de Rham Laplacian of a p-form ($p=3$ for concreteness)

$$dS_{\alpha_1\cdots\alpha_{p+1}} = \sum_{j=1}^{p+1} (-1)^{j+1} \nabla_{\alpha_j} S_{\alpha_1\cdots\hat{\alpha}_j\cdots\alpha_p}$$

$$dS_{\alpha_1\alpha_2\alpha_3\alpha_4} = \nabla_{\alpha_1} S_{\alpha_2\alpha_3\alpha_4} - \nabla_{\alpha_2} S_{\alpha_1\alpha_3\alpha_4} + \nabla_{\alpha_3} S_{\alpha_1\alpha_2\alpha_4} - \nabla_{\alpha_4} S_{\alpha_1\alpha_2\alpha_3}$$

$$dS_{\alpha_1\alpha_2\alpha_3\alpha_4} = \nabla_{\alpha_1} S_{\alpha_2\alpha_3\alpha_4} - \nabla_{\alpha_2} S_{\alpha_1\alpha_3\alpha_4} + \nabla_{\alpha_3} S_{\alpha_1\alpha_2\alpha_4} - \nabla_{\alpha_4} S_{\alpha_1\alpha_2\alpha_3} \quad \text{(bring backward each index in turn to first position)}$$

$$= \nabla_{\alpha_1} S_{\alpha_2\alpha_3\alpha_4} - \nabla_{\alpha_2} S_{\alpha_3\alpha_4\alpha_1} + \nabla_{\alpha_3} S_{\alpha_4\alpha_1\alpha_2} - \nabla_{\alpha_4} S_{\alpha_1\alpha_2\alpha_3} \quad \text{(cycle first index forward to each position in turn)}$$

$$\delta S_{\alpha_2\alpha_3\alpha_4} = -\nabla^{\alpha} S_{\alpha\alpha_2\alpha_3\alpha_4}$$

$$\delta dS_{\alpha_1\alpha_2\alpha_3} = -\nabla^{\alpha} dS_{\alpha\alpha_1\alpha_2\alpha_3}$$

$$= -\nabla^{\alpha}\nabla_{\alpha} S_{\alpha_1\alpha_2\alpha_3} + \nabla^{\alpha}\nabla_{\alpha_1} S_{\alpha\alpha_2\alpha_3} - \nabla^{\alpha}\nabla_{\alpha_2} S_{\alpha_1\alpha\alpha_3} + \nabla^{\alpha}\nabla_{\alpha_3} S_{\alpha_1\alpha_2\alpha}$$

$$d\delta S_{\alpha_1\alpha_2\alpha_3} = \nabla_{\alpha_1}(\delta S)_{\alpha_2\alpha_3} - \nabla_{\alpha_2}(\delta S)_{\alpha_1\alpha_3} + \nabla_{\alpha_3}(\delta S)_{\alpha_1\alpha_2}$$

$$= -\nabla_{\alpha_1}\nabla^{\alpha} S_{\alpha\alpha_2\alpha_3} + \nabla_{\alpha_2}\nabla^{\alpha} S_{\alpha\alpha_1\alpha_3} - \nabla_{\alpha_3}\nabla^{\alpha} S_{\alpha\alpha_2\alpha_1}$$

$$= -\nabla_{\alpha_1}\nabla^{\alpha} S_{\alpha\alpha_2\alpha_3} + \nabla_{\alpha_2}\nabla^{\alpha} S_{\alpha_1\alpha\alpha_3} - \nabla_{\alpha_3}\nabla^{\alpha} S_{\alpha_1\alpha_2\alpha}$$

$$\Delta_{dR} S_{\alpha_1\alpha_2\alpha_3}$$

$$= (d\delta + \delta d) S_{\alpha_1\alpha_2\alpha_3} = -\nabla^{\alpha}\nabla_{\alpha} S_{\alpha_1\alpha_2\alpha_3}$$

$$+ (\nabla^{\alpha}\nabla_{\alpha_1} - \nabla_{\alpha_1}\nabla^{\alpha}) S_{\alpha\alpha_2\alpha_3} + (\nabla^{\alpha}\nabla_{\alpha_2} - \nabla_{\alpha_2}\nabla^{\alpha}) S_{\alpha_1\alpha\alpha_3} + (\nabla^{\alpha}\nabla_{\alpha_3} - \nabla_{\alpha_3}\nabla^{\alpha}) S_{\alpha_1\alpha_2\alpha}$$

$$\Delta_{dR} S_{\alpha_1\alpha_2\alpha_3} - \Delta S_{\alpha_1\alpha_2\alpha_3}$$

$$= -R^{\beta}{}_{\alpha}{}^{\alpha}{}_{\alpha_1} S_{\beta\alpha_2\alpha_3} - R^{\beta}{}_{\alpha_2}{}^{\alpha}{}_{\alpha_1} S_{\alpha\beta\alpha_3} - R^{\beta}{}_{\alpha_3}{}^{\alpha}{}_{\alpha_1} S_{\alpha\alpha_2\beta}$$

$$- R^{\beta}{}_{\alpha_1}{}^{\alpha}{}_{\alpha_2} S_{\beta\alpha\alpha_3} - R^{\beta}{}_{\alpha}{}^{\alpha}{}_{\alpha_2} S_{\alpha_1\beta\alpha_3} - R^{\beta}{}_{\alpha_3}{}^{\alpha}{}_{\alpha_2} S_{\alpha_1\alpha\beta}$$

$$- R^{\beta}{}_{\alpha_1}{}^{\alpha}{}_{\alpha_3} S_{\beta\alpha_2\alpha} - R^{\beta}{}_{\alpha_2}{}^{\alpha}{}_{\alpha_3} S_{\alpha_1\beta\alpha} - R^{\beta}{}_{\alpha}{}^{\alpha}{}_{\alpha_3} S_{\alpha_1\alpha_2\beta}$$

$$- R^{\beta}{}_{\alpha_1} S_{\beta\alpha_2\alpha_3} + R^{\beta}{}_{\alpha_2} S_{\alpha_1\beta\alpha_3} + R^{\beta}{}_{\alpha_3} S_{\alpha_1\alpha_2\beta}$$

$$+ R^{\alpha}{}_{\alpha_1}{}^{\beta}{}_{\alpha_2} S_{\alpha\beta\alpha_3} + R^{\alpha}{}_{\alpha_1}{}^{\beta}{}_{\alpha_3} S_{\alpha\alpha_2\beta}$$

$$- R^{\alpha}{}_{\alpha_1}{}^{\beta}{}_{\alpha_2} S_{\alpha\beta\alpha_3} - R^{\alpha}{}_{\alpha_2}{}^{\beta}{}_{\alpha_3} S_{\alpha_1\alpha\beta}$$

$$- R^{\alpha}{}_{\alpha_1}{}^{\beta}{}_{\alpha_3} S_{\alpha\alpha_2\beta} - R^{\alpha}{}_{\alpha_2}{}^{\beta}{}_{\alpha_3} S_{\alpha_1\alpha\beta}$$

$$\Big\} = -2\sum_{i<j} R^{\alpha}{}_{\alpha_i}{}^{\beta}{}_{\alpha_j} S_{\alpha_1\cdots\alpha\cdots\beta\cdots\alpha_p}$$