

① a)  $x'' + 20x' + 64x = 32.41 \cos(20t)$

$x_h \sim e^{rt} \rightarrow r^2 + 20r + 64 = 0$

$(r+4)(r+16) = 0$

$r = -4, -16$

$e^{rt} = e^{-4t}, e^{-16t}$

$x_h = c_1 e^{-4t} + c_2 e^{-16t}$

$64 [x_p = c_3 \cos 20t + c_4 \sin 20t]$

$20 [x_p' = -20c_3 \sin 20t + 20c_4 \cos 20t]$

$1 [x_p'' = -400c_3 \cos 20t - 400c_4 \sin 20t]$

$x_p'' + 20x_p' + 64x_p = [(64-400)c_3 + 400c_4] \cos 20t + [-400c_3 + (64-400)c_4] \sin 20t = 32.41 \cos 20t + 0 \sin 20t$

$\begin{bmatrix} -336 & 400 \\ -400 & -336 \end{bmatrix} \begin{bmatrix} c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 32.41 \\ 0 \end{bmatrix}$

$\begin{bmatrix} -336 & 400 & 32.41 \\ -400 & -336 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -21/13 \\ 0 & 1 & 25/13 \end{bmatrix} \quad c_3 = -21/13, c_4 = 25/13$

$x_p = -\frac{21}{13} \cos 20t + \frac{25}{13} \sin 20t$

$x = x_h + x_p = c_1 e^{-4t} + c_2 e^{-16t} - \frac{21}{13} \cos 20t + \frac{25}{13} \sin 20t$

b)  $x' = -4c_1 e^{-4t} - 16c_2 e^{-16t} + 20(\frac{21}{13}) \sin 20t + 20(\frac{25}{13}) \cos 20t$

$x(0) = c_1 + c_2 - 21/13 = 0$

$x'(0) = -4c_1 - 16c_2 + 20 \cdot 25/13 = 64$

$\begin{bmatrix} 1 & 1 \\ -4 & -16 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 21/13 \\ 64 - 20 \cdot 25/13 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 21/13 \\ -4 & -16 & 64 - 500/13 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 167/39 \\ 0 & 1 & -8/13 \end{bmatrix} \quad c_1 = 167/39, c_2 = -8/13$

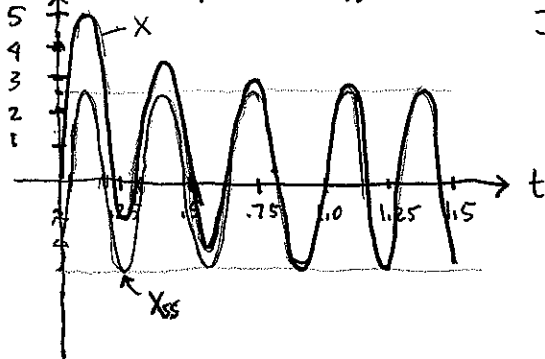
$x = \frac{167}{39} e^{-4t} - \frac{8}{13} e^{-16t} - \frac{21}{13} \cos 20t + \frac{25}{13} \sin 20t$

$\approx 4.282 e^{-4t} - 2.667 e^{-16t} - 1.615 \cos 20t + 1.923 \sin 20t$

c)  $\tau_1 = \frac{1}{4} = .25, \tau_2 = \frac{1}{16} = .0625$  } self-explanatory  
 $\tau = \frac{2\pi}{\omega} = \frac{2\pi}{20} = \frac{\pi}{10} \approx 0.3146$

The second exponential decays away quickly compared to the first. The first exponential decays over several periods of the steady state solution.

d)  $x_{ss} = -\frac{21}{13} \cos 20t + \frac{25}{13} \sin 20t$



In about 4 or 5 characteristic times  $\tau_1$  (about  $t = 1 \sim 1.25$ ) the transient dies away & you only see the steady state:

Amplitude =  $\sqrt{21^2 + 25^2} / 13 \approx 2.51$

② a)  $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}' = \begin{bmatrix} -3 & 0 & 2 \\ 3 & -10 & 0 \\ 0 & -10 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 28 \end{bmatrix}$

b)  $|A - \lambda I| = \begin{vmatrix} -3-\lambda & 0 & 2 \\ 3 & -10-\lambda & 0 \\ 0 & -10 & -2-\lambda \end{vmatrix} = -\lambda^3 - 15\lambda^2 - 56\lambda = -\lambda(\lambda^2 + 15\lambda + 56) = -\lambda(\lambda+7)(\lambda+8) = 0$

$\lambda = 0, -7, -8$

$A - 0I = \begin{bmatrix} -3 & 0 & 2 \\ 3 & -10 & 0 \\ 0 & -10 & -2 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -2/3 \\ 0 & 1 & -1/5 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} x_1 - \frac{2}{3}x_3 = 0 \\ x_2 - \frac{1}{5}x_3 = 0 \\ x_3 = t \end{matrix}$

$x_1 = \frac{2}{3}t, x_2 = \frac{1}{5}t, x_3 = t$

$\langle x_1, x_2, x_3 \rangle = t \langle \frac{2}{3}, \frac{1}{5}, 1 \rangle$

$A + 7I = \begin{bmatrix} 4 & 0 & 2 \\ 3 & -3 & 0 \\ 0 & 10 & 5 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} x_1 + \frac{1}{2}x_3 = 0 \\ x_2 + \frac{1}{2}x_3 = 0 \\ -x_3 = t \end{matrix}$

$x_1 = -\frac{1}{2}t, x_2 = -\frac{1}{2}t, x_3 = t$

$\langle x_1, x_2, x_3 \rangle = t \langle -\frac{1}{2}, -\frac{1}{2}, 1 \rangle$

$A + 8I = \begin{bmatrix} 5 & 0 & 2 \\ 3 & -2 & 0 \\ 0 & 10 & 6 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 2/5 \\ 0 & 1 & 3/5 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} x_1 + \frac{2}{5}x_3 = 0 \\ x_2 + \frac{3}{5}x_3 = 0 \\ -x_3 = t \end{matrix}$

$x_1 = -\frac{2}{5}t, x_2 = -\frac{3}{5}t, x_3 = t$

$\langle x_1, x_2, x_3 \rangle = t \langle -\frac{2}{5}, -\frac{3}{5}, 1 \rangle$

$B = \begin{bmatrix} \frac{2}{3} & -\frac{1}{2} & -\frac{2}{5} \\ \frac{1}{5} & -\frac{1}{2} & -\frac{3}{5} \\ 1 & 1 & 1 \end{bmatrix} \rightarrow \begin{matrix} \vec{x}' = A\vec{x} \\ (\vec{x} = B\vec{y}, \vec{y} = B^{-1}\vec{x}) \\ B^{-1}[(B\vec{y})]' = A(B\vec{y}) \end{matrix}$

$\vec{y}'' = (B^{-1}AB)\vec{y} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & -8 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -7y_2 \\ -8y_3 \end{bmatrix}$

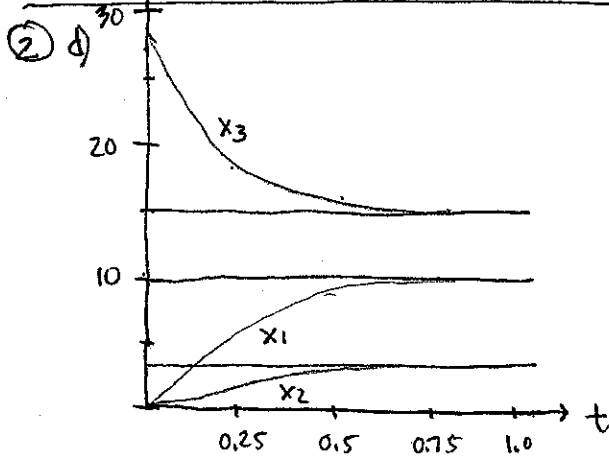
$\begin{matrix} y_1' = 0 & y_1 = c_1 \\ y_2' = -7y_2 & y_2 = c_2 e^{-7t} \\ y_3' = -8y_3 & y_3 = c_3 e^{-8t} \end{matrix} \rightarrow \vec{x} = B\vec{y}$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{2} & -\frac{2}{5} \\ \frac{1}{5} & -\frac{1}{2} & -\frac{3}{5} \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 e^{-7t} \\ c_2 e^{-7t} \\ c_3 e^{-8t} \end{bmatrix} = \begin{bmatrix} \frac{2}{3}c_1 - \frac{1}{2}c_2 e^{-7t} - \frac{2}{5}c_3 e^{-8t} \\ \frac{1}{5}c_1 - \frac{1}{2}c_2 e^{-7t} - \frac{3}{5}c_3 e^{-8t} \\ c_1 + c_2 e^{-7t} + c_3 e^{-8t} \end{bmatrix}$

$t=0: B \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 28 \end{bmatrix} \rightarrow \langle B | \begin{bmatrix} 0 \\ 0 \\ 28 \end{bmatrix} \rangle \xrightarrow{\text{rref}} \begin{bmatrix} 100 & 15 \\ 0 & 10 & 48 \\ 0 & 0 & -35 \end{bmatrix}$

$c_1 = 15, c_2 = 48, c_3 = -35$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 - 24e^{-7t} + 14e^{-8t} \\ 3 - 24e^{-7t} + 21e^{-8t} \\ 15 + 48e^{-7t} - 35e^{-8t} \end{bmatrix}$  IVP solution



$$c) \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & -\frac{1}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -2c_3 \sin 2t + 2c_2 \cos 2t \\ -4c_3 \sin 4t + 4c_4 \cos 4t \end{bmatrix}$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & -\frac{1}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{4}{3} \begin{bmatrix} 1 & \frac{1}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} x_1'(0) \\ x_2'(0) \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & -\frac{1}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2c_2 \\ 4c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{matrix} c_2 = 0 \\ c_4 = 0 \end{matrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2 \cos 2t \begin{bmatrix} \frac{1}{4} \\ 1 \end{bmatrix} - \cos 4t \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \cos 2t + \frac{1}{2} \cos 4t \\ 2 \cos 2t - \cos 4t \end{bmatrix} \quad \begin{matrix} \text{IVP} \\ \text{soln} \end{matrix}$$

e)  $\lim_{t \rightarrow \infty} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 3 \\ 15 \end{bmatrix}$  exactly the 3 horizontal asymptotes!

3) a)  $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}'' = \underbrace{\begin{bmatrix} -12 & 2 \\ 16 & -8 \end{bmatrix}}_A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} x_1'(0) \\ x_2'(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

b)  $|A - \lambda I| = \begin{vmatrix} -12-\lambda & 2 \\ 16 & -8-\lambda \end{vmatrix} = (\lambda+4)(\lambda+16) = \lambda^2 + 20\lambda + \frac{96-32}{64} = 0$

$\lambda = -4, -16$

$A + 4I = \begin{bmatrix} -8 & 2 \\ 16 & -4 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & -\frac{1}{4} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \begin{matrix} x_1 - \frac{1}{4}x_2 = 0 \\ x_2 = t \end{matrix}$

$x_1 = \frac{1}{4}t, x_2 = t \quad \langle x_1, x_2 \rangle = t \langle \frac{1}{4}, 1 \rangle$

$A + 16I = \begin{bmatrix} 4 & 2 \\ 16 & 8 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \begin{matrix} x_1 + \frac{1}{2}x_2 = 0 \\ x_2 = t \end{matrix}$

$x_1 = -\frac{1}{2}t, x_2 = t \quad \langle x_1, x_2 \rangle = t \langle -\frac{1}{2}, 1 \rangle$

$B = \langle \vec{b}_1, \vec{b}_2 \rangle = \begin{bmatrix} \frac{1}{4} & -\frac{1}{2} \\ 1 & 1 \end{bmatrix}, \quad B^{-1} = \frac{4}{3} \begin{bmatrix} 1 & \frac{1}{2} \\ -1 & \frac{1}{4} \end{bmatrix}$

$\vec{x}'' = B\vec{x} \rightarrow \begin{cases} \vec{x} = B\vec{y} \\ \vec{y} = B^{-1}\vec{x} \end{cases} \rightarrow B^{-1}(B\vec{y})'' = B^{-1}AB\vec{y} \rightarrow \vec{y}'' = (B^{-1}AB)\vec{y}$

$\begin{bmatrix} y_1'' \\ y_2'' \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ 0 & -16 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -4y_1 \\ -16y_2 \end{bmatrix}$

$y_1'' + 4y_1 = 0 \quad y_1 = c_1 \cos 2t + c_2 \sin 2t$

$y_2'' + 16y_2 = 0 \quad y_2 = c_3 \cos 4t + c_4 \sin 4t$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & -\frac{1}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \cos 2t + c_2 \sin 2t \\ c_3 \cos 4t + c_4 \sin 4t \end{bmatrix} = \underbrace{(c_1 \cos 2t + c_2 \sin 2t)}_{\text{tandem mode}} \begin{bmatrix} \frac{1}{4} \\ 1 \end{bmatrix} + \underbrace{(c_3 \cos 4t + c_4 \sin 4t)}_{\text{accordian mode}} \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$  gen soln

$= \begin{bmatrix} \frac{1}{4}(c_1 \cos 2t + c_2 \sin 2t) - \frac{1}{2}(c_3 \cos 4t + c_4 \sin 4t) \\ (c_1 \cos 2t + c_2 \sin 2t) + (c_3 \cos 4t + c_4 \sin 4t) \end{bmatrix}$

d) tandem mode:  $\omega = 2$

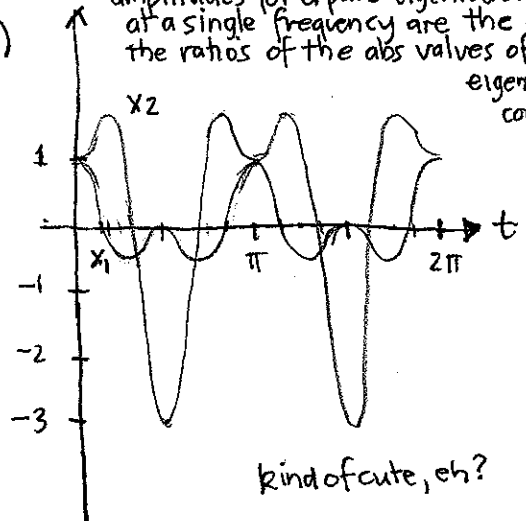
$\frac{x_1}{x_2} = \frac{1}{4}$  ratio of amplitudes

accordian mode:  $\omega = 4$

$\frac{x_1}{x_2} = -\frac{1}{2}$  180° out of phase  
1:2 amplitude ratio

The ratios of the amplitudes for a pure eigenvector solution at a single frequency are the same as the ratios of the abs values of the eigenvector components.

e)



kind of cute, eh?