⑤ DIFFERENTIAL GEOMETRY ON R^n USING CARTESIAN COORDINATES

$R^n = \underbrace{R\times\cdots\times R}_{n \text{ times}} = \{ \underline{r} = (r^1,\ldots,r^n) \mid r^i \in R \}$ $\qquad i,j,\ldots = 1,\ldots,n$

standard cartesian coordinates on R^n: $\{x^i\} \leftrightarrow n$ real valued functions on R^n

$$x^i(\underline{r}) = x^i(r^1,\ldots,r^n) = r^i$$

To avoid confusion later we will use the symbol u^i instead of x^i when convenient.

Note: Recalling R^n is a real vector space with standard basis $\{e_i\}$ and dual basis $\{\omega^i\}$, then $(r^1,\ldots,r^n) = r^i e_i$ and $r^i = \omega^i(r^1,\ldots,r^n) = x^i(r^1,\ldots,r^n)$, ie $x^i = \omega^i$.

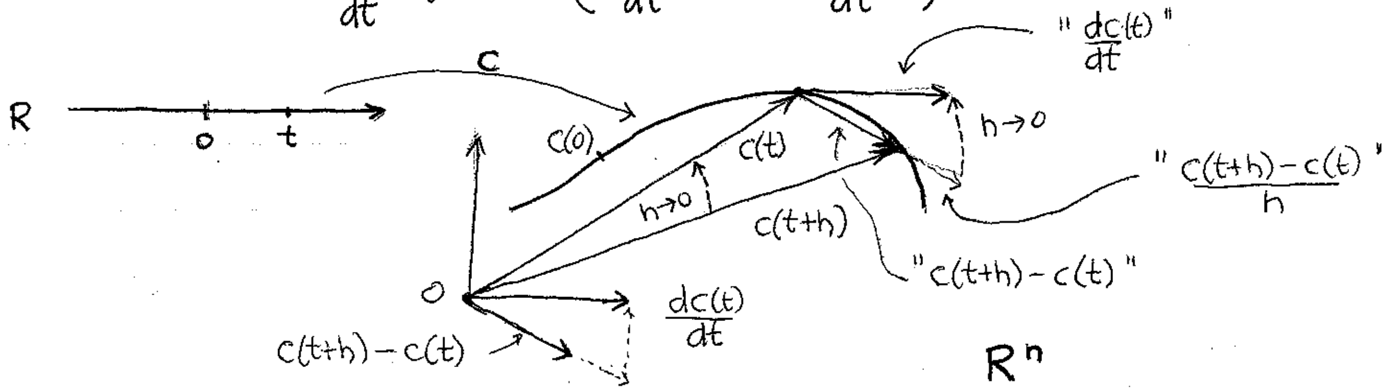
A <u>parametrized curve</u> in R^n is simply a map $c: R \to R^n$ from the real line R into R^n: $\quad t \in R \longrightarrow c(t) = (c^1(t),\ldots,c^n(t)) \in R^n$

where $c^i(t) = x^i(c(t))$ or $c^i = x^i \circ c$ are the component functions. [In physics we usually just write "$x^i(t)$".]

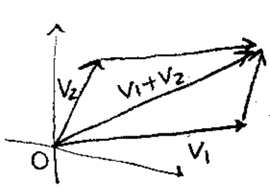
[If we think of t as the time, a parametrized curve is just the trajectory of a particle moving in R^n expressed as a function of time.]

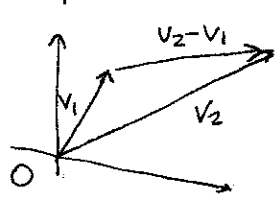
The components of the "<u>tangent vector</u>" to c at $c(t)$ are just the derivatives of the component functions:

$$\frac{d}{dt}c(t) = \left(\frac{dc^1}{dt}(t),\ldots,\frac{dc^n}{dt}(t)\right)$$



Since R^n is a vector space, we can think of each point as an arrow from the origin to the point. Vector addition is accomplished by translating the end of one arrow to the tip of another; their vector sum is the vector whose tip coincides with the tip of the second vector.





Subtraction is similar.

So $c(t+h) - c(t)$ can be pictured as the vector with end at $c(t)$ and tip at $c(t+h)$. The "tangent vector" is just

$$\frac{d}{dt} c(t) = \lim_{h \to 0} \frac{c(t+h) - c(t)}{h},$$

but thought of as a vector having its endpoint at $c(t)$ (call this "$\frac{dc(t)}{dt}$").

The "tangent space" at a point $p \in R^n$ is the collection of all such "tangent vectors" to curves through p, thought of as vectors with ends at p. [In other words we consider a vector space at each point of R^n whose elements consist of difference vectors between other points of R^n and p itself.]

It is also an n-dimensional vector space with basis consisting of the vectors $\{ "e_i|_p" \}$, namely e_i translated to the point p:

$$"\frac{dc(t)}{dt}" = \frac{dc^i(t)}{dt} "e_i|_{c(t)}"$$

Let $\{"\omega^i|_p"\}$ be the dual basis: $"\omega^i|_{c(t)}"\left("\frac{dc(t)}{dt}"\right) = \frac{dc^i(t)}{dt}$.

$"\omega^i|_p"$ picks out the i^{th} component of "tangent vectors" at p.

Note that reparametrizing a parametrized curve leads to a different parametrized curve (although both represent the same curve in R^n) and a new "tangent vector" with the same direction but a different magnitude:

$$t = f(\bar{t}) \qquad \bar{c} = c \circ f \qquad \bar{c}(\bar{t}) = c(f(\bar{t}))$$

$$\frac{d}{d\bar{t}} \bar{c}(\bar{t}) = \underbrace{\frac{df(\bar{t})}{d\bar{t}}}_{\text{proportionality factor}} \underbrace{\left(\frac{dc}{dt}\right)(f(\bar{t}))}_{\text{old tangent at } c(f(\bar{t}))}$$

In the particle analogy, if the particle traces out the same trajectory but as a different function of the time, it has a different speed.

If X is an element of the tangent space at p we can write

$$X = X^i "e_i|_p", \quad X^i = "\omega^i|_p"(X).$$

If every space we dealt with were a vector space, there would be no need to go farther (all we have used is the affine structure of R^n, that differences of points in R^n may be interpreted as vectors). But clearly this is not the case. So we need to define the tangent space at a point of R^n without using the vector space structure of R^n.

The answer has to do with directional derivatives of functions. For every space M we deal with (mostly real manifolds) we can always consider the (∞-dim. vectorspace) space of real valued functions $\mathcal{F}(M)$ on M.

Given a parametrized curve $c: R \to M$ in a space M, then for each function F on M, $F \circ c$ is just a function on R (namely $(F\circ c)(t) = F(c(t))$) and we can take its derivative, i.e. the derivative of the function F along the curve c at the point $c(t)$ or "the directional derivative of F."

Since we cannot discuss manifolds yet, consider R^n.

$F(\underline{r}) = F(r^1, \ldots, r^n)$ is the value of F at $\underline{r}$.

$\left(\frac{\partial F}{\partial u^i}\right)(r^1, \ldots, r^n)$ is the ith partial derivative of F at $\underline{r}$ $\quad \begin{pmatrix}\text{recall} \\ u^i = x^i\end{pmatrix}$

If $c(t) = (c^1(t), \ldots, c^n(t))$ where $c^i(t) = (u^i \circ g)(t)$, then by the chain rule:

$$\frac{d}{dt}(F \circ c(t)) = \left(\frac{\partial F}{\partial u^i}\right)(c(t))\,\frac{dc^i(t)}{dt} .$$

Let $\left.\frac{\partial}{\partial u^i}\right|_p$ be the first order differential operator on R^n which assigns to a function F the value of its ith partial derivative at p:

$$F \to \left.\frac{\partial}{\partial u^i}\right|_p F = \left(\frac{\partial F}{\partial u^i}\right)(p).$$

Then we can write:

$$\frac{d}{dt}(F\circ c)(t) = \underbrace{\left(\frac{dc^i(t)}{dt}\left.\frac{\partial}{\partial u^i}\right|_{c(t)}\right)}_{\substack{\text{directional derivative} \\ \text{along } c \text{ at } c(t)}} F$$

The vector space of directional derivatives at p

$\left\{ X^i \left.\frac{\partial}{\partial u^i}\right|_p \,\middle|\, X^i \in R \right\}$ is clearly isomorphic to the "tangent space" at p

$$X^i \text{"} e_i|_p \text{"} \leftrightarrow X^i \left.\frac{\partial}{\partial u^i}\right|_p ,$$

but does not use the vector space structure of R^n for its definition. We will call this space the tangent space to R^n at p and denote it by TR^n_p : $\underset{\text{tangent}}{T}\,M^{\text{on space } M}_{p \text{ at } p}$

We denote the tangent vector to a parametrized curve c by

$$c'(t) = \frac{dc^i(t)}{dt} \left. \frac{\partial}{\partial u^i} \right|_{c(t)} .$$

Defining $\left. e_i \right|_p = \left. \frac{\partial}{\partial u^i} \right|_p$, then $\{\left. e_i \right|_p\}$ is a basis of TR^n_p called the coordinate basis associated with the cartesian coordinates $\{u^i\}$. Note that for a tangent vector $X = X^i \left. \frac{\partial}{\partial u^i} \right|_p$, we can obtain its i^{th} "coordinate component" simply by letting X act on the cartesian coordinate function u^i:

$$X u^i = X^j \left. \frac{\partial}{\partial u^j} \right|_p u^i = X^j \underbrace{\left(\frac{\partial u^i}{\partial u^j} \right)(p)}_{\delta^i{}_j} = X^j \delta^i{}_j = X^i .$$

Not only that, a tangent vector X is independent of the cartesian coordinates used to define it.

Suppose $e_{i'} = e_j A^{-1j}{}_i$ is a new basis of R^n, then $\omega^{i'} = A^i{}_j \omega^j$ so $u^{i'} = A^i{}_j u^j$ ($u^i = \omega^i$ and $u^{i'} = \omega^{i'}$ by definition) or $u^i = A^{-1i}{}_j u^{j'}$.

The chain rule says $\left. \frac{\partial}{\partial u^{i'}} \right|_p = \underbrace{\left(\frac{\partial u^j}{\partial u^{i'}} \right)(p)}_{A^{-1j}{}_i} \left. \frac{\partial}{\partial u^j} \right|_p$

but $c^{i'}(t) = u^{i'}(c(t)) = A^i{}_j u^j(c(t)) = A^i{}_j c^j(t)$

so $\frac{dc^{i'}(t)}{dt} = A^i{}_j \frac{dc^j(t)}{dt}$ and therefore if $p = c(t_0)$:

$$\underbrace{\frac{dc^{i'}}{dt}(t_0)}_{X^{i'}} \left. \frac{\partial}{\partial u^{i'}} \right|_p = A^i{}_j \frac{dc^j(t_0)}{dt} A^{-1k}{}_i \left. \frac{\partial}{\partial u^k} \right|_p = \underbrace{\frac{dc^j(t_0)}{dt}}_{X^j} \left. \frac{\partial}{\partial u^j} \right|_p .$$

Note: $X^{i'} = A^i{}_j X^j$ is the transformation law for the cartesian coordinate components of the tangent vector X at p.

Since TR^n_p is a vectorspace we may take a basis of any n linearly independent tangent vectors: $\left. e_{i'} \right|_p = \underbrace{e^j{}_i(p)}_{A^{-1j}{}_i(p)} \left. \frac{\partial}{\partial u^j} \right|_p$ (This is an independent change of basis at each point p)

If X is a tangent vector at p, then $X^{i'} = A^i{}_j(p) X^j$ are the new "noncoordinate components."

If we identify the tangent space with directional derivatives, what is its dual space? By definition this space is the space of real linear forms on $T\mathbb{R}^n_p$.

Call it the cotangent space: $(T\mathbb{R}^n_p)^*$

If σ is a tangent covector (or cotangent vector, take your pick) (or even 1-form at p) then

$$\sigma(X) \in \mathbb{R} \quad \text{if} \quad X \in T\mathbb{R}^n_p .$$

<u>Differential of a function</u> For each function f on $\mathbb{R}^n$ then the map $X \in T\mathbb{R}^n_p \longrightarrow Xf \in \mathbb{R}$ (the directional derivative at p of f by X) is in fact a linear form on the tangent space at p

$$aX + bY \to (aX+bY)f = a(Xf) + b(Yf).$$

Denote this linear form by $df|_p$, "the <u>differential</u> of f at p":

$$df|_p\,(X) \equiv Xf \qquad \text{for } X \in T\mathbb{R}^n_p .$$

By definition the basis $\{\omega^i|_p\}$ dual to $\{e_i|_p \equiv \frac{\partial}{\partial u^i}|_p\}$ picks out the i^{th} coordinate component X^i of the tangent vector X; but

$$du^i|_p\,(X) \equiv X u^i = X^i ,$$

i.e. the differentials of the cartesian coordinate functions at p form the basis dual to the basis of cartesian coordinate derivatives at p:

$$\omega^i|_p = du^i|_p \qquad \left[\begin{array}{l}\text{equivalently:} \\ du^i|_p\left(\frac{\partial}{\partial u^j}\Big|_p\right) = \frac{\partial}{\partial u^j}\Big|_p u^i = \delta^i_j\end{array}\right]$$

Expressing $df|_p$ in this basis gives:

$$df|_p\,(X) = Xf = X^i \frac{\partial}{\partial u^i}\Big|_p f = du^i|_p(X)\, \frac{\partial}{\partial u^i}\Big|_p f$$

$$\text{or} \qquad df|_p = \left(\frac{\partial f}{\partial u^i}\right)(p)\; du^i|_p .$$

(The partial derivatives of f are the coordinate components of df)

A general 1-form at p is expressed in the same way:

$$\sigma = \sigma_i(p)\, du^i|_p \quad , \quad \sigma_i(p) = \sigma\left(\frac{\partial}{\partial u^i}\Big|_p\right).$$

df, the function on R^n whose value at p is the element $df|_p$ of the cotangent space at p, is called a <u>differential 1-form</u> on R^n, namely a 1-form <u>field</u> on R^n, or a tangent covector valued function on R^n which smoothly picks out an element of the cotangent space at all points of R^n.

A vector <u>field</u> $\underline{X}$ on R^n is a tangent vector valued function on R^n, that is a smooth choice of an element of the tangent space at all points of R^n.

A <u>frame</u> on R^n is a smooth choice of basis for the tangent spaces on R^n, i.e. a choice of n vector fields which are linearly independent at each point of R^n.

Its <u>dual frame</u> consists of the corresponding n differential 1-forms which form the basis dual to this basis at each point of R^n.

The vector fields $\{\frac{\partial}{\partial u^i}\}$ form a <u>coordinate frame</u> with dual frame $\{dx^i\}$.

If $e_{i'} = e^j{}_i \frac{\partial}{\partial u^j}$ form a frame $(\det(e^j{}_i) \neq 0)$ with dual frame $\omega^{i'} = \omega^i{}_j du^j$ then by duality:

$$\delta^i{}_j = \omega^{i'}(e_{j'}) = \omega^i{}_k du^k (e^\ell{}_j \tfrac{\partial}{\partial u^\ell}) = \omega^i{}_k e^\ell{}_j \underbrace{du^k(\tfrac{\partial}{\partial u^\ell})}_{\delta^k{}_\ell} =$$
$$= \omega^i{}_k e^k{}_j \,,$$

ie the matrices of coordinate components of the frame and dual frame are inverse matrices (these matrices transform the coordinate frame to the general frame and vice versa).

Note if $\underline{X} = \underline{X}^i \frac{\partial}{\partial u^i} = \underline{X}^{i'} e_{i'}$, then $\underline{X}^{i'} = \omega^i{}_j \underline{X}^j$, etc.

Let $V = TR^n_p$, $V^* = (TR^n_p)^*$.

We can introduce $\binom{k}{q}$ tensors at each point p of R^n as in the previous 4 weeks of linear algebra which dealt with a vector space V.

Then we can introduce fields of such tensors on R^n or $\binom{k}{q}$ tensor fields on R^n. All the algebra has already been done.

$\Lambda^k((TR^n_p)^*)$ is the space of tangent k-forms at p or differential k-forms at p. A field of such forms is called a differential k-form on R^n. Such a form can be expressed in a coordinate frame

$$F = \frac{1}{k!} F_{i_1 \cdots i_k}\, dx^{i_1} \wedge \cdots \wedge dx^{i_k} \equiv \frac{1}{k!} F_{i_1 \cdots i_k}\, dx^{i_1 \cdots i_k}$$

or in a noncoordinate frame

$$F = \frac{1}{k!} F_{i_1' \cdots i_k'}\, \omega^{i_1' \cdots i_k'}.$$

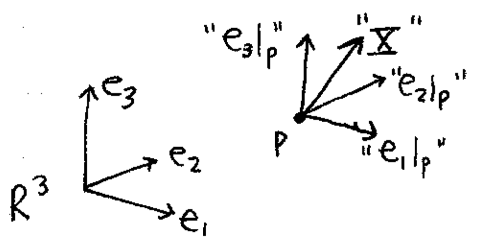
R^n with the natural inner product is Euclidean space E^n (the natural basis $\{e_i\}$ of R^n is orthonormal), i.e. as a vector space the 2-tensor over R^n $\quad \delta = \delta_{ij}\, \omega^i \otimes \omega^j$ makes R^n into a Euclidean inner product space.

But there is a natural identification of each tangent space to R^n with R^n itself: $\quad \frac{\partial}{\partial u^i}\big|_p \leftrightarrow e_i \in R^n$

So equipping each tangent space with the Euclidean inner product using this identification gives us the Euclidean metric tensor field on R^n: $\quad g = \delta_{ij}\, dx^i \otimes dx^j$.

In older notation one writes instead the "line element"

$$ds^2 = \delta_{ij}\, dx^i dx^j = (dx^1)^2 + (dx^2)^2 + (dx^3)^2.$$



old picture

tangent vector X identified with corresponding "tangent vector" "X"

$$dx^i|_p(X) = X^i = \text{"}dx^i\text{"}$$

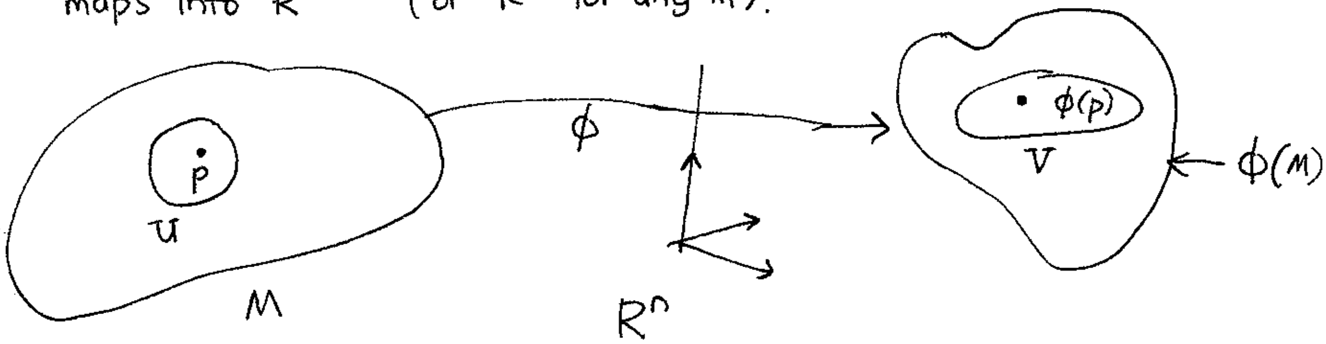
ds^2 = square of length of small or "infinitesimal" displacement "X" of components "dx^i"

$$= \delta_{ij} X^i X^j = \delta_{ij} \text{"}dx^i\text{""}dx^j\text{"}$$

We could go on to introduce derivatives of tensor fields on R^n and see how to integrate differential forms on R^n, but we might as well first introduce manifolds.

We handle manifolds by locally mapping them onto R^n where we know what to do (or can easily figure out what to do).

An <u>n-dimensional manifold</u> M is a space which is everywhere locally like R^n. It must be a topological space so we know what open sets are and can therefore introduce the notion of <u>continuous</u> maps into R^n (or R^m for any m).



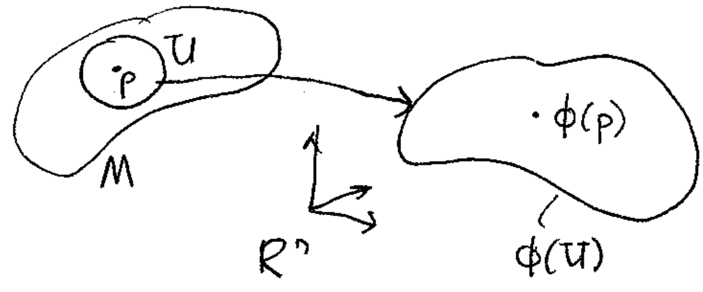
A map $\phi: M \to R^n$ is <u>continuous</u> if it maps nearby points of M to nearby points of R^n, i.e. for every neighborhood V of $\phi(p)$ there exists a neighborhood U of p such that $\phi(U)$ is contained in V.

In particular we can consider continuous 1-1 maps ϕ from open sets $U \subset M$ to R^n. By 1-1 we mean that ϕ establishes a unique correspondence between points of U and $\phi(U)$. (Each point in the image comes from a unique point in U.)

Suppose $\{U_\alpha\}$ is collection of open sets (covering) of M which cover M, i.e. $\bigcup_\alpha U_\alpha = M$ (every point of M belongs to at least 1 set U_α).

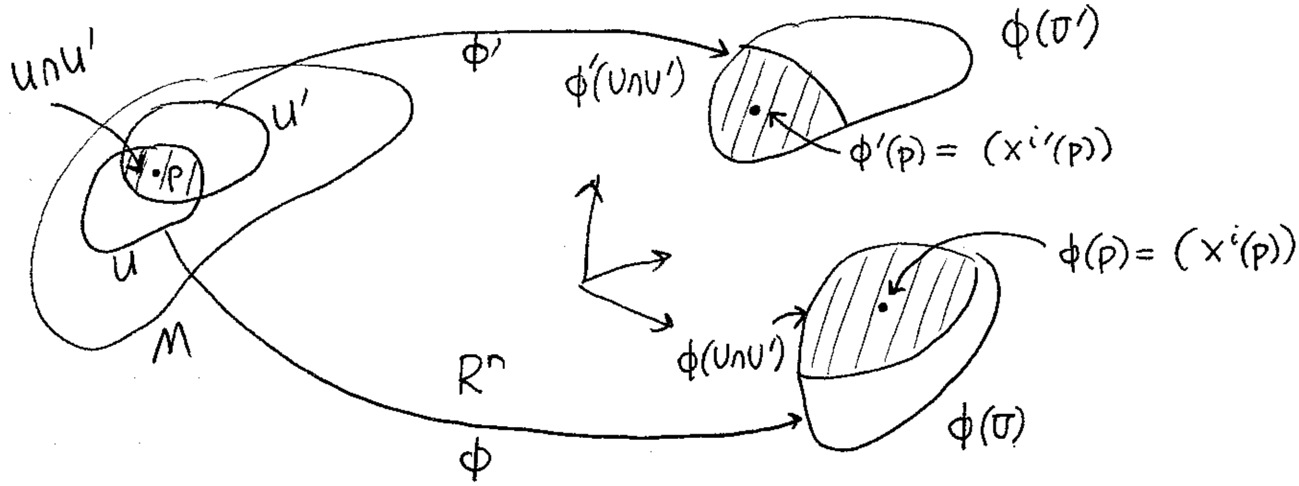
Each pair (U, ϕ) is called a local coordinate chart (or patch) and it enables us to assign local coordinates to each point p of U:

$$X^i(p) \equiv \underbrace{u^i(\underbrace{\phi(p)}_{\text{point in } R^n})}_{\substack{\text{standard cartesian}\\ \text{components of this}\\ \text{point}}}$$



The n functions $x^i = u^i \circ \phi$ on U are called local coordinates on U.

Suppose (U, ϕ) and (U, ϕ') are two overlapping coordinate patches on M (i.e. $U \cap U'$ is nontrivial).



Each point $p \in U \cap U'$ has two sets of coordinates:

$$x^i = u^i \circ \phi \qquad \text{and} \qquad x^{i\prime} = u^i \circ \phi'$$

$$x^i(p) = u^i(\phi(p)) \qquad x^{i\prime}(p) = u^i(\phi'(p))$$

We then get a map F from R^n into itself by expressing the primed coordinates of each point $p \in U \cap U'$ as functions of the unprimed coordinates

$$x^{i\prime}(p) = F^i(x^1(p), \ldots, x^n(p))$$

This is called a coordinate transformation on $U \cap U'$.

A <u>differentiable manifold</u> is essentially a space M together with a coordinate covering $\{(U_\alpha, \phi_\alpha)\}$ such that the coordinate transformation resulting from every pair of overlapping coordinate charts is differentiable (the functions F^i on R^n are differentiable.

(Note we can interchange (U, ϕ) and (U', ϕ') above so the inverse must also be differentiable.)