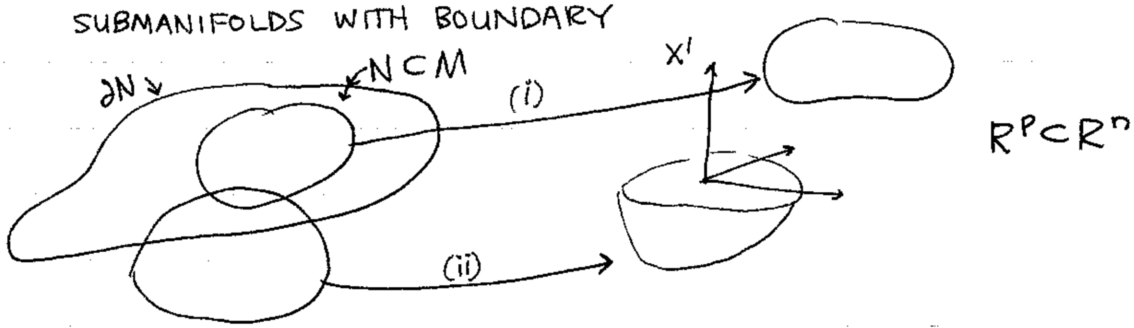
⑩ COVARIANT DIFFERENTIATION, PARALLEL TRANSPORT

NOTE ON LOCAL PARAMETRIZATIONS OF SUBMANIFOLDS OR SUBMANIFOLDS WITH BOUNDARY



An adapted coordinate chart (ϕ, U) on M maps $U \cap N$ into R^p. Its inverse ϕ^{-1} maps $V = \phi(U \cap N)$ back onto $U \cap N$ and is a diffeomorphism since the local coordinates $x^\alpha = u^\alpha \circ \phi$ are differentiable functions of the parameters u^α:

$$x^\alpha \circ \phi^{-1}(u) = u^\alpha \circ \phi \circ \phi^{-1}(u) = u^\alpha$$

since the identity map is trivially infinitely differentiable.

Thus every adapted local coordinate chart has a canonical local parametrization associated with it which uses the coordinates of the submanifold as the parameters.

<u>exercise</u>: Go back to the exercise on p. 90. Now let $N = \{(x,y,z) \in M \mid 0 \le z \le 1\}$ be a submanifold with boundary. Complete $\{\rho, \varphi\}$ to coordinates $\{\rho, \varphi, \zeta\}$ adapted to M. Make a trivial modification to get adapted coordinates $\{y^1, y^2, y^3\} = \{\rho - 1, \varphi, \zeta\}$ for N (assume ζ increases in the direction of the inward normal to M).

Express the unit normal $n_N = (1 + 4x^2 + 4y^2)^{-1/2}\left(\frac{\partial}{\partial z} - 2x\frac{\partial}{\partial x} - 2y\frac{\partial}{\partial y}\right)$ in the new coordinates. Obtain the unit normal 2-vector $n_{\partial N}$ by normalizing $\frac{\partial}{\partial y^1} \wedge \frac{\partial}{\partial y^3}$. Express in cartesian coordinates.

Express $\eta_{\partial N} = \frac{1}{2}\eta \,\lrcorner\, n_{\partial N}|_{\partial N}$ and $\eta_N = \eta \,\lrcorner\, n_N|_N$ in terms of the new coordinates.

Evaluate the area of N and the circumference of ∂N:

$$A = \int_N \eta_N, \quad C = \int_{\partial N} \eta^{\text{ind}}_{\partial N}.$$

Show that the inner orientation of ∂N induced by $n_{\partial N}$ is the opposite of the induced orientation for which $\partial/\partial y^2$ is positively oriented.

Express the Euclidean metric $g = \delta_{ij}\, dx^i \otimes dx^j$ and its restriction to N and ∂N in terms of the new coordinates. Compare $g^{1/2}_{\partial N}$ and $g^{1/2}_N$ with $\eta_{\partial N}$ and η_N.

Let $\beta = x\,dy$, $d\beta = dx \wedge dy$. Evaluate $\int_{\partial N} \beta$ and $\int_N d\beta$ directly in the new coordinates. Show that you get the same result by applying the metric formulas of p 99 ($z = n$) with $X = {*\beta^\#}$.

Repeat for $\beta = x^3 dy$ if you feel motivated.

BUT FIRST MATRIX GROUPS in a hurry

a matrix group $G \subset \begin{matrix} GL(n,\mathbb{R}) \\ \text{or} \\ GL(n,\mathbb{C}) \end{matrix} = n \times n$ real or complex nonsingular matrices (nonzero determinant)

is just a subgroup of one of these groups, i.e. $A_1A_2 \in G$ if $A_1 \in G, A_2 \in G$.
[The group multiplication is just matrix multiplication]

a matrix Lie algebra $\mathfrak{g} \subset \begin{matrix} \mathfrak{gl}(n,\mathbb{R}) \\ \text{or} \\ \mathfrak{gl}(n,\mathbb{C}) \end{matrix} = n \times n$ real or complex matrices

is just a Lie subalgebra of one of these Lie algebras, i.e. a linear subspace s.t. $[A_1, A_2] \in \mathfrak{g}$ if $A_1 \in \mathfrak{g}, A_2 \in \mathfrak{g}$.
[The commutator multiplication $A, B \to [A,B] \equiv AB - BA$ makes the vector space of $n \times n$ matrices into a Lie algebra.]

Every matrix group G has a matrix Lie algebra $\mathfrak{g}$ such that at least locally near $\underline{1} \in G$ (the identity matrix) points of G can be obtained by exponentiating elements of $\mathfrak{g}$:
$$e^B = A \in G, \quad B \in \mathfrak{g}$$
The matrix exponential is
$$e^B \equiv \sum_{k=0}^{\infty} \frac{B^k}{k!}, \quad B^0 \equiv \underline{1}.$$

If $\{E_a\}$ is a basis of $\mathfrak{g}$, with dual basis $\{W^a\}$, then
$[E_a, E_b] = C^c{}_{ab} E_c$ defines the components of the "structure constant tensor" of $\mathfrak{g}$ in this basis
and we can write: $A = e^{\theta^a E_a} \in G.$ (locally)

A <u>representation of G</u> is just a map from G into the group $GL(V)$ of nonsingular linear transformations of a vector space V into itself
$$\rho : G \to GL(V) \qquad \rho(g_1 g_2) = \rho(g_1)\rho(g_2) \qquad g_1, g_2 \in G$$
which preserves the multiplicative structure of G.

A <u>representation of $\mathfrak{g}$</u> is just a map from G into the Lie algebra $\mathfrak{gl}(V)$ of linear transformations of a vector space V into itself
$$\sigma : \mathfrak{g} \to \mathfrak{gl}(V) \qquad \sigma([A_1, A_2]) = [\sigma(A_1), \sigma(A_2)] \qquad A_1, A_2 \in \mathfrak{g}$$
which preserves the commutator structure of $\mathfrak{g}$.

We get matrix representations of G and $\mathfrak{g}$ respectively by expressing the linear transformations in terms of a basis $\{E_a\}$:

$$\rho^a{}_b(g) = W^a(\rho(g)e_b) \qquad g \in G \qquad \underline{\rho}(g) \equiv (\rho^a{}_b(g)) \in GL(m,\mathbb{R}) \text{ or } GL(m,\mathbb{C})$$

$$\sigma^a{}_b(A) = W^a(\sigma(A)e_b) \qquad A \in \mathfrak{g} \qquad \underline{\sigma}(A) \equiv (\sigma^a{}_b(A)) \in \underbrace{\mathfrak{gl}(m,\mathbb{R}) \text{ or } \mathfrak{gl}(m,\mathbb{C})}_{\text{if } m = \dim V.}$$

$\underline{\rho}(G)$ and $\underline{\sigma}(\mathfrak{g})$ are a matrix subgroup and matrix Lie algebra in m dimensions.

Given any representation of G, there is a corresponding representation of $\mathfrak{g}$ such that $\underline{\sigma}(\mathfrak{g})$ is the matrix Lie algebra of $\underline{\rho}(G)$. [namely if $A \in G$, $B \in \mathfrak{g}$ and $A = e^B$ then $\rho(A) = e^{\sigma(B)}$]

<u>EXAMPLE</u> A choice of basis $\{E_a\}$ for V maps its general linear group and its Lie algebra $GL(V)$ and $\mathfrak{gl}(V)$ onto $GL(m, \mathbb{R} \text{ or } \mathbb{C})$ and $\mathfrak{gl}(m, \mathbb{R} \text{ or } \mathbb{C})$ in this way in the same way V itself is mapped onto $\mathbb{R}^m$.
[i.e. by expressing everything in terms of the basis.]

There are two important representations of a pair $(G, \mathfrak{g})$:

(i) <u>The identity representation</u> (Id, id) of $(G, \mathfrak{g})$:

A matrix group G acts on $\mathbb{R}^n(\mathbb{C}^n)$ as a group of linear transformations so

$$\begin{aligned} Id(A) = \rho(A) &= \\ id(A) = \sigma(A) &= \end{aligned} \quad \text{linear transformation} \quad \begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix} \to A \begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix} \qquad \begin{aligned} &A \in G \\ &A \in \mathfrak{g} \end{aligned}$$

The associated matrix representations with respect to the natural basis $\mathbb{R}^n(\mathbb{C}^n)$ map every matrix A onto itself, i.e. are just the identity map.
[Okay a bit trivial but useful to have as terminology].

(ii) <u>The adjoint representation</u> (Ad, ad) of $(G, \mathfrak{g})$

Let $V = \mathfrak{g}$ and define for $B \in \mathfrak{g}$. Then $m = \dim G = \dim \mathfrak{g}$.

$$\rho(A)B \equiv Ad(A)B = ABA^{-1} = \text{conjugation by } A \in G$$

$$\sigma(A)B \equiv ad(A)B = [A,B] = \text{left bracketing by } A \in \mathfrak{g}$$

($[\,,\,]$ is called a Lie bracket)

Picking a basis $\{E_a\}$ gives the associated matrix representations. If $X = X^a E_a \in \mathfrak{g}$

$$W^a(ad(X)E_b) = W^a([X, E_b]) = W^a(X^c[E_c, E_b]) = X^c W^a(C^d{}_{cb} E_d) = C^a{}_{cb} X^c$$

$$\underline{ad}(X) = (C^a{}_{cb} X^c) \equiv X^c \underline{k}_c \quad , \quad \underline{k}_c \equiv (C^a{}_{cb}) \equiv \underline{ad}(E_c).$$

The matrices $\{\underline{k}_c\}$ satisfy $[k_a, k_b] = C^c{}_{ab} k_c$ and give an $m \times m$ representation of the original $n \times n$ matrix Lie algebra.

One can show $\underline{Ad}(e^{X^a E_a}) = e^{X^a \underline{k}_a}$ giving an $m \times m$ representation of G.

EXAMPLE $(G, \mathfrak{g}) = (SU(2), su(2))$

special unitary group in 2 dimensions $\quad SU(2) = \{ U \in GL(3,C) \mid U^\dagger = U^{-1}, \det U = 1 \}$

$$= \{ U^\alpha \hat{e}_\alpha \mid \delta_{\alpha\beta} U^\alpha U^\beta = 1 , (U^\alpha) \in R^4 \} \qquad \begin{matrix} \alpha, \beta = 1,2,3,4 \\ a, b = 1,2,3 \end{matrix}$$

where $\quad \hat{e}_4 = 1 \;, \hat{e}_a = -i\sigma_a \qquad \{\sigma_1, \sigma_2, \sigma_3\} = \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$ *Pauli matrices*

$$U = U^\alpha \hat{e}_\alpha = \begin{pmatrix} u^4 - iu^3 & -iu^1 - u^2 \\ -iu^1 + u^2 & u^4 + iu^3 \end{pmatrix}$$

Manifold of $SU(2)$ is diffeomorphic to $S^3 = \{ (u^\alpha) \in R^4 \mid \delta_{\alpha\beta} u^\alpha u^\beta = 1 \}$.

Matrix Lie algebra $\quad su(2) = \text{span}\{\hat{e}_a\}$

canonical basis $\{\frac{\hat{e}_a}{2}\}$: $\qquad [\frac{\hat{e}_a}{2}, \frac{\hat{e}_b}{2}] = C^c{}_{ab} \frac{\hat{e}_c}{2} \qquad C^c{}_{ab} = \epsilon_{cab}$

Relation between $SU(2)$ and $su(2)$: $\quad$ (set $\theta^a = \Theta n^a$; $n^a n^b \delta_{ab} = 1$:)

$$U(\theta) = e^{\theta^a \frac{\hat{e}_a}{2}} = e^{-\frac{i\theta^a}{2}\sigma_a} \quad \left(= \underline{1} \cos\frac{\Theta}{2} - i n^a \sigma_a \sin\frac{\Theta}{2} \equiv u^\alpha(\theta) \hat{e}_\alpha \right)$$

$SU(2)$ is 3-dimensional submanifold of the 16 dimensional real manifold $SL(3,C)$
This relation becomes a local parametrization of $SU(2)$ when the ranges of the θ^a are specified:

$$\theta^a = \Theta n^a \;, \quad \Theta = \sqrt{\delta_{ab}\theta^a\theta^b} \;, (n^a) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$$

$$\theta \in [0, \pi) \;, \quad \varphi \in [0, 2\pi) \;, \quad \Theta \in [0, 4\pi) \qquad [\text{note } e^{-2\pi i n^a \hat{e}_a} = -\underline{1}]$$

Identity representation

Let $SU(2)$ act on C^2 by matrix multiplication. Elements of C^2 are called 2-spinors.

Adjoint representation

$$\{\underline{k}_1, \underline{k}_2, \underline{k}_3\} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\} \qquad [\underline{k}_a, \underline{k}_b] = \epsilon_{abc} \underline{k}_c$$

span $\{\underline{k}_a\} = so(3,R) =$ antisymmetric 3×3 matrices (real)

$$Ad\left(e^{\theta^a \frac{\hat{e}_a}{2}}\right) = e^{\theta^a k_a} \equiv R(\theta) \;\in SO(3,R) = Ad(SU(2))$$

special orthogonal group in 3 dimensions (real).

This just reflects the familiar (?) fact that conjugating the Pauli matrices by an element of $SU(2)$ rotates them by the corresponding rotation

$$U(\theta)\, \sigma_a\, U(\theta)^{-1} = \sigma_b \left(e^{-\theta^c k_c}\right)^b{}_a$$

The parametrization

$$\underline{R}(\theta) = e^{\theta^a k_a} = e^{\Theta n^a k_a} \quad \in SO(3,R)$$

$$\Theta \in [0, 2\pi) \;, \quad \theta \in [0, \pi) \;, \varphi \in [0, 2\pi) \qquad \text{covers } SO(3,R) \text{ once.}$$

Two matrices $\pm U(\theta)$ correspond to a single rotation $R(\theta)$.

REPRESENTATIONS OF THE GENERAL LINEAR GROUP

The transformation law for the components of a $\binom{p}{q}$-tensor density of weight W over a vector space V under a change of basis is an example of a matrix representation of the general linear group $GL(n,R)$:

$$e_{\alpha'} = e_\beta A^{-1\beta}{}_\alpha \quad , \quad \omega^{\alpha'} = A^\alpha{}_\beta \omega^\beta$$

$$T'^{\alpha_1\cdots\alpha_p}{}_{\beta_1\cdots\beta_q} = (\det A^{-1})^W A^{\alpha_1}{}_{\gamma_1}\cdots A^{\alpha_p}{}_{\gamma_p} A^{-1\delta_1}{}_{\beta_1}\cdots A^{-1\delta_q}{}_{\beta_q} T^{\gamma_1\cdots\gamma_p}{}_{\delta_1\cdots\delta_q}$$

$$\equiv [\rho^{p,q}_W(A)\, T]^{\alpha_1\cdots\alpha_p}{}_{\beta_1\cdots\beta_q}$$

$$\rho^{p,q}_W : GL(n,R) \longrightarrow GL(\underbrace{(\otimes^p V)\otimes(\otimes^q V^*)}_{\text{vector space of representation}})$$

basis: $\{ e_{\alpha_1}\otimes\cdots\otimes e_{\alpha_p}\otimes\omega^{\beta_1}\otimes\cdots\otimes\omega^{\beta_q}\}$

"matrix of $\rho^{p,q}_W(A)$":

$$\rho^{p,q}_W(A)^{\overbrace{\alpha_1\cdots\alpha_p}{}}{}_{\underbrace{\beta_1\cdots\beta_p}_{\sim\text{ single contravariant index}}}{}_{\underbrace{\gamma_1\cdots\gamma_p}_{\sim\text{ single covariant index}}}{}^{\delta_1\cdots\delta_q} = (\det A^{-1})^W A^{\alpha_1}{}_{\gamma_1}\cdots A^{-1\delta_1}{}_{\beta_1}\cdots$$

Integral weight densities arise by taking the natural dual of a "tensor-valued form", i.e. a tensor with a subset of covariant indices which are antisymmetric [ditto for "tensor-valued p-vectors"]

EX. $\quad T = \frac{1}{2} T^\alpha{}_{\underbrace{\beta_1\beta_2}_{\text{antisym}}} e_\alpha\otimes\omega^{\beta_1\beta_2}$, $\quad {}^{\circledast}T = {}^{\circledast}T^{\alpha\beta_3\cdots\beta_n} e_\alpha\otimes e_{\beta_3\cdots\beta_n}$

$$\underbrace{{}^{\circledast}T^{\alpha\beta_3\cdots\beta_n}}_{\text{components of weight 1 tensor}} = \frac{1}{2} T^\alpha{}_{\beta_1\beta_2}\epsilon^{\beta_1\beta_2\beta_3\cdots\beta_n}$$

Note ${}^{\circledast}T$ is a basis dependent tensor over V.

Suppose V is an oriented vector space with metric (innerproduct) g such that η is positively oriented. Then

$${}^{\circledast}\eta = \frac{1}{n!}\eta_{\alpha_1\cdots\alpha_n}\epsilon^{\alpha_1\cdots\alpha_n} = \eta_{1\cdots n} = \pm g^{1/2} \qquad (\text{sgn} = \text{orientation of basis})$$

is a weight 1 scalar density, but

$$g^{1/2} = |{}^{\circledast}\eta| \quad \text{satisfies} \quad g^{1/2\,\prime} = |\det A^{-1}|\, g^{1/2} .$$

"Oriented densities" transform by the absolute value of $\det A^{-1}$; $g^{1/2}$ is an oriented scalar density (weight = 1 if not specified).

We can get oriented densities of any real weight W by multiplying a tensor by $(g^{1/2})^W$.

Suppose we consider a parametrized curve through the identity of the general linear group: $\quad A(t) = (A^{\nu}{}_{\beta}(t))\ , \quad A(0) = \mathbb{1}\ , \quad A'(0) \equiv \frac{dA}{dt}(0) = B.$

Then define:

$$\frac{d}{dt}\Big|_{t=0} T'^{\alpha_1\cdots\alpha_p}{}_{\beta_1\cdots\beta_q} = \frac{d}{dt}\left[\rho^{p,q}_{W}(A(t))T\right]^{\alpha_1\cdots\alpha_p}{}_{\beta_1\cdots\beta_q}$$

$$= B^{\alpha_1}{}_{\gamma_1}T^{\gamma_1\cdots\alpha_p}{}_{\beta_1\cdots\beta_q} + \cdots \underbrace{- B^{\delta_1}{}_{\beta_1}}T^{\alpha_1\cdots\alpha_p}{}_{\delta_1\cdots\beta_q} - \cdots \underbrace{- W B^{\gamma}{}_{\gamma}}T^{\alpha_1\cdots\alpha_p}{}_{\beta_1\cdots\beta_p}$$

$$\equiv \left[\sigma^{p,q}_{W}(B)T\right]^{\alpha_1\cdots\alpha_p}{}_{\beta_1\cdots\beta_q}$$

$$\boxed{\begin{aligned} dA^{-1} &= -A^{-1}dA\,A^{-1} \\ \frac{d}{dt}A^{-1} &= -A^{-1}\frac{dA}{dt}A^{-1} \end{aligned}}$$

$$\boxed{\begin{aligned} &A^{-1}A = 1 \\ &[dA^{-1}A + A^{-1}dA = 0]\,A^{-1} \end{aligned}}$$

$$\boxed{\frac{d}{dt}(\ln\det A^{-1}) = -\mathrm{Tr}\,A\frac{dA^{-1}}{dt} = -\mathrm{Tr}\frac{dA}{dt}A^{-1}}$$

$$\boxed{d\ln\det A = \mathrm{Tr}\,A^{-1}dA = \mathrm{Tr}\,dA\,A^{-1}}$$

$$\boxed{\begin{aligned} \det A &= \frac{1}{n!}\delta^{\gamma_1\cdots\gamma_n}_{\delta_1\cdots\delta_n}A^{\delta_1}{}_{\gamma_1}\cdots A^{\delta_n}{}_{\gamma_n} \equiv \frac{1}{n}\Delta^{\gamma_1}{}_{\delta_1}(A)A^{\delta_1}{}_{\gamma_1} \\ A^{-1\gamma}{}_{\delta} &= (\det A)^{-1}\Delta^{\gamma}{}_{\delta}(A) \\ d(\det A) &= \frac{1}{n!}\delta^{\gamma_1\cdots\gamma_n}_{\delta_1\cdots\delta_n}\left[(dA^{\delta_1}{}_{\gamma_1})A^{\delta_2}{}_{\gamma_2}\cdots A^{\delta_n}{}_{\gamma_n} + \cdots \right. \\ &\qquad \left. + A^{\delta_1}{}_{\gamma_1}\cdots A^{\delta_{n-1}}{}_{\gamma_{n-1}}dA^{\delta_n}{}_{\gamma_n}\right] \\ &= \frac{1}{(n-1)!}\delta^{\gamma_1\cdots\gamma_n}_{\delta_1\cdots\delta_n}(dA^{\delta_1}{}_{\gamma_1})A^{\delta_2}{}_{\gamma_2}\cdots A^{\delta_n}{}_{\gamma_n} \\ &= \Delta^{\gamma_1}{}_{\delta_1}dA^{\delta_1}{}_{\gamma_1} = \det A\,A^{-1\gamma_1}{}_{\delta_1}dA^{\delta_1}{}_{\gamma_1} \end{aligned}}$$

$$\sigma^{p,q}_{W} : gl(n,\mathbb{R}) \longrightarrow gl((\otimes^p V)\otimes(\otimes^q V^*))$$

is the associated representation of $gl(n,\mathbb{R})$.

<u>Remark</u> For any representation $\rho : G \to GL(V)$ of a matrix group G, the associated representation of its matrix Lie algebra $\sigma : g \to gl(V)$ can be defined by: $\boxed{\sigma(B) = \frac{d}{dt}\Big|_0 \rho(A(t))}$ for $A(t) = e^{tB}$.

[since $\rho(A(t)) = e^{\sigma(tB)} = e^{t\sigma(B)}$; $\frac{d}{dt}\rho(A(t)) = \sigma(B)e^{t\sigma(B)} = e^{t\sigma(B)}\sigma(B)$, set $t=0$]

Now you say, so what?

The notation $\rho^{p,q}_{W}$ and $\sigma^{p,q}_{W}$ saves writing out long fomulas when transforming fields or differentiating fields. That's all.

Looks like I've got room to sneak in an

ERRATA. p 80: "$, c > 0$" is missing at the end of the 3rd line from the bottom.

p 83: $\eta_W = (-1)^{\frac{n-s}{2}}(*^{-1}\eta)^{\flat}$

exercise Suppose A is a matrix valued function, for example, a function on the real line as above (a function of t). Then observe the following.

First note that the transformation law for a vector is the identity representation of $GL(n,\mathbb{R})$
$$[\rho^{1,0}_{0}(A)Y]^\alpha = A^\alpha{}_\beta Y^\beta = [\rho^{id}(A)Y]^\alpha.$$

Then $d[\rho^{id}(A)Y]^\alpha = dA^\alpha{}_\beta Y^\beta = \underbrace{A^\alpha{}_\gamma}_{\rho^{id}(A)^\alpha{}_\gamma} \underbrace{A^{-1\gamma}{}_\delta\, dA^\delta{}_\beta Y^\beta}_{[\sigma^{id}(A^{-1}dA)Y]^\gamma}$

Since Y is arbitrary, $\quad d[\rho^{id}(A)] = \rho^{id}(A)\,\sigma^{id}(A^{-1}dA)$

Multiply on left by $\rho^{id}(A^{-1}) = \rho^{id\,-1}(A)$:
$$\rho^{id\,-1}(A)\, d[\rho^{id}(A)] = \sigma^{id}(A^{-1}dA)$$

Or suppressing the superscript "id"
$$\boxed{\rho^{-1}(A)\, d[\rho(A)] = \sigma(A^{-1}dA).}$$

So far Ive done all the work. Now you convince yourself that this formula holds for all of the tensor density representations of $GL(n,\mathbb{R})$. This remains true for any differential operator in place of d since it only uses the product rule for derivatives.

In fact this relation is true for any pair of associated representations (ρ,σ) of a Lie group and its Lie algebra $(G,\mathfrak{g})$. If we get to Lie groups maybe we'll prove this.

We have used a property of representations above.

Note: $\quad \rho(AB) = \rho(A)\rho(B)$

(i) set $B=1$ $\quad \rho(A) = \rho(A)\rho(1) \rightarrow \rho(1) = 1$ = identity transformation on V

(ii) set $B=A^{-1}$: $\quad 1 = \rho(1) = \rho(AA^{-1}) = \rho(A)\rho(A^{-1}) \rightarrow \rho(A^{-1}) = \rho(A)^{-1} \equiv \rho^{-1}(A)$

The matrix representing A^{-1} is the inverse of the matrix representing A.

This is a long exercise, huh? For me not for you.

One last thing, replace A by A^{-1} in the BOXED EQUATION:
$$\boxed{\rho(A)\, d[\rho(A^{-1})] = \sigma(A\, dA^{-1})}$$

CONNECTION ON A MANIFOLD

The value of a function at a point is just a real number and can easily be transported from that point of a manifold to any other point along a given curve between the two points: the transported value is just the value at the original point and it doesn't matter what curve one takes between the two points, the transported value of the function is always the same. The result is path independent.

A function is called covariantly constant along a curve if its value at one point of the curve coincides with the value transported to that point from every other point of the curve, i.e. if the function has the same value all along the curve. A function is covariantly constant if the transported value of the function always agrees with its value at the new point, i.e. if the function is a constant function.

Suppose we try to repeat this for vector fields and the value of such fields at a point, namely tangent vectors. The closest thing we might try would be to choose a particular frame on the manifold and just require that the transported value of a tangent vector along a curve have the same components in this frame as at the initial point. Clearly this too is path independent. A vector field is covariantly constant along a curve if its transported value from any initial point of the curve agrees with the value of the vector field at every other point of the curve. A vector field is covariantly constant if this is true for all curves.

But suppose we work in a new frame. The components of the transported vector will not remain constant along a curve, but will change in a way that can be determined by transforming from the original frame to the new frame (unless the new frame is related to the original one by a constant transformation matrix, in which case no change occurs). However, in order to talk about such a rule for transport we have already singled out a class of privileged frames, i.e. we have imposed more structure on the manifold. This structure is called a <u>connection on the manifold</u> and there are many ways of describing it mathematically. It amounts to a rule for transporting tangent vectors along curves which therefore gives a correspondence between tangent spaces connected by curves, i.e. a way to compare tangent vectors at different points. [It is called a connection since it provides a way to relate tangent spaces at different points of a manifold, i.e. it "connects" tangent spaces.] When the result is path independent, as it is in our construction, the connection is called flat. In our construction, all vector fields with constant components in a privileged frame are covariantly constant. In particular the n^2-parameter family ($n^2 = \dim GL(n,\mathbb{R})$, where $n = \dim M$) of privileged frames are covariantly constant.

Suppose we have a pseudo-Riemannian manifold M with metric g. The metric itself is an additional piece of structure on the manifold. If we insist that all inner products of tangent vectors remain constant under transport, it turns out that the rule for transport is then determined uniquely (if one imposes an additional condition called symmetry or zero torsion as we will later see). This connection is called the <u>metric connection</u>.

Once we specify a rule to transport tangent vectors, then since values of functions (including the components of tensor fields) are trivially transported, we can transport any tensor field. Tangent 1-forms are transported so that their evaluations on tangent vectors are transported as real numbers, i.e. remain constant. Thus if one knows how to transform a frame along a curve from one tangent space to another, then the dual frame must be transported so that it remains dual to the transported frame. A tangent tensor is then transported along a curve so that its components with respect to the transported frame remain constant (since they are just numbers).

For example, suppose $c(t)$ is a parametrized curve through $x = c(0) \in M$ and $\{e_\alpha\}$ is a frame on M with dual frame $\{\omega^\alpha\}$. Let $\{e_\alpha(t)\}$ be the value of the frame transported to $c(t)$ along the curve from $c(0)$ where $e_\alpha(0) = e_\alpha|_x$.

Define a matrix $A(t)$ by $\quad e_\alpha(t) = A^{-1\,\beta}{}_\alpha(t)\, e_\beta|_{c(t)}$.

By duality the transported dual frame must be $\quad \omega^\alpha(t) = A^\alpha{}_\beta(t)\, \omega^\beta|_{c(t)}$

If $X = X^\alpha e_\alpha$ is a vector field, then its value at $c(t)$ transported along the curve from $c(0)=x$ is just

$$X(t) = X^\alpha|_x\, e_\alpha(t) = X^\alpha|_x\, A^{-1\,\beta}{}_\alpha(t)\, e_\beta|_{c(t)} \in TM_{c(t)}.$$

Conversely, the value at x transported along the curve back from $c(t)$ is

$$A^\alpha{}_\beta(t)\, X^\beta|_{c(t)}\, e_\beta \in TM_x.$$

Similarly for any tensor field

$$\begin{aligned} T(t) &= T^{\alpha_1\cdots\alpha_p}{}_{\beta_1\cdots\beta_q}|_x\, e_{\alpha_1}(t)\otimes\cdots\otimes\omega^{\beta_1}(t)\otimes\cdots \\ &= [\rho^{p,q}_{\ 0}(A^{-1}(t))\, T|_x]^{\alpha_1\cdots\alpha_p}{}_{\beta_1\cdots\beta_q}\, e_{\alpha_1}|_{c(t)}\otimes\cdots\otimes\omega^{\beta_1}|_{c(t)}\otimes\cdots \end{aligned}.$$

R^n has a privileged class of frames, namely the coordinate frames associated with any global cartesian coordinate chart. We can therefore introduce a flat connection such that these frames are covariantly constant.

On R^n it is clear what we mean by translating a tangent vector. We just move it around keeping its length and direction fixed; in particular it always remains parallel to its original value. The transport associated with the flat connection is just this "parallel translation", so in analogy with this we call transport "parallel transport" in general. Note that all inner products automatically remain constant under parallel transport on R^n, since they involve only the cartesian components which remain constant, so this is the connection associated with the usual flat metric of R^n (the connection is symmetric as we will see). In fact this remains true for any metric on R^n with constant cartesian components. Thus both the Euclidean and Minkowski metrics on R^4 have the same flat connection.

Suppose we wish to work in a noncartesian frame, like the one associated with spherical coordinates on R^3, for example. How can we perform parallel transport working with such a frame?

Let $\quad e_\alpha = e^i{}_\alpha \frac{\partial}{\partial x^i}$, $\omega^\alpha = \omega^\alpha{}_i \, dx^i$, $\quad A \equiv (\omega^\alpha{}_i)$, $A^{-1} \equiv (e^i{}_\alpha)$.

The matrix A transforms components from the cartesian frame to the noncartesian frame.

Well, here it is useful to go to a differential rule: if the cartesian components remain constant along the curve under parallel transport, their derivative along the curve should be zero.

But we also know how to differentiate tensor fields on R^n as long as we work in cartesian frames. We define the covariant derivative of a tensor field T along a tangent vector X to be a tangent tensor $\nabla_X T$ of the same type as T at the location of X whose cartesian components are just ordinary derivative of the cartesian components of T by X:

$$(\nabla_X T)^{i_1 \cdots i_p}{}_{j_1 \cdots j_q} \equiv X \, T^{i_1 \cdots i_p}{}_{j_1 \cdots j_q} \, .$$

When X is a vector field, $\nabla_X T$ is a tensor field of the same type as T. Its components in a noncartesian frame are obtained by transforming from a cartesian frame. The covariant derivative of a function f by X is just Xf.

EXAMPLE. Recall $\partial_\alpha f \equiv e_\alpha f$. Then

$$Y^\alpha{}_{;\beta} X^\beta \equiv (\nabla_X Y)^\alpha = \omega^\alpha{}_i (\nabla_X Y)^i = \omega^\alpha{}_i (X^j \partial_j Y^i) = \omega^\alpha{}_i (X^\beta \partial_\beta Y^i)$$

$$= \omega^\alpha{}_i [X^\beta \partial_\beta (Y^\gamma e^i{}_\gamma)] = \omega^\alpha{}_i [(X^\beta \partial_\beta Y^\gamma) e^i{}_\gamma + Y^\gamma X^\beta \partial_\beta e^i{}_\gamma]$$

$$= X Y^\alpha + \underbrace{(\omega^\alpha{}_i \partial_\beta e^i{}_\gamma)}_{} X^\beta Y^\gamma$$

"components of the connection" $\equiv \Gamma^\alpha{}_{\beta\gamma} = \omega^\alpha(\nabla_{e_\beta} e_\gamma) \leftrightarrow \nabla_{e_\beta} e_\gamma = \Gamma^\alpha{}_{\beta\gamma} e_\alpha$

"connection 1-forms" $\omega^\alpha{}_\gamma \equiv \Gamma^\alpha{}_{\beta\gamma} \omega^\beta \quad \rightarrow \quad \underbrace{\nabla_X e_\gamma = \omega^\alpha{}_\gamma(X) e_\alpha}$

$$= X^\beta \underbrace{(\partial_\beta Y^\alpha + \Gamma^\alpha{}_{\beta\gamma} Y^\gamma)}_{\equiv Y^\alpha{}_{;\beta}}$$

$$= \underbrace{X Y^\alpha + \omega^\alpha{}_\gamma(X) Y^\gamma}$$

The covariant derivative along X induces a linear transformation of the frame vectors whose matrix is the value of the connection 1-form matrix on X.

Since the frame vectors are not covariantly constant, the covariant derivative is not only the derivative of the components but includes the additional linear transformation of those components due to the covariant derivatives of the frame vectors.

<u>exercise</u> From the definition we immediately get the relation which follows, once we recall the definition of the structure functions of a frame:

$$[e_\alpha, e_\beta] = C^\gamma{}_{\alpha\beta} e_\gamma \quad \Leftrightarrow \quad d\omega^\alpha = -\tfrac{1}{2} C^\alpha{}_{\beta\gamma} \omega^{\beta\gamma}.$$

Then $\Gamma^\alpha{}_{\beta\gamma} - \Gamma^\alpha{}_{\gamma\beta} = \omega^\alpha{}_i (\partial_\beta e^i{}_\gamma - \partial_\gamma e^i{}_\beta) = \omega^\alpha{}_i \, [e_\beta, e_\gamma]^i$.

$$= \omega^\alpha([e_\beta, e_\gamma]) = C^\alpha{}_{\beta\gamma} \qquad \text{or} \qquad \boxed{\Gamma^\alpha{}_{[\beta\gamma]} = \tfrac{1}{2} C^\alpha{}_{\beta\gamma}}$$

Thus if $\{e_\alpha\}$ is a noncartesian coordinate frame, then since partial derivatives commute, $C^\gamma{}_{\alpha\beta} = 0$ and the covariant components of the connection are symmetric. This is what is meant by a <u>symmetric connection</u>.

Show that the boxed equality can be written

$$T^\alpha \equiv d\omega^\alpha + \omega^\alpha{}_\beta \wedge \omega^\beta = 0 \qquad \text{[vanishing torsion]}.$$

Since $\omega^\alpha(e_\beta) = \delta^\alpha{}_\beta$ and ∇_X obeys the product rule (by its definition as the ordinary derivative in a cartesian frame): $(\nabla_X \omega^\alpha)(e_\beta) + \underbrace{\omega^\alpha(\nabla_X e_\beta)}_{\omega^\alpha{}_\beta(X) = \omega^\alpha{}_\gamma(X)\,\omega^\gamma(e_\beta)} = \nabla_X \delta^\alpha{}_\beta = 0$

so $\boxed{\nabla_X \omega^\alpha = -\omega^\alpha{}_\beta(X)\,\omega^\beta}$.

The calculation we did to evaluate $\nabla_X Y$ can be expressed in a different way using the transformation matrix A following the groundwork of page 108.

First $\quad \omega^\alpha{}_i \partial_\beta e^i{}_\gamma X^\beta = \omega^\alpha{}_\gamma(X)$ can be written $A \nabla_X A^{-1} = \omega(X)$.

$$\left[\text{ note also } \quad (\nabla_X A)\, A^{-1} = -\omega(X)\,\right]$$

suppressing $\rho^{1,0}_{0} = \rho^{\mathrm{id}}$

Then
$$Y^\alpha = \rho(A)^\alpha{}_i Y^i$$
$$\begin{aligned}
[\nabla_X Y]^\alpha &= \rho(A)^\alpha{}_i (\nabla_X Y)^i \\
&= \rho(A)^\alpha{}_i \,\nabla_X\left[\rho(A^{-1})^i{}_\beta Y^\beta\right] \\
&= \underbrace{\rho(A)^\alpha{}_i\, \rho(A^{-1})^i{}_\beta}_{\delta^\alpha{}_\beta} \nabla_X Y^\beta + \underbrace{\rho(A)^\alpha{}_i \nabla_X\left[\rho(A^{-1})^i{}_\beta\right]}_{\sigma(A\nabla_X A^{-1})^\alpha{}_\beta} Y^\beta
\end{aligned}$$

$\sigma(A\nabla_X A^{-1})^\alpha{}_\beta\, Y^\beta$ by oblique boxed equation on page 108, letting $d \to \nabla_X$

$$\begin{aligned}
&= X Y^\alpha + [\sigma(A\nabla_X A^{-1}) Y]^\alpha \\
&= X Y^\alpha + [\sigma(\omega(X))\, Y]^\alpha .
\end{aligned}$$

This same calculation can be repeated for any $\binom{p}{q}$- tensor density of weight w representation of $GL(n,\mathbb{R})$:

$$(\nabla_X T)^{\alpha_1\cdots\alpha_p}{}_{\beta_1\cdots\beta_q} = X T^{\alpha_1\cdots\alpha_p}{}_{\beta_1\cdots\beta_q} + \left[\sigma^{p,q}_{w}(\omega(X))\,T\right]^{\alpha_1\cdots\alpha_p}{}_{\beta_1\cdots\beta_q} .$$

Since the covariant derivative is linear in X we can introduce a $\binom{p}{q+1}$- tensor density of weight w called the <u>covariant derivative</u> of T

$$\begin{aligned}
(\nabla T)^{\alpha_1\cdots\alpha_p}{}_{\beta_1\cdots\beta_q\beta_{q+1}} &\equiv T^{\alpha_1\cdots\alpha_p}{}_{\beta_1\cdots\beta_p;\beta_{p+1}} \\
&\equiv \partial_{\beta_{q+1}} T^{\alpha_1\cdots\alpha_p}{}_{\beta_1\cdots\beta_q} + \left[\sigma^{p,q}_{w}(\omega_{\beta_{q+1}})\,T\right]^{\alpha_1\cdots\alpha_p}{}_{\beta_1\cdots\beta_q} .
\end{aligned}$$

Closely related to the covariant derivative of a tensor field by a tangent vector X is the covariant derivative of a tensor defined along a parametrized curve.

First let $T\circ c(t)$ be the value of a tensor density along $c(t)$ and define

$$\frac{D}{dt}[T\circ c(t)] = \nabla_{c'(t)} T = \left\{c'(t)\, T^{\alpha_1\cdots\alpha_p}{}_{\beta_1\cdots\beta_q} + \left[\sigma^{p,q}_{w}(\omega(c'(t)))\,T\right]^{\alpha_1\cdots\alpha_p}{}_{\beta_1\cdots\beta_q}\right\} e_{\alpha_1}\otimes\cdots\omega^{\beta_1}\otimes\cdots .$$

Then correspondingly if $T(t) \in$ "$T^{p,q}_{w} M_{c(t)}$" is a tangent tensor density defined along $c(t)$

define
$$\frac{D}{dt} T^{\alpha_1\cdots\alpha_p}{}_{\beta_1\cdots\beta_q}(t) = \frac{d}{dt} T^{\alpha_1\cdots\alpha_p}{}_{\beta_1\cdots\beta_q}(t) + \left[\sigma^{p,q}_{w}(\omega(c'(t)))\,T(t)\right]^{\alpha_1\cdots\alpha_p}{}_{\beta_1\cdots\beta_q} .$$

$\frac{D}{dt}$ is exactly the transformed version of $\frac{d}{dt}$ acting on the cartesian components of a tangent tensor density defined along $c(t)$.

Note for example: $\omega^\alpha{}_\beta(c'(t)) = \omega^\alpha\left(\frac{D}{dt}(e_\beta \circ c(t))\right)$.

[e_β is the vector with components $\delta^\alpha_\beta \to$ plug into formula]

This allows us to connect the covariant derivative with parallel transport. A tensor field which is parallelly transported along a curve $c(t)$ from $c(0)$ has constant cartesian components and hence satisfies

$$\frac{D}{dt} T(t) = 0 .$$

Return to the discussion of page 110 and let

$$T^{\alpha_1\cdots\alpha_p}{}_{\beta_1\cdots\beta_q}(t) = \left[\rho^{p,q}_{\ 0}(\underset{\sim}{A}^{-1}(t))\, T|_x\right]^{\alpha_1\cdots\alpha_p}{}_{\beta_1\cdots\beta_q}$$

be the components of the tensor field $T(t)$ parallelly transported along the curve $c(t)$ from $x=c(0)$. The notation $\underset{\sim}{A}(t)$ is used so this matrix is not confused with the frame transformation matrix.

We can consider the inverse operation, namely take the value of T at $c(t)$ and parallel transport it back to x. Let $X = c'(0)$ and now let $T(t)$ stand for the tensor at x with components:

$$T^{\alpha_1\cdots\alpha_p}{}_{\beta_1\cdots\beta_q}(t) = \left[\rho^{p,q}_{\ 0}(\underset{\sim}{A}(t))\, T\circ c(t)\right]^{\alpha_1\cdots\alpha_p}{}_{\beta_1\cdots\beta_q}$$

Now we can take the ordinary derivative of a 1-parameter family of tensors at x:

$$\left.\frac{d}{dt}\right|_{t=0} T^{\alpha_1\cdots\alpha_p}{}_{\beta_1\cdots\beta_q}(t) = \underbrace{\left.\frac{d}{dt}\right|_{t=0} T^{\alpha_1\cdots\alpha_p}{}_{\beta_1\cdots\beta_q}\circ c(t)}_{\underset{X}{\underline{c'(0)}}\, T^{\alpha_1\cdots\alpha_p}{}_{\beta_1\cdots\beta_q}} + \underbrace{\left[\left.\frac{d}{dt}\right|_{t=0}\{\rho^{p,q}_{\ 0}(\underset{\sim}{A}(t))\}\, T|_x\right]^{\alpha_1\cdots\alpha_p}{}_{\beta_1\cdots\beta_q}}_{\left[\sigma^{p,q}_{\ 0}\left(\underbrace{\left.\frac{d}{dt}\right|_0 \underset{\sim}{A}(t)}_{=\,\omega(c'(0))\,=\,\omega(X)}\right) T|_x\right]^{\alpha_1\cdots\alpha_p}{}_{\beta_1\cdots\beta_q}}$$

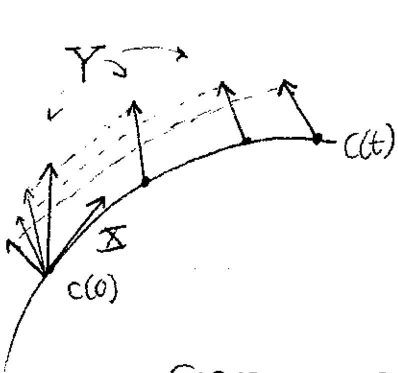
$$= [\nabla_X T]^{\alpha_1\cdots\alpha_p}{}_{\beta_1\cdots\beta_q}$$

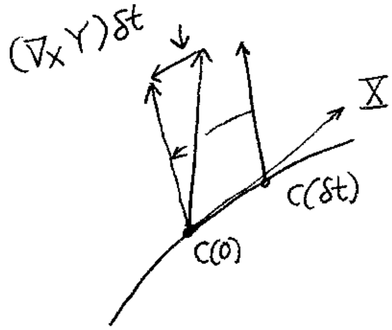
$e_\alpha(t) = e_\beta|_{c(t)}\, \underset{\sim}{A}^{-1\beta}{}_\alpha(t)$ is covariant constant along $c(t)$ so

$$0 = \frac{D}{dt}\left[e_\beta\, \underset{\sim}{A}^{-1\beta}{}_\alpha(t)\right] = \left.\frac{D}{dt}\right|_{t=0}\left[e_\beta \underset{\sim}{A}^{-1\beta}{}_\alpha(t)\right]$$

$$= \left[\frac{De_\beta}{dt}\underset{\sim}{A}^{-1\beta}{}_\alpha(t) - e_\beta \underset{\sim}{A}^{-1\beta}{}_\gamma(t)\, \frac{d}{dt}\underset{\sim}{A}^\gamma{}_\delta(t)\, \underset{\sim}{A}^{-1\delta}{}_\alpha(t)\right]\Big|_{t=0}$$

$$= \underbrace{\left.\frac{De_\beta}{dt}\right|_{t=0}}_{\omega^\alpha{}_\beta(X)e_\alpha} - e_\beta \frac{d\underset{\sim}{A}^\beta{}_\alpha(0)}{dt} \quad\rightarrow\quad \omega(X) = \left.\frac{d}{dt}\right|_{t=0}\underset{\sim}{A}(t)$$





Since we cannot compare tensors at different points of a manifold directly, we have to move them all to the same point first, which is what parallel transport allows us to do. For the values of a tensor field along a curve, we can transport them all back to the point $c(0)$ where we can take the ordinary t derivative of the one parameter family of tensors we obtain. This gives us the covariant derivative of the tensor field along the tangent $c'(0) = \mathbb{X}$ at $c(0)$.

METRIC CONNECTION (the metric is covariantly constant)

Suppose $X(t)$ and $Y(t)$ are parallelly transported along $c(t)$ from their values $\mathbb{X}(0)$ and $Y(0)$ at $c(0)$: $\quad \dfrac{D\mathbb{X}(t)}{dt} = 0 = \dfrac{DY(t)}{dt}$.

For a metric connection their inner product also remains constant:

$$0 = \frac{d}{dt}\left[g_{\alpha\beta}\circ c(t)\,\mathbb{X}^\alpha(t)\,Y^\beta(t)\right] = \frac{D}{dt}\left[g_{\alpha\beta}\circ c(t)\,\mathbb{X}^\alpha(t)\,Y^\beta(t)\right]$$

$$= \left[\frac{D}{dt}g_{\alpha\beta}\circ c(t)\right]\mathbb{X}^\alpha(t)Y^\beta(t) + g_{\alpha\beta}\circ c(t)\left[\underbrace{\frac{D\mathbb{X}^\alpha}{dt}}(t)\,Y^\beta(t) + \mathbb{X}^\alpha(t)\frac{DY^\beta}{dt}(t)\right]$$

$$= \left[\frac{D}{dt}g_{\alpha\beta}\circ c(t)\right]\mathbb{X}^\alpha(t)Y^\beta(t)$$

shorthand for $\left(\dfrac{D\mathbb{X}}{dt}\right)^\alpha = 0$ etc

shorthand for $\left[\dfrac{D\,\mathbb{g}\circ c(t)}{dt}\right]_{\alpha\beta}$

Since X and Y are arbitrary we must have

$$\nabla_{c'(t)}\,\mathbb{g} = 0 \quad \text{for every } c(t)$$

hence $\boxed{\nabla \mathbb{g} = 0}$ In components: $g_{\beta\gamma;\alpha} = \partial_\alpha g_{\beta\gamma} - \underbrace{g_{\delta\gamma}\Gamma^\delta{}_{\alpha\beta}}_{\equiv \Gamma_{\delta\alpha\beta}} - \underbrace{g_{\beta\delta}\Gamma^\delta{}_{\alpha\gamma}}_{\equiv \Gamma_{\beta\alpha\gamma}} = 0$

Or
$$\partial_\alpha g_{\beta\gamma} = \Gamma_{\gamma\alpha\beta} + \Gamma_{\beta\alpha\gamma}$$
$$\partial_\beta g_{\gamma\alpha} = \Gamma_{\alpha\beta\gamma} + \Gamma_{\gamma\beta\alpha}$$
$$-\partial_\gamma g_{\alpha\beta} = -\Gamma_{\beta\gamma\alpha} - \Gamma_{\alpha\gamma\beta}$$

$$\{\gamma\alpha\beta\} \equiv \tfrac{1}{2}(\partial_\alpha g_{\beta\gamma} + \partial_\beta g_{\gamma\alpha} - \partial_\gamma g_{\alpha\beta}) = \underbrace{\Gamma_{\gamma(\alpha\beta)}}_{\Gamma_{\gamma\alpha\beta} - \Gamma_{\gamma[\alpha\beta]}} - \Gamma_{\beta[\gamma\alpha]} - \Gamma_{\alpha[\gamma\beta]}$$

Use symmetry $\Gamma^\gamma{}_{[\alpha\beta]} = \tfrac{1}{2}C^\gamma{}_{\alpha\beta}$

$$\Gamma_{\gamma\alpha\beta} = \{\gamma\alpha\beta\} + \Gamma_{\gamma[\alpha\beta]} + \Gamma_{\beta[\gamma\alpha]} + \Gamma_{\alpha[\gamma\beta]} = \{\gamma\alpha\beta\} + \tfrac{1}{2}(C_{\gamma\alpha\beta} + C_{\beta\gamma\alpha} + C_{\alpha\gamma\beta})$$

$$\Gamma^\gamma{}_{\alpha\beta} = g^{\gamma\delta}\{\delta\alpha\beta\} + \tfrac{1}{2}(C^\gamma{}_{\alpha\beta} + C_\beta{}^\gamma{}_\alpha + C_\alpha{}^\gamma{}_\beta)$$

$$\Gamma^\gamma{}_{\alpha\beta} = \left\{ {\gamma \atop \alpha\beta} \right\} + \tfrac{1}{2} C^\gamma{}_{\nu\beta} + C_{(\beta}{}^\gamma{}_{\alpha)}$$

$\left\{ {\gamma \atop \alpha\beta} \right\}$ and $\{\gamma\alpha\beta\}$ are called Christoffell symbols of the first and second kind but which is first and which is second means looking up the answer in a book and I'm not in the mood. These are the components of the connection in a coordinate frame ($C^\alpha{}_{\beta\gamma}=0$).
On the other hand in an orthonormal frame ($dg_{\alpha\beta}=0$), these vanish and the structure functions determine the connection components.

In fact suppose we have an arbitrary pseudo-Riemannian manifold (M, g). The same formula defines the metric connection which tells us how to transport tensors or tensor densities along curves in such a way that all inner products remain invariant. The only thing we cannot do if the connection is not flat (parallel transport path dependent) is go back to a covariantly constant frame (cartesian frames in the case of R^n).

CURVATURE

Returning to R^n, note that the connection formula may be written in a suggestive way: $\Gamma^\alpha{}_{\beta\gamma} = \omega^\alpha{}_i \partial_\beta e^i{}_\gamma \to \omega = A dA^{-1}$.

Note that $\Omega \equiv d\omega + \omega\wedge\omega = \underbrace{d(AdA^{-1})}_{dA\wedge dA^{-1}} + AdA^{-1}\wedge AdA^{-1} = 0$

$$\underbrace{dA}_{-AdA^{-1}dA}\wedge dA^{-1} = -AdA^{-1}\wedge AdA^{-1}$$

Now suppose we let $A = A_2 A_1$, i.e. first we transform from the cartesian frame $\{\partial_i\}$ to the frame $e_{1\alpha}$ and then to another frame $\{e_{2\alpha}\}$:

$$e_{2\alpha} = \partial_i (A_2 A_1)^{-1\,i}{}_\alpha = \underbrace{(\partial_i A_1^{-1\,i}{}_\beta)}_{e_{1\beta}} A_2^{-1\,\beta}{}_\alpha$$

no derivative occurring!!

The connection 1-form $\omega_{(2)} \equiv \omega$ in the final frame is:

$$\omega_{(2)} = AdA^{-1} = A_2 A_1 d(A_2 A_1)^{-1} = A_2 A_1 d(A_1^{-1} A_2^{-1})$$
$$= A_2 A_1 dA_1^{-1} A_2^{-1} + A_2 A_1 A_1^{-1} dA_2^{-1} = A_2 \omega_{(1)} A_2^{-1} + A_2 dA_2^{-1}$$

$$\boxed{\omega_{(2)} = A_2 \omega_{(1)} A_2^{-1} + A_2 dA_2^{-1}}$$

or
$$\Gamma_{(2)}{}^\alpha{}_{\beta\gamma} = \underbrace{A_2{}^\alpha{}_\delta A_2^{-1\,\mu}{}_\beta A_2^{-1\,\nu}{}_\gamma \Gamma_{(1)}{}^\delta{}_{\mu\nu}}_{[\rho^{1,2}_0(A_2)\Gamma_{(1)}]^\alpha{}_{\beta\gamma}} + A_2{}^\alpha{}_\delta e_{(2)\beta} A_2^{-1\,\delta}{}_\gamma$$

So the connection components do not transform as a $\binom{1}{2}$ tensor under a change of frame but have an additional inhomogeneous term involving the derivative of the transformation matrix itself.

exercise: using only the definitions

$$\Gamma'^{\alpha}{}_{\beta\gamma} = \omega'^{\alpha}(\nabla_{e'_\beta} e'_\gamma) \ , \ \Gamma^{\alpha}{}_{\beta\gamma} = \omega^{\alpha}(\nabla_{e_\beta} e_\gamma) \ , \ e'_\alpha = A^{-1\beta}{}_{\alpha} e_\beta$$

derive the formula $\qquad \omega' = A\omega A^{-1} + A\,dA^{-1}$,

so this transformation law holds for any connection, not just the flat connection on R^n.

Letting $\Omega' = d\omega' + \omega'\wedge\omega'$, using instead the notation of the exercise:

$$\Omega' = d(A\omega A^{-1} + A\,dA^{-1}) + (A\omega A^{-1} + A\,dA^{-1})\wedge(A\omega A^{-1} + A\,dA^{-1})$$

$$= \underbrace{dA\wedge\omega A^{-1}}_{= -A\,dA^{-1}A \ (1)} + A\,d\omega A^{-1} - A\omega\wedge dA^{-1} \ (2) + \underbrace{dA\wedge dA^{-1}}_{= -A\,dA^{-1}A \ (3)} + A\omega\wedge\omega A^{-1} + A\,dA^{-1}\wedge A\omega A^{-1} \ (1) + \underbrace{A\omega A^{-1}\wedge A\,dA^{-1}}_{= A\omega\wedge dA^{-1} \ (2)} + A\,dA^{-1}\wedge A\,dA^{-1} \ (3)$$

$\boxed{\text{exercise: why?}}$

$$= A(d\omega + \omega\wedge\omega)A^{-1}$$

$$\boxed{\Omega' = A\Omega A^{-1}}$$

Putting back the indices, we see that Ω is a matrix-valued 2-form

$$\Omega^{\alpha}{}_{\beta} = \Omega^{\alpha}{}_{\beta\gamma\delta}\,\omega^{|\gamma\delta|} \equiv R^{\alpha}{}_{\beta\gamma\delta}\,\omega^{|\gamma\delta|}$$

and the transformation law means

$$R'^{\alpha}{}_{\beta\gamma\delta} = A^{\alpha}{}_{\epsilon}\,A^{-1\sigma}{}_{\beta}\,A^{-1\mu}{}_{\gamma}\,A^{-1\nu}{}_{\delta}\,R^{\epsilon}{}_{\sigma\mu\nu}$$

are the components of a $\binom{1}{3}$ tensor field called the <u>curvature tensor</u>.

Ω is called the <u>curvature 2-form</u>.

We saw that in the case of R^n, $\Omega = 0 = R^{\alpha}{}_{\beta\gamma\delta}$.

exercise.

Transform the Euclidean metric on R^3 into spherical coordinates

$$g = \delta_{ij}\, dx^i \otimes dx^j = dr\otimes dr + r^2 (d\theta\otimes d\theta + \sin^2\theta\, d\varphi\otimes d\varphi).$$

Let $\{e_1, e_2, e_3\}$ be the orthonormal frame on R^3 (singular on the z-axis) obtained by normalizing the spherical coordinate frame $\{\partial_\theta, \partial_\varphi, \partial_r\}$. Let $\{y^3, y^1, y^2\} = \{r, \theta, \varphi\}$.

Write out the matrix $\quad e^i{}_a = \dfrac{\partial x^i}{\partial y^a} \cdot \dfrac{1}{(g'_{aa})^{1/2}}$ $\quad$ [prime refers to y-coordinate components]

Evaluate the connection 1-form matrix in this frame.

Notice that it is an antisymmetric matrix

Evaluate the curvature 2-form matrix. Verify that it vanishes.

Now consider the 2x2 subblock of the connection matrix with indices 1 and 2. Convince yourself that this is the connection matrix associated with the restriction of the metric g to a sphere of radius r :

$$^{(2)}g \equiv g\big|_{dr=0} = r^2 (d\theta\otimes d\theta + \sin^2\theta\, d\varphi\otimes d\varphi).$$

Evaluate $\;{}^{(2)}\Omega = d\,{}^{(2)}\omega + {}^{(2)}\omega \wedge {}^{(2)}\omega\;$, where ${}^{(2)}\omega$ is the 2x2 block.

This is a 2x2 matrix valued 2-form on a 2-manifold (a 2-form on a 2-manifold has 1 independent component). Notice that it is antisymmetric so only one of two nonzero components is independent.

This is $R^1{}_{212} = R_{1212}$. What is its value?