① a) $X'' + 20X' + 64X = 32.41\cos(20t)$

$x_h \sim e^{rt} \rightsquigarrow r^2 + 20r + 64 = 0$

$(r+4)(r+16) = 0$

$r = -4, -16$

$e^{rt} = e^{-4t}, e^{-16t}$

$X_h = c_1 e^{-4t} + c_2 e^{-16t}$

$64[X_p = c_3\cos 20t + c_4\sin 20t]$

$20[X_p' = -20c_3\sin 20t + 20c_4\cos 20t]$

$1[X_p'' = -400c_3\cos 20t - 400c_4\sin 20t]$

$X_p'' + 20X_p' + 64X_p = [(64-400)c_3 + 400c_4]\cos 20t$
$\qquad + [-400c_3 + (64-400)c_4]\sin 20t$
$\qquad = 32.41\cos 20t + 0\sin 20t$

$\begin{bmatrix} -336 & 400 \\ -400 & -336 \end{bmatrix}\begin{bmatrix} c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 32.41 \\ 6 \end{bmatrix}$

$\begin{bmatrix} -336 & 400 & 32.41 \\ -400 & -336 & 0 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & -21/13 \\ 0 & 1 & 25/13 \end{bmatrix}$ $c_3 = -21/13$
$c_4 = 25/13$

$X_p = -\frac{21}{13}\cos 20t + \frac{25}{13}\sin 20t$

$\boxed{X = X_h + X_p = c_1 e^{-4t} + c_2 e^{-16t} - \frac{21}{13}\cos 20t + \frac{25}{13}\sin 20t}$

b) $X' = -4c_1 e^{-4t} - 16c_2 e^{-16t} + 20(\frac{21}{13})\sin 20t + 20(\frac{25}{13})\cos 20t$

$X(0) = c_1 + c_2 - 21/13 = 0$
$X'(0) = -4c_1 - 16c_2 + 20\cdot 25/13 = 64$

$\begin{bmatrix} 1 & 1 \\ -4 & -16 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 21/13 \\ 64 - 20\cdot 25/13 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & +21/13 \\ -4 & -16 & 64-500/13 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 167/39 \\ 0 & 1 & -8/3 \end{bmatrix}$ $c_1 = 167/39$
$c_2 = -8/3$

$\boxed{\begin{aligned} X &= \frac{167}{39}e^{-4t} - \frac{8}{3}e^{-16t} + -\frac{21}{13}\cos 20t + \frac{25}{13}\sin 20t \\ &\approx 4.282e^{-4t} - 2.667e^{-16t} - 1.615\cos 20t + 1.923\sin 20t \end{aligned}}$
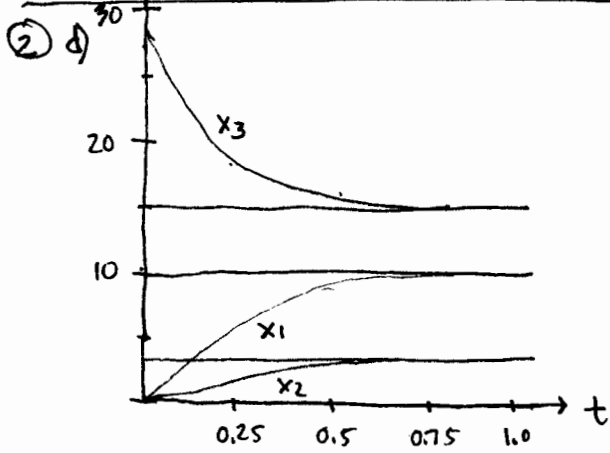
c) $\tau_1 = \frac{1}{4} = .25$, $\tau_2 = \frac{1}{16} = .0625$ } self-explanatory

$T = \frac{2\pi}{\omega} = \frac{2\pi}{20} = \frac{\pi}{10} \approx 0.3146$

(BONUS) The second exponential decays away quickly compared to the first. The first exponential decays over several periods of the steady state solution.

d) $X_{ss} = -\frac{21}{13}\cos 20t + \frac{25}{13}\sin 20t$
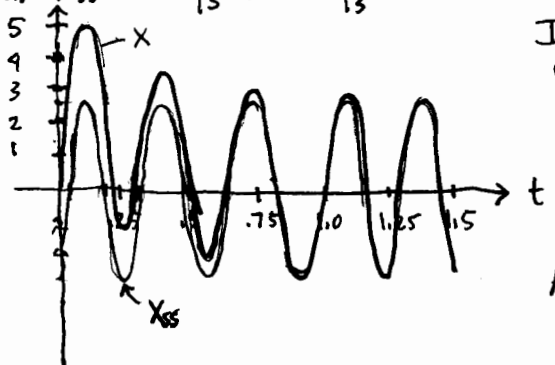
In about 4 or 5 characteristic times τ_1 (about $t = 1 \sim 1.25$) the transient dies away & you only see the steady state:

Amplitude = $\sqrt{21^2 + 25^2}/13 \approx 2.51$

② a) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}' = \begin{bmatrix} -3 & 0 & 2 \\ 3 & -10 & 0 \\ 0 & 10 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $\begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 28 \end{bmatrix}$

b) $|A - \lambda I| = \begin{vmatrix} -3-\lambda & 0 & 2 \\ 3 & -10-\lambda & 0 \\ 0 & 10 & -2-\lambda \end{vmatrix} = -\lambda^3 - 15\lambda^2 - 56\lambda$
$= -\lambda(\lambda^2 + 15\lambda + 56)$
$= -\lambda(\lambda+7)(\lambda+8) = 0$

$\lambda = 0, -7, -8$

$A - 0I = \begin{bmatrix} -3 & 0 & 2 \\ 3 & -10 & 0 \\ 0 & 10 & -2 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & -2/3 \\ 0 & 1 & -1/5 \\ 0 & 0 & 0 \end{bmatrix}$ $x_1 - \frac{2}{3}x_3 = 0$
$x_2 - \frac{1}{5}x_3 = 0$
$x_3 = t$

$x_1 = \frac{2}{3}t$, $x_2 = \frac{1}{5}t$, $x_3 = t$
$\langle x_1, x_2, x_3 \rangle = t\langle \underbrace{\frac{2}{3}, \frac{1}{5}, 1}_{\vec{b_1}} \rangle$

$A + 7I = \begin{bmatrix} 4 & 0 & 2 \\ 3 & -3 & 0 \\ 0 & 10 & 5 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$ $x_1 + \frac{1}{2}x_3 = 0$
$x_2 + \frac{1}{2}x_3 = 0$
$x_3 = t$

$x_1 = -\frac{1}{2}t$, $x_2 = -\frac{1}{2}t$, $x_3 = t$
$\langle x_1, x_2, x_3 \rangle = t\langle \underbrace{-\frac{1}{2}, -\frac{1}{2}, 1}_{\vec{b_2}} \rangle$

$A + 8I = \begin{bmatrix} 5 & 0 & 2 \\ 3 & -2 & 0 \\ 0 & 10 & 6 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 2/5 \\ 0 & 1 & 3/5 \\ 0 & 0 & 0 \end{bmatrix}$ $x_1 + \frac{2}{5}x_3 = 0$
$x_2 + \frac{3}{5}x_3 = 0$
$x_3 = t$

$x_1 = -\frac{2}{5}t$, $x_2 = -\frac{3}{5}t$, $x_3 = t$
$\langle x_1, x_2, x_3 \rangle = t\langle \underbrace{-\frac{2}{5}, -\frac{3}{5}, 1}_{\vec{b_3}} \rangle$

$\underline{B = \begin{bmatrix} \frac{2}{3} & -\frac{1}{2} & -\frac{2}{5} \\ \frac{1}{5} & -\frac{1}{2} & -\frac{3}{5} \\ 1 & 1 & 1 \end{bmatrix}}$ $\rightarrow \vec{x}' = A\vec{x}$
$\underset{\vec{x} = B\vec{y}, \vec{y} = B^{-1}\vec{x}}{\curvearrowleft\curvearrowright}$
$B^{-1}[(B\vec{y})' = A(B\vec{y})]$

$\vec{y}' = (B^{-1}AB)\vec{y} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & -8 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -7y_2 \\ -8y_3 \end{bmatrix}$

$y_1' = 0$ $y_1 = c_1$
$y_2' = -7y_2$ $y_2 = c_2 e^{-7t}$ $\Big\} \longrightarrow \vec{x} = B\vec{y}$:
$y_3' = -8y_3$ $y_3 = c_3 e^{-8t}$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{2} & -\frac{2}{5} \\ \frac{1}{5} & -\frac{1}{2} & -\frac{3}{5} \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 e^{-7t} \\ c_3 e^{-8t} \end{bmatrix} = \begin{bmatrix} \frac{2}{3}c_1 - \frac{1}{2}c_2 e^{-7t} - \frac{2}{5}e^{-8t} \\ \frac{1}{5}c_1 - \frac{1}{2}c_2 e^{-7t} - \frac{3}{5}e^{-8t} \\ c_1 + c_2 e^{-7t} + c_3 e^{-8t} \end{bmatrix}$

general soln.

$t = 0$: $B\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 28 \end{bmatrix} \rightarrow \langle B | \begin{bmatrix} 0 \\ 0 \\ 28 \end{bmatrix} \rangle \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 0 & 15 \\ 0 & 1 & 0 & 48 \\ 0 & 0 & 1 & -35 \end{bmatrix}$

$c_1 = 15, c_2 = 48, c_3 = -35$

$\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 - 24e^{-7t} + 14e^{-8t} \\ 3 - 24e^{-7t} + 21e^{-8t} \\ 15 + 48e^{-7t} - 35e^{-8t} \end{bmatrix}}$ IVP solution

② d)

e) $\lim\limits_{t\to\infty} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 3 \\ 15 \end{bmatrix}$ exactly the 3 horizontal asymptotes!

c) $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & -\frac{1}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -2c_1\sin 2t + 2c_2\cos 2t \\ -4c_3\sin 4t + 4c_4\cos 4t \end{bmatrix}$

$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & -\frac{1}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\begin{bmatrix} c_1 \\ c_3 \end{bmatrix} = \frac{4}{3}\begin{bmatrix} 1 & \frac{1}{2} \\ -1 & \frac{1}{4} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$\begin{bmatrix} x_1'(0) \\ x_2'(0) \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & -\frac{1}{2} \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2c_2 \\ 4c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ → $\begin{aligned} c_2 &= 0 \\ c_4 &= 0 \end{aligned}$

$\boxed{\begin{aligned} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= 2\cos 2t \begin{bmatrix} \frac{1}{4} \\ 1 \end{bmatrix} - \cos 4t \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2}\cos 2t + \frac{1}{2}\cos 4t \\ 2\cos 2t - \cos 4t \end{bmatrix} \quad \begin{matrix} \text{IVP} \\ \text{soln} \end{matrix} \end{aligned}}$

d) tandem mode: $\omega = 2$

$\frac{x_1}{x_2} = \frac{1}{4}$ ratio of amplitudes

accordian mode: $\omega = 4$

$\frac{x_1}{x_2} = -\frac{1}{2}$ 180° out of phase   1:2 amplitude ratio.

The ratios of the amplitudes for a pure eigenvector solution at a single frequency are the same as the ratios of the abs values of the eigenvector components.

③ a) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}'' = \underbrace{\begin{bmatrix} -12 & 2 \\ 16 & -8 \end{bmatrix}}_{A}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} x_1'(0) \\ x_2'(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

b) $|A - \lambda I| = \begin{vmatrix} -12-\lambda & 2 \\ 16 & -8-\lambda \end{vmatrix} = (\lambda+8)(\lambda+12) - 32 = \lambda^2 + 20\lambda + \frac{96-32}{64}$
$= (\lambda+4)(\lambda+16) = 0$

$\lambda = -4, -16$

$A + 4I = \begin{bmatrix} -8 & 2 \\ 16 & -4 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} \overset{L}{1} & \overset{F}{-\frac{1}{4}} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $x_1 - \frac{1}{4}x_2 = 0$ $x_2 = t \uparrow$

$x_1 = \frac{1}{4}t$, $x_2 = t$ $\langle x_1, x_2 \rangle = t\underbrace{\langle \frac{1}{4}, 1 \rangle}_{\vec{b_1}}$

$A + 16I = \begin{bmatrix} 4 & 2 \\ 16 & 8 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} \overset{L}{1} & \overset{F}{\frac{1}{2}} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $x_1 + \frac{1}{2}x_2 = 0$ $x_2 = t \uparrow$

$x_1 = -\frac{1}{2}t$, $x_2 = t$ $\langle x_1, x_2 \rangle = t\underbrace{\langle -\frac{1}{2}, 1 \rangle}_{\vec{b_2}}$

$B = \langle \vec{b_1} | \vec{b_2} \rangle = \begin{bmatrix} \frac{1}{4} & -\frac{1}{2} \\ 1 & 1 \end{bmatrix}$, $B^{-1} = \frac{4}{3}\begin{bmatrix} 1 & \frac{1}{2} \\ -1 & \frac{1}{4} \end{bmatrix}$

$\vec{X}'' = B\vec{X} \longrightarrow \left(\begin{matrix} \vec{x} = B\vec{y} \\ \vec{y} = B^{-1}\vec{x} \end{matrix}\right) \rightarrow B^{-1}(B\vec{y})'' = B^{-1}AB\vec{y}$
$\vec{y}'' = (B^{-1}AB)\vec{y}$

e)

kind of cute, eh?

$\begin{bmatrix} y_1'' \\ y_2'' \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ 0 & -16 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -4y_1 \\ -16y_2 \end{bmatrix}$ $\begin{aligned} y_1'' + 4y_1 = 0 \quad & y_1 = c_1\cos 2t + c_2\sin 2t \\ y_2'' + 16y_2 = 0 \quad & y_2 = c_3\cos 4t + c_4\sin 4t \end{aligned}$

$\boxed{\begin{aligned} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} \frac{1}{4} & -\frac{1}{2} \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_1\cos 2t + c_2\sin 2t \\ c_3\cos 4t + c_4\sin 4t \end{bmatrix} = \underbrace{(c_1\cos 2t + c_2\sin 2t)\begin{bmatrix} \frac{1}{4} \\ 1 \end{bmatrix}}_{\text{tandem mode}} + \underbrace{(c_3\cos 4t + c_4\sin 4t)\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}}_{\text{accordian mode}} \quad \begin{matrix} \text{gen} \\ \text{soln} \end{matrix} \\[2mm] &= \begin{bmatrix} \frac{1}{4}(c_1\cos 2t + c_2\sin 2t) - \frac{1}{2}(c_3\cos 4t + c_4\sin 4t) \\ (c_1\cos 2t + c_2\sin 2t) + (c_3\cos 4t + c_4\sin 4t) \end{bmatrix} \end{aligned}}$