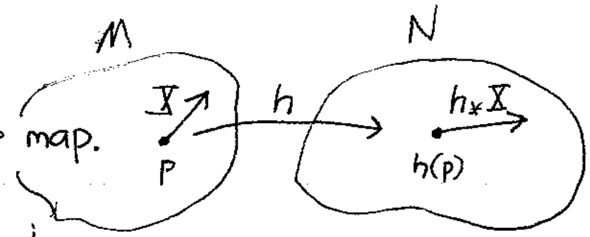
12. Lie Derivatives and Lie Dragging

Suppose $h: M \to N$ is a differentiable map.



local coord's: $\{x^i\}$, $\{y^\alpha\}$

$y^\alpha \circ h$ = differentiable functions of x^i on the appropriate domain

We can define the push forward of tangent vectors from TM_p to $TN_{h(p)}$

If $\mathbb{X} \in TM_p$, then $h_*\mathbb{X} \in TN_{h(p)}$ is defined by its action on functions on N:

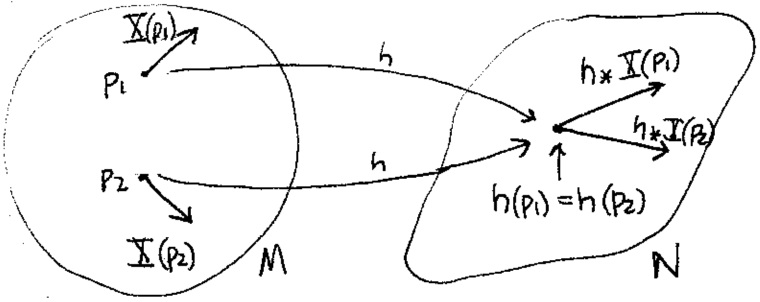
$$\underset{\underset{F(N)}{\cap}}{(h_*\mathbb{X}) f} \equiv \mathbb{X}(\underset{\underset{F(M)}{\cap}}{f \circ h})$$

← "smooth functions on N and M"

Letting $f = y^\alpha$ gives the component version : $(h_*\mathbb{X})^\alpha = \dfrac{\partial(y^\alpha \circ h)}{\partial x^i}(p)\, \mathbb{X}^i$

or $h_* \dfrac{\partial}{\partial x^i}\Big|_p = \dfrac{\partial(y^\alpha \circ h)}{\partial x^i}(p) \dfrac{\partial}{\partial y^\alpha}\Big|_{h(p)}$ (JUST THE CHAIN RULE)

IF h is 1-1 (maps distinct points of M to distinct points of N), then we can push forward a vector field on M to a vector field on the image $h(M)$ of the map: $(h_*\mathbb{X})(h(p)) \equiv h_*(\mathbb{X}(p))$



← If h is not 1-1 this can happen, i.e. one doesn't have a unique tangent vector at $h(p_1) = h(p_2)$

exercise: show that Lie brackets of vector fields are preserved under pushing forward: $h_*[\mathbb{X}, \mathbb{Y}] = [h_*\mathbb{X}, h_*\mathbb{Y}]$.

Clearly this can be extended to all contravariant tangent tensors and tensor fields. ($h_*\, \mathbb{X} \otimes \mathbb{Y} = h_*\mathbb{X} \otimes h_*\mathbb{Y}$, etc.)

On the other hand covariant tensor fields go the opposite direction, one can always pull back a covariant tensor field on N to one on M

$$h^*(T_{\alpha\ldots}\, dy^\alpha \otimes \ldots) = h^* T_{\alpha\ldots}\; h^* dy^\alpha \otimes \ldots = T_{\alpha\ldots} \circ h\; \underbrace{d(y^\alpha \circ h)}_{\frac{\partial(y^\alpha \circ h)}{\partial x^i} dx^i} \otimes \cdots$$

(JUST PLUG IN THE TRANSFORMATION "$y^\alpha = y^\alpha(x)$")

exercise: By its definition $h^* f = f \circ h$, $h^* df = d(h^* f)$ the exterior derivative commutes with h^* for functions and 1-forms. Show that this is true in general using the above coordinate formula:

$$d \circ h^* = h^* \circ d .$$

pushing forward and pulling back are related by

$$\underbrace{(h^*\beta)(X,\ldots,Y)}\Big|_p = \beta(h_*X,\ldots,h_*Y)\Big|_{h(p)}$$

pf. $\beta_{\alpha\ldots\gamma}\circ h \; \underbrace{d(y^\alpha\circ h)(X)} \cdots d(y^\gamma\circ h)(Y)$

$df(X) \equiv Xf \qquad X(y^\alpha\circ h) = \underbrace{(h_*X)y^\alpha}_{(h_*X)^\alpha} \qquad (h_*Y)^\beta$

$$\beta_{\alpha\ldots\gamma}\circ h\,(h_*X^\alpha)\cdots(h_*Y)^\gamma = \beta(h_*X,\ldots,h_*Y)$$

This is rather trivial in component language, for example:

$$\text{"}\quad (h^*\sigma)(X) = \underbrace{\left(\beta_\alpha \frac{\partial y^\alpha}{\partial x^i}\right)}_{(h^*\beta)_i} X^i = \beta_\alpha \underbrace{\left(\frac{\partial y^\alpha}{\partial x^i} X^i\right)}_{(h_*X)^\alpha} = \beta(h_*X) \quad\text{"}$$

So if h is 1-1 we've got all the contravariant tensor fields going in the same direction as h and all of the functions and covariant tensor fields are coming back. When h is a diffeomorphism this traffic flow problem can be solved. The inverse $h^{-1}: N \to M$ goes in the opposite direction so we can push forward contravariant tensor fields with h and pull back covariant tensor fields and functions with h^{-1} so everybody is going the same direction, forward from M to N. We can then extend this to mixed tensor fields (and densities).

It is called <u>dragging along by h</u> and is exactly what one does in transforming fields under point transformations. Since h is a diffeomorphism we can use the same indices.

$$\text{Let}\begin{cases} h^i = h^* y^i = y^i\circ h \in F(M) \\ h^{-1\,i} = h^{-1*}x^i = x^i\circ h^{-1} \in F(N) \end{cases}$$

If $T^{i\ldots}{}_{j\ldots}$ are the components of a weight W tensordensity field on M, then the components of the <u>dragged along field</u> hT on N are

$$(hT)^{i\ldots}{}_{j\ldots}(h(p)) = \left(\det\left(\frac{\partial h^{-1\,k}}{\partial y^\ell}(h(p))\right)\right)^W \frac{\partial h^i}{\partial x^m}(p)\cdots\frac{\partial h^{-1\,n}}{\partial y^j}(h(p))\cdots T^{m\ldots}{}_{n\ldots}(p)$$

This looks something like a coordinate transformation.

In fact $\{h^{-1i}\}$ are local coordinates on N (why?) called the <u>dragged along coordinates</u>. Since $\left(\frac{\partial h^{-1i}}{\partial y^j}(h(p))\right)$ and $\left(\frac{\partial h^i}{\partial x^j}(p)\right)$ are inverse matrices and $\left(\frac{\partial h^{-1i}}{\partial y^j}(h(p))\right)$ and $\left(\frac{\partial y^i}{\partial h^{-1j}}(h(p))\right)$ are also inverse matrices, then $\frac{\partial y^i}{\partial h^{-1j}}(h(p))$ $\frac{\partial h^i}{\partial x^j}(p)$ and we can write

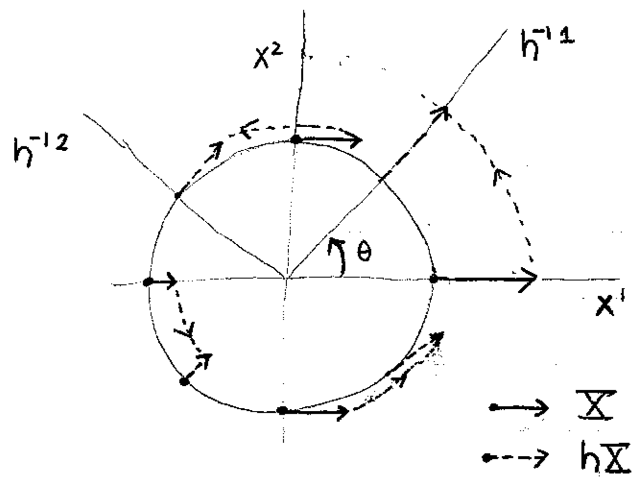
$$(hT)^{i\dots}{}_{j\dots}(h(p)) = \left[\rho^{p,q}_{w}\left(\left(\frac{\partial h^{-1k}}{\partial y^\ell}(h(p))\right)^{-1}\right) T(p)\right]^{i\dots}{}_{j\dots}$$

This is just the coordinate transformation from the $\{h^{-1i}\}$ coordinates to the $\{y^i\}$ coordinates and says that $T^{i\dots}{}_{j\dots}(p)$ are the $\{h^{-1i}\}$ components of hT, i.e. the dragged along field has the same components with respect to the dragged along coordinates as the original field with respect to the original coordinates.

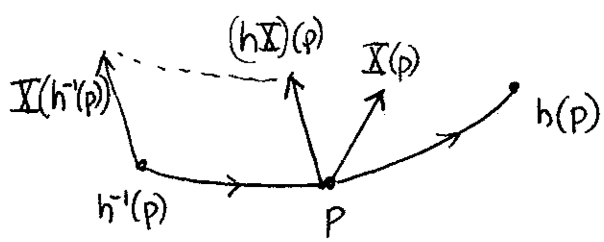
<u>example</u>

Suppose we rotate the plane R^2 by $45°$. Then we get the following picture if we dragalong selected values of a vector field:

A translation would just shift the entire field.

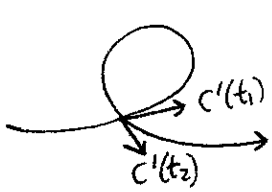


In this example $N = M$, i.e. we have a diffeomorphism of a manifold into itself. In that case hT is a new field on M of the same type as T, whose value at a point p is obtained from the point $h^{-1}(p)$ which is mapped onto p:



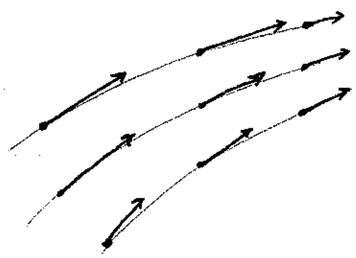
<u>example</u>

The tangent to a parametrized curve $c: R \to M$ is just the push forward of the natural coordinate vector field on the real line R



The curve cannot intersect itself if we are to have a unique tangent at each point of the curve.

THE FLOW OF A SMOOTH VECTOR FIELD (1-parameter group of diffeomorphisms)



Suppose we have a nice vector field $\mathbb{X}$ on M.

An <u>integral curve</u> of $\mathbb{X}$ is just a parametrized curve $c(t)$ whose tangent $c'(t)$ coincides with $\mathbb{X}$ at each point of the curve, i.e. a "flow line" of $\mathbb{X}$. (line of force if we interpret $\mathbb{X}$ as a force field):

$$c'(t) = \mathbb{X} \circ c(t)$$

In local coordinates: $$\frac{dc^i(t)}{dt} = \mathbb{X}^i(c(t)) \qquad c^i(t) \equiv x^i \circ c(t), \quad \mathbb{X}^i \equiv \mathbb{X} x^i$$

This ordinary differential equation has the formal solution
$$c^i(t) = [e^{t\mathbb{X}} x^i](c(0)) \equiv \left(\left[\sum_{k=0}^{\infty} \frac{t^k \mathbb{X}^k}{k!} \right] x^i \right)(c(0))$$
valid in some interval about zero of the parameter t

[This is just the Taylor series expansion of $c^i(t)$ about $t=0$:
$$\left.\frac{dc^i(t)}{dt}\right|_{t=0} = (\mathbb{X}^i x)(c(0))$$
$$\left.\frac{d^2c^i(t)}{dt^2}\right|_{t=0} = \frac{\partial \mathbb{X}^i}{\partial x^j}(c(0)) \underbrace{\frac{dc^j(c(0))}{dt}}_{\mathbb{X}^j(c(0))} = (\mathbb{X}^2 x^i)(c(0))$$
etc..]

Note that the parameter t on any integral curve is completely fixed up to an additive constant, so intervals of this parameter have an invariant meaning.

Suppose we move each point of M a parameter distance t along the integral curve of $\mathbb{X}$ which passes through it ($c(t_0) \to c(t_0+t)$). We then get a point transformation of M which for a nice $\mathbb{X}$ is a diffeomorphism. This is true for each real value of the parameter t for a nice $\mathbb{X}$, so we get a 1-parameter family of diffeomorphisms designated by $\mathbb{X}_t$.

Note that $\mathbb{X}_0$ is just the identity diffeomorphism (no points move), while $\mathbb{X}_{-t}$ returns the points to the original positions they had before being moved by $\mathbb{X}_t$, i.e. $\mathbb{X}_t^{-1} = \mathbb{X}_{-t}$. Clearly $\mathbb{X}_{t_1} \circ \mathbb{X}_{t_2} = \mathbb{X}_{t_1+t_2}$, since first moving a parameter distance t_2 (note $\mathbb{X}_{t_2}$ acts first) and then t_1 is clearly equivalent to moving a parameter distance t_1+t_2, so we have a 1-parameter abelian group of diffeomorphisms of M associated with X called the <u>flow</u> of X.

If X were the velocity field of a stationary fluid flow, the diffeomorphism X_t would correspond to letting the points of the fluid move for a time interval t.

<u>example</u> $X = a^i \frac{\partial}{\partial x^i}$ on R^3 with cartesian coordinates $\{x^i\}$ and a^i constants. Then $X x^i = a^i$, $X^n x^i = 0,\ n>1$ so

$$X_t: \quad c^i(t) = (e^{t a^i \partial_i} x^i)(c(0)) = (x^i + t a^i)(c(0))$$
$$= c^i(0) + t a^i$$

or "$x^i(t) = x^i + t a^i$".

This is a translation by the distance $|t|(\delta_{ij}a^ia^j)^{1/2}$ in the direction $n^k = \frac{t a^k}{|t|(\delta_{ij}a^ia^j)^{1/2}}$. X_1 is just a translation by (a^1, a^2, a^3).

<u>example</u> $X = n^i \epsilon_{ijk} x^j \frac{\partial}{\partial x^k}$, $\delta_{ij} n^i n^j = 1$. Again on R^3.

$$X^i = X x^i = n^k (\mathcal{J}_k)^i{}_j x^j \qquad (\mathcal{J}_k)^l{}_j \equiv -\epsilon_{klj}$$
$$X^n x^i = ([n^k \mathcal{J}_k]^n)^i{}_j x^j$$
$$e^{tX} x^i = [e^{t(n^k \mathcal{J}_k)}]^i{}_j x^j$$
$$c^i(t) = [e^{t n^k \mathcal{J}_k}]^i{}_j c^j(0)$$

This is a rotation of angle t about the direction n (right hand rule)

The vector field X is said to "generate" the 1-parameter group of diffeomorphisms $\{X_t \mid t \in R\}$.

Given such a nice vector field X, one can drag along a tensor field on M by its one-parameter group of diffeomorphisms

$$T \to X_t T.$$

At each point p of M we have a 1-parameter family of tensor fields (or densities) $(X_t T)(p)$ with $(X_0 T)(p) = T(p)$.
We can take the t-derivative and define the Lie derivative of T with respect to X:

$$\boxed{\mathcal{L}_X T = -\frac{d}{dt}\Big|_{t=0} X_t T.}$$

This field, of the same type as T, tells how T begins to change at p under dragging along by X (short for dragging along by the flow of X)

Note that $\mathcal{L}_X X_t T = -\frac{d}{ds}\Big|_{s=0} X_s X_t T = -\frac{d}{ds}\Big|_{s=0} X_{s+t} T$

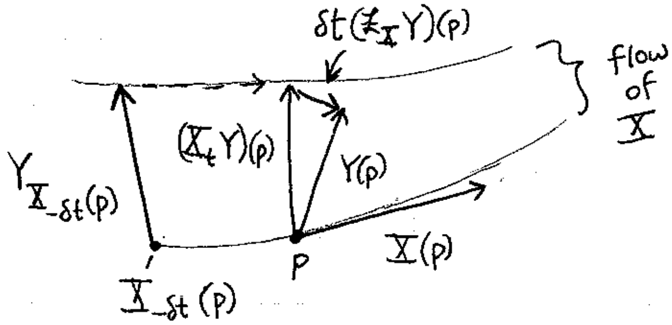
$$= -\frac{d}{dt} X_t T$$

or $\frac{d}{dt}(X_t T) = -\mathcal{L}_X (X_t T)$, $X_0 T = T$

This is an ordinary differential equation for $X_t T$ which, exactly as above, has the power series solution

$$\boxed{X_t T = e^{-t\mathcal{L}_X} T,}$$

valid for t in some interval about zero.



For small δt,

$$(X_{\delta t} Y)(p) \simeq Y(p) - \delta t (\mathcal{L}_X Y)(p)$$

so we get this picture for the "infinitesimal" dragging along of a vector field Y by the flow of X.

<u>exercise</u>: what is wrong with this picture?

Clearly tensor fields which are invariant under dragging along by X have vanishing Lie derivative by X.

$$X_t T = T \text{ for all } t \iff \mathcal{L}_X T = 0.$$

When T is a metric tensor field g, vector fields X for which $\mathcal{L}_X g = 0$ are called killing vector fields.

Remark. For a function f,

$$\mathcal{L}_X f = -\frac{d}{dt}\Big|_0 \underbrace{X_t f}_{\substack{\text{pull back of}\\ f \text{ by } X_{-t}}} = \underbrace{-\frac{d}{dt}\Big|_0 f\circ X_{-t}}_{\substack{\text{derivative of } f\\ \text{by tangent to curve}\\ \text{at } t=0}} = -(-X)f = Xf$$

$$\boxed{\mathcal{L}_X f = Xf}$$

so the Lie derivative of a function is just the ordinary directional derivative.

Consider the formula for $(hT)^{i\cdots}{}_{j\cdots}(h(p))$ on page 133.

Set p equal to $h^{-1}(p)$ and assume we can work in a single coordinate patch so that we can also set $y^i = x^i$:

$$(hT)^{i\cdots}{}_{j\cdots}(p) = \left(\det\left(\frac{\partial h^{-1k}}{\partial x^{\ell}}(p)\right)\right)^{W} \frac{\partial h^i}{\partial x^m}\circ h^{-1}(p) \cdots \frac{\partial h^{-1n}}{\partial x^j}(p)\cdots T^{m\cdots}{}_{n\cdots}\circ h^{-1}(p)$$

or $\quad (hT)^{i\cdots}{}_{j\cdots} = [\rho^{p,q}_{W}(A)\, T\circ h^{-1}]^{i\cdots}{}_{j\cdots} \quad , \quad A^i{}_j = \frac{\partial h^i}{\partial x^j}\circ h^{-1}$

Now set $h = X_t$ and compute

$$(\mathcal{L}_X T)^{i\cdots}{}_{j\cdots} = -\frac{d}{dt}\Big|_0 (X_t T)^{i\cdots}{}_{j\cdots} = -\frac{d}{dt}\Big|_0 [\rho^{p,q}_{W}(A(t))\, T\circ X_{-t}]^{i\cdots}{}_{j\cdots}$$

$$= \underbrace{-\frac{d}{dt}\Big|_0 (T\circ X_{-t})^{i\cdots}{}_{j\cdots}}_{X\,T^{i\cdots}{}_{j\cdots}} - [\sigma^{p,q}_{W}(\underbrace{\tfrac{dA}{dt}(0)})\, T]^{i\cdots}{}_{j\cdots}$$

$$\frac{d}{dt}\Big|_0 A^i{}_j(t) = \frac{d}{dt}\Big|_0 \left(\frac{\partial}{\partial x^j} x^i\circ X_t\right)\circ X_{-t}$$

$$= \frac{\partial}{\partial x^j}\underbrace{\left(\frac{d}{dt}\Big|_0 x^i\circ X_t\right)}_{X x^i = X^i}\circ X_0 + \underbrace{\frac{d}{dt}\Big|_0 \left(\underbrace{\frac{\partial x^i}{\partial x^j}}_{\delta^i{}_j}\circ X_{-t}\right)}_{0} = \frac{\partial X^i}{\partial x^j}$$

$$\boxed{(\mathcal{L}_X T)^{i\cdots}{}_{j\cdots} = X T^{i\cdots}{}_{j\cdots} - [\sigma^{p,q}_{W}((\tfrac{\partial X^k}{\partial x^{\ell}}))\, T]^{i\cdots}{}_{j\cdots}}\ .$$

↑ note the minus sign

For a vector field we get $(\mathcal{L}_X Y)^i = X Y^i - \frac{\partial X^i}{\partial x^j} Y^j = [X,Y]^i$

so $\boxed{\mathcal{L}_X Y = [X,Y].}$

The dragging along operation for a diffeomorphism h clearly respects tensorproducts, evaluations and contraction:

$$h(T\otimes S) = hT \otimes hS\,, \quad h(T(\beta,\dots,Y,\dots)) = hT(h\beta,\dots,hY,\dots),\ \text{etc.}$$

so setting $h = X_t$ and taking $\frac{d}{dt}\big|_0$ means that the Lie derivative inherits from $\frac{d}{dt}$ a product rule for tensor products and evaluations and commutes with contractions:

$$\mathcal{L}_X\, T\otimes S = \mathcal{L}_X T\otimes S + T\otimes \mathcal{L}_X S$$

$$\mathcal{L}_X(T(\beta,\dots,Y,\dots)) = (\mathcal{L}_X T)(\beta,\dots,Y,\dots) + T(\mathcal{L}_X\beta,\dots,Y,\dots)+\dots + T(\beta,\dots,\mathcal{L}_X Y,\dots)+\dots$$

etc.

This gives us a way to derive an invariant formula for the Lie derivative of a 1-form β:

$$\begin{aligned}\mathcal{L}_X\, \beta(Y) &= X\beta(Y) \qquad \text{(just a function)}\\ &= (\mathcal{L}_X\beta)(Y) + \beta(\underbrace{\mathcal{L}_X Y}_{[X,Y]}) \qquad \text{(product rule)}\end{aligned}$$

$$\boxed{(\mathcal{L}_X\beta)(Y) = X\beta(Y) - \beta([X,Y])}$$

One can derive an invariant formula for any tensor field, but let's not.

Remark. For a frame $\{e_i\}$ with dual frame $\{\omega^i\}$

$$(\mathcal{L}_X\omega^i)_j = (\mathcal{L}_X\omega^i)(e_j) = \underbrace{X\delta^i_j}_{0} - \omega^i(\mathcal{L}_X e_j) = -(\mathcal{L}_X e_j)^i$$

so one quickly obtains in this way the frame formula:

$$(\mathcal{L}_X T)^{i\dots}{}_{j\dots} = X T^{i\dots}{}_{j\dots} - [\sigma_W^{p,q}(((\mathcal{L}_X e_\ell)^k))T]^{i\dots}{}_{j\dots}$$

Comma to semicolon rule

Suppose ∇ is a symmetric connection, i.e. $\Gamma^k{}_{m\ell} = \Gamma^k{}_{\ell m}$ in a coordinate frame; then one can rewrite the coordinate formula by changing all ordinary derivatives to covariant derivatives

$$(\nabla_X T)^{i\dots}{}_{j\dots} = X T^{i\dots}{}_{j\dots} + [\sigma^{p,q}_{W}(\omega(X))T]^{i\dots}{}_{j\dots}$$

or
$$X T^{i\dots}{}_{j\dots} = (\nabla_X T)^{i\dots}{}_{j\dots} - [\sigma^{p,q}_{W}(\omega(X))T]^{i\dots}{}_{j\dots}$$

but
$$(\mathcal{L}_X T)^{i\dots}{}_{j\dots} = X T^{i\dots}{}_{j\dots} - [\sigma^{p,q}_{W}((\partial_m X^k))T]^{i\dots}{}_{j\dots}$$

$$= (\nabla_X T)^{i\dots}{}_{j\dots} - [\sigma^{p,q}_{W}((\underbrace{\partial_m X^k + \underbrace{\Gamma^k{}_{\ell m}}_{\Gamma^k{}_{m\ell}} X^\ell}_{\nabla_m X^k}))T]^{i\dots}{}_{j\dots}$$

$$(\mathcal{L}_X T)^{i\dots}{}_{j\dots} = (\nabla_X T)^{i\dots}{}_{j\dots} - [\sigma^{p,q}_{W}(\nabla_m X^k)T]^{i\dots}{}_{j\dots}$$

If one uses commas to indicate ordinary derivatives and semicolons for covariant derivatives, this formula is obtained by replacing commas by semicolons:

$$\underbrace{(\mathcal{L}_X T)^{i\dots}{}_{j\dots}}_{\text{sometimes written } \mathcal{L}_X T^{i\dots}{}_{j\dots}} = X^k T^{i\dots}{}_{j\dots,k} - [\sigma^{p,q}_{W}((X^k{}_{,m}))T]^{i\dots}{}_{j\dots}$$

$$= X^k \underbrace{T^{i\dots}{}_{j\dots;k}}_{\text{this means } (\nabla T)^{i\dots}{}_{j\dots;k}} - [\sigma^{p,q}_{W}((X^k{}_{;m}))T]^{i\dots}{}_{j\dots}$$

This formula is covariant and holds in a frame also.

For a covariant constant metric

$$\mathcal{L}_X g_{ij} = g_{ij,k}X^k + 2g_{k(i}X^k{}_{,j)} = \underbrace{g_{ij;k}}_{0}X^k + 2g_{k(i}X^k{}_{;j)}$$
$$= 2(g_{k(i}X^k)_{;j)} = 2X_{(i;j)}$$

Killing vector fields X satisfy: $\quad 0 = \mathcal{L}_X g_{ij} = 2X_{(i;j)}$

which just says that $\nabla X^\flat$ is antisymmetric, i.e. a 2-form.

These are called <u>Killing's Equations</u>.

example For a flat metric g on R^n in cartesian coordinates $(g_{ij}) = (\eta_{ij}) = \mathrm{diag}(1,\dots,-1,\dots)$ and the covariant derivative is just the ordinary derivative. Killing's equations are:

$$\partial_{(i} X_{j)} = 0$$

Now play a bit: $0 = \partial_k(\partial_i X_j + \partial_j X_i) = \partial_k \partial_i X_j + \partial_j \partial_k X_i$

$$0 = -\partial_i \partial_j X_k - \partial_k \partial_i X_j$$
$$0 = -\partial_j \partial_k X_i - \partial_i \partial_j X_k$$

add: $0 = -2\partial_i \partial_j X_k$

solution of this equation: $X_j = a_{jk} x^k + b_j$ $\quad a_{jk}, b_j$ constants

plug back in K.EQ: $\partial_{(i} X_{j)} = a_{(ij)} = 0$

So $X^i = a^i{}_j x^j + b^i$ is the general solution of these equations ($a_{(ij)} = 0$).
(all indices are raised and lowered with the constant metric component matrix)

The condition $a_{(ij)} = 0$ just says that the matrix $a = (a^i{}_j)$ belongs to the matrix Lie algebra of the orthogonal group of this signature. Recall page 22.

If $e^{ta} \in SO(p,q)$ then: $\frac{d}{dt}\Big|_0 \left[(e^{ta})^T \eta (e^{ta}) = \eta \right]$

$$a^T \eta + \eta a = 0$$
$$a^k{}_i \eta_{kj} + \eta_{ik} a^k{}_j = 0$$
$$a \in so(p,q) = o(p,q) \longleftarrow a_{ji} + a_{ij} = 0$$

We can pick a canonical basis of this matrix Lie algebra:

$J_{ij} \equiv ((J_{ij})^k{}_\ell)$ where $(J_{ij})^k{}_\ell = -(\delta^k_i \eta_{j\ell} - \delta^k_j \eta_{i\ell})$

or more simply: $(J_{ij})^{k\ell} = -\delta^{k\ell}_{ij}$.

exercise. Show that the commutation relations $[J_{ij}, J_{k\ell}] = C^{mn}{}_{ij\,k\ell} J_{mn}$ can be concisely written $[J^{ij}, J_{k\ell}] = 4\,\delta^{[i}{}_{[k} J^{j]}{}_{\ell]}$.

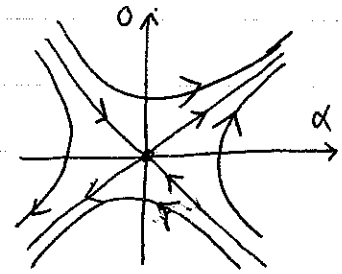
The corresponding Killing vector fields are: $\gamma_{ij} = (J_{ij})^k{}_\ell x^\ell \frac{\partial}{\partial x^k} \equiv \xi(J_{ij})$

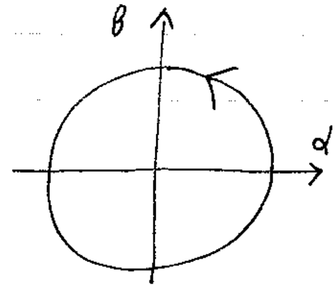
exercise. Show that $[\gamma^{ij}, \gamma_{k\ell}] = -4\,\delta^{[i}{}_{[k} \gamma^{j]}{}_{\ell]}$.

Introduce the n coordinate Killing vector fields $p_i = \partial/\partial x^i$ which generate the translations of R^n into itself.

exercise. Show that $[\gamma^{ij}, p_k] = -(J_{ij})^m{}_k p_m$
$[p_i, p_j] = 0$ (freebee).

If 0 stands for any timelike index ($g_{00} = -1$) and (α, β) any spacelike indices
(I can't seem to get my latin & greek conventions straight, can I?)
then we get the following picture for active boosts and rotations (for the canonical generators all the action takes place on 2-planes)





$$e^{t\, \mathcal{J}_{\alpha 0}} \sim \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \qquad e^{t\, \mathcal{J}_{\alpha\beta}} \sim \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

A diffeomorphism which leaves the metric invariant under dragging along is called an <u>isometry</u>. The full <u>isometry group</u> of the flat metric is the inhomogeneous pseudo-orthogonal group $IO(p,q)$ which consists of the pseudo-orthogonal transformations and the translations. It is "<u>generated by</u>" the Lie algebra of Killing vector fields we obtained above, i.e. locally about the identity the isometries are just "flowing along the integral curves of the Killing vector fields."

Note. The Killing vector field commutators must close because of the identity $[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]}$
which one can prove, showing that

$$\mathcal{L}_X g = 0 = \mathcal{L}_Y g \to \mathcal{L}_{[X,Y]}\, g = 0$$

, i.e. the linear space of solutions of Killing's equations is closed under the Lie bracket operation.

<u>Inadequately explained remark</u>. The group $IO(p,q)$ can be represented by a matrix group in $p+q+1$ dimensions:

$$\begin{pmatrix} \boxed{O(p,q)} & \begin{matrix} b^1 \\ \vdots \\ b^n \end{matrix} \\ 0 \cdots 0 & 1 \end{pmatrix} \sim IO(p,q)$$

For spacetime we have $O(3,1) =$ Lorentz group
$IO(3,1) =$ Poincare group .

For R^3 we have $\quad O(3,R) =$ 3dim rotation group

$\qquad\qquad\qquad\qquad IO(3,R) =$ Euclidean group in 3 dimensions

In this case we can define $\quad J_i = \epsilon_{ijk} J^{jk}, \quad \mathcal{J}_i = (J_i)^k{}_\ell x^\ell \partial_k = \epsilon_{ijk} x^j \frac{\partial}{\partial x^k}$

and $\qquad [J_i, J_j] = \epsilon_{ijk} J_k \qquad [\mathcal{J}_i, \mathcal{J}_j] = -\epsilon_{ijk} \mathcal{J}_k$

$$[\mathcal{J}_i, \wp_j] = -\epsilon_{ijk} \wp_k .$$

If $\; X = n^i \mathcal{J}_i \;, \; \delta_{ij} n^i n^j = 1 \;$ then $\; \Psi_\theta = e^{\theta n^k \mathcal{J}_k}$

$$\Psi_\theta x^i = (e^{\theta n^k J_k})^i{}_j x^j$$

is a rotation of angle θ about n :

right hand rule.

NOTIME FOR EXPLANATION $\qquad$ In cartesian coordinates (orthonormal):

$$\underbrace{\mathcal{L}_{\mathcal{J}_k}}_{\substack{\text{total}\\ \text{angular}\\ \text{momentum}\\ \text{operator}}} T^{i\dots}{}_{j\dots} = \underbrace{\mathcal{J}_k}_{\substack{\text{orbital}\\ \text{angular}\\ \text{momentum}\\ \text{operator}}} T^{i\dots}{}_{j\dots} \underbrace{-[\sigma^{p,q}_{W}(J_k) T]}_{\substack{\text{spin}\\ \text{angular}\\ \text{momentum}\\ \text{operator}}}{}^{i\dots}{}_{j\dots}$$

In order to get hermitian operators in QM we define

$$\mathbb{J}_k = -i \mathcal{L}_{\mathcal{J}_k} \qquad L_k = -i \mathcal{J}_k \qquad S_k = \text{"}\, i\sigma(J_k) \text{"}$$

$$\mathbb{J}_k = L_k + S_k$$

square of total angular momentum: $\quad \mathbb{J}^2 = \delta^{ab} \mathbb{J}_a \mathbb{J}_b = L^2 + S^2 + \underbrace{2\delta^{ab} S_a L_b}_{\text{spin orbit}}$

All of this naturally extends to half-integral spin fields (spinors).

<u>exercise</u> : Show $\quad [\mathbb{J}_i, \mathbb{J}_j] = i \epsilon_{ijk} \mathbb{J}_k$

$\qquad\qquad\qquad\qquad [\mathbb{J}_i, p_j] = i \epsilon_{ijk} p_k \quad (p_i \equiv -i \wp_i).$

Show $\; S^2 X = 1(1+1) X \;$ for a vector field X (spin 1)

$\qquad S^2 T = 2(2+1) T \;$ for a symmetric tracefree $\binom{0}{2}$ tensor T (spin 2)

Lie transformation groups, Lie group Theory

Suppose we introduce an r-dim real vector space $\xi(g) = \text{span}\{X^a e_a\}$ where X^a are constants and $\{e_a\}$ are r vector fields (linearly independent over the real numbers). For each $X \in \xi(g)$ we get a 1-parameter group of diffeomorphisms of the manifold M into itself. What is the condition that these are all subgroups of an r-dimensional group of diffeomorphisms of M into itself? Simply that $\xi(g)$ be closed under the Lie bracket:

$$X, Y \in \xi(g) \longrightarrow [X, Y] \in \xi(g) \quad \text{i.e. } [e_a, e_b] = C^c{}_{ab} e_c \,, \quad C^c{}_{ab} \text{ constants,}$$

i.e. $\xi(g)$ is a Lie subalgebra of the infinite dimensional Lie algebra of vector fields on the manifold. The elements of $\xi(g)$ are said to <u>generate</u> the r-dim. group of diffeomorphisms (an r-dim. subgroup of the ∞-dim. group of diffeomorphisms of the manifold into itself).

Conversely, given such an r-dim. group, one can introduce a corresponding Lie algebra of generators and one can compute all the local properties of the group from the constants $C^a{}_{bc}$, the components of the structure constant tensor of the Lie algebra.

On any matrix group (or any "Lie group") we can consider 2 diffeomorphism subgroups called left and right translations (left or right multiplication of the group by fixed elements). These turn out to be important in the local differential geometry of fiber bundles which play a fundamental role in physics.

The classical mechanics of a rigid body involve a left invariant metric on the special orthogonal group $SO(3,\mathbb{R})$ (invariant under all left translations). The Euler equations are just the geodesic equations for this metric. (Geodesic?) The corresponding quantum problem describes the rotational modes of molecules.

Because of the fundamental role of groups in most areas of physics, it is worthwhile understanding them at a higher level than time usually permits in physics courses.

Okay already, I quit.

I hope I have inspired at least one of you to continue what we have begun here. Thanks for giving me the opportunity to do this, even if it did cost me a lot of time.

TABLE OF CONTENTS

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(room for a continuation?)

This convention of putting the table of contents at the end is Italian.