4

~~ERRATA~~ ERRATA:

p.12 $(S\wedge T)_{\alpha\dots\alpha} = \dots\delta^{\beta\dots}_{\alpha\dots}S_{[\alpha\dots}T_{\dots\alpha]}$ (obvious from index conventions)

should be β's

p.16 line 1 p-forms $\longrightarrow$ (n−p)-forms

p.28 $\pi(1,\dots,n) = (\pi^{-1}(1),\dots,\pi^{-1}(n))$

$$(1,2,3,4) \to (4,3,2,1) = (\pi^{-1}(1),\pi^{-1}(2),\pi^{-1}(3),\pi^{-1}(4))$$
$$= (\pi(4),\pi(3),\pi(2),\pi(1))$$

$4=\pi(1)$, $1=\pi^{-1}(4)$ thanks John Bruno.

SO THIS WEEK WE FINISH OFF LINEAR ALGEBRA (for now)

all we need to remember about "$\wedge$": $\omega^{\alpha_1\dots\alpha_p}\wedge\omega^{\alpha_{p+1}\dots\alpha_{p+q}} = \omega^{\alpha_1\dots\alpha_{p+q}}$.

Then: p-form $S = \frac{1}{p!}S_{\alpha_1\dots\alpha_p}\omega^{\alpha_1\dots\alpha_p} = S_{|\alpha_1\dots\alpha_p|}\omega^{\alpha_1\dots\alpha_p}$

q-form $T = \frac{1}{q!}T_{\alpha_1\dots\alpha_q}\omega^{\alpha_1\dots\alpha_q} = T_{|\alpha_1\dots\alpha_q|}\omega^{\alpha_1\dots\alpha_q}$

(p+q)-form $S\wedge T = \frac{1}{p!q!}S_{\alpha_1\dots\alpha_p}T_{\alpha_{p+1}\dots\alpha_{p+q}}\underbrace{\omega^{\alpha_1\dots\alpha_p}\wedge\omega^{\alpha_{p+1}\dots\alpha_{p+q}}}_{=\omega^{\alpha_1\dots\alpha_{p+q}}}$

$$= \frac{1}{p!q!}\underbrace{S_{[\alpha_1\dots\alpha_p}T_{\alpha_{p+1}\dots\alpha_{p+q}]}}_{\frac{1}{(p+q)!}\delta^{\beta_1\dots\beta_{p+q}}_{\alpha_1\dots\alpha_{p+q}}S_{\beta_1\dots\beta_p}T_{\beta_{p+1}\dots\beta_{p+q}}}\omega^{\alpha_1\dots\alpha_{p+q}}$$

$$= \frac{1}{p!q!}\delta^{\beta_1\dots\beta_{p+q}}_{|\alpha_1\dots\alpha_{p+q}|}S_{\beta_1\dots\beta_p}T_{\beta_{p+1}\dots\beta_{p+q}}\omega^{\alpha_1\dots\alpha_{p+q}}$$

$$= \underbrace{\delta^{\beta_1\dots\beta_{p+q}}_{|\alpha_1\dots\alpha_{p+q}|}S_{|\beta_1\dots\beta_p|}T_{|\beta_{p+1}\dots\beta_{p+q}|}}_{[S\wedge T]_{|\alpha_1\dots\alpha_{p+q}|}}\omega^{\alpha_1\dots\alpha_{p+q}}$$

$$\left(= \frac{(p+q)!}{p!q!}S_{[\alpha_1\dots\alpha_p}T_{\alpha_{p+1}\dots\alpha_{p+q}]}\right)$$

So let's do some examples.

Oops, not much room here.

$\underline{n=3}$: $\dim \Lambda^1(V^*) = 3 \quad \dim \Lambda^2(V^*) = 3 \quad \dim \Lambda^3(V^*) = 1 \qquad (V = E^3)$

$(p,q) = (1,1)$

$$B \wedge C = (B_i \omega^i) \wedge (C_j \omega^j) = B_{[i} C_{j]} \, \omega^{ij} = 2 B_{[i} C_{j]} \, \omega^{|ij|}$$

$$= \underset{"\epsilon_{1ij} B_i C_j"}{(B_2 C_3 - B_3 C_2)} \omega^{23} + \underset{"\epsilon_{2ij} B_i C_j"}{(B_3 C_1 - B_1 C_3)} \omega^{31} + \underset{"\epsilon_{3ij} B_i C_j"}{(B_1 C_2 - B_2 C_1)} \omega^{12}$$

$$= (B \times C)_1 \omega^{23} + (B \times C)_2 \omega^{31} + (B \times C)_3 \omega^{12}$$

$$\text{"}B \times C\text{"} = (*B \wedge C) \equiv \delta_{il} \epsilon^{ljk} B_j C_k \, \omega^i \qquad \| \quad (*B \wedge C)_i \, {*\omega^i}$$

$$*\omega^i = \tfrac{1}{2} \delta^{ij} \epsilon_{jkl} \, \omega^{kl}$$

$p,q = (1,2)$

$$A \wedge J = (A_i \omega^i) \wedge (\tfrac{1}{2} J_{jk} \omega^{jk}) = (A_1 \underbrace{J_{23}}_{(*J)^1} + A_2 \underbrace{J_{31}}_{(*J)^2} + A_3 \underbrace{J_{12}}_{(*J)^3}) \, \omega^{123}$$

$$= A_i (*J)^i \, \omega^{123} \qquad *J^i = \tfrac{1}{2} \epsilon^{ijk} J_{jk}$$

Now A, B, C 1-forms ($J = B \wedge C$):

$$A \wedge (B \wedge C) = [A_1 (B_2 C_3 - B_3 C_2) + \dots] \, \omega^{123} = \underbrace{A_i (*B \wedge C)^i}_{\text{"}A \cdot (B \times C)\text{"}} \, \omega^{123}$$

$$= \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} \omega^{123}$$

You see the star operator is forced on us if we want to represent our results in the simplest way.

$\underline{n=4}$: $\dim \Lambda^1(V^*) = 4, \quad \dim \Lambda^2(V^*) = 6, \quad \dim \Lambda^3(V^*) = 4, \quad \dim \Lambda^4(V^*) = 1$

$V = M^4$ (Minkowski spacetime) $\qquad \alpha = 0, 1, 2, 3$ and $i = 1, 2, 3$.

$(p,q) = (1,1)$

$$B \wedge C = (B_0 \omega^0 + B_i \omega^i) \wedge (C_0 \omega^0 + C_j \omega^j)$$

$$= (B_i C_0 - C_i C_0) \, \omega^{i0} + \underbrace{B_{[i} C_{j]} \omega^{ij}}_{\frac{1}{2} (B \times C)_{ij} \omega^{ij}}$$

$(p,q)=(1,2)$ $\quad A\wedge F = (A_0\omega^0 + A_\ell\omega^\ell)\wedge(\underbrace{F_{i0}}_{E_i}\omega^{i0} + \frac{1}{2}\underbrace{F_{ij}}_{\epsilon_{ijk}B^k}\omega^{ij})$

$$= \underbrace{A_0B^i(\tfrac{1}{2}\epsilon_{ijk}\,\omega^{0jk})}_{\substack{(-\frac{1}{2}\epsilon_{i0jk}\,\omega^{0jk}) \\ {}^*\omega_i}} + \underbrace{A_{[i}E_{j]}\,\omega^{ij0}}_{\omega^{0ij}} + (A_1F_{23}+A_2F_{31}+A_3F_{12})\omega^{123}$$

$$= [A_0B^i + \epsilon^{ijk}A_jE_k]\ \tfrac{1}{2}\epsilon_{imn}\omega^{0mn} + A_iB^i\omega^{123}.$$

$(p,q) = (2,2)$

$$F = E_i\omega^{i0} + \tfrac{1}{2}B^i\epsilon_{ijk}\omega^{jk}, \qquad G = D_i\omega^{i0} + \tfrac{1}{2}H^i\epsilon_{ijk}\omega^{jk}$$

$$F\wedge G = \tfrac{1}{2}E_i\,H^\ell\,\epsilon_{\ell mn}\underbrace{\omega^{i0mn}}_{\substack{-\omega^{0imn} \\ -\epsilon^{imn}\omega^{0123}}} + \tfrac{1}{2}D_i\,B^\ell\epsilon_{\ell mn}\underbrace{\omega^{mni0}}_{-\omega^{0imn}}$$

$$2\delta^i_\ell$$

$$= -(E_iH^i + D_iB^i)\,\omega^{0123}$$

$$G = {}^*F \equiv -B_i\omega^{i0} + \tfrac{1}{2}E^i\epsilon_{ijk}\omega^{jk}$$

$$F\wedge{}^*F = -(E_iE^i - B_iB^i)\,\omega^{0123} \qquad \left(= \tfrac{1}{2}F^{\alpha\beta}F_{\alpha\beta}\ \omega^{0123} \right.$$

$$\text{or}\quad F\wedge F = -2E_iB^i\ \omega^{0123} \qquad \left. = \tfrac{1}{2}F^{\alpha\beta}\,{}^*F_{\alpha\beta}\ \omega^{0123} \right)$$

$E^2 - B^2$ and $E\cdot B$ are the only 2 invariants of the EM field under Lorentz transformations.

These examples show the utility of introducing the star operation as done on pp 16-17.

The star operation has 2 parts.

One uses the epsilon-indicators to go from a p-form to an (n-p)-vector (or p-vector to an (n-p)-form)

$$T_{\alpha_1\cdots\alpha_p} \longrightarrow *T^{\alpha_{p+1}\cdots\alpha_n} = \frac{1}{p!} T_{\alpha_1\cdots\alpha_p} \epsilon^{\alpha_1\cdots\alpha_p\alpha_{p+1}\cdots\alpha_n} \qquad \left(\begin{matrix}\text{natural}\\ \text{dual}\end{matrix}\right)$$

This we can do without a metric. But one needs a metric to get back to a form, which can be obtained in this case by lowering all the indices. BUT we get into trouble if we raise and lower indices on the ϵ-indicators because they are "not the components of tensors."

So we have to use the η-tensors instead:

$$T_{\alpha_1\cdots\alpha_p} \longrightarrow *T^{\alpha_{p+1}\cdots\alpha_n} = \frac{1}{p!} T_{\alpha_1\cdots\alpha_p} \eta^{\alpha_1\cdots\alpha_p\alpha_{p+1}\cdots\alpha_n} \qquad \left(\begin{matrix}\text{metric}\\ \text{dual}\end{matrix}\right)$$

$$\boxed{*T_{\alpha_{p+1}\cdots\alpha_n} = \frac{1}{p!} T_{\alpha_1\cdots\alpha_p} \eta^{\alpha_1\cdots\alpha_p}{}_{\alpha_{p+1}\cdots\alpha_n}}$$

$*$ is a linear operation so

$$*T = *\left(\frac{1}{p!} T_{\alpha_1\cdots\alpha_p}\omega^{\alpha_1\cdots\alpha_p}\right) = \frac{1}{p!} T_{\alpha_1\cdots\alpha_p} *\omega^{\alpha_1\cdots\alpha_p}$$

but $*T = \frac{1}{(n-p)!} *T_{\alpha_{p+1}\cdots\alpha_n}\omega^{\alpha_{p+1}\cdots\alpha_n} = \frac{1}{p!} T_{\alpha_1\cdots\alpha_p} \frac{1}{(n-p)!} \eta^{\alpha_1\cdots\alpha_p}{}_{\alpha_{p+1}\cdots\alpha_n}\omega^{\alpha_{p+1}\cdots\alpha_n}$

$$\therefore \boxed{*\omega^{\alpha_1\cdots\alpha_p} = \frac{1}{(n-p)!} \eta^{\alpha_1\cdots\alpha_p}{}_{\alpha_{p+1}\cdots\alpha_n}\omega^{\alpha_{p+1}\cdots\alpha_n}} \boxed{\equiv \eta^{\alpha_1\cdots\alpha_p}}$$

$$\boxed{*T = \frac{1}{p!} T_{\alpha_1\cdots\alpha_p}\eta^{\alpha_1\cdots\alpha_p}}$$

$p=0$, T function $\quad *T = T*1 = \frac{1}{n!} T\eta_{\alpha_1\cdots\alpha_n}\omega^{\alpha_1\cdots\alpha_n}$

$= T\eta \quad$ ie $\quad \boxed{\begin{matrix}*1 = \eta \\ *\eta = 1\end{matrix}}$

$p=n$, $\quad *T = \frac{1}{n!} T_{\alpha_1\cdots\alpha_n}\eta^{\alpha_1\cdots\alpha_n}$

$$*\eta = \frac{1}{n!}\eta_{\alpha_1\cdots\alpha_n}\eta^{\alpha_1\cdots\alpha_n} = 1$$

Now what is $**T$?

$$*(*T)_{\beta_1\cdots\beta_p} = \frac{1}{(n-p)!}\underbrace{\left(*T_{\alpha_1\cdots\alpha_{n-p}}\right)}_{\frac{1}{p!}T_{\gamma_1\cdots\gamma_p}\eta^{\gamma_1\cdots\gamma_p}{}_{\alpha_1\cdots\alpha_{n-p}}}\eta^{\alpha_1\cdots\alpha_{n-p}}{}_{\beta_1\cdots\beta_p}$$

$$= \frac{1}{p!(n-p)!}\,T_{\gamma_1\cdots\gamma_p}\;\underbrace{\eta^{\gamma_1\cdots\gamma_p\alpha_1\cdots\alpha_{n-p}}\;\underbrace{\eta_{\alpha_1\cdots\alpha_{n-p}\beta_1\cdots\beta_p}}_{\eta_{\beta_1\cdots\beta_p\alpha_1\cdots\alpha_{n-p}}\,(-1)^{p(n-p)}}}_{(-1)^{\frac{n-s}{2}}\,|\det g|^{-1/2}\,|\det g|^{1/2}\;\underbrace{\delta^{\gamma_1\cdots\gamma_p\alpha_1\cdots\alpha_{n-p}}_{\beta_1\cdots\beta_p\alpha_1\cdots\alpha_{n-p}}}_{(n-p)!\;\delta^{\gamma_1\cdots\gamma_p}_{\beta_1\cdots\beta_p}}}$$

$$p!\,T_{\beta_1\cdots\beta_p}$$

$$= (-1)^{\frac{n-s}{2}}(-1)^{p(n-p)}\,T_{\beta_1\cdots\beta_p}$$

$$\boxed{** = (-1)^{\frac{n-s}{2}}(-1)^{p(n-p)}\,\mathrm{id}}$$

S, T p-forms:
$$\overbrace{\underset{p}{S}\wedge \underset{n-p}{*T}}^{n\text{-form}} = (?)\eta$$

$$S\wedge *T = \left(\frac{1}{p!}S_{\alpha_1\cdots\alpha_p}\omega^{\alpha_1\cdots\alpha_p}\right)\wedge\left(\frac{1}{p!}T^{\beta_1\cdots\beta_p}\,\overbrace{\eta_{\beta_1\cdots\beta_p}}^{*\omega_{\beta_1\cdots\beta_p}}\right)$$

$$= \frac{1}{(p!)^2}S_{\alpha_1\cdots\alpha_p}T^{\beta_1\cdots\beta_p}\;\underbrace{\omega^{\alpha_1\cdots\alpha_p}\wedge\eta_{\beta_1\cdots\beta_p}}_{\frac{1}{(n-p)!}\underbrace{\eta_{\beta_1\cdots\beta_p\beta_{p+1}\cdots\beta_n}}_{g^{1/2}\epsilon_{\beta_1\cdots\beta_p\beta_{p+1}\cdots\beta_n}}\underbrace{\omega^{\alpha_1\cdots\alpha_p\beta_{p+1}\cdots\beta_n}}_{\epsilon^{\alpha_1\cdots\alpha_p\beta_{p+1}\cdots\beta_n}\,\omega^{1\cdots n}}}$$

$$\underbrace{\epsilon_{\beta_1\cdots\beta_p\beta_{p+1}\cdots\beta_n}\epsilon^{\alpha_1\cdots\alpha_p\beta_{p+1}\cdots\beta_n}}_{(n-p)!\;\delta^{\alpha_1\cdots\alpha_p}_{\beta_1\cdots\beta_p}},\qquad g^{1/2}\,\omega^{1\cdots n} = \eta$$

$$= \delta^{\alpha_1\cdots\alpha_p}_{\beta_1\cdots\beta_p}\,\eta$$

$$= \underbrace{\left(\frac{1}{p!}S_{\alpha_1\cdots\alpha_p}T^{\alpha_1\cdots\alpha_p}\right)}_{\equiv\langle S,T\rangle}\eta$$

$$\boxed{S\wedge *T = \langle S,T\rangle\,\eta}$$

$n=3$, $V=E$, $g_{ij}=\delta_{ij}$

$p=1$ $\quad *\omega^i = \frac{1}{2}\epsilon_{ijk}\,\omega^{jk}$

$*\omega^1 = \omega^{23}$

$*\omega^2 = \omega^{31}$

$*\omega^3 = \omega^{12}$

$p=2$ $\quad *\omega^{ij} = \epsilon^{ijk}\omega_k$

$*\omega^{23} = \omega^1$

$*\omega^{31} = \omega^2$

$*\omega^{12} = \omega^3$

$** = 1$ $\quad \binom{n=3}{s=n}$

vectors: $\overset{*}{[}(B^i e_i)\wedge(C^j e_j)] = \epsilon^{ijk}B_j C_k e_i$

$*(B\wedge C) = B\times C$

$n=4$, $V=M^4$, $g_{ij}=\eta_{ij}$ $(-+++)$ $\quad g_{00}=-1$, $g_{ij}=\delta_{ij}$

$\eta_{0123}=1 \quad \eta^{0123}=-1 \quad \eta_{0ijk}=\epsilon_{ijk} \quad \eta^{0ijk}=-\epsilon^{ijk}$

$p=1$ $\quad *\omega^\alpha = \eta^{\alpha\beta\gamma\delta}\,\omega_{|\beta\gamma\delta|}$

$*\omega^0 = \eta^{0123}\,\omega_{123} = -\omega^{123}$

$*\omega^i = \underbrace{\eta^{i0jk}}_{-\eta^{0ijk}}\,\underbrace{\omega_{|0jk|}}_{-\omega^0{}_{jk}} = -\frac{1}{2}\epsilon^{ijk}\omega^0{}_{jk}$

$*\omega^1 = -\omega^{023}$

$*\omega^2 = -\omega^{031}$

$*\omega^3 = -\omega^{012}$

$p=2$ $\quad *\omega^{\alpha\beta} = \eta^{\alpha\beta\gamma\delta}\,\omega_{|\gamma\delta|}$

$*\omega^{i0} = \underbrace{\eta^{i0jk}}_{\epsilon^{ijk}}\,\omega_{|jk|}$

$*\omega^{10} = \omega^{23}$

$*\omega^{20} = \omega^{31}$

$*\omega^{30} = \omega^{12}$

$p=3$ $\quad *\omega^{\alpha\beta\gamma} = \eta^{\alpha\beta\gamma\delta}\,\omega_\delta$

$*\omega^{123} = \underbrace{\eta^{1230}}_{\underbrace{-\eta^{0123}}_{1}}\,\underbrace{\omega_0}_{-\omega^0} = -\omega^0$

$*\omega^{0ij} = \eta^{0ijk}\omega_k = -\epsilon^{ijk}\omega_k$

$*\omega^{023} = -\omega^1$

$*\omega^{031} = -\omega^2$

$*\omega^{012} = -\omega^3$

Final remarks : "Densities" = "nontensor components of tensors"

Suppose $T = \frac{1}{n!} T_{\alpha_1 \dots \alpha_n} \omega^{\alpha_1 \dots \alpha_n} = T_{1\dots n}\, \omega^{1 \dots n}$ is an n-form.

Under a change of basis $e_{\alpha'} = e_\beta A^{-1\beta}{}_\alpha$, the component of T in the induced basis of $\Lambda^n(V^*)$ transforms as follows:

$$\tau = T_{1' \dots n'} = (\det A^{-1})\, T_{1\dots n} \qquad \text{" scalar density of weight 1 "}$$

On the other hand $^*T$ is a "scalar", independent of basis

$$^*T = T_{1\dots n}\, {}^*\omega^{1\dots n} = T_{1\dots n}\, \eta^{1\dots n} = T_{1\dots n} (-1)^{\frac{n-S}{2}} g^{-1/2} \underbrace{\epsilon^{1\dots n}}_{1} \operatorname{sgn}(\text{orientation of } e)$$

or $$\tau = (-1)^{\frac{n-S}{2}} \operatorname{sgn}(\text{orientation})\, g^{1/2} (\underbrace{^*T}_{\text{scalar}})$$

this provides the sign to make τ a weight-1 scalar density

"oriented density of weight 1"

$$g_{..}' = A^{-1T} g_{..} A^{-1}$$

$$\det g_{..}' = (\det A^{-1})^2 \det g_{..}$$

$$g'^{1/2} \equiv |\det g_{..}'|^{1/2} = |\det A^{-1}|\, g^{1/2}$$

doesnt change sign with $\det A^{-1}$

Can make other densities:

$$\tau_W \equiv (g^{1/2})^W\, {}^*T \qquad \text{"oriented scalar density of weight } W\text{"}$$

$$\tau_W' = |\det A^{-1}|^W \tau_W$$

This can also be done with other rank tensors.

For example the "natural dual" of a 2-form on E^3 produces a "vector density"

$$B^i = \tfrac{1}{2} \epsilon^{ijk} F_{jk}$$

since not a tensor