g^{ab}\,\Pi \wedge {}^*\Pi \qquad \delta(\Pi^{ab} g_{ab}) = (\delta\Pi^{ab})\, g_{ab} + \Pi^{ab}\,\delta g_{ab}$$

$$= (\delta\Pi^{ab}) \wedge (2 g_{ab}\,{}^*\Pi) + (\delta g_{ab})(2\Pi^{ab} \wedge {}^*\Pi - \tfrac{1}{2} g^{ab}\,\Pi \wedge {}^*\Pi)$$

Using (5) we get exactly analogous formulas except for a minus sign:

this minus from g^{ab} π^{ab} λ_{ab}

$$\delta[\lambda_{ab}\wedge{}^*\lambda^{ab}] = (\delta\lambda_{ab})\wedge 2{}^*\lambda^{ab} + (\delta g_{ab})[-2\lambda^{c(a}\wedge{}^*\lambda^{b)}{}_c + \tfrac{1}{2}g^{ab}\lambda_{cd}\wedge{}^*\lambda^{cd}]$$

$$\delta[\lambda\wedge{}^*\lambda] = (\delta\lambda_{ab})\wedge 2g^{ab}{}^*\lambda + (\delta g_{ab})[-2\lambda^{ab}\wedge{}^*\lambda + \tfrac{1}{2}g^{ab}\lambda\wedge{}^*\lambda]$$

$$\delta[\chi_{ab}\wedge\chi^{ab}] = \delta\chi_{ab}\wedge{}^*2\chi^{ab} + (\delta g_{ab})[2\chi^{c(a}\wedge{}^*\chi^{b)}{}_c + \tfrac{1}{2}g^{ab}\chi_{cd}\wedge{}^*\chi^{cd}]$$

$$\delta[\rho^{ab}\wedge{}^*\chi_{ab}] = (\delta\chi_{ab})\wedge{}^*\rho^{ab} + (\delta\rho_{ab})\wedge{}^*\chi^{ab} + \delta g_{ab}(2\rho^{c(a}\wedge{}^*\chi^{b)}{}_c + \tfrac{1}{2}g^{ab}\rho^{cd}\wedge{}^*\chi_{cd})$$

Now we have to vary $\mathcal{J}^a{}_b\wedge(\Omega^b{}_a - 2q_a\Theta^b) \equiv \mathcal{J}^a{}_b\wedge S^b{}_a$

$$\delta[\mathcal{J}^a{}_b\wedge S^b{}_a] = (\delta\mathcal{J}^a{}_b)\wedge S^b{}_a + \mathcal{J}^a{}_b\wedge\delta S^b{}_a = G^{acmn}\delta g_{mn}\mathcal{J}_{cb}\wedge S^b{}_a + \mathcal{J}^a{}_b\wedge S^b{}_a$$

$$\delta S^b{}_a = \delta\Omega^b{}_a - 2(\delta q_a)\Theta^b - 2q_a\delta(D\theta^b) = D\delta\omega^b{}_a - 2\delta q_a\Theta^b - 2q_a\delta\omega^b{}_c\wedge\theta^c$$

$$\delta[\mathcal{J}^a{}_b\wedge(\Omega^b{}_a - 2q_a\Theta^b)] = (\delta g_{ab})G^{cdab}\mathcal{J}_{ce}\wedge(\Omega^b{}_a - 2q_a\Theta^b)$$
$$+ \mathcal{J}^a{}_b\wedge D\delta\omega^b{}_a - 2(\delta q_a)\mathcal{J}^a{}_b\wedge\Theta^b - 2q_a\mathcal{J}^a{}_b\wedge\delta\omega^b{}_c\wedge\theta^c.$$

Finally, the last term in $\mathcal{H}$:

$$\delta d[\mathcal{J}^a{}_b\wedge T^b{}_a] = d[\delta(\mathcal{J}^a{}_b\wedge T^b{}_a)] = d[\delta\mathcal{J}^a{}_b\wedge T^b{}_a + \mathcal{J}^a{}_b\wedge\delta T^b{}_a] = d[G^{acmn}\delta g_{mn}\mathcal{J}_{cb}\wedge T^b{}_a + \mathcal{J}^a{}_b\wedge\delta T^b{}_a]$$

$\delta T^b{}_a$ is easily computed when the frames are fixed:

$$\delta T^b{}_a = \delta[e_a\lrcorner\Theta^b] = e_a\lrcorner\delta D\theta^b = e_a\lrcorner(\delta\omega^b{}_c\wedge\theta^c) \qquad (7)$$

$$\delta d(\mathcal{J}^a{}_b\wedge T^b{}_a) = d[G^{acmn}\delta g_{mn}\mathcal{J}_{cb}\wedge T^b{}_a + \mathcal{J}^a{}_b\wedge e_a\lrcorner(\delta\omega^b{}_c\wedge\theta^c)]$$

Collecting our results:

$$N\delta\mathcal{H} = [\delta\pi^{ab}]\,2N({}^*\pi_{ab} - \tfrac{1}{2}g_{ab}{}^*\pi) + [\delta\lambda_{ab}]\,2N({}^*\lambda^{ab} - g^{ab}{}^*\lambda) + [\delta\chi_{ab}]\,2N({}^*\chi^{ab} + {}^*\rho^{ab})$$
$$+ [\delta\rho_{ab}]\,2N{}^*\chi^{ab} + [\delta q_a](-2N\mathcal{J}^a{}_b\wedge\Theta^b)$$
$$+ [\delta g_{ab}]\Big\{ NG^{cdab}\mathcal{J}_{de}\wedge(\Omega^e{}_c - 2q_c\Theta^e) + 2N(\pi_c{}^{(a}\wedge{}^*\pi^{b)c} - \tfrac{1}{2}\pi^{ab}\wedge{}^*\pi) - \tfrac{1}{2}Ng^{ab}(\pi^{cd}\wedge{}^*\pi_{cd} - \tfrac{1}{2}\pi\wedge{}^*\pi)$$
$$-2N(\lambda_c{}^{(a}\wedge{}^*\lambda^{b)c} - \lambda^{ab}\wedge{}^*\lambda) + \tfrac{1}{2}Ng^{ab}(\lambda_{cd}\wedge{}^*\lambda^{cd} - \lambda\wedge{}^*\lambda)$$
$$+2N\chi_c{}^{(a}\wedge{}^*\chi^{b)c} + \tfrac{1}{2}Ng^{ab}\chi_{cd}\wedge{}^*\chi^{cd} + 4N\rho^{c(a}\wedge{}^*\chi^{b)}{}_c + N\rho^{cd}\wedge{}^*\chi_{cd}\Big\}$$
$$+ 2N\delta\omega^b{}_c\wedge q_a\mathcal{J}^a{}_b\wedge\theta^c + N\mathcal{J}^a{}_b\wedge D(\delta\omega^b{}_a) - 2Nd\{G^{acmn}\delta g_{mn}\mathcal{J}_{cb}\wedge T^b{}_a + \mathcal{J}^a{}_b\wedge e_a\lrcorner(\delta\omega^b{}_c\wedge\theta^c)\}$$

$$\equiv C + [\delta g_{ab}]\,NB^{ab} + 2N\delta\omega^a{}_b\wedge q_c\mathcal{J}^c{}_a\wedge\theta^b + N\mathcal{J}^a{}_b\wedge D(\delta\omega^b{}_a) - 2Nd\{G^{acmn}\delta g_{mn}\mathcal{J}_{cb}\wedge T^b{}_a + \mathcal{J}^a{}_b\wedge e_a\lrcorner(\delta\omega^b{}_c\wedge\theta^c)\}$$

By the Liebnitz rule this can be written

$$N\delta\mathcal{H} = C + [\delta g_{ab}](NB^{ab} + 2dN\wedge G^{cdab}\mathcal{J}_{de}\wedge T^e{}_c) + [\delta\omega^a{}_b]\wedge\{2(Nq_c + N_{,c})\mathcal{J}^c{}_a\wedge\theta^b + D(N\mathcal{J}^b{}_a)\} + (\text{exact form})$$

Define $M^{ab} = NB^{ab} + 2\,dN\wedge G^{cdab}\mathcal{J}_{de}\wedge T^e{}_c$ ■

$$N^a\delta\mathcal{H}_a = -N^a\delta\tilde{D}\pi_a = -N^a\{\tilde{D}\delta\pi_a - \delta\tilde{\omega}^b{}_a\wedge\pi_b\} = -d(N^a\delta\pi_a) + (\tilde{D}N^a)\wedge\delta\pi_a + \delta\omega^a{}_b\wedge N^b\pi_a + \underbrace{\delta T^b{}_a}_{e_a\lrcorner(\delta\omega^b{}_c\wedge\theta^c)}\wedge N^a\pi_b$$

$$N^a\delta\mathcal{H}_a = (\tilde{D}N^a)\wedge\delta\pi_a + \underbrace{\delta\omega^a{}_b\wedge[N\lrcorner(\theta^b\wedge\pi_a)]}_{\delta\omega^a{}_b\wedge[N\lrcorner(\theta^b\wedge\pi_a)]} - d(N^a\delta\pi_a)$$

But $\delta\pi_a = \delta[2e_b\lrcorner\pi^b{}_a] = 2e_b\lrcorner\delta\pi^b{}_a$ so:

$$\tilde{D}N^a\wedge\delta\pi_a = 2(\tilde{D}N^a)\wedge e_b\lrcorner\delta\pi^b{}_a = 2\tilde{\nabla}_b N^a\delta[g_{ac}\pi^{bc}] = 2g_{ac}\tilde{\nabla}_b N^a\delta\pi^{bc} + 2\tilde{\nabla}_b N^a\pi^{bc}\delta g_{ac}$$
$$= [\delta\pi^{ab}]\{g_{ac}\tilde{\nabla}_b N^c + g_{cb}\tilde{\nabla}_a N^c\} + [\delta g_{ab}]\{\pi^{ac}\tilde{\nabla}_c N^b + \pi^{cb}\tilde{\nabla}_c N^a\}$$

Using this and $2\pi^{ab} = \pi^a\wedge\theta^b$ we have:

$$N^a\delta\mathcal{H}_a = [\delta\pi^{ab}]\{g_{ac}\tilde{\nabla}_b N^c + g_{cb}\tilde{\nabla}_a N^c\} + [\delta g_{ab}]\{\pi^{ac}\tilde{\nabla}_c N^b + \pi^{cb}\tilde{\nabla}_c N^a\} + [\delta\omega^a{}_b]\wedge\{2N\lrcorner\pi^b{}_a\}$$ ■

So far we have varied the connection one-forms as if they were all independent, but we cant do this because the connection is metric and constrained by 18 equations $Dg_{ab}=0$. These constraints are introduced in the Lagrangian by a multiplier in the Hamiltonian. Then we may vary all the $\omega^a{}_b$'s freely. The total Hamiltonian $\mathcal{H}_T$ is written as $\mathcal{H}_T = \mathcal{H}_G + \chi^{ab}\wedge Dg_{ab}$ where χ^{ab} is a $\overset{sym}{\wedge}$ (2) type 2-form. The variation of this term is:

$$\delta[\chi^{ab}\wedge Dg_{ab}] = \delta\chi^{ab}\wedge Dg_{ab} + [\delta g_{ab}](-D\chi^{ab}) + [\delta\omega^a{}_b]\wedge(-2\chi_a{}^b) + d(\chi^{ab}\,\delta g_{ab})$$

$$\begin{cases} \delta(Dg_{ab}) = \delta(dg_{ab} - 2\omega^\gamma{}_{(a}g_{\gamma b)}) = d(\delta g_{ab}) - 2\,\delta\omega^\gamma{}_{(a}g_{b)\gamma} - \omega^\gamma{}_{(a}\,\delta g_{b)\gamma} \\ \chi^{ab}\wedge\delta(Dg_{ab}) = [d(\chi^{ab}\,\delta g_{ab}) - d\chi^{ab}\,\delta g_{ab}] - 2\,\delta\omega^\gamma{}_a\chi_\gamma{}^a - \chi^{ab}\wedge\omega^\gamma{}_a\,\delta g_{b\gamma} \\ \qquad = d(\chi^{ab}\delta g_{ab}) - \delta g_{ab}\,D\chi^{ab} + \delta\omega^a{}_b\wedge(-2\chi_a{}^b) \end{cases}$$

So the terms involving $\delta g_{ab}, \delta\omega^a{}_b$ are modified by the addition of the multiplier terms.

$$\delta g_{ab}: \quad \{M^{ab}\} + \{\Pi^{ac}\tilde{\nabla}_c N^b + \Pi^{cb}\tilde{\nabla}_c N^a\} - D\chi^{ab}$$

$$\delta\omega^a{}_b: \quad \{2(Nq_c + N_{,c})\,\mathcal{J}^c{}_a\wedge\theta^b + D(N\mathcal{J}^b{}_a)\} + \{2\underline{N}\lrcorner\Pi^b{}_a\} - 2\chi_a{}^b$$

Since there is no momentum conjugate to $\omega^a{}_b$ we may set this latter expression equal to zero $\left(\frac{\delta L}{\delta\omega} = \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\omega}}\right) = 0\right)$

Taking the symmetric part: $\quad 2(Nq_c + N_{,c})\,\mathcal{J}^{c(a}\wedge\theta^{b)} + 2\underline{N}\lrcorner\Pi^{ab} - 2\chi^{ab} = 0$

ie: $$\chi^{ab} = (Nq_c + N_{,c})\,\mathcal{J}^{c(a}\wedge\theta^{b)} + \underline{N}\lrcorner\Pi^{ab}$$

Reinserting this into δg_{ab}: $\quad M^{ab} + \Pi^{ac}\tilde{\nabla}_c N^b + \Pi^{cb}\tilde{\nabla}_c N^a - D[(Nq_c + N_{,c})\,\mathcal{J}^{c(a}\wedge\theta^{b)}] - D(\underline{N}\lrcorner\Pi^{ab})$

Now $\mathcal{J}^{c[a}\wedge\theta^{b]} = -g^{[ab]}\mathcal{J}^c + g^{c[b}\mathcal{J}^{a]}$ so considering the antisymmetric part of the same:

$$2(Nq_c + N_{,c})\,\mathcal{J}^{c[a}\wedge\theta^{b]} + D(N\mathcal{J}^{ba}) = 0$$

And $$D(N\mathcal{J}^{ba}) = N_{,c}\,\theta^c\wedge\mathcal{J}^{ba} + ND\mathcal{J}^{ba} = N_{,c}[g^{ca}\mathcal{J}^b - g^{cb}\mathcal{J}^a] + N\mathcal{J}^{ba}{}_c\,\Theta^c$$

$$= -2N_{,c}\,g^{c[b}\mathcal{J}^{a]} + N\mathcal{J}^{ba}{}_c\,\Theta^c$$ so the antisymmetric part becomes:

$$N[2q^{[b}\mathcal{J}^{a]} + \mathcal{J}^{ba}{}_c\,\Theta^c] = 0$$ This is a constraint equation.

We gather together all the constraint equations; these are obtained by setting to zero the coefficients of the variations of all variables not having conjugate momenta:

$$\left.\begin{array}{lll} [\delta\rho_{ab}] & (a) & 0 = 2N\,{}^*\mathcal{J}^{ab} \\ [\delta\mathcal{L}_{ab}] & (b) & 0 = 2N({}^*\mathcal{J}^{ab} + {}^*\rho^{ab}) \\ [\delta\lambda_{ab}] & (c) & 0 = 2N({}^*\lambda^{ab} - g^{ab}\,{}^*\lambda) \\ [\delta q_a] & (d) & 0 = -2N\mathcal{J}^a{}_b\wedge\Theta^b \\ [\delta\chi_{ab}] & (e) & 0 = N[q^b\mathcal{J}^a - q^a\mathcal{J}^b + \mathcal{J}^{ba}{}_c\,\Theta^c] \end{array}\right\} \text{algebraic constraints}$$

$$\left.\begin{array}{l} 0 = \mathcal{H}_a = -2\tilde{D}[e_b\lrcorner\Pi^b{}_a] \\ 0 = \mathcal{H} = \Pi^{ab}\wedge{}^*\Pi_{ab} - \frac{1}{2}\Pi\wedge{}^*\Pi + \lambda_{ab}\wedge{}^*\lambda^{ab} - \lambda\wedge{}^*\lambda + \mathcal{L}_{ab}\wedge{}^*\mathcal{L}^{ab} + 2\rho_{ab}\,{}^*\mathcal{J}^{ab} \\ \qquad - \mathcal{J}^{ab}\wedge[\mathcal{L}_{ab} + 2q_a\,\Theta_b] - 2\,d[\mathcal{J}^a{}_b\wedge Q^b{}_a] \end{array}\right\} \text{Hamiltonian constraints.}$$

The dynamical equations determined the $\underline{t}$-derivatives of the canonical variables g_{ab} and their conjugate momenta Π^{ab} by:

$$\dot{g}_{ab} = \frac{\delta\mathcal{H}_T}{\delta\Pi^{ab}} \qquad -\dot{\Pi}^{ab} = \frac{\delta\mathcal{H}_T}{\delta g_{ab}}$$

$$\dot{g}_{ab} = 2N({}^*\Pi_{ab} - \tfrac{1}{2} g_{ab} {}^*\Pi) + g_{ac}\tilde{\nabla}_b N^c + g_{cb}\tilde{\nabla}_a N^c$$

$$-\dot{\Pi}^{ab} = N G^{cdab} \varepsilon_{de} \wedge [\Omega^e{}_c - 2 q_c \theta^e] + 2N[\Pi_c{}^{(a} \wedge {}^*\Pi^{b)c} - \tfrac{1}{2}\Pi^{ab} \wedge {}^*\Pi]$$
$$- \tfrac{1}{2} N g^{ab}[\Pi^{cd} \wedge {}^*\Pi_{cd} - \tfrac{1}{2}\Pi \wedge {}^*\Pi] - 2N[\lambda_c{}^{(a} \wedge {}^*\lambda^{b)c} - \lambda^{ab} \wedge {}^*\lambda]$$
$$+ \tfrac{1}{2} N g^{ab}[\lambda_{cd} \wedge {}^*\lambda^{cd} - \lambda^{ab} \wedge {}^*\lambda] + 2N \mathcal{X}_c{}^{(a} \wedge {}^*\mathcal{X}^{b)c} + \tfrac{1}{2} N g^{ab} \mathcal{X}_{cd} \wedge {}^*\mathcal{X}^{cd}$$
$$+ 4N \rho^{c(a} \wedge {}^*\mathcal{X}^{b)c} + \cancel{\tfrac{1}{2} N g^{ab} \mathcal{X}_{cd} \wedge {}^*\mathcal{X}^{cd} + 4N} + N \rho^{cd} \wedge {}^*\mathcal{X}_{cd}$$
$$- D[(N q_c + N_{,c}) \varepsilon^{c(a} \wedge \theta^{b)}] - [D(\underline{N} \lrcorner \Pi^{ab}) - \Pi^{ac}\tilde{\nabla}_c N^b - \Pi^{cb}\tilde{\nabla}_c N^a]$$

Note that where the partial derivatives of the lapse appear one can replace $N q_c + N_{,c}$ by $N q_c$.

"The strategy is to solve the algebraic constraints for the torsion variables, and find (g_{ab}, Π^{ab}) such that the Hamiltonian constraints are satisfied. At this stage one has a good initial data set. The physicist now decides how to continue the surface by choosing the lapse, and the shift. Using the dynamical equations he can compute the values of (g_{ab}, Π^{ab}) on the next surface."

Set $N=0$:
$$\dot{g}_{ab} = g_{ac}\tilde{\nabla}_b N^c + g_{cb}\tilde{\nabla}_a N^c = [\mathcal{L}_{\underline{N}} g]_{ab}$$
$$\dot{\Pi}^{ab} = D(N \lrcorner \Pi^{ab}) + \Pi^{ac}\tilde{\nabla}_c N^b + \Pi^{cb}\tilde{\nabla}_c N^a = [\mathcal{L}_{\underline{N}} \Pi]^{ab}$$

This is expected since the surface is just being dragged along the shift vector $\underline{N}$.

Note that the choice of $N, \underline{N}$ amounts to a choice of the generator $\underline{t}$ of the surface deformation.

SOLUTION OF THE VACUUM ALGEBRAIC CONSTRAINTS. RETURN TO V_4 THEORY

The U_4 Theory is a matter theory, ie in vacuum it reduces to V_4 theory as will be shown. But the solution of the vacuum algebraic constraints is instructive not only because it gives us this result but because with matter terms present the reduction occurs basically along the same lines.

Division by N is allowed even if matter is present since N would appear as a factor.

Constraint equations (a), (b), (c) yield: $\mathcal{H}_{ab}=0$, $\rho_{ab}=0$, $\lambda_{ab}=0$.

(a) $0=\mathcal{J}^a{}_{bA}\theta^{cd}\,\mathbf{O}\,T^b{}_{cd} = (g^{ac}\delta^d_b - g^{ad}\delta^c_b)T^b{}_{cd}\,\mathcal{J} = 2g^{ac}T^d{}_{cd}$, $\quad 0 = T^a{}_{ab}$

Now

$$\mathbf{O}^c\mathcal{J}^{ba}{}_c = \tfrac{1}{2}T^c{}_{em}\theta^e\wedge\theta^m\wedge\mathcal{J}^{ba}{}_c = \tfrac{1}{2}T^c{}_{em}\theta^e\wedge[\delta^m_c\mathcal{J}^{ba} + g^{ma}\mathcal{J}_c{}^b + g^{mb}\mathcal{J}^a{}_c]$$

$$= \tfrac{1}{2}T^c{}_{em}[\delta^m_c g^{ea}\mathcal{J}^b - \delta^m_c g^{eb}\mathcal{J}^a + g^{ma}g^{eb}\mathcal{J}_c - g^{ma}\delta^e_c\mathcal{J}^b + (g^{mb}\delta^e_c\mathcal{J}^a - g^{mb}g^{ea}\mathcal{J}_c]$$

$$= T^c{}_{em}[\delta_c{}^{[m}g^{e]a}\mathcal{J}^b + g^{a[m}g^{e]b}\mathcal{J}_c + g^{b[m}\delta^{e]}_c\mathcal{J}^a]$$

$$= T^{ca}{}_c\mathcal{J}^b + T^{cba}\mathcal{J}_c + T^c{}_c{}^b\mathcal{J}^a = [T^{ca}{}_c\delta^b_e + T_e{}^{ba} + Q^c{}_c{}^b\delta^a_e]\mathcal{J}^e$$

Hence

(e) $0 = q^b\delta^a_e - q^a\delta^b_e + Q^{ca}{}_c\delta^b_e + Q_e{}^{ba} + Q^c{}_c{}^b\delta^a_e$. $\qquad$ Now use (d) $T^a{}_{ab}=0$ $\qquad$ $Q\equiv T$

$q^b\delta^a_e - q^a\delta^b_e + \mathbf{O}\,T_e{}^{ba} = 0$ $\qquad$ Set $b=e$, use trace $T=0$ to get: $\quad -2q_a=0$

which inserted back into ↑ gives $\qquad T^a{}_{bc}=0$.

Thus in vacuum all the torsion terms are zero. If matter were present, essentially the same manipulations would yield the torsion.

STRUCTURE OF THE ALGEBRAIC CONSTRAINTS, arising from the form of the Hamiltonian

$\mathcal{H}_{ab}$ is determined by $\dfrac{\delta\mathcal{H}_T}{\delta\rho_{ab}}$, ρ_{ab} is the leftover of $\dfrac{\delta\mathcal{H}_T}{\delta\mathcal{H}_{ab}}$ when one subtracts off $\mathcal{H}_{ab}$.

The trace of $\mathbf{O}$ $\underline{T}$ is determined by $\dfrac{\delta\mathcal{H}_T}{\delta q_a}$.

$$2q^a = 3\,\mathbf{O}\,T^{ca}{}_c + T_c{}^{ca} + T^c{}_c{}^a = -3T^c{}_c{}^a + T^c{}_c{}^a + T^c{}_c{}^a = -T^c{}_c{}^a$$

$$q_a = -\tfrac{1}{2}T^c{}_{ca}$$

$\underline{q}$ is just the trace of T plus the trace of a matter term if present. (recall $q_a = \overline{T}^{\perp}{}_{\perp a}$)

The torsion T is then the leftover when one subtracts q away from (e) the Lagrange multiplier constraint.

Interestingly enough $\lambda_{ab} = \overline{T}_{(a\parallel b)}$ is not coupled at all to $\rho_{ab} = \overline{T}_{[a\perp b]}$

Lagrangian to Hamiltonian

$\boxed{\text{neglecting the exact 4-form}}$

$\downarrow$

$$\mathcal{L} = T - V \qquad \begin{cases} T \equiv -N\,\mathcal{S}^{ab}\wedge\{K_a\wedge K_b\} \\ V \equiv -N\,\mathcal{S}^{ab}\wedge\Omega_{ab} \end{cases} \qquad \begin{cases} K_a \equiv K_{ab}\theta^b \\ K_{ab} = -\frac{1}{2}(\mathcal{L}_{\hat{n}}\,g)_{ab} = \frac{1}{2N}\left[(\mathcal{L}_N g)_{ab} - \dot{g}_{ab}\right] \\ \dot{} \equiv \mathcal{L}_t \equiv \partial_t \text{ in comoving ADM frame.} \end{cases}$$

$$\delta_{\dot{g}} T = -N\,\mathcal{S}^{ab}\wedge[\,(\delta_{\dot{g}} K_a)\wedge K_b + K_a\wedge(\delta_{\dot{g}} K_b)\,]$$

$$= -2N\,\mathcal{S}^{ab}\wedge K_a\wedge\underbrace{\delta_{\dot{g}} K_b}_{\theta^c\left(-\frac{1}{2N}\right)\delta\dot{g}_{bc}} = \mathcal{S}^{ab}\wedge K_a\wedge\theta^c\,\delta\dot{g}_{cb}$$

$$\delta_{\dot{g}} K_{ab} = -\frac{1}{2N}\delta\dot{g}_{ab}$$

$$\therefore\ \Pi^{ab} \equiv \mathcal{S}^{c(a}\wedge K_c\wedge\theta^{b)} = \underline{\mathcal{S}_{ce}\wedge\theta^d\wedge\theta^b}\,g^{ae}K^c{}_d = \mathcal{S}\,\delta^{db}_{ce}\,g^{ae}K^c{}_d = \mathcal{S}\left[g^{ab}K^c_c - K^{ab}\right]$$

$$\Pi = g_{ab}\Pi^{ab} = 2K\mathcal{S}\ , \quad K = \tfrac{1}{2}\Pi^*$$

$$*\left(\tfrac{1}{2}\Pi g^{ab} - \Pi^{ab}\right) = K^{ab} \qquad \dot{g}_{ab} = (\mathcal{L}_N g)_{ab} - 2N K_{ab} \qquad \Pi^{ab}\wedge K_{ab} = \tfrac{1}{2}\Pi\wedge{}^*\Pi - \Pi^{ab}\wedge{}^*\Pi_{ab}$$

$$\mathcal{H}_G \equiv \Pi^{ab}\dot{g}_{ab} - \mathcal{L} = \Pi^{ab}\dot{g}_{ab} - T + V$$

$$\Pi^{ab}\dot{g}_{ab} = \Pi^{ab}\left[\underbrace{(\mathcal{L}_N g)_{ab}}_{2\nabla_{(a}N_{b)}} - 2N K_{ab}\right] = 2\Pi_a{}^b\nabla_b N^a + 2N\left(\Pi^{ab}\wedge{}^*\Pi_{ab} - \tfrac{1}{2}\Pi\wedge{}^*\Pi\right)$$

$$-T = N(K_a\wedge K_b\wedge\mathcal{S}^{ab}) = N(K^2 - K^{ab}K_{ab})\mathcal{S} = N\left\{\tfrac{1}{4}({}^*\Pi)^2 - \left(-\tfrac{1}{4}({}^*\Pi)^2 + {}^*\Pi^{ab}\wedge{}^*\Pi_{ab}\right)\right\}$$

$$= N\left\{\tfrac{1}{2}\Pi\wedge{}^*\Pi - \Pi^{ab}\wedge{}^*\Pi_{ab}\right\}$$

$$\mathcal{H}_G = \underbrace{2\Pi_a{}^b\nabla_b N^a} + V + N\left(\Pi^{ab}\wedge{}^*\Pi_{ab} - \tfrac{1}{2}\Pi\wedge{}^*\Pi\right)$$

$$2\Pi_a{}^b e_b\lrcorner\, DN^a = -\underbrace{(e_b\lrcorner\Pi_a{}^b)}_{\equiv\Pi_a}\wedge DN^a \qquad \begin{cases} C_X(\alpha\wedge\beta) = (C_X\alpha)\wedge\beta + (-1)^{\beta}\alpha\wedge C_X\beta \\ 0 = e_b\lrcorner(\underset{3}{\Pi_a{}^b}\wedge\underset{1}{DN^a}) = (e_b\lrcorner\Pi_a{}^b)\wedge DN^a - \Pi_a{}^b\wedge(e_b\lrcorner DN^a) \end{cases}$$

$$d(\Pi_a\wedge N^a) = D(\Pi_a\wedge N^a) = N^a\underbrace{D\Pi_a} + \underline{\Pi_a\wedge DN^a}$$

$$\mathcal{H}_a \equiv -D\Pi_a$$

$$\boxed{\begin{aligned} \mathcal{H}_G &= N^a\mathcal{H} + N^a\mathcal{H}_a + d(N^a\Pi_a) \\ \mathcal{L} &= \Pi^{ab}\dot{g}_{ab} - \mathcal{H}_G \end{aligned}} \qquad \boxed{\begin{cases} \mathcal{H} = \Pi^{ab}\wedge{}^*\Pi_{ab} - \tfrac{1}{2}\Pi\wedge{}^*\Pi - \mathcal{S}^{ab}\wedge\Omega_{ab} \\ \mathcal{H}_a = -D\Pi_a \qquad \Pi_a \equiv e_b\lrcorner\,\Pi_a{}^b \end{cases}}$$

VARYING THE HAMILTONIAN

σ three form $\quad s \equiv {}^*\sigma\,,\ \sigma = s\mathcal{J}$:

$$\delta\sigma = \delta(s\mathcal{J}) = (\delta s)\mathcal{J} + s\,\delta\mathcal{J} = \delta({}^*\sigma)\mathcal{J} + s\,\tfrac{1}{2}\delta g_{ab}\,g^{ab}\mathcal{J}$$

$${}^*\delta\sigma = \delta({}^*\sigma) + \delta g_{ab}\,\tfrac{1}{2}g^{ab}\,{}^*\sigma \quad\longrightarrow\quad \delta({}^*\sigma) = {}^*\delta\sigma - \tfrac{1}{2}\delta g_{ab}\,g^{ab}\,{}^*\sigma. \quad (1)$$

But $\alpha\wedge{}^*\beta = \beta\wedge{}^*\alpha$ for 3forms α,β so:

$$\delta(\sigma^A\wedge{}^*\sigma_A) = \delta\sigma^A\wedge{}^*\sigma_A + \sigma^A\wedge\delta{}^*\sigma_A = \delta\sigma^A\wedge{}^*\sigma_A + \delta\sigma_A\wedge{}^*\sigma^A - \tfrac{1}{2}g^{ab}\delta g_{ab}\,\sigma^A\wedge{}^*\sigma_A. \quad (2)$$

$$\delta\sigma_{ab} = \delta(g_{ac}g_{bd}\sigma^{cd}) = \delta g_{ac}\,\sigma^c{}_b + \delta g_{bd}\,\sigma^d{}_a + g_{ac}g_{bd}\,\delta\sigma^{bd} \quad (3)$$

ω_A zero form : ${}^*\omega^A = \omega^A\mathcal{J}$,

$$\delta(\omega_A\wedge{}^*\omega^A) = \delta(\omega_A\omega^A\mathcal{J}) = \delta\omega_A\wedge{}^*\omega^A + \delta\omega^A\wedge{}^*\omega_A + \tfrac{1}{2}\delta g_{ab}\,g^{ab}\,\omega_A\wedge{}^*\omega^A \quad (5)$$

$$\delta\,\Pi^{ab}\wedge{}^*\Pi_{ab} = \delta\Pi^{ab}\wedge{}^*\Pi_{ab} + \left\{\delta\Pi_{ab} = 2\,\delta g_{c(a}\Pi^c{}_{b)} + g_{ac}g_{bd}\,\delta\Pi^{cd}\right\}\wedge{}^*\Pi^{ab} - \tfrac{1}{2}g^{ab}\delta g_{ab}\,\Pi^{cd}\wedge{}^*\Pi_{cd}$$

$$= \delta\Pi^{ab}\wedge 2\,{}^*\Pi_{ab} + \delta g_{ab}\left[2\Pi_c{}^{(a}\wedge{}^*\Pi^{b)c} - \tfrac{1}{2}g^{ab}\Pi^{cd}\wedge{}^*\Pi_{cd}\right]$$

$$\delta(\Pi\wedge{}^*\Pi) = 2\delta\Pi\wedge{}^*\Pi - \tfrac{1}{2}\delta g_{ab}g^{ab}\,\Pi\wedge{}^*\Pi \qquad \delta(\Pi^{ab}g_{ab}) = \delta\Pi^{ab}g_{ab} + \Pi^{ab}\delta g_{ab}$$

$$= \delta\Pi^{ab}\wedge(2g_{ab}{}^*\Pi) + \delta g_{ab}\left(2\Pi^{ab}\wedge{}^*\Pi - \tfrac{1}{2}g^{ab}\,\Pi\wedge{}^*\Pi\right)$$

$$\delta(\mathcal{J}^a{}_b\wedge\Omega^b{}_a) = +\,G^{ab}\delta g_{ab}\mathcal{J} + \underbrace{\mathcal{J}^a{}_b\wedge D\delta\omega^b{}_a}$$

$$\cancel{N\mathcal{J}^a{}_b\wedge D\delta\omega^b{}_a} \quad d(N\mathcal{J}^a{}_b\wedge\delta\omega^b{}_a) = D(N\mathcal{J}^a{}_b)\wedge\delta\omega^b{}_a - N\mathcal{J}^a{}_b\wedge D\delta\omega^b{}_a$$

$$N\delta\mathcal{H} = \delta\Pi^{ab}\left\{2N({}^*\Pi_{ab} - \tfrac{1}{2}g_{ab}{}^*\Pi)\right\} + \cancel{\delta g_{ab}NG^{ab}\mathcal{J}} +$$

$$+\,\delta g_{ab}\left\{NG^{ab}\mathcal{J} + 2N(\Pi_c{}^{(a}\wedge{}^*\Pi^{b)c} - \tfrac{1}{2}\Pi^{ab}\wedge{}^*\Pi) - \tfrac{1}{2}Ng^{ab}(\Pi^{cd}\wedge{}^*\Pi_{cd} - \tfrac{1}{2}\Pi\wedge{}^*\Pi)\right\}$$

$$+\,\delta\omega^b{}_a\wedge DN\mathcal{J}^a{}_b \qquad - d(N\mathcal{J}^a{}_b\wedge\delta\omega^b{}_a)$$

$$N^a\delta\mathcal{H}_a = -N^a\delta D\Pi_a = -N^a\{D\delta\Pi_a - \delta\omega^b{}_a\wedge\Pi_b\}$$

$$= -d(N^a\delta\Pi_a) + DN^a\wedge\delta\Pi_a + \delta\omega^a{}_b\wedge N^b\Pi_a$$

$$\delta\Pi_a = \delta(2e_b\lrcorner\Pi^b{}_a) = 2e_b\lrcorner\delta\Pi^b{}_a$$

$$DN^a\wedge\delta\Pi_a = 2DN^a\wedge e_b\lrcorner\delta\Pi^b{}_a = 2\nabla_b N^a\,\delta(g_{ac}\Pi^{bc}) = 2g_{ac}\nabla_b N^a\,\delta\Pi^{bc} + 2\nabla_b N^a\Pi^{bc}\delta g_{ac}$$

$$= \delta\Pi^{ab}\{2g_{c(a}\nabla_{b)}N^c\} + \delta g_{ab}\{2\Pi^{c(a}\nabla_c N^{b)}\}$$

$$\delta H = \delta\Pi^{ab}\wedge\{2N({}^*\Pi_{ab} - \tfrac{1}{2}g_{ab}{}^*\Pi) + 2g_{c(a}\nabla_{b)}N^c\}$$

$$+\,\delta g_{ab}\wedge\{NG^{ab}\mathcal{J} - \tfrac{1}{2}Ng^{ab}(\Pi^{cd}\wedge{}^*\Pi_{cd} - \tfrac{1}{2}\Pi\wedge{}^*\Pi) + 2N\,\Pi_c{}^{(a}\wedge{}^*\Pi^{b)c} - \tfrac{1}{2}\Pi^{ab}\wedge{}^*\Pi) + 2\Pi^{c(a}\nabla_c N^{b)}\}$$

$$+\,\delta\omega^b{}_a\wedge\{\underbrace{DN\mathcal{J}^a{}_b} + N^a\Pi_b\} \qquad + \qquad - d\{N\mathcal{J}^a{}_b\wedge\delta\omega^b{}_a + N^a\Pi_a\}$$

$$= \delta\omega^b{}_a\wedge[N\lrcorner(\underbrace{\theta^a\wedge\Pi_b}_{\tfrac{1}{2}\Pi^a{}_b})] = \delta\omega^b{}_a\wedge[N^b\Pi_a - \theta^b\wedge(N\lrcorner\Pi_a)]$$

$$= \delta\omega^b{}_a\wedge(2N\lrcorner\Pi^a{}_b)$$

$$[\delta\omega]_{ba}\wedge\{\underbrace{DN\mathcal{J}^{ab}}_{2N_{,c}\mathcal{J}^{[a}g^{b]c}} + 2N_{,c}\underbrace{\mathcal{J}^{ca}\wedge\theta^b}_{[\mathcal{J}^a g^{bc} - g^{ab}\mathcal{J}^c]}\} = [\delta\omega]_{ba}\wedge[+2N_{,c}\mathcal{J}^{c(a}\wedge\theta^{b)}]$$

REAL SCALAR FIELD

$$({}^{(4)}g^{\alpha\beta}) = \left(\begin{array}{c|c} -1/N^2 & N^b/N^2 \\ \hline N^a/N^2 & g^{ab} - N^aN^b/N^2 \end{array}\right) \qquad \begin{array}{l} (\hat{n}^\alpha) = (N^{-1}, -N^{-1}N^a) \\ (\hat{n}_\alpha) = (-N, 0) \end{array}$$

$$\mathcal{L}_\phi = -\frac{1}{2}\underbrace{\sqrt{-g}}_{Ng^{1/2}}\ \underbrace{{}^4g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta}}_{-\frac{1}{N^2}\dot{\phi}^2 + \frac{2N^a}{N^2}\phi_{,a}\dot{\phi} + (g^{ab} - N^aN^b/N^2)\phi_{,a}\phi_{,b}}$$

$$= \frac{g^{1/2}}{2N}\dot{\phi}^2 - \frac{g^{1/2}\dot{\phi}}{N}N^a\phi_{,a} - \frac{N}{2}g^{1/2}g^{ab}\phi_{,a}\phi_{,b} + \frac{g^{1/2}}{2N}(N^a\phi_{,a})^2$$

$$\Pi = \frac{\partial\mathcal{L}_\phi}{\partial\dot{\phi}} = N^{-1}g^{1/2}\dot{\phi} - N^{-1}g^{1/2}N^a\phi_{,a} = N^{-1}g^{1/2}(\dot{\phi} - N^a\phi_{,a}) = g^{1/2}\phi_{,\perp}$$

$$\dot{\phi} = Ng^{-1/2}\Pi + N^a\phi_{,a}$$

$$H_\phi = \Pi\dot{\phi} - \mathcal{L}_\phi$$

$$= \underbrace{\Pi(Ng^{-1/2}\Pi + N^a\phi_{,a})}_{Ng^{-1/2}\Pi^2 + \Pi(N^a\phi_{,a})} - \underbrace{\frac{g^{1/2}}{2N}(Ng^{-1/2}\Pi + N^a\phi_{,a})^2}_{-\frac{N}{2}g^{-1/2}\Pi^2 - \Pi N^a\phi_{,a} - \frac{g^{1/2}}{2N}(N^a\phi_{,a})^2} + \underbrace{\frac{g^{1/2}}{N}(Ng^{-1/2}\Pi + N^a\phi_{,a})(N^a\phi_{,a})}_{\Pi(N^a\phi_{,a}) + \frac{g^{1/2}}{N}(N^a\phi_{,a})^2}$$

$$+ \frac{N}{2}g^{1/2}g^{ab}\phi_{,a}\phi_{,b} - \frac{g^{1/2}}{2N}(N^a\phi_{,a})^2$$

$$= N\left\{\frac{1}{2}g^{-1/2}\Pi^2 + \frac{1}{2}g^{1/2}g^{ab}\phi_{,a}\phi_{,b}\right\} + N^a\underbrace{\{\Pi\,\partial_a\phi\}}_{\Pi\,\mathcal{L}_N\phi}$$

$$= N\mathcal{H}^\phi + N^a\mathcal{H}^\phi_a$$

HIGGS FIELD

MAXWELL FIELD

$(A_0, A_a) = (-\phi, A_a)$ $\quad (F_{a0} = -\dot{A}_a - \partial_a \phi \neq E_a$ in flat spacetime$)$

$$\begin{cases} e_0 = N e_\perp + N^a e_a, \quad N\omega^0 = \omega^\perp \\ -\phi = A_0 = N A_\perp + N^a A_a \end{cases}$$

$$-n^a = \mathcal{E}^a = (-{}^4g)^{1/2} \underbrace{F^{0a}}_{E^a} = N g^{1/2} F^{0a} = g^{1/2} F^{\perp a}$$

$$\mathcal{B}^a = (-{}^4g)^{1/2} \underbrace{F^{*0a}}_{B^a} = \epsilon^{abc} F_{bc} = g^{1/2} n^{abc} F_{bc} \qquad \mathcal{B}_a = g^{1/2} n_{abc} F^{bc} = g\, \epsilon_{abc} F^{bc}$$

$$\left.\begin{aligned} F^{bc} &= n^{bca} B_a = g^{-1} \epsilon^{bca} \mathcal{B}_a \\ F_{bc} &= \epsilon_{bca} \mathcal{B}^a \end{aligned}\right\} \quad \tfrac{1}{2} F^{bc} F_{bc} = g^{-1} \mathcal{B}_a \mathcal{B}^a$$

$$\sqrt{-{}^4g}\,(F^{0\beta})_{;\beta} = (\sqrt{-g}\, F^{0\beta})_{,\beta} = \mathcal{E}^b{}_{,b} = g^{1/2}(F^{\perp\beta}{}_{;\beta})$$

LAGRANGIAN $\quad \mathcal{L}^M = -\frac{k}{4}\sqrt{-{}^4g}\left(F^{\alpha\beta}F_{\alpha\beta} = 2\underbrace{F^{\perp a}}_{g^{-1/2}\mathcal{E}^a}\underbrace{F_{\perp a}}_{(-g^{-1/2}\mathcal{E}_a)} + \underbrace{F^{ab}F_{ab}}_{2g^{-1}\mathcal{B}^a\mathcal{B}_a}\right) = +\frac{k}{2} N g^{-1/2}\, g^a{}_{ab}(\mathcal{E}^a\mathcal{E}^b - \mathcal{B}^a\mathcal{B}^b)$

$$\begin{cases} F_{0a} = \partial_0 A_a - \partial_a A_0 \quad (+ C\, A_0 A_a) \\ \dot{A}_a = -\partial_a\phi + \underbrace{F_{0a}}_{\substack{N\underbrace{F_{\perp a}}_{-Ng^{-1/2}\mathcal{E}_a} + N^b \underbrace{F_{ba}}_{\epsilon_{bac}\mathcal{B}^c}}} = -\partial_a\phi - N g^{-1/2}\mathcal{E}_a + N^a \epsilon_{abc}\mathcal{B}^b \quad (- C\, A_0 A_a) \end{cases}$$

$$\Pi^a = \frac{\partial \mathcal{L}^M}{\partial \dot{A}_a} = -k\mathcal{E}^a$$

$$\mathcal{R} = \underbrace{\Pi^a \dot{A}_a}_{-k\mathcal{E}^a\dot{A}_a} - \mathcal{L} = k\mathcal{E}^a(\partial_a\phi + N\mathcal{E}_a g^{-1/2} - N^b\epsilon_{bac}\mathcal{B}^c \quad (- C\,(A_0 A_a))) - \frac{Nk}{2} g^{-1/2}(\mathcal{E}^2 - \mathcal{B}^2)$$

$$= N\underbrace{\left(\frac{k}{2} g^{-1/2}(\mathcal{E}^a\mathcal{E}_a + \mathcal{B}^a\mathcal{B}_a)\right)}_{\mathcal{H}^M} + N^a\underbrace{(-k\epsilon_{abc}\mathcal{E}^b\mathcal{B}^c)}_{\mathcal{H}^M{}_a} + k\partial_a(\mathcal{E}^a\phi) - k\phi\,\mathcal{E}^a{}_{,a} - (k\phi^a\, C A_0 A_a$$

$$\mathcal{R}_{ADM} = N\mathcal{H}^M + N^a\mathcal{H}^M{}_a + \underbrace{k\phi\, n^a{}_{,a}}_{\substack{k\underbrace{A_0}_{\substack{NA_\perp + N^a A_a \\ \equiv -\lambda}}(-n^a{}_{,a})}} \qquad A_0(-\nabla_a \Pi^a)$$

$$\underbrace{N^a(\underbrace{\mathcal{H}^M{}_a - A_a n^b{}_{,b}}_{\mathcal{H}^{M(CAN)}_a \equiv \mathcal{J}_a}) + \lambda k n^a{}_{,a}}$$

$$n^a \mathcal{L}_N A_a = n^a[\underbrace{A_{a|b}}_{\substack{A_{a|b} - A_{b|a} + A_{b|a} \\ \epsilon F_{ba} = \epsilon_{bac}\mathcal{B}^c \\ N^b\epsilon_{bac} n^a \mathcal{B}^c}} N^b + N^b{}_{|b} A_b] = N^b\epsilon_{bac} n^a \mathcal{B}^c + \underbrace{n^a A_{b|a} N^b}_{\substack{n^a (A_b N^b)_{|a} \\ (n^a A_b N^b)_{|a} - A_b N^b n^a{}_{|a}}} + N^b{}_{|a} A_b n^a$$

$$= N^a(\underbrace{\epsilon_{abc} n^b \mathcal{B}^c - A_a n^b{}_{|b}}_{\mathcal{H}^{M(CAN)}_a \equiv \mathcal{J}_a}) + (n^a A_b N^b)_{|a}$$

YANG-MILLS-HIGGS

$$\mathcal{L} = k\left[-\tfrac{1}{4} F^i_{\alpha\beta}F_i^{\alpha\beta} - \tfrac{1}{2}\overset{G}{\nabla}_\alpha\phi^A \overset{G}{\nabla}^\alpha\phi_A - P(\phi)\right]\sqrt{-g}$$

$$F^i_{\alpha\beta} = \partial_\alpha A^i_\beta - \partial_\beta A^i_\alpha + C^i{}_{jk} A^j_\alpha A^k_\beta \qquad -iT_i = \rho(e_i) \qquad [e_a, e_b] = C^c_{ab} e_c$$

$$\overset{G}{\nabla}_\alpha \Pi^i = \partial_\alpha \Pi^i + A^j_\alpha C^i{}_{jk}\Pi^k \qquad [ie_a, ie_b] = i C^c_{ab}(ie_c)$$

$$\overset{G}{\nabla}_\alpha \phi^A = \partial_\alpha\phi^A - iA^j_\alpha T_j{}^A{}_B\phi^B$$

$$\Pi_i^a = \frac{\partial\mathcal{L}}{\partial \dot{A}^i_a} = -k\mathcal{E}^a_i = k\mathcal{H}^a_i \qquad \Pi_A = \frac{\partial\mathcal{L}}{\partial\dot\phi^A} = N^{-1}g^{1/2}(\nabla_0\phi_A - N^a\nabla_a\phi_A)$$

$$\nabla_0\phi_A = Ng^{-1/2}\Pi_A + N^a\nabla_a\phi_A$$

$$\mathcal{R}_\phi = \Pi_A\dot\phi^A - \mathcal{L}_\phi = \Pi_A(\nabla_0\phi^A + iA^A_0{}_B\phi^B) - \mathcal{L}_\phi$$

$$= \text{"}\mathcal{R}_\phi\text{"} - A^i_0\underbrace{(\Pi_A(-iT_i{}^A{}_B)\phi^B)}_{Q_i^{Higgs}}$$

$$\mathcal{R}^{YMH} = N\mathcal{H}^{YMH} + N^a\mathcal{H}^{YMH}_a + A^i_0 C_{(i)} \qquad A_0 = NA_\perp + N^aA_a$$

$$\mathcal{H}^{YMH} = \tfrac{k}{2}g^{-1/2}(\mathcal{E}^a_i\mathcal{E}^i_a + \mathcal{B}^a_i\mathcal{B}^i_a) + \tfrac{k}{2}g^{-1/2}(\Pi^A_i\Pi^i_A) + \tfrac{k}{2}g^{1/2}\nabla_i\phi^A\nabla^i\phi_A$$

$$\mathcal{H}^{YMH}_a = -k\epsilon_{abc}\mathcal{E}^b_i\mathcal{B}^{ic} + \Pi_A\nabla_a\phi^A \qquad \mathcal{J}_a = \mathcal{H}^{YMH}_a - A^i_a C_{(i)}$$

$$C_{(i)} = -\nabla_a\mathcal{H}^a_i - Q_i^{Higgs}$$

$$\frac{\delta\mathcal{L}}{\delta A^i_\mu} = 0$$

$$\delta\mathcal{L}^{YM} = -\tfrac{k\sqrt{-g}}{2}F^{\alpha\beta}_i\delta F^i_{\alpha\beta} - k\sqrt{-g}\,F^{\alpha\beta}_i\,\delta A^i_{\beta;\alpha} \sim k\sqrt{-g}\,F^{\alpha\beta}_i{}_{;\alpha}\,\delta A^i_\beta = -\underbrace{k\sqrt{-g}\,F^{\alpha\beta}_i{}_{;\beta}}_{\mathcal{J}^\alpha_i}\delta A^i_\alpha$$

$$\delta\mathcal{L}^H = -\nabla^\alpha\phi_A\,\underbrace{\delta(\nabla_\alpha\phi^A)}_{-iT_i{}^A{}_B\phi^B\delta A^i_\alpha}\,k\sqrt{-g} \qquad \frac{\delta\mathcal{L}^H}{\delta A^i_\alpha} = k\sqrt{-g}\,\underbrace{i\nabla^\alpha\phi_A T_i{}^A{}_B\phi^B}_{\mathcal{J}^{H\alpha}_i}$$

YM equations $\left\{ -F^{\alpha\beta}_i{}_{;\beta} + J^\alpha_i = 0 \right.$

$$-(F^{\alpha\beta}_i{}_{;\beta} - A^j_\beta C^k_{ij}F^{\alpha\beta}_k) + J^\alpha_i$$

without G-connection

$$I^\alpha_i$$

$$Q^H_i = \mathcal{J}^{H0}_i = i\underbrace{\nabla^0\phi_A}_{g^{00}\nabla_0\phi_A + g^{0a}\nabla_0\phi_A - \frac{1}{N^2}(\nabla_0\phi_A - N^a\nabla_a\phi_A)} T_i{}^A{}_B\phi^B = -\frac{i\sqrt{g}}{N}\Pi_A T_i{}^A{}_B\phi^B$$

$$N g^{-1/2}\Pi_A$$

$$\sqrt{-g}\,Q_i^H = Q^H_i$$

$$F^{\alpha\beta}_i{}_{;\beta} = I^\alpha_i$$

$$0 = F^{\alpha\beta}_i{}_{;\beta\alpha} = I^\alpha_i{}_{;\alpha}$$

$$I^\alpha_i = J^{H\alpha}_i - C^k_{ij}A^j_\beta F^{\alpha\beta}_k$$

$$I^0_i = J^{H0}_i - C^k_{ij}A^j_\beta F^{0\beta}_k$$

$$- C^k_{ij}A^j_b E^b_k$$

$$I^0_i = \langle\Pi^H, \rho(e_i)\phi^H\rangle + \langle\Pi^Y, k_i A^Y\rangle$$

inverse trans's

$g \to \mathcal{L}_t g$

$g' = \mathcal{L}_N g$

$A' =$

$A \mapsto S^{-1}(AS + dS)$ $\quad S = e^{\lambda^i k_i}$

$A' = -\lambda^i k_i A + d\lambda^i k_i$

$\phi \to e^{-\lambda^i \rho(e_i)} \phi$

$\phi' = -\lambda^i \rho(e_i)$

$N \lrcorner \gamma \sim \langle \Pi, \mathcal{L}_N g \rangle \sim N \lrcorner \delta\Pi$

$\lambda^i C_i \sim \langle \Pi_{ij} - \lambda^i k_i A + \overset{D\lambda^i}{d\lambda^i k_i} \rangle \qquad -\Pi^a_{i;a}\lambda^i - \lambda^i \Pi^a \mathcal{L}^k_{ij} A^j_a = -\Pi^a_{i;a}\lambda^i$

$\lambda^i C^H_i \sim \langle \Pi, -\rho(k_i)\phi \rangle = +\lambda^i \langle \Pi, \rho(e_i)\phi \rangle = \lambda^i Q_{\phi i}$

Marsden - Fischer idea for gauge generators in 3+1 split.