⑥ EXAMPLES OF DIFFERENTIABLE MANIFOLDS

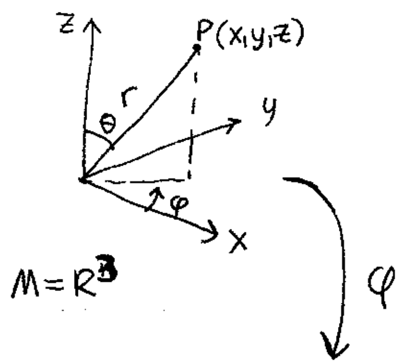
- R^n is itself a differentiable manifold with $U = R^n$, $\phi = id$ (identity map)

  $x^i = u^i \circ id$ are <u>global</u> coordinates on R^n

  similarly any open set $U \subset R^n$ is a differentiable manifold on which the usual coordinates are global.

  other familiar local coordinates on R^3 are

- spherical coordinates. Set $U =$ "$R^3 -$ z-axis $-$ x-z halfplane $(x>0)$"

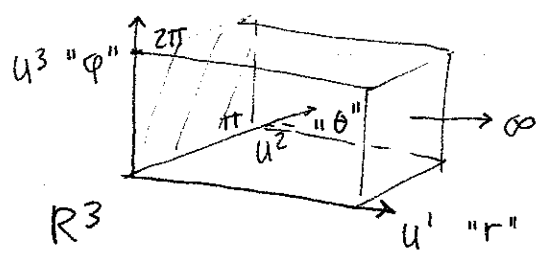


$$\phi(x,y,z) = (r(x,y,z),\ \theta(x,y,z),\ \varphi(x,y,z))$$

$$r(x,y,z) = \sqrt{x^2+y^2+z^2} \qquad \in (0,\infty)$$

$$\theta(x,y,z) = \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) \qquad \in (0,\pi)$$

$$\varphi(x,y,z) : \quad \tan\varphi = \frac{y}{x} \qquad \varphi \in (0,2\pi)$$



$\phi(U) = (0,\infty) \times (0,\pi) \times (0,2\pi)$

on U, ϕ is a 1-1 continuous map. In fact ϕ is a differentiable function of the cartesian coordinates on R^n.

So if we take V to be any open set containing the positive x-z plane, then $U \cup V = R^n$ and we get a coordinate covering of R^n by taking $\{(U,\phi), (V, id)\}$.

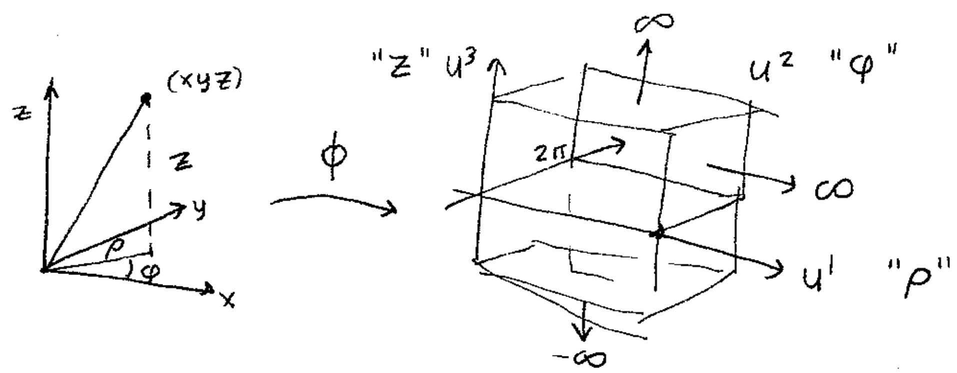
Then the above equations define a coordinate transformation on $U \cap V$. (We could take $V = R^n$ for example.)

Note that the usual spherical coordinates <u>do not cover</u> R^3. When we solve Laplace's equation in spherical coordinates by separation of variables, we get solutions which are regular on $\phi(U)$, but then we pick out the solutions which are regular on R^3 by remembering the above coordinate transformation near the "coordinate singularities" of spherical coordinates (periodicity in φ, regularity in r)

Expressing (x,y,z) in terms of (r,θ,φ) gives

$$\begin{aligned} x &= r\sin\theta\cos\varphi \\ y &= r\sin\theta\sin\varphi \\ z &= r\cos\theta \end{aligned} \qquad \text{or} \qquad \begin{aligned} F^1(u^1,u^2,u^3) &= u^1 \sin u^2 \cos u^3 \\ F^2(u^1,u^2,u^3) &= u^1 \sin u^2 \sin u^3 \\ F^3(u^1,u^2,u^3) &= u^1 \cos u^2 \end{aligned}$$

■ cylindrical coordinates. Same $\mathcal{U}$ as above



$\rho(x,y,z) = \sqrt{x^2+y^2}$

$\varphi(x,y,z) =$ same as above

$z(x,y,z) = z$

$$\phi(x,y,z) = (\rho(x,y,z),\ \phi(x,y,z),\ z(x,y,z))$$

(we're already overusing "z")

same comments as above.

Let's look at some simpler cases:

■ $M_1 = R$ with global chart $(U,\phi_1) = (R, id)$ — global coordinate: $x(t) \equiv u(t) = t$

$M_2 = R$ with global chart (U,ϕ_2), $U = R$, $\phi_2(t) = t^3 \equiv \overline{x}(t)$.

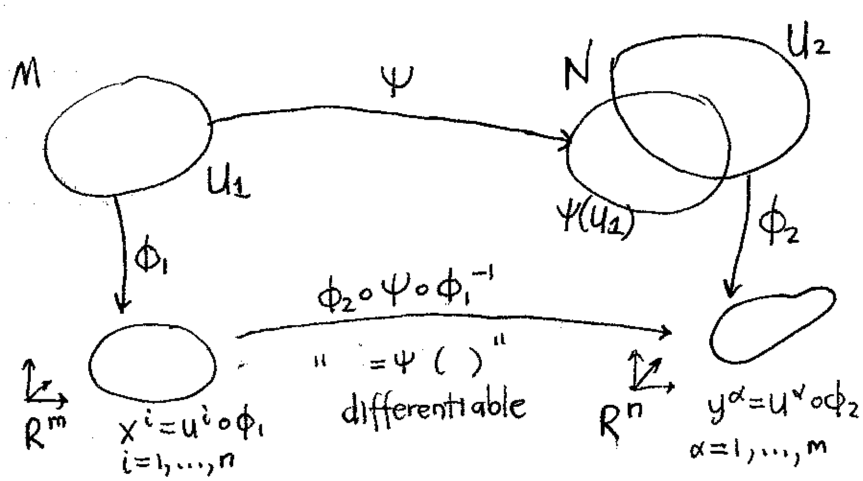
expressing $\overline{x}(t)$ and $x(t)$ in terms of each other:

$$\overline{x}(t) = (x(t))^3 = \phi_2(\phi_1^{-1}(t)) \equiv F(x(t)) \qquad F(u) = u^3$$

$$x(t) = (\overline{x}(t))^{1/3} = \phi_1(\phi_2^{-1}(t)) \equiv F^{-1}(\overline{x}(t)) \qquad F^{-1}(u) = u^{1/3}$$

But F^{-1} is not a differentiable function, so M_1 and M_2 are different differentiable manifolds. [Differentiable functions of x are not differentiable functions of $\overline{x}$: M_1 and M_2 are the same sets but with different differentiable structures.] [By definition, a function F on a manifold is differentiable if it is a differentiable function of the local coordinates on each chart or patch i.e. if $F \circ \phi^{-1}$ is differentiable on $\phi(U) \subset R^n$.]

A map $\Psi: M \to N$ from one manifold ($\dim M = m$) into another ($\dim N = n$) is <u>differentiable</u> if its expression in terms of local coordinates on M and N is differentiable on each local patch:



$\phi_2 \circ \Psi \circ \phi_1^{-1} : R^m \to R^n$ so we know when this is differentiable.

A 1-1 map $\psi : M \to N$ with inverse ψ^{-1} is called a <u>diffeomorphism</u> if <u>both</u> ψ and ψ^{-1} are differentiable. M and N are called <u>diffeomorphic</u>.

■ Returning to our example, M_1 and M_2 are different differentiable manifolds but they are diffeomorphic: let $\psi : M_1 \to M_2$ be defined by $\psi(t) = t^{1/3}$. Then

$$\phi_2(\psi(\phi_1^{-1}(t)) = \phi_2(\psi(t)) = (\psi(t))^3 = t \qquad \text{ie } \phi_2 \circ \psi \circ \phi_1^{-1} = \mathrm{id}$$

which is trivially differentiable, and

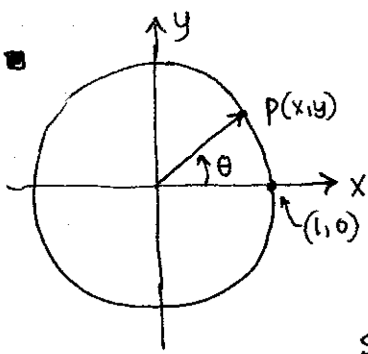
$$\phi_1(\psi^{-1}(\phi_2^{-1}(t)) = \psi^{-1}(t^{1/3}) = t \quad \text{so } \phi_1 \circ \psi \circ \phi_2^{-1} = \mathrm{id}$$

, ditto, so ψ is a diffeomorphism. In terms of local coordinates:

$$\bar{x}(\psi(t)) = t = x(t)$$
$$x(\psi^{-1}(t)) = t^3 = \bar{x}(t)$$

This diffeomorphism identifies points with the same coordinates "$\bar{x} = x$".

In fact all 1-dimensional manifolds are diffeomorphic to either the real line (R with its natural manifold structure) or the circle:

■ 

$S^1 = \{(x,y) \in R^2 \mid x^2+y^2=1\}$

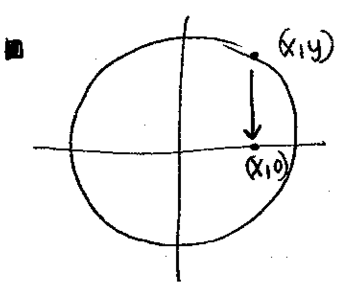
Set $U = S^1 - \{(1,0)\}$

$\theta(x,y) =$ radian measure angle measured counterclockwise from x-axis $\in (0, 2\pi)$

We need 2 such local patches to cover S^1, the second could be $U_2 = S^1 - \{(0,1)\}$ with θ_2 measured from the y-axis.

The two local patches <u>together</u> define the "usual" differentiable structure on S^1 (such that restricting differentiable functions on R^2 to S^1 gives differentiable functions on the manifold S^1; S^1 is said to be a <u>submanifold</u> of R^2)

other useful coordinates on S^1:

■ 

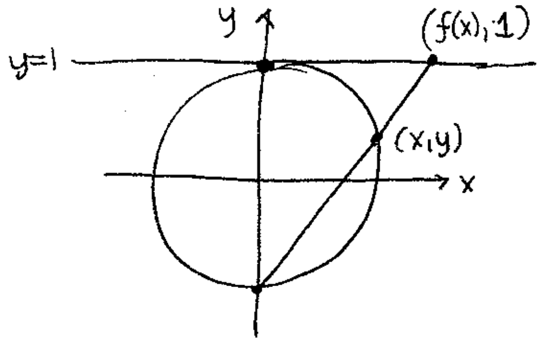
$U_+ = \{(x,y) \in S^1 \mid y > 0\}$ $\qquad U_- = \{(x,y) \in S^1 \mid y < 0\}$

$X_+(x,y) = x \in (-1,1)$ $\qquad X_-(x,y) = x \in (-1,1)$

but $U_+ \cup U_- = S^1 - \{(1,0), (0,-1)\}$.

In fact we need 2 more patches $(x>0, x<0)$ to cover S^1 with these kinds of coordinates

■ projective coordinates



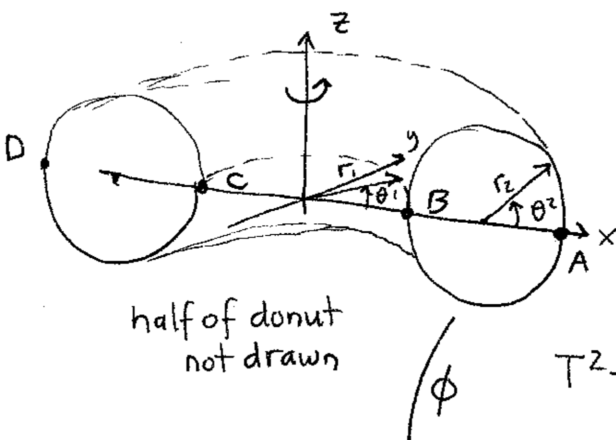
$U = S^2 - \{(0,-1)\}$

$\phi(x,y) = f(x) = \text{do some trig} = \dfrac{2x}{1+y} \in R$

This maps every point except the south pole onto R, but its that single point which makes S^1 different from R.

We need 2 such patches to cover S^1, another patch projecting from the north pole to the line $y=-1$.

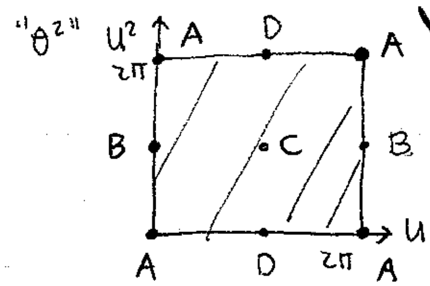
■ The 2-torus: $T^2 = S^1 \times S^1$ (as a submanifold of R^3)



revolve the circle in the z-x plane centered at $(r_1+r_2, 0, 0)$ around the z-axis. Call this T^2.

$(\theta^1, \theta^2): T^2 \to (0,2\pi) \times (0,2\pi) \in R^2$ are the local coords defined in the diagram. The corresponding <u>coordinate domain</u> $U =$

$T^2 - \{\text{outer circumference in x-y plane}\} - \{\text{circle described above}\}$.



■ still more global coords on M_1 = real line

$\phi_3(t) = \tanh t \in (-1,1)$

$\phi_4(t) = 2\sin^{-1}(\tanh t) \in (-\pi, \pi)$.

Notice that ϕ_4 maps the real line onto the circle, sending the points at ∞ to the same point $\theta = \pm\pi$ of S^1! (Not diffeo's since ϕ_3 and ϕ_4 cant be inverted, one point of S^1 has no inverse)

■

graph of a function on R^n in R^{n+1}:

$u^{n+1} = f(u^1, \dots, u^n)$

Let $\phi(u_1, \dots u_{n+1}) = (u_1, \dots, u_n)$ and $U = $ "R^n" $(u^{n+1}=0)$

$x^i(u_1, \dots, u_{n+1}) = u_i \quad , \quad i < n+1$

This is a global coordinate patch. (f must be differentiable)

■ Let V be any vector space with basis $\{e_i\}$ and dual basis $\{\omega^i\}$. Then $U = V$ and $\phi(X) = (\omega^1(X), \dots, \omega^n(X)) \in R^n$ gives a global chart with global coordinates $x^i = u^i \circ \phi = \omega^i$ making V a differentiable manifold.

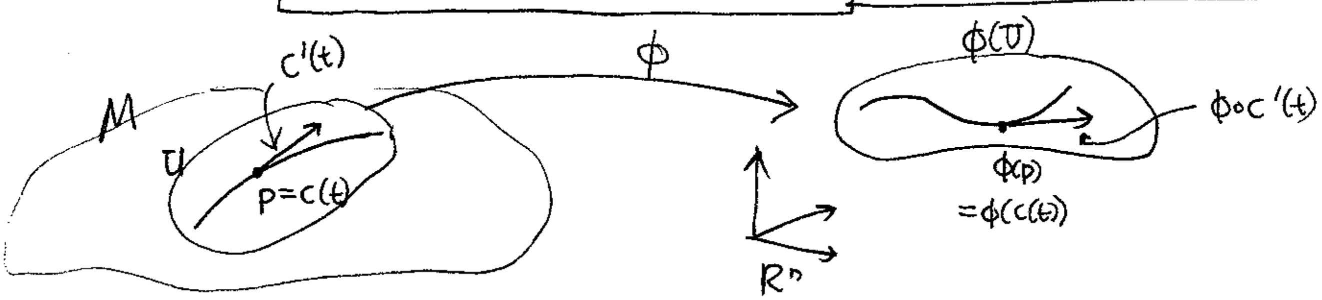
So now that we have an idea what manifolds are and how to map them onto R^n, how do we do differential geometry on them? All we have to do is define the tangent and cotangent spaces.

Let $\mathcal{F}(M)$ be the space of real valued differentiable functions on M. Given $f \in \mathcal{F}(M)$, for each local patch (U, ϕ) we get a function $F = f \circ \phi^{-1}$ on $\phi(U) \subset R^n$ by "expressing f in terms of the local coordinates":
$$f(p) = F(\phi(p)) = F(x^1(p), \ldots, x^n(p))$$

If $c : R \to M$ is a parametrized curve in M (differentiable, now that we know what that means), then $\phi \circ c$ is a parametrized curve in R^n and we can <u>define</u> the tangent to c at $c(t)$ acting on f to be equal to the tangent to $\phi \circ c$ at $\phi(c(t))$ acting on F

i.e. express c and f in terms of local coordinates and just pretend we're on R^n.

$$\boxed{c'(t)\, f \equiv (\phi \circ c)'(t)\, F} \boxed{= \frac{d}{dt} F(\phi(c(t)))}$$
"derivative of F along $\phi \circ c$ at $\phi(c(t))$"



The derivative of f along c at $c(t)$ equals the derivative of $F = f \circ \phi^{-1}$ along $c \circ \phi$ at $\phi(c(t))$ in R^n.

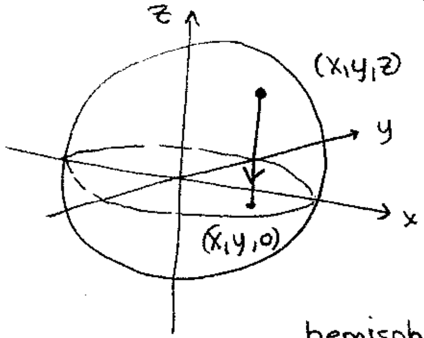
What about coordinate derivatives?

Define $$\left.\frac{\partial}{\partial x^i}\right|_p f \equiv \left.\frac{\partial}{\partial u^i}\right|_{\phi(p)} F$$ i.e. "express f as a function of the coordinates x^i and differentiate with respect to x^i"

Then using the chain rule in the above equation:
$$c'(t)\, f = \frac{d(\phi\circ c)^i(t)}{dt} \left(\frac{\partial F}{\partial u^i}\right)(\phi(c(t))) = \frac{d(\phi\circ c)^i(t)}{dt} \left.\frac{\partial}{\partial u^i}\right|_{\phi(c(t))} F = \frac{d(\phi\circ c)^i(t)}{dt} \left.\frac{\partial}{\partial x^i}\right|_{c(t)} f$$

$$c'(t) = \underbrace{\frac{d(\phi\circ c)^i(t)}{dt}}_{\text{"}\frac{d\bar{x}^i(t)}{dt}\text{"}} \left.\frac{\partial}{\partial x^i}\right|_{c(t)} .$$

EXAMPLE $\quad S^2 = \{(x,y,z) \in R^3 \mid x^2+y^2+z^2=1\}$



$$\phi(x,y,z) = (x,y) \in R^2$$

$$U_\pm = \{(x,y,z) \in S^2 \mid \pm y > 0\}$$

Note $\quad U_+ \cup U_- = S^2 -$ equator so either need 2 more local patches of any kind to cover the equator or a total of 6 of these projective coordinate patches (based on the 6 hemispheres associated in pairs with the coordinate axes)

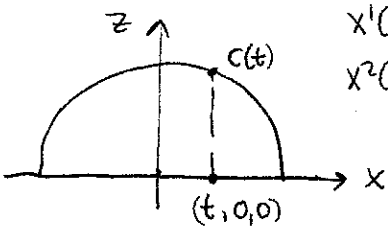
Let's work on U_+ : $\quad x^1(x,y,z) = x, \quad x^2(x,y,z) = y$

Let $f(x,y,z) = x^2+y^2-z^2$. On U_+ :

$$f(x,y,z) = x^2+y^2-(1-x^2-y^2) = 2(x^2+y^2)-1 = F(x^1(xyz), x^2(xyz))$$

$$F(u^1,u^2) = 2[(u^1)^2+(u^2)^2-1].$$

Let $c(t) = (t, 0, \sqrt{1-t^2})$ for $t \in (-1,1)$. $\quad (\phi\circ c)(t) = (t,0)$



$$x^1(c(t)) = u^1(\phi(c(t))) = t$$
$$x^2(c(t)) = u^2(\phi(c(t))) = 0$$

" just translation along x^1 coordinate axis on S^2: "

" $(x^1(t), x^2(t)) = (t,0)$ "

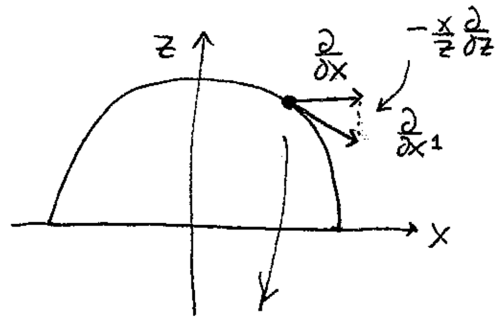
$$c'(t) = \frac{d(\phi\circ c)^i(t)}{dt} \frac{\partial}{\partial x^i}\Big|_{c(t)} = \frac{\partial}{\partial x^1}\Big|_{c(t)}$$

$$c'(t) f = \frac{\partial}{\partial x^1}\Big|_{c(t)} f = \frac{\partial}{\partial u^1} F(u^1,u^2)\Big|_{\phi(c(t))} = 4u^1\Big|_{\phi(c(t))} = 4t = 4x^1(c(t))$$

Note that this is not the same as

$$\frac{\partial}{\partial x}\Big|_{(t,0,\sqrt{1-t^2})} f = \frac{\partial}{\partial x}\Big|_{(t,0,\sqrt{1-t^2})} (x^2+y^2-z^2) = 2t$$

even though " $x^1 = x$ " , " $x^2 = y$ ".

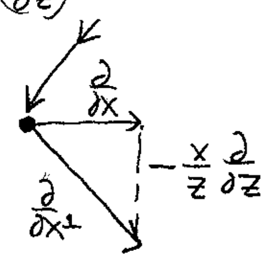


In terms of the chain rule :

$$\frac{\partial}{\partial x} f(x,y,z(x,y)) = \left(\frac{\partial f}{\partial x}\right)(x,y,z(x,y)) + \frac{\partial z(x,y)}{\partial x}\left(\frac{\partial f}{\partial z}\right)(x,y,z(x,y))$$

$$z(x,y) = \sqrt{1-(x^2+y^2)}, \quad \frac{\partial z(x,y)}{\partial x} = -\frac{x}{z(x,y)}$$

or $$\frac{\partial f}{\partial x^1} = \left(\frac{\partial}{\partial x} - \frac{x}{z}\frac{\partial}{\partial z}\right) f$$



Notice that the tangent vector $\frac{\partial}{\partial x^1}$ which has Euclidean norm $g\left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^1}\right) = 1 + \frac{x^2}{z^2}$ is running into trouble at the end of the coordinate patch $(z \to 0)$. This is a " coordinate singularity".

THIS example is a good example of the need to consider submanifolds. Consider a subspace N of M which is a k-dimensional manifold. Then N is a submanifold of M if one can find a set of local coordinate charts $\{U_\alpha, \phi_\alpha\}$ of M (not necessarily all of M) such that

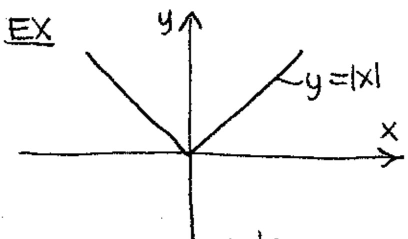
$$N \subset \bigcup_\alpha U_\alpha \quad \text{and} \quad u^i(\phi_\alpha(U_\alpha \cap N)) = 0, \quad i = k+1, \ldots, n$$

i.e. the intersection of each coordinate patch with N is described by the vanishing of the last (or any) $n-k$ coordinates:

$$x^i_\alpha(p) = 0, \quad i = k+1, \ldots, n \qquad p \in N \cap U_\alpha.$$

Such coordinates are called adapted coordinates. The restrictions of x^i_α to $U_\alpha \cap N$ are local coordinates on the submanifold.

EX

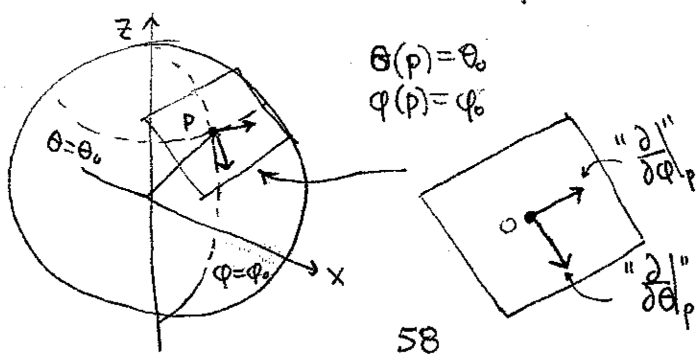


Consider the graph N of the function $f(x) = |x|$ on the x-y plane (R^2).
Let, $U = N$, $\phi((x, |x|)) = x$.
This is a global coordinate chart making N into a manifold but it is clearly not a submanifold of R^2 since f is not differentiable at the origin, so no adapted coordinates exist for a neighborhood of that point.

EX For $S^2 \subset R^3$, spherical coordinates are nearly adapted coordinates. Let $x^1 = \theta$, $x^2 = \varphi$, $x^3 = r - 1$. These are adapted coordinates on R^3 which induce local coordinates $\{\theta, \varphi\}$ on "$S^2 - \{$semicircle from north to south pole through $(1,0,0)\}$". Another set of spherical coordinates based on the y-axis say, with the angle φ measured from the negative x-axis, would lead to a covering of S^2 by two adapted local coordinate patches.

With this definition the tangent space to a point of a submanifold may be identified with an obvious subspace of the full tangent space.

In particular for submanifolds of R^n where the full tangent space may be identified with R^n itself, the tangent space to a k-dimensional submanifold can be visualized as a k-dimensional plane tangent to the submanifold



"tangent space" to S^2 at p

$x = r\sin\theta\cos\varphi$
$y = r\sin\theta\sin\varphi$
$z = r\cos\theta$

"$\frac{\partial}{\partial\theta}\Big|_p$" $= (r_0\cos\theta_0\cos\varphi_0, r_0\cos\theta_0\sin\varphi_0, -r_0\sin\theta_0)$

"$\frac{\partial}{\partial\varphi}\Big|_p$" $= (-r_0\sin\theta_0\sin\varphi_0, r_0\sin\theta_0\cos\varphi_0, 0)$

$r_0 = 1$

Getting back to our discussion of tangent and cotangent spaces on a manifold M, any tangent vector at $p \in M$ can be expressed as a linear combination of the coordinate derivatives which are a basis of the tangent space

$$X = X^i \frac{\partial}{\partial x^i}\Big|_p \in TM_p = \text{the tangent space to } M \text{ at } p$$

where $X^i \in R$ are constants.

The differential of a function f on M at p is defined as before

$$df|_p(X) = Xf \qquad X \in TM_p .$$

$$dx^j|_p\left(\frac{\partial}{\partial x^i}\Big|_p\right) = \frac{\partial}{\partial x^i}\Big|_p x^j = \frac{\partial}{\partial u^i}\Big|_{\phi(p)} u^j = \delta^j{}_i$$

so $\{dx^i|_p\}$ is the basis dual to $\{\partial/\partial x^i|_p\}$:

$$X^i = dx^i|_p(X) .$$

They are a basis of $\quad T\ M_p = $ the cotangent space to M at p

We can introduce an arbitrary frame and dual frame on M exactly as on R^n but the existence of a <u>global</u> frame is not guaranteed. (For example S^2 has no everywhere nonvanishing vector field.)

A vector field X on M is a tangent vector valued function on M. In each local coordinate patch (U, ϕ) it has the expression

$X = X^i \frac{\partial}{\partial x^i}$ where X^i are smooth functions on U (the components of X with respect to the local coordinates $\{x^i\}$).

Consider the overlap of such a local chart with another $(\bar U, \bar\Phi)$, $x^i = u^i \circ \bar\Phi$. Then on $U \cap \bar U$ we have a coordinate transformation. We can express each of the coordinates as functions of the others:

$\bar x^i = F^i(x^1, \dots, x^n)$ where $F^i = u^i \circ \phi \circ \bar\Phi^{-1}$ are functions on $\phi(\bar U \cap \bar U) \subset R^n$

$x^i = F^{-1\,i}(\bar x^1, \dots, \bar x^n)$ " $F^{-1\,i} = u^i \circ \phi \circ \bar\Phi^{-1}$ " " " " " " "

[Note $F = \bar\Phi \circ \phi^{-1}$ is a diffeomorphism of $\phi(U \cap \bar U)$ onto $\bar\Phi(U \cap \bar U)$ with inverse $F^{-1} = \phi \circ \bar\Phi^{-1}$: a coordinate transformation is a transformation on R^n not M.]

Introduce the matrix of the transformation between the two coordinate frames on $\bar{U} \cap U$

$$d\bar{x}^i\left(\frac{\partial}{\partial x^j}\right) = \frac{\partial \bar{x}^i}{\partial x^j} = (\partial_j F^i)\circ \phi^{-1} \equiv \mathcal{J}^{\bar{i}}{}_j =$$ Jacobian matrix of the transformation from $\{x^i\}$ to $\{\bar{x}^i\}$

$$dx^i\left(\frac{\partial}{\partial \bar{x}^j}\right) = \frac{\partial x^i}{\partial \bar{x}^j} = (\partial_j F^{-1\,i})\circ \phi^{-1} \equiv \mathcal{J}^{i}{}_{\bar{j}} =$$ ditto with x^i and $\bar{x}^i$ interchanged

Then
$$\frac{\partial}{\partial \bar{x}^i} = dx^j\left(\frac{\partial}{\partial \bar{x}^i}\right)\frac{\partial}{\partial x^j} = \mathcal{J}^{i}{}_{\bar{j}}\frac{\partial}{\partial x^j}$$
$$\frac{\partial}{\partial x^i} = d\bar{x}^j\left(\frac{\partial}{\partial x^i}\right)\frac{\partial}{\partial \bar{x}^j} = \mathcal{J}^{\bar{j}}{}_i\frac{\partial}{\partial \bar{x}^j}$$

(dual transformation
$$d\bar{x}^i = \mathcal{J}^{\bar{i}}{}_j\, dx^j$$
$$dx^i = \mathcal{J}^{i}{}_{\bar{j}}\, d\bar{x}^j \, .)$$

is a transformation of the basis of the tangent space at each point of $U \cap \bar{U}$ so the components of all tensors over this vector space transform accordingly at each point.

For example if $X = X^i \frac{\partial}{\partial x^i} = \bar{X}^i \frac{\partial}{\partial \bar{x}^i}$ on $U \cap \bar{U}$,

$$\bar{X}^i = d\bar{x}^i(X) = \mathcal{J}^{\bar{i}}{}_j X^j \; , \quad X^i = dx^i(X) = \mathcal{J}^{i}{}_{\bar{j}}\bar{X}^j \, .$$

1-forms transform by the inverse of this transformation.
[Note $(\mathcal{J}^{\bar{i}}{}_j)$ and $(\mathcal{J}^{i}{}_{\bar{j}})$ are inverse matrices!]

Now that we've got the tangent and cotangent spaces, we can introduce $\binom{p}{q}$-tensors and tensor fields and differential forms on M in an obvious way.
(define $dx^{i_1 \cdots i_k} = dx^{i_1}\wedge \ldots \wedge dx^{i_k}$)
Under a coordinate transformation, the coordinate components undergo the obvious transformation law involving the Jacobian matrix. We can also consider noncoordinate frames and transformations between them, just as we did on R^n.

For example, the coordinate basis of $\Lambda^n(T^*M_p) = \{n\text{-forms at } p\}$ transforms by the determinant of the Jacobian matrix:

$$\mathcal{J} \equiv \det(\mathcal{J}^{\bar{i}}{}_j) = \text{Jacobian of transformation from } \{x^i\} \text{ to } \{\bar{x}^i\}$$
$$\bar{\mathcal{J}} \equiv \det(\mathcal{J}^{i}{}_{\bar{j}}) = \mathcal{J}^{-1} = \quad \cdots \quad \cdots \quad \{\bar{x}^i\} \text{ to } \{x^i\}$$

$$d\bar{x}^{1\cdots n} = d\bar{x}^1\wedge\ldots\wedge d\bar{x}^n = \frac{\partial \bar{x}^1}{\partial x^{i_1}}dx^{i_1}\wedge\ldots\wedge\frac{\partial \bar{x}^n}{\partial x^{i_n}}dx^{i_n} =$$
$$= \frac{1}{n!}\frac{\partial \bar{x}^1}{\partial x^{i_1}}\cdots\frac{\partial \bar{x}^n}{\partial x^{i_n}}\,\delta^{i_1\cdots i_n}_{j_1\cdots j_n}\, dx^{j_1\cdots j_n} = \frac{1}{n!}\det(\mathcal{J}^{\bar{i}}{}_j)\,\epsilon_{j_1\cdots j_n} dx^{j_1\cdots j_n} = \mathcal{J} dx^{1\cdots n}$$

The components of an n-form

$$L = \frac{1}{n!} L_{i_1\cdots i_n} dx^{i_1\cdots i_n} = L_{1\cdots n} dx^{1\cdots n} = \frac{1}{n!} \bar{L}_{i_1\cdots i_n} d\bar{x}^{i_1\cdots i_n} = \bar{L}_{1\cdots n} d\bar{x}^{1\cdots n}$$

then undergo the following transformation on $U \cap \bar{U}$:

$$\bar{L}_{i_1\cdots i_n} = \mathcal{J}^{-1} L_{i_1\cdots i_n}$$

Suppose we define a "scalar density" $\mathcal{L}$ on U and $\bar{\mathcal{L}}$ on $\bar{U}$ (or in the domain of any noncoordinate frame) as the $1\cdots n$ component of L in the associated basis of $\Lambda^n(T^*M_p)$ at each p in the appropriate domain:

$$\mathcal{L} \equiv L_{1\cdots n} \text{ in } U, \quad \bar{\mathcal{L}} \equiv \bar{L}_{1\cdots n} \text{ in } \bar{U}, \text{ etc.}$$

Then $\quad \bar{\mathcal{L}} = \mathcal{J}^{-1}\mathcal{L}$ is the resulting transformation law on $U \cap \bar{U}$.

This scalar density is an object on M which only has meaning with respect to a particular local frame (coordinate or noncoordinate) and has no invariant meaning like a tensor field. But it is quite useful.

Suppose $V \subset U \cap \bar{U}$. Then we can integrate L on V by making the definition

$$\int_V L = \int_V \mathcal{L}\, dx^{1\cdots n} \equiv \int_{\phi(V)} \mathcal{L}\circ\phi^{-1}\, du^1\cdots du^n$$

where the integral on the right is just an ordinary integral of the function $\mathcal{L}\circ\phi^{-1}$ on a subset of R^n. This definition is coordinate independent due to the formula:

$$\int_{F(A)} f\, du^1\cdots du^n = \int_A f\circ F \det(\partial_i F^j)\, du^1\cdots du^n$$

for a diffeomorphism $F: A \to F(A)$ on R^n

Then setting $F = \bar{\Phi}\circ\phi^{-1}$ and $A = \phi(V)$, we get

$$\int_V \bar{\mathcal{L}}\, d\bar{x}^{1\cdots n} = \int_{\bar{\Phi}(V)} \bar{\mathcal{L}}\circ\bar{\Phi}^{-1} du^1\cdots du^n = \int_{\phi(V)} \underbrace{\bar{\mathcal{L}}\circ\phi^{-1} \underbrace{\det(\partial_i F^j)}_{\mathcal{J}\circ\phi^{-1}}}_{\mathcal{L}\circ\phi^{-1}} du^1\cdots du^n$$

$$= \int_V \mathcal{L}\, dx^{1\cdots n}$$

For example if $\mathcal{L}$ is a Lagrangian density of a field on Minkowski spacetime expressed in standard cartesian ("inertial") coordinates then the action I is really the integral of the corresponding 4-form L which is independent not only of the choice of inertial coordinates (Lorentz invariance) but of the choice of coordinates period.

If we want to integrate over an open set $V \subset M$ which cannot be covered with a single coordinate patch, then we have to play games integrating over overlapping patches and getting rid of the unwanted contributions from overlaps (partitions of unity).

WHAT CAN WE DO NEXT? We've already polished off all the algebra we need to do with tensors. What's left is differentiation. We know how to differentiate functions, but what about tensor fields and differential forms?

In fact there are 3 kinds of derivatives which generalize the differential of a function

(1) exterior differentiation of forms → Stoke's and Gauss's Thms → [Electromagnetism, Gauge Theories]

(2) covariant derivatives → (pseudo)Riemannian geometry [nonrelativistic mechanics, special relativity, general relativity]

(3) Lie derivatives → diffeomorphisms, Lie Groups → SYMMETRIES