(11) GAUGE FIELDS

exercise a) evaluate the curvature tensor components in terms of the connection components starting from the definition

$$\Omega^\alpha{}_\beta = d\omega^\alpha{}_\beta + \omega^\alpha{}_\gamma \wedge \omega^\gamma{}_\beta = \tfrac{1}{2} R^\alpha{}_{\beta\gamma\delta}\, \omega^{\gamma\delta} \qquad \text{and} \qquad d\omega^\alpha = -\tfrac{1}{2} C^\alpha{}_{\beta\gamma}\, \omega^{\beta\gamma}$$

The answer you should get is

$$R^\alpha{}_{\beta\gamma\delta} = \partial_\gamma \Gamma^\alpha{}_{\delta\beta} - \partial_\delta \Gamma^\alpha{}_{\gamma\beta} - C^\epsilon{}_{\gamma\delta}\Gamma^\alpha{}_{\epsilon\beta} + \Gamma^\alpha{}_{\gamma\epsilon}\Gamma^\epsilon{}_{\delta\beta} - \Gamma^\alpha{}_{\delta\epsilon}\Gamma^\epsilon{}_{\gamma\beta}$$

The third term drops out in a coordinate frame.

(b) You have already shown in a previous exercise (have you?) that if we define the torsion 2-forms

$$T^\alpha = \tfrac{1}{2} T^\alpha{}_{\beta\gamma}\,\omega^{\beta\gamma} \equiv d\omega^\alpha + \omega^\alpha{}_\beta \wedge \omega^\beta = \Gamma^\alpha{}_{\gamma\beta}\,\omega^{\gamma\beta} - \tfrac{1}{2} C^\alpha{}_{\gamma\beta}\,\omega^{\gamma\beta}$$

$$= \left(\Gamma^\alpha{}_{[\beta\gamma]} - \tfrac{1}{2} C^\alpha{}_{\beta\gamma}\right)\omega^{\beta\gamma} \qquad \text{or} \qquad T^\alpha{}_{\beta\gamma} = 2\Gamma^\alpha{}_{[\beta\gamma]} - C^\alpha{}_{\beta\gamma}$$

then for a symmetric connection one has $T^\alpha{}_{\beta\gamma} = 0$.

Suppose the connection is not symmetric, i.e. the torsion is nonzero and given by this formula so that $\Gamma^\alpha{}_{[\beta\gamma]} = \tfrac{1}{2}(C^\alpha{}_{\beta\gamma} + T^\alpha{}_{\beta\gamma})$.
How does the formula for $\Gamma^\alpha{}_{\beta\gamma}$ on page 115 change (also page 116)?

Answer: $$\Gamma^\gamma{}_{\alpha\beta} = \left\{ {\gamma \atop \alpha\beta} \right\} + \tfrac{1}{2}\left(C^\gamma{}_{\alpha\beta} + C_\beta{}^\gamma{}_\alpha + C_\alpha{}^\gamma{}_\beta\right) + \underbrace{\tfrac{1}{2}\left(T^\gamma{}_{\alpha\beta} + T_\beta{}^\gamma{}_\alpha + T_\alpha{}^\gamma{}_\beta\right)}_{\equiv K^\gamma{}_{\alpha\beta} \text{ "contorsion tensor"}}$$

Notice that by construction the metric is covariantly constant with respect to this connection, which is said to be "a metric connection" as opposed to "the metric connection" which is the unique such symmetric connection (Christoffel symbols in a coordinate frame).

(c) Since the connection components transform with an inhomogeneous term depending only on the frame transformation, the difference of two different connections transforms homogeneously, i.e. as a $\binom{1}{2}$-tensor field.
The difference between the symmetric metric connection and the most general (nonsymmetric) metric connection is called the contorsion.
Show that the torsion components also transform as a $\binom{1}{2}$-tensor following the exercise on page 117. Many current theories involving gravity coupled to half-integral spin fields have nonvanishing torsion.

(d) Show that the curvature tensor can be invariantly defined by the formula

$$\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \equiv R(X,Y)Z \equiv R^\alpha{}_{\beta\gamma\delta}\, Z^\beta X^\gamma Y^\delta e_\alpha .$$

[Evaluate this in a frame and compare the result with (a).]

Let us summarize the way we arrived at the covariant derivative and parallel transport for $\binom{p}{q}$-tensor density fields of weight W.

We started with the vector space TM_x attached to each point x of the manifold M. A choice of frame $\{e_\alpha\}$ on M gave us a way to identify each of these spaces with R^n (by expressing tangent vectors in terms of their components with respect to the frame); the frame vectors themselves are identified with the natural basis of R^n:

$$X = X^\alpha e_\alpha \quad \rightarrow \quad (X^\alpha) \in R^n .$$

Elements of the general linear group $GL(n,R)$ act on $V = R^n$ corresponding to a change of the basis; this is just the identity representation of $GL(n,R)$. A change of frame on M is accomplished by the action of a $GL(n,R)$-valued function which acts on each space R^n (identified with TM_x) to transform the components of tangent vectors or vector fields

$$\underbrace{e_\alpha \rightarrow e_\alpha{}' = L^{-1\,\beta}{}_\alpha e_\beta}_{\text{change of frame}} \qquad\qquad \underbrace{(X^\alpha) \rightarrow (X^{\alpha'}) = (L^\alpha{}_\beta X^\beta) \in R^n}_{\text{change of components}}$$

The components of a $\binom{p}{q}$-tensor density of weight W transform by the corresponding representation of $GL(n,R)$

$$T^{\alpha\cdots}{}_{\beta\cdots} \rightarrow T^{\alpha\cdots}{}_{\beta\cdots}{}' = [\rho^{p,q}_W(L)\, T\,]^{\alpha\cdots}{}_{\beta\cdots} .$$

The connection was specified by a matrix-valued 1-form, i.e. a 1-form with values in $gl(n,R)$, the Lie algebra of the general linear group. Its value on a tangent vector X specifies the covariant derivatives along X of the frame vectors, or if you will, of the natural basis of R^n with respect to that choice of frame.

$$\nabla_X e_\alpha = \omega^\beta{}_\alpha(X)\, e_\beta \qquad\qquad \omega^\beta{}_\alpha(X) = \Gamma^\beta{}_{\gamma\alpha} X^\gamma$$

The components of the covariant derivative of a $\binom{p}{q}$ tensor density of weight W are then

$$(\nabla_X T)^{\alpha\cdots}{}_{\beta\cdots} = \underbrace{X\, T^{\alpha\cdots}{}_{\beta\cdots}}_{\substack{\text{ordinary derivative}\\ \text{of components}}} + \underbrace{[\sigma^{p,q}_W(\omega(X))\, T\,]^{\alpha\cdots}{}_{\beta\cdots}}_{\substack{\text{compensating term since}\\ \text{basis not covariant constant}}}$$

These formulas hold for a specific choice of frame on the manifold. Under the above change of frame the connection 1-form matrix changes by

$$\omega' = L\omega L^{-1} + L\,dL^{-1}$$

which is an inhomogeneous adjoint transformation, while the curvature 2-form matrix

$$\Omega \equiv d\omega + \omega\wedge\omega$$

transforms under the adjoint representation (conjugation by L):

$$\Omega' = L\Omega L^{-1}.$$

The covariant derivative itself is "covariant" under a change of frame, i.e. it transforms by the same representation as the field before the covariant derivative was taken:

$$(\nabla_X T)^{\alpha\cdots}{}_{\beta\cdots}{}' = [\rho^{p,q}_{W}(L)(\nabla_X T)]^{\alpha\cdots}{}_{\beta\cdots}\,.$$

This was the whole point of introducing the covariant derivative, to obtain a derivative operator which did not depend on which frame we allowed it to operate.

A choice of frame in which to express the components of fields is called a "choice of gauge" and the change of frame a "gauge transformation".

Note that a global frame may not exist on the manifold in the same way that global coordinates may not exist. On the 2-sphere, no everywhere nonzero vector field exists, so no global frame exists and one is forced to use a local frame on each member of a set of open sets covering it, such as the coordinate frame on each coordinate patch of a covering of the 2-sphere by local coordinate patches. The Mobius strip also has no global frame (if one existed one could orient the manifold).

In this case one says that there is no "global gauge". Each local patch on which a smooth local frame is chosen is called a "local gauge", and one has a covering of the manifold by "local gauge patches". One does not necessarily need to use coordinate patches. For example, global frames exist on the circle S^1 and the 3-sphere S^3 so a global patch exists which is the manifold itself.

We can carry through every step by replacing the general linear group $GL(n,\mathbb{R})$ and its matrix Lie algebra $gl(n,\mathbb{R})$ by any $m\times m$ real or complex matrix group $G \subset GL(m,\mathbb{R})$ or $GL(m,\mathbb{C})$ and its matrix Lie algebra $\mathfrak{g} \subset gl(m,\mathbb{R})$ or $gl(m,\mathbb{C})$.

If $\{E_a\} = \{E_a{}^A{}_B\}$ is a basis of this matrix Lie algebra considered as a real vector space, ie any element of $\mathfrak{g}$ can be expressed as $\theta = \theta^a E_a$ with θ^a real, then its real structure constants are defined by $[E_a, E_b] = E_a E_b - E_b E_a = C^c{}_{ab} E_c$ and any element of the matrix group close enough to the identity can be obtained by exponentiating an element of the Lie algebra

$$S = e^{\theta^a E_a} \in G.$$

That is we consider fields on the manifold with values in some vector space V on which a representation of the matrix group G acts ($\mathbb{R}^n$ or one of its tensor product spaces for the general linear group), we fix a basis of V (the natural basis of $\mathbb{R}^n$ or its tensor product spaces for $GL(n,\mathbb{R})$), and then consider changing the components of these V valued fields under the action of the group.

If G is a real or complex matrix group, its identity representation acts on $\mathbb{R}^m$ or $\mathbb{C}^m$ with natural basis $\{E_A\}$:

$$\psi = (\psi^1, \dots, \psi^m) = \psi^A E_A \quad \in \quad \mathbb{R}^m \text{ or } \mathbb{C}^m \quad \equiv V$$

$$\psi \to \psi' = S\psi = S^A{}_B \psi^B E_A$$

If ψ is a V-valued field on the manifold, $\psi = \psi^A E_A$, with components ψ^A which are functions on the manifold, we can introduce the G-covariant derivative of ψ or "gauge covariant derivative" by specifying the covariant derivatives of the basis vectors, which is done by giving a $\mathfrak{g}$-valued connection 1-form

$$\nabla_X E_A = A^B{}_A(X) E_B$$

$$A = (A^B{}_A) = A^a E_a \qquad A^B{}_A = A^B{}_{\alpha A}\omega^\alpha = A^a{}_\alpha \omega^\alpha E_a{}^A{}_B$$

$$A(X) = A^a_\alpha X^\alpha E_a \in \mathfrak{g}$$

evaluation on a tangent vector X gives a matrix in the Lie algebra

$$\text{Then}\quad \overset{G}{\nabla}_X \psi = \overset{G}{\nabla}_X (\psi^A E_A) = (\underbrace{\overset{G}{\nabla}_X \psi^A}_{\text{just functions}}) E_A + \psi^A (\overset{G}{\nabla}_X E_A)$$

$$= (\underbrace{X\psi^A}_{\substack{\text{ordinary derivative}\\ \text{of components}}} + \underbrace{A^A{}_B(X)\psi^B}_{\substack{\text{compensating term}\\ \text{since basis not}\\ \text{covariantly constant}}}) E_A \equiv (\overset{G}{\nabla}_X \psi)^A E_A$$

If (ρ, σ) is any representation of $(G, \mathfrak{g})$, then if ϕ is a function on the manifold with values in the vector space of the representation then

$$\overset{G}{\nabla}_X \phi = X\phi + \sigma(A(X))\phi \,.$$

Suppose we transform the components of all such fields by a function on the manifold with values in G

$$\psi^{A'} = S^A{}_B \psi \quad \text{or} \quad \psi' = S\psi$$

$$A' = SAS^{-1} + SdS^{-1}$$

$$\phi' = \rho(S)\phi$$

"change of gauge"
or
"gauge transfomation"

The connection form satisfies the same transformation law as in the general linear group case by its definition as the matrix of covariant derivatives of the basis vectors $\{E_A\}$ in each choice of local gauge.

There is a basic difference between tensor densities and fields which transform under a representation of $(G, \mathfrak{g})$. A choice of gauge in the general linear group case was a frame on the manifold used to map the tangent spaces onto R^n, the frame itself identified with the natural basis of R^n. The equation $\nabla_X e_B = \omega^\alpha{}_B(X) e_\alpha$ may then be interpreted as specifying the covariant derivatives of the natural basis of R^n in each choice of gauge, i.e. when identified with a particular frame.

However, for a general matrix group G, there is no relation between the vector space R^m or C^m of the identity representation of the group and the manifold itself, so one cannot identify its natural basis $\{E_A\}$ with some invariant object living on the manifold, that is elements of V do not come from expressing invariant objects on the manifold in a particular basis.

One just says that $\overset{G}{\nabla}_X E_A = A^B{}_A(X) E_B$
specifies the covariant derivatives of the natural basis in that particular gauge in which the connection 1-form is A.
In short ONE ONLY HAS COMPONENTS AND NO INVARIANT FIELDS.

The components of the covariant derivatives $(\overset{G}{\nabla}_X \Psi)^A$ or $(\overset{G}{\nabla}_X \phi)^{\cdots}_{\cdots}$ transform exactly as the fields themselves.
[In physics developments this condition is used to determine the transformation law for the connection.]

$$(\overset{G}{\nabla}_X \phi)' = \rho(S)(\overset{G}{\nabla}_X \phi).$$

If we define the curvature 2-form matrix by the same formula as in the general linear group case but now call it F

$$F \equiv dA + A \wedge A$$

then exactly as in that case it transforms by the adjoint representation

$$F' = SFS^{-1}.$$

Expressing $F = F^a E_a$ *and* $A = A^a E_a$ *in a basis of the Lie algebra:*

$$F^a E_a = \underbrace{d(A^a E_a)}_{dA^a E_a} + \underbrace{A^b E_b \wedge A^c E_c}_{A^b \wedge A^c \underbrace{E_{[b} E_{c]}}_{\frac{1}{2}(E_b E_c - E_c E_b) = \frac{1}{2} C^a{}_{bc} E_a}}$$

or
$$F^a = dA^a + \tfrac{1}{2} C^a{}_{bc} A^b \wedge A^c.$$

But F is a matrix valued 2-form, i.e. displaying all of its components we have $F^A{}_{B\,\alpha\beta}$

these transform under $GL(n,\mathbb{R})$

these transform under adjoint representation of G

How do we take covariant derivatives of fields with some indices associated with representations of the matrix group G and others associated with the general linear group?

Simple. We start off taking the covariant derivative of the field with respect to the metric connection and then we add on the terms associated with the particular representation of G.

For example, the curvature 2-form $F = \frac{1}{2} F_{\alpha\beta}\, \omega^\alpha \wedge \omega^\beta$ is a matrix-valued 2-form which transforms under the adjoint representation of G

$$F' = \rho(S) F = S F S^{-1} \quad , \qquad \sigma(A(X))F = [A(X), F]$$

so
$$(\overset{G}{\nabla}_X F)_{\alpha\beta} = \underbrace{X F_{\alpha\beta} - F_{\gamma\beta}\, \omega^\gamma{}_\alpha(X) - F_{\alpha\gamma}\, \omega^\gamma{}_\beta(X)}_{(\nabla_X F)_{\alpha\beta}} + [A(X), F_{\alpha\beta}]$$

In other words we first take the ordinary derivative of the components and then add on the terms arising from the appropriate linear transformation of each index. In our example the G-indices have been suppressed by the matrix notation, but we could write explicitly

$$(\overset{G}{\nabla}_X F)^A{}_{B\,\alpha\beta} = X F^A{}_{B\alpha\beta} - F^A{}_{B\gamma\beta}\, \omega^\gamma{}_\alpha(X) - F^A{}_{B\alpha\gamma}\, \omega^\gamma{}_\beta(X) + A^A{}_C(X) F^C{}_{B\alpha\beta} - F^A{}_{C\alpha\beta} A^C{}_B(X)$$

or using the components with respect to a basis $\{E_a\}$ of the Lie algebra $\mathfrak{g}$

$$(\overset{G}{\nabla}_X F)^a{}_{\alpha\beta} = (\nabla_X F)^a{}_{\alpha\beta} + C^a{}_{bc} A^b(X) F^c{}_{\alpha\beta}.$$

We can also introduce $\overset{G}{\nabla}$ and the semicolon notation exactly as on p. 113 for ordinary covariant derivatives and we can drop parentheses with the understanding that we mean the components of the covariant derivative and not the covariant derivative of the components (which is just the ordinary derivative). For example

$$\overset{G}{\nabla}_\gamma F^a{}_{\alpha\beta} = F^a{}_{\alpha\beta;\gamma} = \partial_\gamma F^a{}_{\alpha\beta} - F^a{}_{\delta\beta} \Gamma^\delta{}_{\gamma\alpha} - F^a{}_{\alpha\delta} \Gamma^\delta{}_{\gamma\beta} + \underbrace{C^a{}_{bc} A^b_\gamma}_{\text{“}A^a{}_{\gamma c}\text{”}} F^c.$$

or if $\psi = \psi^A E_A$ transforms under the identity representation

$$\overset{G}{\nabla}\psi = d\psi + \sigma(A)\psi = (\partial_\alpha \phi^A + A^A_\alpha{}_B \phi^B)\, \omega^\alpha E_A \ .$$

COVARIANT AND GAUGE COVARIANT EXTERIOR DERIVATIVE

A very useful thing to do is introduce the covariant exterior derivative and the gauge covariant exterior derivative of tensor-valued differential forms and V-valued differential forms respectively or differential forms with extra indices of both tensor and G-type.

Example. The curvature tensor (components $R^\alpha{}_{\beta\gamma\delta}$) may be considered as a $\binom{1}{1}$ tensor-valued 2-form since its last two covariant indices are antisymmetric. Ω is in fact the matrix of components of this $\binom{1}{1}$-tensor valued 2-form. If we evaluate the form arguments $\Omega(X,Y)$, we get the components of a $\binom{1}{1}$-tensor.

Let $\rho^{p,q}_{W}$ be the representation under which the additional indices transform.

$$\underbrace{DT^{\alpha\cdots}{}_{\beta\cdots}}_{\text{extra indices}} = dT^{\alpha\cdots}{}_{\beta\cdots} + [\rho^{p,q}_{W}(\omega)\wedge T]^{\alpha\cdots}{}_{\beta\cdots}$$

For example: $$D\Omega^\alpha{}_\beta = d\Omega^\alpha{}_\beta + \omega^\alpha{}_\gamma\wedge\Omega^\gamma{}_\beta - \underbrace{\overset{\text{1-form}}{\omega^\gamma{}_\beta}\wedge\overset{\text{2-form}}{\Omega^\alpha{}_\gamma}}_{-\Omega^\alpha{}_\gamma\wedge\omega^\gamma{}_\beta}$$

or: $$D\Omega = d\Omega + \underbrace{\omega\wedge\Omega - \Omega\wedge\omega}_{\equiv[\omega\wedge\Omega]}$$ combined wedge and commutator of ω and Ω

One can show that for a symmetric connection, D is just the antisymmetrized covariant derivative:

p-form $$(d\sigma)_{\alpha_{p+1}\alpha_1\ldots\alpha_p} = (p+1)\,\partial_{[\alpha_{p+1}}\sigma_{\alpha_1\ldots\alpha_p]} = (p+1)\,\nabla_{[\alpha_{p+1}}\sigma_{\alpha_1\ldots\alpha_p]}$$

tensor-valued p-form $$d\,T^{\overbrace{\beta\ldots}}{}_{\underbrace{\gamma\ldots}_{\text{extra indices}}\,\underbrace{\alpha_{p+1}\alpha_1\ldots\alpha_p}_{(p+1)\text{-form indices}}} = (p+1)\,\nabla_{[\alpha_{p+1}}T^{\beta\ldots}{}_{|\gamma\ldots|\,\alpha_1\ldots\alpha_p]}$$

here means exclude these indices from antisymmetrization

For a V-valued p-form, we just add on the wedge of the connection form

$$\overset{G}{D}T = dT + \sigma(A)\wedge T \qquad (V\text{-indices suppressed})$$

For example, consider a 2-form which transforms under the adjoint representation like the curvature $\mathfrak{g}$-valued 2-form

in a basis of $\mathfrak{g}$: $$\overset{G}{D}F^a = dF^a + C^a{}_{bc}A^b\wedge F^c$$

without a basis: $$\overset{G}{D}F = dF + A\wedge F - \underbrace{(-1)^2 F\wedge A} = dF + \underbrace{A\wedge F - F\wedge A}_{\equiv[A\wedge F]}$$ combined wedge & commutator

since we are using matrix multiplication, A must be on the right here so we have to move 1-form across the 2-form and include the sign change (trivial for a 2-form)

For p-forms with extra indices of both types, the generalization is obvious

$$\overset{G}{D}T = DT + \sigma(A)\wedge T \qquad \text{(all indices suppressed)}$$

$\overset{G}{D}$ is an operator which maps a stuff-valued p-form into a stuff-valued (p+1)-form of the same stuff type which transforms under a gauge transformation and change of frame in the same way as the original stuff-valued p-form.

For example any $\binom{p}{q}$-tensor density of weight W can be considered as a tensor density valued 0-form (same for a V-valued field)

In which case D and $\overset{G}{D}$ reduce to the covariant derivative and gauge covariant derivative

$$DT^{\alpha\cdots}{}_{\beta\cdots} = dT^{\alpha\cdots}{}_{\beta\cdots} + [\rho^{p,q}_{W}(\omega)T]^{\alpha\cdots}{}_{\beta\cdots} = (\nabla_\gamma T^{\alpha\cdots}{}_{\beta\cdots})\,\omega^\gamma$$

$$\overset{G}{D}\phi = d\phi + \rho(A)\phi = \overset{G}{\nabla}\phi$$

Again we leave off parentheses with the understanding that we mean the components of the covariant exterior derivative rather than the covariant exterior derivative of the components (which is just the ordinary exterior derivative)

EXAMPLE. $\quad F = dA + A\wedge A \qquad dA = F - A\wedge A$

$$0 = d^2A = dF - \underbrace{dA}_{F-A\wedge A}\wedge A + A\wedge\underbrace{dA}_{F-A\wedge A} = \underbrace{dF + A\wedge F - F\wedge A}_{\overset{G}{D}F} + \cancel{A\wedge A\wedge A} - \cancel{A\wedge A\wedge A}$$

so $\quad \overset{G}{D}F \equiv 0 \qquad$ (this is an identity)

For $GL(n,\mathbb{R})$, $\quad D\Omega = 0$ is called the 2nd Bianchi identity.

For a symmetric connection

$$0 = (D\Omega^\alpha{}_\beta)_{\epsilon\gamma\delta} = 3\nabla_{[\epsilon}R^\alpha{}_{|\beta|\gamma\delta]} = 3\,R^\alpha{}_{\beta[\gamma\delta;\epsilon]}$$

EXAMPLE. $\quad \Theta^\alpha = d\omega^\alpha + \omega^\alpha{}_\beta\wedge\omega^\beta = D\omega^\alpha \qquad d\omega^\alpha = \Theta^\alpha - \omega^\alpha{}_\beta\wedge\omega^\beta$

$$0 = d^2\omega^\alpha = d\Theta^\alpha - d\omega^\alpha{}_\beta\wedge\omega^\beta + \omega^\alpha{}_\beta\wedge\underbrace{d\omega^\beta}_{\Theta^\beta - \omega^\beta{}_\gamma\wedge\omega^\gamma}$$

$$= \underbrace{d\Theta^\alpha + \omega^\alpha{}_\beta\wedge\Theta^\beta}_{D\Theta^\alpha} \underbrace{- d\omega^\alpha{}_\beta\wedge\omega^\beta - \omega^\alpha{}_\beta\wedge\omega^\beta{}_\gamma\wedge\omega^\gamma}_{-\Omega^\alpha{}_\beta\wedge\omega^\beta}$$

Thus when $\Theta = 0$ (symmetric connection like the metric connection)

$$\Omega^\alpha{}_\beta\wedge\omega^\beta = R^\alpha{}_{\beta\gamma\delta}\,\omega^{\gamma\delta\beta} = R^\alpha{}_{[\beta\gamma\delta]}\,\omega^{\beta\gamma\delta} = 0$$

Thus $\quad R^\alpha{}_{[\beta\gamma\delta]} = 0 \qquad$ (first Bianchi identity)

Torsion is only defined for the $GL(n,\mathbb{R})$ connection on a manifold since it involves the manifold structure.

NO TIME NO SPACE SO IGNORE THIS REMARK. RICCI IDENTITIES $\Leftrightarrow$ $D^2T^{\alpha\cdots}{}_{\beta\cdots} = [\sigma^{p,q}_{W}(\Omega)\wedge T]^{\alpha\cdots}{}_{\beta\cdots}$ (+ analog for $\overset{G}{D}{}^2$). metric = covariant constant $\binom{0}{2}$ tensor valued 0-form:

$$0 = D^2 g_{\alpha\beta} = -g_{\gamma\beta}\Omega^\gamma{}_\alpha - g_{\alpha\gamma}\Omega^\gamma{}_\beta = -(\Omega_{\alpha\beta} + \Omega_{\beta\alpha}) = -2\,\Omega_{(\alpha\beta)} \quad \text{so}$$

$R_{\alpha\beta\gamma\delta} = R_{[\alpha\beta]\gamma\delta}$ The curvature tensor is also antisymmetric in its first two indices

Recall Maxwell's equations for $F = dA$:

$$dF \equiv 0 \quad (\text{consequence of } d^2A = 0)$$

$$\underbrace{-{*d*}}_{\delta = \operatorname{div}} F = -\underset{\substack{\uparrow \\ \text{current 1-form}}}{J}$$

$$\underbrace{-{*d*}}_{\delta = -\operatorname{div}} J = {*d^2*F} = 0 \quad \text{charge conservation}$$

We saw on page 96–97 that this meant from Stoke's Thm

$$\int_\Sigma {*J} = Q = \text{constant}$$

Σ ← spacelike slice of spacetime $t = \text{constant}$.

Now consider a general G-connection. A is called the gauge potential and $F = dA + A \wedge A$ the gauge field strength

$$\overset{G}{D}F \equiv 0 \quad (\text{consequence of } d^2A = 0, \text{ previous page})$$

generalizes the first Maxwell equation

$$\underbrace{-{*\overset{G}{D}*}}_{\overset{G}{\delta} = -\operatorname{div}_G} F = -J$$

gauge covariant divergence

generalizes the second, where $J = J^a E_a$ is a Lie algebra valued 1-form, the current density 1-form which is a source of the gauge field

Then
$$d{*J} = -d\,(\overset{G}{D}{*F}) = -d\,(d{*F} + [A \wedge {*F}])$$
$$-{*d}({*J} + [A \wedge {*F}]) = 0$$
$$\underbrace{\operatorname{div}}_{\substack{\text{ordinary} \\ \text{divergence}}} (J - {*[A \wedge {*F}]}) = 0$$

$$\int_\Sigma ({*J} + \underset{\uparrow}{[A \wedge {*F}]}) = Q = Q^a E_a = \text{constant}$$

the nonabelian gauge field itself carries charge which contributes to the total conserved charge (while the photon has zero charge)

EXAMPLE. ELECTROMAGNETISM

Let $G = U(1) =$ 1×1 unitary matrices $(U^\dagger = U^{-1})$

$= \{ S = e^{-i\theta} \mid \theta \in \mathbb{R} \}$

Lie algebra $\quad \mathfrak{g} = \mathfrak{u}(1) = \{ \underset{\sim}{-i}\,\theta \mid \theta \in \mathbb{R} \}$

$-i = E_1 =$ basis of Lie algebra

The 1-dimensional representations of $U(1)$ are specified by the integers

$\rho_n(e^{-i\theta}) = e^{-in\theta}$, $\sigma_n(-i\theta) = -in\theta$

n must be an integer since $1 = \rho_n(1) = \rho_n(e^{-i2\pi}) = e^{-i2\pi n}$.

let ψ be a complex scalar field on Minkowski spacetime, i.e. a wavefunction.

ψ is allowed to undergo arbitrary phase transformations since only relative phases matter in QM.

$\psi \to e^{-in\theta}\psi$ so ψ transforms under the representation ρ_n of $U(1)$.

We can specify a $U(1)$-connection by a pure imaginary 1-form

$$A = -i\mathbb{A} = -i\mathbb{A}_\alpha \omega^\alpha$$

$$\overset{G}{\nabla}_X \psi = X\psi - in\mathbb{A}(X)\psi = X^\alpha(\partial_\alpha - in\mathbb{A}_\alpha)\psi$$

Now suppose we pick a new basis of the Lie algebra of $U(1)$

$E_1 = -ie$, define $q = ne$, i.e. $e^{-in\theta} = e^{-iq\theta e^{-1}}$

Now: $A = \mathbb{A}E_1 = -ie\mathbb{A}$ and $\sigma_n(A) = -iq\mathbb{A}$

so $\overset{G}{\nabla}_X \psi = X\psi - iq\mathbb{A}(X)\psi = X^\alpha(\partial_\alpha - iq\mathbb{A}_\alpha)\psi$.

Call $q = ne$ the charge of the field ψ (*It must be an integral multiple of e*)

Since G is abelian, $A\wedge A = 0$ and

$$\mathbb{F}E_1 = F = dA = d\mathbb{A}\,E_1 \longrightarrow \mathbb{F} = d\mathbb{A}.$$

Thus a $U(1)$ gauge theory involves a real 1-form $\mathbb{A}$ with corresponding curvature $\mathbb{F} = d\mathbb{A}$ (the components of the connection and curvature in the basis $\{E_1\}$) which transform under a gauge transformation as follows

$$A \to \underbrace{SAS^{-1}}_{A} + \underbrace{SdS^{-1}}_{e^{-i\theta}d(e^{i\theta}) = id\theta} = A - e\,d\theta\,E_1$$

or $\mathbb{A} \to \mathbb{A} + d(-e\theta)$ (adding a differential of a function to $\mathbb{A}$)

$\mathbb{F} \to S\mathbb{F}S^{-1} = \mathbb{F}$ (gauge invariant)

We can identify $\mathbb{A}$ with the vector potential of the electromagnetic field and $\mathbb{F}$ with the electromagnetic field 2-form. If e is the electronic charge (positive), then q is the charge of the complex scalar field ψ. Thus electromagnetism has the structure of a $U(1)$ gauge theory.

EXAMPLE. STRONG INTERACTIONS

We can go through the same steps for $G = SU(3)$ = special unitary group in 3 dimensions $= \{U \in GL(3,\mathbb{C}) \mid \det U = 1,\ U^\dagger = U^{-1}\}$

This is an 8-dim. matrix group whose Lie algebra consists of antihermitian matrices

$$H \in su(3) \rightarrow H^\dagger = -H \quad \text{or} \quad H = -iK \text{ with } K^\dagger = K \text{ (hermitian)}$$

Let $\{E_a\}$ be a basis of $su(3)$: $[E_a, E_b] = C^c{}_{ab} E_c$

or $E_a = -i\mathcal{E}_a$, $\mathcal{E}_a$ hermitian and $[\mathcal{E}_a, \mathcal{E}_b] = iC^c{}_{ab}\mathcal{E}_c$

The identity representation acts on $\mathbb{C}^3$.

We can write $\quad S = e^{\theta^a E_a} = e^{-i\theta^a \mathcal{E}_a}$

$$A = A^a E_a = -iA^a\mathcal{E}_a \quad , \quad F^a = dA^a + \tfrac{1}{2}C^a{}_{bc}A^b \wedge A^c$$

If we now introduce a new basis $\{\mathcal{E}_a\}$ such that $E_a = -ig\,\mathcal{E}_a$

then $\quad A = -ig\,\mathbb{A}^a\mathcal{E}_a \quad , \quad F = -ig\,\mathbb{F}^a\mathcal{E}_a$

$$\mathbb{F}^a = d\mathbb{A}^a + gC^a{}_{bc}\mathbb{A}^b \wedge \mathbb{A}^c.$$

A scalar field which transforms under the adjoint representation of $SU(3)$ is called a Higgs field:

$$\phi \rightarrow S\phi = e^{-ig\theta^a \operatorname{ad}(\mathcal{E}_a)}\phi$$

$\phi = \phi^a E_a$

8 Higgs fields

$$\overset{G}{\nabla}_X\phi = X\phi - ig\operatorname{ad}(\mathbb{A}(X))\phi = X\phi - ig[\mathbb{A}(X),\phi]$$

A Dirac spinor field (What is that?) which transforms under the identity representation is called a quark field

$$\psi \rightarrow S\psi$$
$$\overset{G}{\nabla}_X\psi = X\psi - ig\,\mathbb{A}(X)\psi$$

$\psi = \underbrace{(\psi^1, \psi^2, \psi^3)}$

3-quark fields

each of which is a Dirac spinor field

The matrix valued 1-form $\mathbb{A}$ is called the gluon field

$$\mathbb{A} = \underbrace{\mathbb{A}^a}\mathcal{E}_a$$

8 1-forms $\rightarrow$ 8 gluon fields

The electroweak interactions are described by an $SU(2) \times U(1)$ gauge theory. The group has 4 dimensions so we get 4 connection 1-forms which are associated with the photon (e&m gauge potential) and the 3 vector bosons.

See Abers & Lee, Phys. Rep. 9, no. 1, 1–141, 1973

or many other discussions

WHAT WE DON'T HAVE TIME FOR : FIBER BUNDLES, ETC.

We must consider a covering of the manifold (spacetime) by local gauge patches, with different local gauges related by a gauge transformation on the overlap. Or on a manifold with no global frame we have the analogous covering by local frames. This is elegantly combined with the manifold M into a larger manifold which is locally the product manifold of M with a group (principal fiber bundle) or vector space (vector bundle). We then get the frame bundle and tensor bundles over the manifold ($GL(n,\mathbb{R})$ case) or principal G-bundles or vector bundles over M in the G-gauge theory case.

These are extremely useful in physics. The velocity and momentum phase spaces associated with a classical mechanical configuration space (= manifold M) are just the tangent and cotangent bundles over the manifold, namely $2n$-dimensional manifolds whose points are points of M plus tangent vectors or covectors at those points (position and velocity or position and momentum). The frame bundle (a $GL(n,\mathbb{R})$ principal bundle whose points are points of M plus frames at those points) is essential to discuss spinors & spinor analysis required for fields of half integral spin. The study of 2-spinors (group $G = SL(2,C)$) or Dirac spinors (G = group associated with Dirac algebra, with Lie algebra = span $\{\gamma_{[\alpha}\gamma_{\beta]}\}$) is carried out as above except the group objects are related to the manifold structure (spinors are rigidly connected to orthonormal frames on spacetime).

No time for Lagrangians.

No time for Laplacians, geodesics, gravitational theory

I could go on listing all of the things we don't have time for at this point. I hope some of you are interested enough to find the time to read about this for yourself.