②

Summary of Summary (Part 1, pp 1–9)

summation convention : repeated upper and lower indices are summed over all possible values

V real vector space, basis $\{e_\alpha\}$, vector $X = X^\alpha e_\alpha \in V$ $\qquad X^\alpha = \omega^\alpha(X)$

V^* dual vector space of real-valued linear functions on V

dual basis $\{\omega^\alpha\}$ defined by $\omega^\alpha(e_\beta) = \delta^\alpha{}_\beta$ $\qquad$ components wrt basis $\{e_\alpha\}$

"1-form" or covector $\quad \sigma = \sigma_\alpha \omega^\alpha \qquad \sigma_\alpha = \sigma(e_\alpha)$

$(V^*)^* \sim V$ we can identify real-valued linear maps on the dual space with evaluation on vectors:

$$\underbrace{X}_{\text{linear form on } V^*}(\sigma) \equiv \sigma(X) \qquad (= \sigma_\alpha X^\alpha \text{ in component form})$$

$\binom{p}{q}$-tensors over V are just multilinear real-valued functions with p covector arguments and q vector arguments

$$T = T^{\alpha_1\cdots\alpha_p}{}_{\beta_1\cdots\beta_q}\, e_{\alpha_1}\otimes\cdots\otimes e_{\alpha_p}\otimes\omega^{\beta_1}\otimes\cdots\otimes\omega^{\beta_p} \in T^{p,q}(V)$$

value of function T :

$$\underbrace{T(\sigma^1,\ldots,\sigma^p, X_1,\ldots,X_q)}_{\substack{\text{not component indices}\\ \text{just labels of a set of}\\ \text{covectors and vectors}}} = T^{\alpha_1\cdots\alpha_p}{}_{\beta_1\cdots\beta_q}\,\sigma^1(e_{\alpha_1})\cdots\sigma^p(e_{\alpha_p})\,\omega^{\beta_1}(X_1)\cdots\omega^{\beta_q}(X_q)$$

$$= T^{\alpha_1\cdots\alpha_p}{}_{\beta_1\cdots\beta_q}\,\sigma^1{}_{\alpha_1}\cdots\sigma^p{}_{\alpha_p}\,X^{\beta_1}{}_1\cdots X^{\beta_q}{}_q \in R$$

$\otimes$ = "tensor product" is just the natural multiplication of linear functions (covectors and vectors interpreted as linear functions on V and V^* respectively).

In component form:

$$T, S \to (T\otimes S)^{\alpha_1\cdots\alpha_{p+r}}{}_{\beta_1\cdots\beta_{q+s}} = T^{\alpha_1\cdots\alpha_p}{}_{\beta_1\cdots\beta_q}\, S^{\alpha_{p+1}\cdots\alpha_{p+r}}{}_{\beta_{q+1}\cdots\beta_{q+s}}$$

value of T on basis vectors and covectors gives its components

$$T(\omega^{\alpha_1},\ldots,\omega^{\alpha_p}, e_{\beta_1},\ldots,e_{\beta_q}) = T^{\alpha_1\cdots\alpha_p}{}_{\underbrace{\beta_1\cdots\beta_q}}$$

these indices associated with the covector arguments are called contravariant

these indices associated with the vector arguments are called covariant

Contraction is just the natural evaluation of a single vector argument of a tensor on a vector (or covector argument on a covector) or of that argument on one of the vector factors associated with a covector argument (etc.)

$$T = T^{\alpha_1\cdots\alpha_p}{}_{\beta_1\cdots\beta_q}\, e_{\alpha_1}\otimes\cdots\otimes e_{\alpha_p}\otimes\omega^{\beta_1}\otimes\cdots\otimes\omega^{\beta_q}$$

$$\rightarrow T^{\alpha_1\cdots\alpha_{p-1}\alpha_p}{}_{\beta_1\cdots\beta_{q-1}\beta_q}\, e_{\alpha_1}\otimes\cdots\otimes e_{\alpha_{p-1}}\otimes\omega^{\beta_1}\otimes\cdots\otimes\omega^{\beta_{q-1}}\,\underbrace{\omega^{\beta_q}(e_{\alpha_p})}_{=\delta^{\beta_q}{}_{\alpha_p}}$$

$$= T^{\alpha_1\cdots\alpha_{p-1}\alpha}{}_{\beta_1\cdots\beta_{q-1}\alpha}\, e_{\alpha_1}\otimes\cdots\otimes e_{\alpha_{p-1}}\otimes\omega^{\beta_1}\otimes\cdots\otimes\omega^{\beta_{q-1}}$$

$$\text{or} \rightarrow T \lrcorner X = T^{\alpha_1\cdots\alpha_p}{}_{\beta_1\cdots\beta_q}\, e_{\alpha_1}\otimes\cdots\otimes e_{\alpha_p}\otimes\omega^{\beta_1}\otimes\cdots\otimes\omega^{\beta_{q-1}}\,\omega^{\beta_q}(X)$$

$$= T^{\alpha_1\cdots\alpha_p}{}_{\beta_1\cdots\beta_{q-1}\beta}X^{\beta}\, e_{\alpha_1}\otimes\cdots\otimes e_{\alpha_p}\otimes\omega^{\beta_1}\otimes\cdots\otimes\omega^{\beta_{q-1}}$$

$\binom{1}{1}$-tensors $\sim$ linear maps from V into V

$$B = B^{\alpha}{}_{\beta}\, e_\alpha\otimes\omega^\beta \qquad X\in V \rightarrow B\lrcorner X = B^{\alpha}{}_{\beta}X^\beta e_\alpha$$

$\binom{0}{2}$-tensors $\sim$ linear maps from V into V^*

$$\Omega = \Omega_{\alpha\beta}\,\omega^\alpha\otimes\omega^\beta \qquad X\in V \rightarrow \Omega\lrcorner X = \Omega_{\alpha\beta}X^\beta\omega^\alpha \in V^*$$

$\binom{2}{0}$- tensors $\sim$ linear maps from $V^*\rightarrow V$

$$\Lambda = \Lambda^{\alpha\beta}e_\alpha\otimes e_\beta \qquad \sigma\in V^* \rightarrow \Lambda\lrcorner\sigma = \Lambda^{\alpha\beta}\sigma_\beta e_\alpha \in V$$

EXAMPLE $\Omega_{\alpha\beta} = -\Omega_{\beta\alpha}$, $\det(\Omega_{\alpha\beta})\neq 0$ (antisymmetric nondegenerate matrix $\rightarrow$ even dimension)

Let $(\Lambda^{\alpha\beta}) = (\Omega_{\alpha\beta})^{-1} \equiv (\Omega^{\alpha\beta})$. Then

$$X_\alpha \equiv \Omega_{\alpha\beta}X^\beta \qquad \sigma^\alpha \equiv \Omega^{\alpha\beta}\sigma_\beta$$

lowering and raising indices with "antisymmetric metric" or "symplectic form". This is important in Hamiltonian dynamics.

EXAMPLE. $\Omega_{\alpha\beta} = g_{\alpha\beta} = g_{\beta\alpha}$, $\det(g_{\alpha\beta})\neq 0$ (symmetric nondegenerate matrix)

Let $(\Lambda^{\alpha\beta}) = (g^{\alpha\beta}) = (g_{\alpha\beta})^{-1}$. Then

$$X_\alpha \equiv g_{\alpha\beta}X^\beta \qquad \sigma^\alpha \equiv g^{\alpha\beta}\sigma_\beta$$

lowering and raising indices with a symmetric matrix. This is important period.

Such a $\binom{0}{2}$ tensor g gives an inner product on V and on all the spaces of $\binom{p}{q}$-tensors :

$$X \cdot Y = g(X, Y) = g_{\alpha\beta} X^{\alpha} Y^{\beta}$$

$$T \cdot S = T^{\alpha_1 \dots \alpha_p}{}_{\beta_1 \dots \beta_q}\, g_{\alpha_1 \gamma_1} \cdots g_{\alpha_p \gamma_p}\, g^{\beta_1 \delta_1} \cdots g^{\beta_q \delta_q}\, S^{\gamma_1 \dots \gamma_p}{}_{\delta_1 \dots \delta_q}$$

$$= T^{\alpha_1 \dots \alpha_p}{}_{\beta_1 \dots \beta_q}\, S_{\alpha_1 \dots \alpha_p}{}^{\beta_1 \dots \beta_q} = T_{\gamma_1 \dots \gamma_p}{}^{\delta_1 \dots \delta_q}\, S^{\gamma_1 \dots \gamma_p}{}_{\delta_1 \dots \delta_q}$$

Now that we're raising and lowering indices indiscriminately the ordering of individual covariant and contravariant indices (or arguments) becomes important.

$$T = T^{\alpha}{}_{\beta\gamma}\, e_{\alpha} \otimes \omega^{\beta} \otimes \omega^{\gamma}$$

$$\rightarrow \quad T^{\alpha}{}_{\beta}{}^{\gamma}\, e_{\alpha} \otimes \omega^{\beta} \otimes e_{\gamma}$$

$$\rightarrow \quad T_{\alpha\beta}{}^{\gamma}\, \omega^{\alpha} \otimes \omega^{\beta} \otimes e_{\gamma} \qquad \text{etc.}$$

CHANGE OF BASIS

$$e_{\alpha'} = e_{\beta} A^{-1\beta}{}_{\alpha} \qquad \omega^{\alpha'} = A^{\alpha}{}_{\beta}\, \omega^{\beta}$$

all covariant indices transform by A^{-1}

all contravariant indices transform by A

$$T^{\alpha'_1 \dots \alpha'_p}{}_{\beta'_1 \dots \beta'_q} = A^{\alpha_1}{}_{\gamma_1} \cdots A^{\alpha_p}{}_{\gamma_p}\, A^{-1\delta_1}{}_{\beta_1} \cdots A^{-1\delta_q}{}_{\beta_q}\, T^{\gamma_1 \dots \gamma_p}{}_{\delta_1 \dots \delta_q}$$

$$\left(T(\omega^{\alpha'_1}, \dots, \omega^{\alpha'_p}, e_{\beta'_1}, \dots, e_{\beta'_q}) = T(A^{\alpha_1}{}_{\gamma_1} \omega^{\gamma_1}, \dots, e_{\delta_q} A^{-1\delta_q}{}_{\beta_q}) \right)$$

EXAMPLE

$$g_{\alpha'\beta'} = A^{-1\gamma}{}_{\alpha'}\, g_{\gamma\delta}\, A^{-1\delta}{}_{\beta'}$$

$$(g_{\alpha'\beta'}) = (A^{-1})^{T} (g_{\alpha\beta})\, A^{-1} \qquad \text{matrix form.}$$

ORTHONORMAL BASES

For a given metric g, one can always find bases such that

$$e_\alpha \cdot e_\beta = g(e_\alpha, e_\beta) = \eta_{\alpha\beta}$$

$$\eta = (\eta_{\alpha\beta}) = \mathrm{diag}\,(\underbrace{1,\dots,1}_{p}, \underbrace{-1,\dots,-1}_{q})$$

$s = p - q =$ signature
only $|s|$ important

linear transformations which map ON bases onto ON bases are called pseudoorthogonal transformations

$$\eta = (O^{-1})^T \eta\, O^{-1} \qquad \text{or} \qquad \eta = O^T \eta\, O$$

$$\left.\begin{array}{l} O \in O(p,q) \\ \det O = 1 \to \ \in SO(p,q) \end{array}\right\} \text{pseudo- and special pseudoorthogonal groups}$$

EX.

$E^n = (\mathbb{R}^n, \delta)$ Euclidean space

δ : usual metric $\quad \delta_{ab}\omega^a \otimes \omega^b$ in natural basis $\{e_\alpha\}$

$O(n,0) = O(n,\mathbb{R})$ orthogonal groups
$S(n,0) = SO(n,\mathbb{R})$ special orthogonal groups

$M^n = (\mathbb{R}^n, \eta)$ Minkowski spacetime (n-dim)

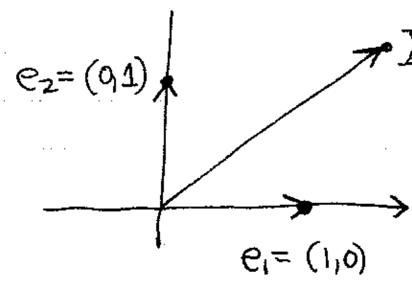
η : signature $\pm[(n-1) - 1] = \pm(n-2)$

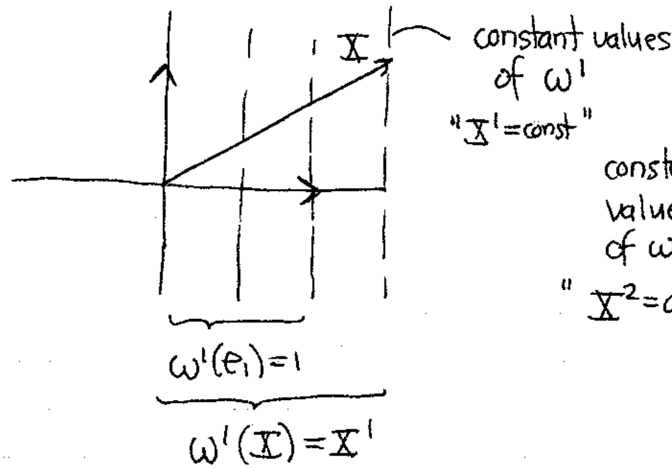
$O(n-1, 1)$ n-dim Lorentz group

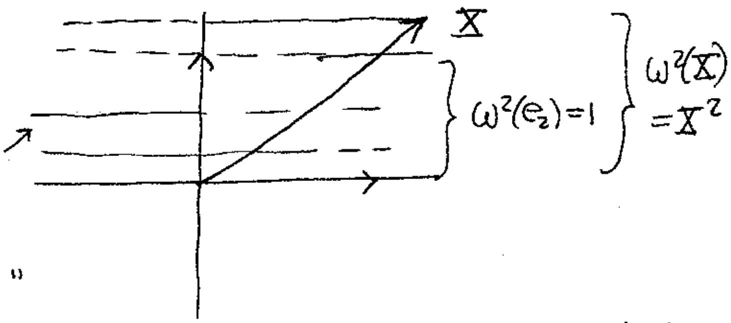
$SO(n-1, 1)$ subgroup

EXAMPLE R^2 $\{e_1, e_2\}$ standard basis

$$X = (X^1, X^2) = X^1 e_1 + X^2 e_2 \qquad \begin{cases} e_1 = (1,0) \\ e_2 = (0,1) \end{cases}$$



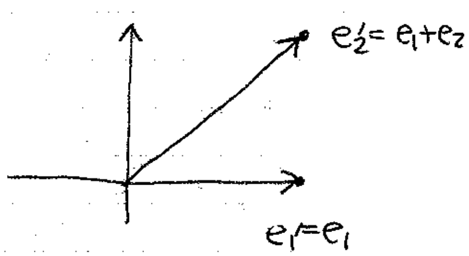




$\omega^1(X) = X^1 = 0$ means X is parallel to constant value lines of ω^1

Now change the basis :

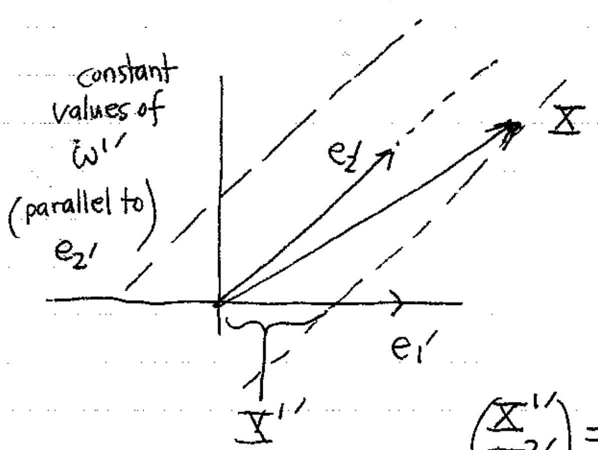
$$e_1' = (1,0) = e_1 \qquad e_1 = e_1'$$
$$e_2' = (1,1) = e_1 + e_2 \qquad e_2 = e_2' - e_1'$$

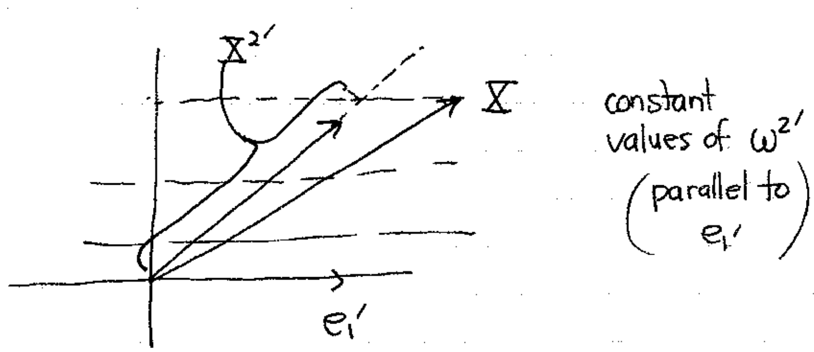


$$(e_1, e_2) = (e_1', e_2') \underbrace{\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}}_{A} \leftrightarrow \quad e_a = e_{b'} A^b{}_a$$

$$\begin{pmatrix} \omega^{1'} \\ \omega^{2'} \end{pmatrix} = \overbrace{\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}}^{A} \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix} \quad \leftrightarrow \quad \omega^{a'} = A^a{}_b \, \omega^b$$

$$\omega^{1'} = \omega^1 - \omega^2 \qquad \omega^{2'} = \omega^2$$





$$\begin{pmatrix} X^{1'} \\ X^{2'} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X^1 \\ X^2 \end{pmatrix} = \begin{pmatrix} X^1 - X^2 \\ X^2 \end{pmatrix}$$

OBLIQUE CARTESIAN COORDINATES

$$(e_{1'}, e_{2'}) = (e_1, e_2)\underbrace{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}_{A^{-1}} \qquad e_{a'} = e_b A^{-1b}{}_a$$

$$g = \delta_{ab}\,\omega^a \otimes \omega^b$$

$$(g_{a'b'}) = (A^{-1})^T \, 1 \, (A^{-1}) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

or equivalently

$$e_{1'} \cdot e_{1'} = (1,0)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 = g_{1'1'}$$

$$e_{1'} \cdot e_{2'} = (1,0)\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1 = g_{1'2'}$$

$$e_{2'} \cdot e_{1'} = (1,1)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 = g_{2'1'}$$

$$e_{2'} \cdot e_{2'} = (1,1)\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2 = g_{2'2'}$$

$$g = \omega^{1'} \otimes \omega^{1'} + \omega^{1'} \otimes \omega^{2'} + \omega^{2'} \otimes \omega^{1'} + 2\,\omega^{2'} \otimes \omega^{2'}$$

$$g(X,X) = \underbrace{(X^1)^2 + (X^2)^2}_{= (X^1, X^2)\begin{pmatrix} X^1 \\ X^2 \end{pmatrix}} = \underbrace{(X^{1'})^2 + 2X^{1'}X^{2'} + 2(X^{2'})^2}_{= (X^{1'}, X^{2'})\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} X^{1'} \\ X^{2'} \end{pmatrix}}$$

$$X^{a'} \, g_{a'b'} \, X^{b'}$$

normal matrix multiplication:

matrix times column vector

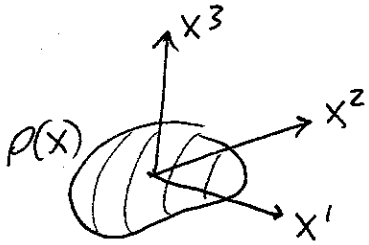
row vector times matrix

$a' \leftrightarrow a'$ transpose

$b' \leftrightarrow b'$ usual

EXAMPLE

$E^3 = (\mathbb{R}^3, \delta) \qquad g = \delta_{\alpha\beta}\,\omega^\alpha \otimes \omega^\beta \qquad X \cdot Y = \delta_{\alpha\beta} X^\alpha Y^\beta$



compact charge distribution

let x^α be the ~~fun~~ 1-forms ω^α considered as ordinary functions on E^3, ie cartesian coordinates.

Define $\quad T^{\alpha_1 \ldots \alpha_r} = \int \rho\, x^{\alpha_1} \ldots x^{\alpha_r}\, d^3x = T^{(\alpha_1 \ldots \alpha_r)}$ (totally symmetric)

potential $\quad \phi(x) = \int \frac{\rho(x')\, d^3x'}{|x - x'|} = $ Taylor expand about $x - x' = 0$ $= \ldots$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} T^{\alpha_1 \ldots \alpha_n} \partial_{\alpha_1} \cdots \partial_{\alpha_n} \left(\frac{1}{r}\right)$$

$r \equiv \sqrt{\delta_{\alpha\beta} x^\alpha x^\beta}$

$\delta^{\alpha\beta} \partial_\alpha \partial_\beta \left(\frac{1}{r}\right) = 0 \quad (r \neq 0)$

can actually subtract off trace.

multipole moments.
∞ number of symmetric (trace free) contravariant tensors on E^3.

$x^\alpha \to x^{\alpha'} = A^\alpha{}_\beta x^\beta$ $\qquad$ change of cartesian coordinates

$T^{\alpha'_1 \ldots \alpha'_r} = A^{\alpha_1}{}_{\beta_1} \cdots A^{\alpha_r}{}_{\beta_r} T^{\beta_1 \ldots \beta_r}$.

EXAMPLE FOR PAGE 15

Let $\{e_\alpha\}_{\alpha=0,1,2,3}$ be the relabled standard basis on R^4 with the Lorentz inner product of components

$$e_\alpha \cdot e_\beta = \eta_{\alpha\beta}, \quad \eta = (\eta_{\alpha\beta}) = \mathrm{diag}(-1, 1, 1, 1)$$

i.e. just $M^{3,1}$ with a permuted choice of basis vectors.

If $\{\omega^\alpha\}$ is the dual basis

$$\omega^{0123} = \omega^0 \wedge \omega^1 \wedge \omega^2 \wedge \omega^3$$

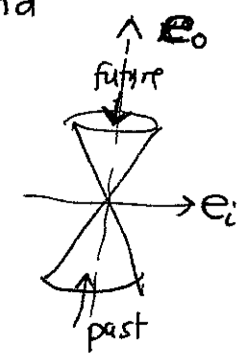
defines the natural orientation of Minkowski spacetime.

Lorentz transformations with matrices

$$T = \mathrm{diag}(-1, 1, 1, 1) \quad \text{and} \quad P = \mathrm{diag}(1, -1, -1, -1)$$

$$\det T = -1 = \det P$$

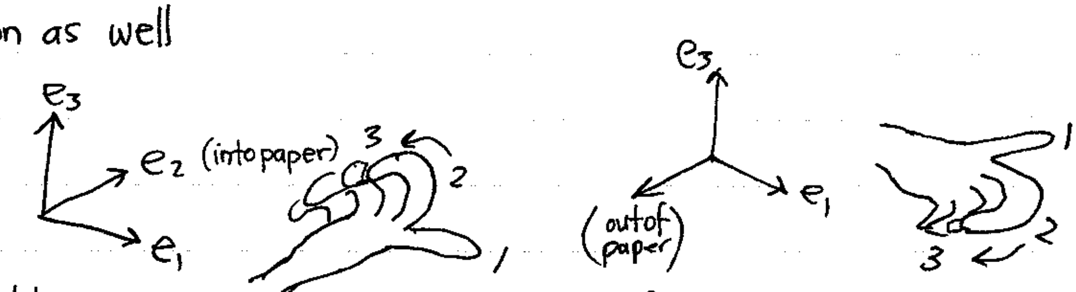
change the orientation. These represent time reversal and a spatial reflection about the origin (parity).



Minkowski spacetime also has a natural time orientation, namely a choice of one of the 2 disconnected components of the interior of the light cone as the future of the origin.

EXAMPLE.

On E^3 one has "lefthanded" and "righthanded" ~~frames~~ bases. The righthanded bases are those which are rotated from the natural basis, the left handed ones undergo a total reflection as well



righthanded left handed.

EXAMPLE FOR PAGE 18

□ Let $W = \ker n = \{X \in V \mid n_\alpha X^\alpha = n(X) = 0\}$

where $n = n_\alpha \omega^\alpha \in V^*$. is a covector.

No metric is needed to define W.

Now let $n^\# = g^{\alpha\beta} n_\beta e_\alpha = n^\alpha e_\alpha$ be the vector obtained by raising the index of n.

Then

$$W = \{X \in V \mid g(X, n^\#) = g_{\alpha\beta} X^\alpha n^\beta = 0\}$$

i.e. W consists of all vectors orthogonal to the vector $n^\#$, called a normal vector to W.

□ Let $W = \{X \in V \mid \omega^a(X) = 0 \ , \ a = p+1, \ldots, n\}$ (no metric)

$= \{X \in V \mid g(\omega^{a\#}, X) = 0 \ , \ a = p+1, \ldots, n\}$ (metric)

The $(n-p)$ vectors $\omega^{a\#}$ are obtained from ω^a by raising their indices (by index we mean component index) are orthogonal to W and called normals to W.

Each p-dimensional linear subspace is therefore specifiable by giving $(n-p)$ normal vectors.

SUMMARY OF SUMMARY (PART 2: pp 10-18)

There are only 3 important ideas introduced for antisymmetric tensors

$[\ ]$ = antisymmetrization

$\wedge = [\otimes]$ = antisymmetrized tensor product

$*$: defined so that $*T$ is an $(n-p)$-form obtained from a p-form T such that $T \wedge {*T} = \frac{1}{p!}(T \cdot T)\, \eta$

unit n-form

The rest are just details.