§17. Concluding Remarks

The conclusion to be drawn from the examples of the preceding section and indeed from our whole approach is that no fixed choice of lapse, shift, configuration space variables, structure constant tensor components or dynamical formulation is suited to all situations arising in spatially homogeneous cosmology. Versatility is instead demanded. In each case the symmetries present must be exploited rather than ignored.

However, in all but the most specialized subsystems, the equations one is faced with are exceedingly complicated and one must settle for a qualitative analysis for which various techniques are available. From the Hamiltonian point of view the most natural of these is the Misner-Ryan approach in which the free (vacuum type I or "Kasner") dynamics is used to evolve in Ω-time the approximate solution curve on $\beta^+\beta^-$ space (in the general case) between collisions with idealized moving potential walls due to the gravitational, effective and fluid potentials. This is essentially equivalent to the Lifshitz-Khalatnikov approach based on the form of the equations given at the end of the last section,(33,73) but the two descriptions involve rather different language. Although they have so far only been applied to type IX and some diagonal cases, we have seen how ideas that apply in these situations generalize. All subsystems that can be reduced to first order equations in two variables have been studied very effectively by Collins using qualitative differential equation techniques, and he points out the existence of certain exceptional solutions not recoverable from the moving potential wall description.(47,72,88) Various other techniques have been employed to elucidate properties of the solutions.(83,85,93,97)

As is usual in SH cosmology we have emphasized the geometry rather than the perfect fluid source by initially expressing the Einstein equations in a normal comoving SH ADM frame adapted to the SH foliation. The structure functions for the frame are then canonical constants and the inner products evolve; the frame also automatically determines a coordinate system on the spacetime modulo one on the homogeneity group as discussed in §12. A similar picture accompanies the choice of any SH comoving ADM frame. Taub prefers one whose ADM generator is $\mu^{-1}u$ where u is the fluid four-velocity and μ the chemical potential.[26] This has the advantage that v_a and ℓ are constants but the disadvantage that one must work with nondiagonal metric matrices $\underline{g}$ and the lapse and shift complicate the equations further. In certain cases lightlike SH ADM generators prove valuable, particularly in studying the regions in which the SH hypersurfaces lose their spatial character and which cannot be reached by a normal ADM generator.[59, 92, 95] This was illustrated with the Taub-Nut spacetimes.

An alternative approach taken by Ellis and coworkers relies on orthonormal frame techniques. Usually an orthonormal frame containing the normal and a "spatial triad" tangent to the SH foliation is adapted to the properties of the fluid.[45] The spatial triad cannot be comoving and so has time-dependent structure functions. For example, $\{e_\perp, e''_a\}$ is an orthonormal frame whose time-dependent structure functions $C''^a{}_{bc}$ remain in diagonal form:

(17.1) $$e''_a = e_b (e^{\underline{\beta}}\underline{S})^{-1\,b}{}_a \,, \quad C''^a{}_{bc} = (e^{\underline{\beta}})^a{}_d\, C^d{}_{fg}\, (e^{-\underline{\beta}})^f{}_b (e^{-\underline{\beta}})^g{}_c \,.$$

The Jacobi identities, field equations and conservation equations then involve the triad rotation and structure functions, the extrinsic curvature of the SH foliation as well as the fluid variables and kinematical quantities. At this stage many general statements can be made by examination of these equations, but to actually complete the Einstein equations, the Jacobi identities must be used in constructing a coordinate system and hence a comoving frame from which the usual equations follow for the new inner products. Ellis and others have also used rigorous techniques to study the global structure of SH perfect fluid spacetimes.[46]

The Newman-Penrose formalism has been exploited by Siklos in studying the global behavior of SH spacetimes.[99, 106, 107] This formalism is particularly well adapted to the problem of the change of the character of the homogeneous hypersurfaces from spacelike to timelike across a bounding null hypersurface, as occurs in the Taub-Nut spacetimes. The connection with algebraically special vacuum solutions which admit a slicing by SH hypersurfaces is also greatly clarified in this approach.[107]

It should be noted that almost every analysis of SH perfect fluid spacetimes assumes the equation of state $p = (\gamma - 1)\rho$ where $\gamma \in [1,2]$ is a constant parameter. For models which attempt to take into account fluid viscosity, see the article by Belinsky and Khalatnikov in this book. Apart from fluid sources, one may consider electromagnetic fields using appendix C and cosmological torsion models may be studied once reformulated in terms of a Riemannian theory with additional sources. The spatially homogeneous "Lichnerowicz universes" treated by Ozsvath[90] are also amenable to our methods although they involve a much more complicated source behavior.

Spinor sources may also be considered.[110-112] To introduce spinors on a SH spacetime, the orthonormal frame $\{e_\perp, e_{a''}\}$ of (17.1) is natural. The structure constant tensor components $C''^{a}{}_{bc}$, although time-dependent, remain in standard diagonal form. From the Hamiltonian point of view,[110] this choice of gauge for the "tetrad" might be called "standard diagonal form gauge."

Topological questions have largely been ignored in our discussion.[71] We have tacitly assumed that the homogeneous 3-geometries from which our SH spacetimes are constructed are connected 3-dimensional Lie groups with left invariant metrics. Given a Lie algebra $\mathfrak{g}$, any connected Lie group with that Lie algebra may be obtained from a unique "simply connected covering group" G by taking the quotient G/D of G by a discrete normal subgroup D of G.(1) The existence of such subgroups therefore permits a choice of topologies for a homogeneous 3-geometry of a given Bianchi type. The various quotient groups may be visualized as certain (nonunique) submanifolds of the covering group G with appropriate identifications made on their boundaries. For example, type I geometries may be open with the topology of $\mathbb{R}^3$ or closed up into a torus $T^3 = S^1 \times S^1 \times S^1$ by "periodic boundary conditions", etc. The underlying manifold of the unit quaternion group $SL(1, Q_1)$ (which is isomorphic to $SU(2)$) is the 3-sphere S^3 according to (C.18). Its only discrete normal subgroup is its center $\{\hat{e}_0, -\hat{e}_0\}$ consisting of the north and south poles of S^3. The corresponding quotient group is $SO(3,\mathbb{R})$ whose underlying manifold is real projective 3-space P^3, i.e. S^3 with antipodal points identified (or just the upper hemisphere with antipodal points of the equator identified). As discussed at the end of Appendix C, the canonical type VII_0 and VIII unit Bianchi quaternion groups have the topology $\mathbb{R}^2 \times S^1$ with the range of the canonical coordinate of the second kind x^3 being the interval $[0, 4\pi)$; the simply connected covering groups (with the topology of $\mathbb{R}^3$) are obtained be extending the range of this coordinate to the real line. (The canonical type VIII unit Bianchi quaternion group is isomorphic to $SL(2,\mathbb{R})$, the double covering group of $SO(2,1)$.)

However, our definition of a homogeneous 3-geometry is really too strict. We should relax the simply transitive condition to allow discrete isotropy groups, permitting a much larger choice of topologies. These additional homogeneous 3-geometries are then of the form G/D where D is a discrete but not normal subgroup of the simply connected covering group G. (G acts transitively on the manifold of

Correction for 17.4. Insert new paragraph after the first:

In order to interpret the action of the automorphism group of the Lie algebra $\mathfrak{g}$ in terms of the action of the automorphism group of the Lie group G via a time-dependent shift vector field, it is necessary that $\mathrm{Aut}(G) \cong \mathrm{Aut}(\mathfrak{g})$, i.e. that every automorphism of $\mathfrak{g}$ is induced by the dragging along action of an automorphism of G. (More precisely, the components of the two groups connected to the identity must be isomorphic.) This imposes certain topological restrictions on G. For example, it is sufficient that G be simply connected.$^{(1)}$ If $\mathrm{Aut}(G)$ is smaller than $\mathrm{Aut}(\mathfrak{g})$, the shift vector field required to complete a canonical reduced frame to a canonical comoving ADM frame will not exist globally. The underlying manifold of the simply connected covering group of each Bianchi type except type IX is $\mathbb{R}^3$. In the extreme case of Bianchi type I with the closed topology of T^3, the automorphism group of G is trivial apart from possible discrete automorphisms ($\mathrm{Aut}(G)^+ = \{\mathrm{Id}\}$) while the automorphism group of $\mathfrak{g}$ is maximal ($\mathrm{Aut}(\mathfrak{g}) = GL(\mathfrak{g})$).

left cosets of D in G by left translation; D is then the isotropy group at the identity coset. Furthermore a left invariant frame on G projects down to a frame on the quotient which is invariant under the action of G.) Other topologies may be introduced (say to artificially close a 3-geometry) by making arbitrary identifications, i.e. by taking the quotient of G under the action of some discrete group. However, if the latter group is not a subgroup of G, G will no longer act on the identified manifold as a global group and Killing vector fields and invariant frames can only be introduced locally, being incompatible with the identifications. (Consider the rotations of $\mathbb{R}^2$ and what happens when it is converted into T^2 by identifications.) Such 3-geometries have been called "globally inhomogeneous" in connection with Friedmann spacetimes where interesting physical effects due to nontrivial topology have been described.[64,65] More detailed remarks on topology may be found in the excellent review by Ellis.[91]

Finally we should relax our definition of a homogeneous 3-geometry (and hence of a SH spacetime) to allow any multiply transitive isometric action since the metric will still be determined by its value at any point. The only new examples this allows according to Bianchi's original work are 3-dimensional manifolds $M = G_1 \times S^2$ or $G_1 \times \mathbb{R}^2$ on which the 4-dimensional direct product group $G_4 = G_1 \times G_3$ acts as an isometry group. G_1 may be either $\mathbb{R}$ or S^1 with its natural Lie group structure and G_3 is either $SO(3,\mathbb{R})$ or $SO(2,1)$ respectively. M is a 1-parameter family of geodesically parallel 2-surfaces of constant positive or negative curvature respectively (the copies of S^2 or $\mathbb{R}^2$ in the product manifold) on which the group G_3 acts multiply transitively while G_1 acts by translation on the orthogonal geodesics (the copies of G_1 in M).

The metric ${}^3g$ and Killing vector fields $\{\xi_a, \xi_4\}$ may be written in the following form:

(17.2) $$ {}^3g = \bar{g}_{22}\, dy^2 \otimes dy^2 + \bar{g}_{33}\, {}^2g_\kappa $$

$$ {}^2g_\kappa = dy^3 \otimes dy^3 + S_3{}^2\, dy^1 \otimes dy^1 $$

$$ S_3 = \kappa^{-1}\sinh \kappa y^3 \qquad ct_3 = \cosh \kappa y^3 / S_3 \qquad s_1 = \sin y^1 \quad c_1 = \cos y^1 $$

$$ \xi_1 = s_1\, \partial/\partial y^3 + c_1\, ct_3\, \partial/\partial y^1 \qquad [\xi_2, \xi_3] = \xi_1 $$

$$ \xi_2 = c_1\, \partial/\partial y^3 - s_1\, ct_3\, \partial/\partial y^1 \qquad [\xi_3, \xi_1] = \xi_2 $$

$$ \xi_3 = \partial/\partial y^1 \qquad [\xi_1, \xi_2] = -\kappa^2 \xi_3 $$

$$ \xi_4 = \partial/\partial y^2 \qquad [\xi_a, \xi_4] = 0 \quad . $$

$\kappa = 1$ corresponds to the $SO(2,1)$ case and $\kappa = i$ to the $SO(3,\mathbb{R})$ case. The 2-metric ${}^2g_\kappa$ has constant Gaussian curvature $-\kappa^2 = R^{12}{}_{12}({}^2g_\kappa)$. The most general symmetric second rank covariant tensor field which is annihilated by the Killing vector fields under Lie derivation must be a constant linear combination of $dy^2 \otimes dy^2$ and ${}^2g_\kappa$. The spacetimes with this class of 3-geometries are called Kantowski-Sachs spacetimes, the spacetime metric involving three functions of time $\{N_t, \bar{g}_{22}(t), \bar{g}_{33}(t)\}$:

(17.3) $$ g = -N_t{}^2\, dt \otimes dt + {}^3g_t \; . $$

Since these spacetimes are so different mathematically, we have omitted a study of them here.

Note, however, that the coordinates $\{y^a\}$ are singular at $y^3 = 0$. For the case $\kappa = 1$, one may introduce new coordinates $\{\bar{x}^1, \bar{x}^3\}$ on the hypersurfaces of constant $y^2 = \bar{x}^2$ such that ${}^2g_\kappa$ assumes the form:(23,113)

(17.4) $$ {}^2g_\kappa = d\bar{x}^3 \otimes d\bar{x}^3 + e^{-2\bar{x}^3}\, d\bar{x}^1 \otimes d\bar{x}^1 \, . $$

The metric ${}^3g$ then has the component matrix $\underline{\bar{g}} = \mathrm{diag}(\bar{g}_{33}, \bar{g}_{22}, \bar{g}_{33})$ with respect to the frame (10.26) with $q = a = \frac{1}{2}$, i.e. this is just the Taublike case for Bianchi type $\mathrm{III} = \mathrm{VI}_{-1}$ with respect to a noncanonical frame. The spatial curvature matrix is obtained by evaluating ((16.90):

(17.5) $$ \bar{g}_{33}\, \underline{\bar{R}}^* = -\kappa^2 (\underline{\hat{e}}^1{}_1 + \underline{\hat{e}}^3{}_3) \qquad \kappa = 1 \, . $$

The equations for $\{N, \bar{g}_{22}, \bar{g}_{33}\}$ and the fluid variables in the $SO(3,\mathbb{R})$ case are identical with the $SO(2,1)$ case with the exception that the sign of the spatial curvature tensor switches. In other words, if one solves the Taublike type III equations with κ left as an arbitrary parameter, one may obtain the positive curvature Kantowski-Sachs solutions by the analytic continuation $\kappa = 1 \mapsto \kappa = i$. The vacuum and dust cases are exactly integrable.(87)

As a final note, we hope to have encouraged some readers not familiar with Lie groups to pursue their own study of a subject which finds extensive application throughout physics and to fill the many gaps and omissions in our elementary introduction. For those unfamiliar with SH cosmology, we hope to have presented a somewhat unified view of the background and dynamical situation.

A. Brief Summary of Notation and Formulas.

(A.1) For a manifold M, TM_x is the tangent space at $x \in M$, T^*M the cotangent space at x, and in general, $TM_x^{r,s}$ is the space of $\binom{r}{s}$-tensors at x. Let $TM^{r,s} = \{TM_x^{r,s} \mid x \in M\}$ be the $\binom{r}{s}$-tensor bundle over M, where $TM^{1,0} = TM$ and $TM^{0,1} = T^*M$ are the tangent and cotangent bundles. Let $T^{r,s}(M)$ be the real vector space of $\binom{r}{s}$-tensor fields over M, with $T^{0,0}(M) = \mathfrak{F}(M)$ the functions on M, $T^{1,0}(M) = \mathfrak{X}(M)$ the vector fields and $T^{0,1}(M) = \mathfrak{X}^*(M)$ the 1-forms. Although differentiability assumptions have not been mentioned, because of the nature of the subject everything in sight is usually assumed to be analytic. We tend to use the symbols f, g for functions, X, Y, Z for tangent vectors and vector fields, ω and σ for 1-forms and α and β for p-forms, and g for metric tensor fields. $\mathfrak{D}(M)$ is the infinite-dimensional group of diffeomorphisms of M into itself; we often use the symbol h for an element of this group. Id_M, abbreviated to Id when no confusion can arise, is the identity element of $\mathfrak{D}(M)$.

(A.2) Let $h: M \to N$ be a differentiable map between manifolds, with $x \in M \mapsto h(x) \in N$. Composition with h pulls back functions f on N to functions on M by $(f \circ h)(x) = f(h(x))$. The differential of h at x, $dh(x): TM_x \to TN_{h(x)}$, pushes forward tangent vectors at x to $h(x)$:

$$[dh(x) X(x)] f = X(x) f \circ h \quad ; \quad X(x) \in TM_x .$$

The transpose of the differential of h at x, $dh(x)^*: T^*N_{h(x)} \to T^*M_x$, pulls back covectors from $h(x)$ to x:

$$[dh(x)^* \sigma(h(x))](X(x)) = \sigma(h(x))(dh(x) X(x)) ; \quad \sigma \in T^*N_{h(x)} .$$

If $\{x^\mu\}, \{y^i\}$ are local coordinates about x, $h(x)$, then the matrix of the linear transformation $dh(x)$ with respect to the bases $\{\partial/\partial x^\mu\}, \{\partial/\partial y^i\}$ (evaluated at x and $h(x)$) has components $(\partial h^i/\partial x^\mu)(x)$, where $h^i = y^i \circ h$:

$$dh(x)(\partial/\partial x^\mu)(x) = (\partial h^i/\partial x^\mu)(x)\,(\partial/\partial y^i)(h(x)) ,$$

$$dh(x) X(x) = (\partial h^i/\partial x^\mu)(x)\, X^\mu(x)\,(\partial/\partial y^i)(h(x)) .$$

where $X = X^\mu \partial/\partial x^\mu$. Similarly with $\sigma = \sigma_i dy^i$:

$$dh(x)^* dy^i(h(x)) = (\partial h^i/\partial x^\mu)(x)\ dx^\mu(x)$$
$$dh(x)^* \sigma(h(x)) = \sigma_i(h(x))\ (\partial h^i/\partial x^\mu)(x)\ dx^\mu(x).$$

Except for the explicit arguments, these are familiar classical formulas. $dh(x)$ may be extended to contravariant tensors at x and dh^* to covariant tensor fields. When h is a diffeomorphism, tensor fields of any type may be pushed forward or pulled back as discussed in §3.

(A.3) The Chain Rule. If $h: M \to N$ and $k: N \to P$ are differentiable maps and $k \circ h: M \to P$ their composition, then:

$$d(k\circ h)(x) = dk(h(x)) \circ dh(x)$$
$$d(k\circ h)(x)^* = dh(x)^* \circ dk(x)^*.$$

In local coordinates $\{x^\mu\}, \{y^i\}, \{z^a\}$ on M, N, P with $k^a = z^a \circ k$ and $h^i = y^i \circ h$, this just the old fashioned chain rule:

$$\partial(k\circ h)^a/\partial x^\mu\ (x) = \partial k^a/\partial y^i\ (h(x))\ \partial h^i/\partial x^\mu(x).$$

(A.4) A parametrized curve c (or sloppily $c(t)$ indicating the parameter explicitly) is a map from the real line $\mathbb{R}$ into M, $c: \mathbb{R} \to M$. Let $c^\mu(t) = x^\mu \circ c(t)$ in local coordinates $\{x^\mu\}$. The tangent vector to the curve c at $c(t) \in M$ is denoted by $c'(t) \in TM_{c(t)}$:

$$c'(t) f = d/dt\ f\circ c(t) \quad , \quad f \in \mathcal{F}(M),$$
$$c'(t) = dc(t)\ d/dt = dc^\mu(t)/dt\ \ \partial/\partial x^\mu(c(t)).$$

We often use the notation $d/dt|_0\, F(t)$ to indicate the derivative at $t=0$ of a function of t.

(A.5) The Lie bracket $[X,Y]$ of two vector fields X and Y is another vector field defined by:

$$[X,Y]f = X(Yf) - Y(Xf) \quad , \quad f \in \mathcal{F}(M),$$
$$[X,Y] = (X^\nu \partial Y^\mu/\partial x^\nu - Y^\nu \partial X^\mu/\partial x^\nu)\ \partial/\partial x^\mu.$$

It satisfies the Jacobi identity:
$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0.$

(A.6) The Lie derivative $\mathcal{L}_X$ with respect to a vector field X maps $T^{r,s}(M)$ into itself linearly (over $\mathbb{R}$) and may be defined inductively by:

$$\mathcal{L}_X f = Xf \quad , \quad \mathcal{L}_X Y = [X,Y],$$
$$\mathcal{L}_X [T(Y,\ldots,\sigma,\ldots)] = (\mathcal{L}_X T)(Y,\ldots,\sigma,\ldots) + T(\mathcal{L}_X Y,\ldots,\sigma,\ldots) + \cdots$$

$$+ T(Y,\dots, \mathcal{L}_X \sigma, \dots) + \dots .$$

It satisfies:

$$[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_X \mathcal{L}_Y - \mathcal{L}_Y \mathcal{L}_X = \mathcal{L}_{[X,Y]} .$$

(A.7) Exterior derivative. Let f be a 0-form (function), σ a 1-form and α and β p- and q- forms respectively:

$$df(X) = Xf$$

$$d\sigma(X,Y) = X\sigma(Y) - Y\sigma(X) - \sigma([X,Y])$$

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta .$$

(A.8) Contraction of forms.

$$(X \lrcorner \alpha)(X_2, \dots, X_p) = \alpha(X, X_2, \dots, X_p)$$

$$X \lrcorner (\alpha \wedge \beta) = (X \lrcorner \alpha) \wedge \beta + (-1)^p \alpha \wedge (X \lrcorner \beta).$$

(A.9) One may show that when acting on any form α, $\mathcal{L}_X$ is given by:

$$\mathcal{L}_X \alpha = X \lrcorner d\alpha + d(X \lrcorner \alpha).$$

(A.10) Product Manifold: $M \times N = \{(x,y) \mid x \in M, y \in N\}$.

If $\{x^\mu\}, \{y^i\}$ are local coordinates on M, N then "$\{x^\mu, y^i\}$" are coordinates on $M \times N$ defined by $x^\mu(x,y) = x^\mu(x)$, etc. If $X = X^\mu \partial/\partial x^\mu$ and $Y = Y^i \partial/\partial y^i$, then fields X, Y are induced on $M \times N$ having the same functional dependence and expression in the coordinates $\{x^\mu, y^i\}$. $[X,Y] = 0$ since $\partial/\partial y^i$ annihilates functions of $\{x^\mu\}$ only and vice versa. Any tensor fields on the component manifolds induce corresponding fields on the product manifold $M \times N$ in this way.

(A.11) Frame, Dual Frame. Let $\{e_a\}$ be a frame on M, i.e. $e_a \in \mathfrak{X}(M)$ such that $\{e_a(x)\}$ is a basis for TM_x for all x. (Global frames, like global coordinate systems, do not always exist, in which case one speaks of local frames.) Let $\{\omega^a\}$ be the dual frame of 1-forms defined by the duality relation:

$$e_b \lrcorner \omega^a = \omega^a(e_b) = \delta^a{}_b .$$

Taking components of a tensor field T in the frame $\{e_a\}$ amounts to the following:

$$T = T^{a\cdots}{}_{b\cdots}\, e_a \otimes \cdots \otimes \omega^b \otimes \cdots \,, \quad T^{a\cdots}{}_{b\cdots} = T(\omega^a,\dots, e_b, \dots).$$

Parentheses and brackets around several indices indicate symmetrization and antisymmetrization. The structure functions $C^a{}_{bc} = -C^a{}_{cb}$ for the frame $\{e_a\}$ are defined by:

$$[e_a, e_b] = C^c{}_{ab}\, e_c \,, \quad d\omega^a = -\tfrac{1}{2} C^a{}_{bc}\, \omega^b \wedge \omega^c.$$

The second relation is a consequence of the first together with the duality relation and the definition (A.7) for $d\omega^a$. For convenience we introduce redundant notations for the components of derivatives of components of fields in the frame $\{e_a\}$:

$$e_a T^{b\cdots}{}_{c\cdots} = \partial_a T^{b\cdots}{}_{c\cdots} = T^{b\cdots}{}_{c\cdots,a}\,.$$

The Jacobi identity then implies:

$$\partial_{[a} C^d{}_{bc]} - C^d{}_{e[a} C^e{}_{bc]} = 0.$$

(A.12). Let g be a metric tensor field on M and g^{-1} its inverse or contravariant form:

$$g = g_{ab}\, \omega^a \otimes \omega^b \,, \quad g_{ab} = g(e_a, e_b) \,, \quad g = |\det(g_{ab})|$$

$$g^{-1} = g^{ab}\, e_a \otimes e_b \,, \quad g^{ab} g_{bc} = \delta^a{}_c .$$

All indices on components taken in the frame $\{e_a\}$ will be "lowered and raised" by the matrices g_{ab}, g^{ab} as usual.

(A.13). The components of the metric connection ∇ in the frame $\{e_a\}$ are defined with a convention opposite to that of MTW and evaluated by the usual formula:

$$\nabla_{e_a} e_b = \Gamma^c{}_{ab}\, e_c$$

$$2\Gamma_{cab} = g_{ca,b} - g_{ab,c} + g_{bc,a} + C_{cab} - C_{abc} + C_{bca}$$

$$\Gamma^c{}_{ab} = \left\{ {c \atop ab} \right\} + K^c{}_{ab} + \tfrac{1}{2} C^c{}_{ab} \,,$$

where $\left\{ {c \atop ab} \right\}$ is the Christoffel combination of frame derivatives and $K^c{}_{ab} = C_{(a}{}^c{}_{b)}$. Various consequences of this formula are:

$$\Gamma_{(bc)a} = \tfrac{1}{2} \partial_a g_{bc} - K_{abc} \qquad \Gamma^c{}_{[ab]} = \tfrac{1}{2} C^c{}_{ab}$$

$$\Gamma^c{}_{ca} = \partial_a \ln g^{1/2} - C^c{}_{ac} = g^{-1/2}\, \tilde{\partial}_a\, g^{1/2}$$

$$\Gamma^c{}_{ac} = \partial_a \ln g^{1/2}$$

We have introduced the notation $\tilde{\partial}_a = \partial_a - C^c{}_{ac}$.

(A.14). An application of (A.6) with $T = g$ leads to the relation

$$(\mathcal{L}_X g)_{ab} = (\mathcal{L}_X g)(e_a, e_b) = X g_{ab} + 2 g_{c(a} \partial_{b)} X^c - 2 X^c K_{cab}.$$

In particular if g_{ab} are constants:

$$(\mathcal{L}_{e_c} g)_{ab} = -2K_{cab}.$$

(A.15). The components of the covariant derivative of a tensor density T of weight W in the frame $\{e_a\}$ are:

$$(\nabla T)^{a\cdots}{}_{b\cdots c} = T^{a\cdots}{}_{b\cdots;c} =$$
$$\partial_c T^{a\cdots}{}_{b\cdots} + \Gamma^a{}_{cd} T^{d\cdots}{}_{b\cdots} + \cdots - \Gamma^d{}_{cb} T^{a\cdots}{}_{d\cdots} - \cdots - W\,\Gamma^d{}_{cd}\, T^{a\cdots}{}_{b\cdots}.$$

The covariant divergences of a vector density $\mathcal{X}$ and a symmetric tensor density Π are given by:

$$\mathcal{X}^a{}_{;a} = \partial_a \mathcal{X}^a + \mathcal{X}^d \Gamma^a{}_{ab} - \mathcal{X}^a \Gamma^c{}_{ac} = \delta_a \mathcal{X}^a$$
$$\Pi_a{}^b{}_{;b} = \delta_b \Pi_a{}^b - \Pi^b{}_c \Gamma^c{}_{ba} = \delta_b \Pi_a{}^b - \Pi^b{}_c C^c{}_{ba} - \tfrac{1}{2}\Pi^{bc}\partial_a g_{bc}$$
$$= g_{ac}\,\delta_c \Pi^{cb} + 2\Pi^{bc}\{ca;b\} - \Pi^b{}_c C^c{}_{ba}.$$

(A.16) The components of the Riemann curvature tensor, the Ricci tensor, the scalar curvature and Einstein tensor (following the conventions of MTW) are:

$$R^a{}_{bcd} = \partial_c \Gamma^a{}_{db} - \partial_d \Gamma^a{}_{cb} - \Gamma^a{}_{eb} C^e{}_{cd} + \Gamma^a{}_{ce}\Gamma^e{}_{db} - \Gamma^a{}_{de}\Gamma^e{}_{cb}$$
$$R_{bd} = R^a{}_{bad} = \partial_a \Gamma^a{}_{db} - \partial_d \Gamma^a{}_{ab} - \Gamma^a{}_{eb} C^e{}_{ad} + \Gamma^a{}_{ae}\Gamma^e{}_{db} - \Gamma^a{}_{de}\Gamma^e{}_{ab}$$
$$R = g^{bd} R_{bd} \qquad G_{ab} = R_{ab} - \tfrac{1}{2} g_{ab} R.$$

(A.17) Choose a coordinate frame so that $C^a{}_{bc} = 0$, $\Gamma^a{}_{bc} = \left\{ {a \atop bc} \right\}$. Consider a 1-parameter family of metrics with components $g_{ab}(\lambda)$, and let $g_{ab} = g_{ab}(0)$ and $g'_{ab} = d/d\lambda|_0\, g_{ab}(\lambda)$. One may easily establish the relations:

$$\Gamma'^a{}_{bc} = \tfrac{1}{2} g^{ad}(g'_{da;b} + g'_{db;a} - g'_{ab;d})$$
$$g^{ab} R'_{ab} = g^{ab}(\Gamma'^c{}_{ab} - \Gamma'^d{}_{ad}\,\delta^c{}_b)_{;c}.$$

Putting these together we obtain a covariant formula valid in any frame:

$$g^{\frac{1}{2}} g^{ab} R'_{ab} = \mathcal{G}^{abcd} g'_{ab;cd}$$
$$\mathcal{G}^{abcd} = g^{1/2}(g^{a(c}g^{d)b} - g^{ab}g^{cd}).$$

A useful consequence of this is:

$$N g^{1/2} g^{ab} R'_{ab} = \mathcal{G}^{abcd} N_{;cd}\, g'_{ab} + \mathcal{X}^a{}_{;a},$$

where $\mathcal{X}^a{}_{;a}$ is a divergence arising from two integrations by parts.

(A.18) Another useful variational formula follows from (6.57):

$$(g^{1/2} g^{ab})' = -g^{1/2}(g^{a(c}g^{d)b} - \tfrac{1}{2} g^{ab} g^{cd})\, g'_{cd}.$$

Together with the previous result we obtain:

$$(g^{1/2} R)' = -g^{1/2} G^{ab} g'_{ab} + \mathcal{G}^{abcd} g'_{ab;cd}.$$

(A.19) In the text we will be studying various manifolds of different dimensions. The range of all indices should always be clear from the context. For spacetime we reserve Greek letters for the range $\{0,1,2,3\}$ and Latin letters for $\{1,2,3\}$.

B. Geodesics of $(\bar{\mathcal{M}},\bar{\mathcal{G}})$.

In this appendix we obtain the geodesics of $(\bar{\mathcal{M}},\bar{\mathcal{G}})$ following the work of DeWitt [37] but using a Lagrangian formulation which is much simpler. Let $\bar{\mathcal{G}}$ here denote the restriction to $\bar{\mathcal{M}}$ of the DeWitt metric rather than the conformally scaled metric of (6.45). According to the discussion following (6.64), a Lagrangian for the geodesics of the positive definite metric $\bar{\mathcal{G}}$ on $\bar{\mathcal{M}}$ is:

(B.1) $$\bar{L} = \bar{g}^{ac}\bar{g}^{bd}\dot{\bar{g}}_{ab}\dot{\bar{g}}_{cd}.$$

Carrying out the differentiations in the Lagrange equations:

$$0 = (\partial\bar{L}/\partial\dot{\bar{g}}_{ab})^{\cdot} - \partial\bar{L}/\partial\bar{g}_{ab}$$

and lowering the indices (which is equivalent to raising the index pair with the metric $\bar{\mathcal{G}}$) yields DeWitt's matrix equation:

(B.2) $$\ddot{\bar{g}} - \dot{\bar{g}}\,\bar{g}^{-1}\dot{\bar{g}} = 0.$$

(One should really include the constraint $\det\bar{g} - 1 = 0$ in the Lagrangian with a Lagrange multiplier λ, and add the equation of motion $\bar{g}^{ab}\dot{\bar{g}}_{ab} = 0$, but the complete set of equations then implies $\lambda = 0$.)

Denote the time variable of this Lagrangian system by $\bar{s}$, so $\dot{\bar{g}}$ corresponds to $d/d\bar{s}\ \bar{g}$ when evaluated on a curve $\bar{g}(\bar{s})$. The kinetic energy Lagrangian is just the inner product of the tangent to the curve with itself and is a constant of the motion since no explicit $\bar{s}$ dependence is present in the Lagrangian. By definition this makes $\bar{s}$ an affine parameter for a solution curve; chosing the constant of energy to be one (a unit tangent) makes $\bar{s}$ the arclength.

Consider a geodesic through $\underline{A}^T\underline{A} \in \bar{\mathcal{M}}$, where $\underline{A} \in SL(3,\mathbb{R})$. A direction at $\underline{A}^T\underline{A}$ is specified by a unit tangent vector X with traceless matrix $\underline{X}$ of the form $\underline{X} = \underline{A}^{-1}\underline{N}\,\underline{A}$, where $\underline{N}$ is symmetric in order for X to be tangent to $\bar{\mathcal{M}}$. N is also traceless and by the unit nature of X satisfies:

$$1 = \bar{\mathcal{G}}(X,X) = \mathrm{Tr}\,\underline{X}^2 = \mathrm{Tr}\,\underline{N}^2.$$

By straightforward differentiation one may verify DeWitt's solution of (B.2) for the arclength parametrized geodesic through $\underline{A}^T\underline{A}$ with initial tangent X:

(B.3) $$\bar{g}(\bar{s},\underline{A},\underline{N}) = \underline{A}^T e^{\bar{s}\underline{N}}\underline{A}.$$

$\underline{A}$ and $\underline{N}$ are not unique for a given initial point and direction as discussed following (6.41).

Now consider the geodesics of $(\mathcal{M},\mathcal{G})$. The corresponding

Lagrangian expressed in the DeWitt coordinates $\{\mathcal{Y}, \bar{g}_{ab}\}$ in which the metric has the form (6.64) is:

(B.4) $$L = -\dot{\mathcal{Y}}^2 + (K\mathcal{Y})^2 \bar{L}.$$

Let s denote the time variable for this system, so the dot is now associated with d/ds. Choosing the constant of energy β to be $1, 0, -1$ in the case of spacelike, null and timelike geodesics respectively makes s an affine parameter for the solution curves which is arclength in the nonnull case.

The Lagrange equations for $\bar{g}$ are exactly as above but with an extra term due to the derivative hitting $\mathcal{Y}^2$:

(B.4) $$-\delta\bar{L}/\delta\bar{g}_{cd}\,\bar{g}_{ca}\bar{g}_{db} + 2(\ln\mathcal{Y})^{\cdot}\,\dot{\bar{g}}_{ab} = 0.$$

This is just the equation for a geodesic $\bar{c}(s)$ of $(\bar{\mathcal{M}}, \bar{\mathcal{G}})$ with tangent $\bar{c}'(s)$ which is not affinely parametrized:

$$\bar{D}\bar{c}'/ds = -2(\ln\mathcal{Y})^{\cdot}\,\bar{c}'.$$

The solution is given by (B.3):

(B.5) $$\bar{g}(s) = \underline{A}^T \exp(\bar{S}(s)\underline{N})\,\underline{A}.$$

Evaluating $\bar{L}$ on this solution gives $\bar{L} = \dot{\bar{S}}^2$. The full Lagrangian is therefore:

(B.6) $$L = -\dot{\mathcal{Y}}^2 + (K\mathcal{Y}\dot{\bar{S}})^2.$$

Since $\bar{S}$ is a cyclic variable, its conjugate momentum $P_{\bar{S}} = 2K^2\mathcal{Y}^2\dot{\bar{S}} = 2K^2\alpha$ is a constant of the motion which may clearly be assumed nonnegative:

(B.7) $$\dot{\bar{S}} = \alpha\mathcal{Y}^{-2}.$$

For a solution, the Lagrangian (kinetic energy) has the constant value:

(B.8) $$\beta = -\dot{\mathcal{Y}}^2 + (K\alpha/\mathcal{Y})^2.$$

Solving this for $\dot{\mathcal{Y}}$, a simple integration yields DeWitt's expression:

(B.9) $$\mathcal{Y}(s) = [s(2K\alpha - \beta s)]^{1/2}.$$

Inserting this into (B.7), another simple integration yields when $\alpha \neq 0$:

(B.10) $$\bar{S}(s) = (2K)^{-1}\ln[\;s/(2K\alpha - \beta s)].$$

For null geodesics ($\beta = 0$) we might as well set $2K\alpha = 1$ since the scale of the affine parameter s is unimportant in that case and since α cannot vanish without making $\bar{S}$ and $\mathcal{Y}$ constants so that the solution represents a point and not a curve. α may vanish only in the timelike case.

By (6.62) and (B.4) the solution may be written:

(B.11) $\quad \underline{g}(s) = (\kappa \varsigma)^{4/3}\, \underline{A}^T e^{\bar{s}\underline{N}}\, \underline{A}.$

Suppose $\underline{O} \in SO(3,\mathbb{R})$ diagonalizes the symmetric traceless matrix $\underline{N}$:

$$\underline{N} = \underline{O}^{-1}\underline{N}_D\underline{O}\ , \ \mathrm{Tr}\,\underline{N}_D = 0\ , \ \mathrm{Tr}\,\underline{N}_D^{\,2} = 1\,.$$

These conditions on the diagonal matrix $\underline{N}_D$ imply:

$$\underline{N}_D = 6^{-1/2}(\cos\theta\,\hat{\underline{e}}_+ + \sin\theta\,\hat{\underline{e}}_-),$$

so that:

(B.12) $\quad \underline{g}(s) = (\underline{O}\underline{A})^T (\kappa\varsigma)^{4/3} e^{\bar{s}\underline{N}_D} (\underline{O}\underline{A}) = \underline{B}^T \underline{g}_D \underline{B}$

$$\underline{g}_D(s) = \varsigma^{4/3} \exp \bar{s}\underline{N}_D \quad , \quad \underline{B} = \kappa^{2/3}\,\underline{O}\underline{A}\,.$$

The geodesics are therefore diagonal apart from the transpose action of fixed elements of $GL(3,\mathbb{R})$. For null and timelike geodesics respectively, $g_D(s)$ is explicitly:

(B.13) $\quad \underline{g}_D(s) = s^{2/3(\hat{\underline{e}}_0 + \cos\theta\,\hat{\underline{e}}_+ + \sin\theta\,\hat{\underline{e}}_-)}$

$$\underline{g}_D(s) = s^{2/3}\,(s+2\kappa\alpha)^{2/3}\,[s/(s+2\kappa\alpha)]^{\frac{2}{3}(\cos\theta\,\hat{\underline{e}}_+ + \sin\theta\,\hat{\underline{e}}_-)}\,.$$

C. Bianchi Quaternions

Let $C^a{}_{bc} = \epsilon_{bcd} n^{da}$ with $\underline{n} = \mathrm{diag}(n^{(1)}, n^{(2)}, n^{(3)})$ be the components of the SCT of a class A Lie algebra in diagonal form. The matrices $\underline{R}_a = C^b{}_{ac} \hat{\underline{e}}^c{}_b$ generate the adjoint matrix Lie algebra. Now make the following definitions:

(C.1) $\gamma_{ab} = \frac{1}{2}\,\mathrm{TR}\,\underline{R}_a \underline{R}_b = -\gamma^*{}_{ab} = -\mathrm{diag}(n^{(2)}n^{(3)}, n^{(3)}n^{(1)}, n^{(1)}n^{(2)})$

$$\gamma_{0\alpha} = \gamma_{\alpha 0} = \gamma^*{}_{0\alpha} = \gamma^*{}_{\alpha 0} = \delta_{0\alpha}$$

$$M^0{}_{\alpha\beta} = \gamma_{\alpha\beta} \qquad M^\alpha{}_{0\beta} = M^\alpha{}_{\beta 0} = \delta^\alpha{}_\beta \qquad M^a{}_{bc} = C^a{}_{bc}$$

$$M^\alpha{}_{[\beta\gamma]} = C^\alpha{}_{\beta\gamma} = \delta^\alpha_a C^a{}_{bc} \delta^b_\beta \delta^c_\gamma \qquad M^\alpha{}_{(bc)} = \delta^\alpha_0 \gamma_{bc}$$

$$\underline{M}_\alpha = M^\gamma{}_{\alpha\beta} \hat{\underline{e}}^\beta{}_\gamma \quad \underline{\tilde{M}}_\alpha = M^\gamma{}_{\beta\alpha} \hat{\underline{e}}^\beta{}_\gamma \quad \underline{C}_\alpha = C^\gamma{}_{\alpha\beta} \hat{\underline{e}}^\beta{}_\gamma \,.$$

We are using the indices 0, 1, 2, 3 rather than 1, 2, 3, 4 for $\mathbb{R}^4$ and $gl(4,\mathbb{R})$. One may easily verify that $M^\gamma{}_{\alpha\beta}$ satisfies the following "associativity property" which may be written in three equivalent ways in matrix notation:

(C.2) $M^\epsilon{}_{\alpha\delta} M^\delta{}_{\beta\gamma} = M^\delta{}_{\alpha\beta} M^\epsilon{}_{\delta\gamma}$

$$\underline{M}_\alpha \underline{M}_\beta = M^\gamma{}_{\alpha\beta} \underline{M}_\gamma \,, \quad \underline{\tilde{M}}_\alpha \underline{\tilde{M}}_\beta = M^\gamma{}_{\beta\alpha} \underline{\tilde{M}}_\gamma \,, \quad [\underline{M}_\alpha, \underline{\tilde{M}}_\beta] = 0 \,,$$

from which it follows that:

(C.3) $[\underline{M}_\alpha, \underline{M}_\beta] = 2C^\gamma{}_{\alpha\beta} \underline{M}_\gamma \qquad [\underline{\tilde{M}}_\alpha, \underline{\tilde{M}}_\beta] = -2C^\gamma{}_{\alpha\beta} \underline{\tilde{M}}_\gamma \,,$

while the Jacobi identity implies:

(C.4) $[\underline{C}_\alpha, \underline{C}_\beta] = C^\gamma{}_{\alpha\beta} \underline{C}_\gamma \,.$

Note that $\underline{M}_0 = \underline{\tilde{M}}_0 = \underline{1}$ and $\mathrm{Tr}\, \underline{M}_a = \mathrm{Tr}\, \underline{R}_a = 0 = \underline{C}_0$. $\{\underline{M}_\alpha\}$ and $\{\underline{\tilde{M}}_\alpha\}$ are linearly independent sets of matrices for each Bianchi type and satisfy:

(C.5) $\underline{M}_{(a} \underline{M}_{b)} = \gamma_{ab} \underline{1} = \underline{\tilde{M}}_{(a} \underline{\tilde{M}}_{b)} \,.$

$\{\underline{M}_\alpha\}$ and $\{\underline{\tilde{M}}_\alpha\}$ generate two mutually commuting isomorphic matrix subalgebras of $gl(4,\mathbb{R})$. They also generate mutually commuting isomorphic matrix Lie algebras, each of which is the direct sum of the trivial Lie algebra generated by $\underline{1}$ and a 3-dimensional Lie algebra of the Bianchi type to which $C^a{}_{bc}$ belongs or simply "of Bianchi type $\underline{n}$." The 3-dimensional Lie algebras generate mutually commuting subgroups of $SL(4,\mathbb{R})$; $\{\frac{1}{2}\underline{M}_a\}$ is the basis of the first in which the components of the SCT are $C^a{}_{bc}$. The matrices $\{\underline{C}_\alpha\}$ generate the adjoint matrix group shared by the two 4-dimensional Lie algebras, but since $C^\gamma{}_{\alpha\beta}$ vanishes if any index is zero, this group is just the direct product of the identity group and the adjoint matrix group of the 3-dimensional Lie algebras:

(C.6) $\quad \underline{\mathcal{R}}(x) = \exp x^a \underline{C}_a \,, \quad \underline{R}(x) = \exp x^a \underline{k}_a$

$\qquad \mathcal{R}^0{}_0 = 1 \,, \mathcal{R}^0{}_a = 0 = \mathcal{R}^a{}_0 \,, \mathcal{R}^a{}_b = R^a{}_b \,.$

Choosing canonical coordinates of the first kind for the two matrix groups we have the parametrizations:

(C.7) $\quad \underline{A}(x) = \exp X^\alpha \underline{M}_\alpha/2 = e^{\frac{1}{2}X^0} \exp X^a \underline{M}_a/2$

$\qquad \underline{\tilde{A}}(x) = \exp X^\alpha \underline{\tilde{M}}_\alpha/2 = e^{\frac{1}{2}X^0} \exp X^a \underline{\tilde{M}}_a/2$

$\qquad \widetilde{(\underline{A}_1 \underline{A}_2)} = \underline{\tilde{A}}_2 \underline{\tilde{A}}_1 \,.$

The adjoint identity and invariant forms for the first group are:

(C.8) $\quad \underline{A}(x)\, \underline{M}_\alpha\, \underline{A}^{-1}(x) = \underline{M}_\beta\, \mathcal{R}^\beta{}_\alpha(x)$

$\qquad \underline{A}^{-1} d\underline{A} = \omega^\alpha \underline{M}_\alpha/2 \;, \quad d\underline{A}\,\underline{A}^{-1} = \tilde{\omega}^\alpha \underline{M}_\alpha/2 \,, \quad \tilde{\omega}^\alpha = \mathcal{R}^\alpha{}_\beta \omega^\beta .$

Because the span of the sets $\{\underline{M}_\alpha\}$ and $\{\underline{\tilde{M}}_\alpha\}$ are each closed under multiplication, $\underline{A}(x)$ is a linear combination of $\{\underline{M}_\alpha\}$ and $\underline{\tilde{A}}(x)$ of $\{\underline{\tilde{M}}_\alpha\}$:

(C.9) $\quad \underline{A}(x) = a^\alpha(x)\, \underline{M}_\alpha \qquad \underline{\tilde{A}}(x) = a^\alpha(x)\, \underline{\tilde{M}}_\alpha \,.$

The parametrized functions $a^\alpha(x)$ may be computed by carrying out the exponentiation using the fact that $X^a X^b \underline{M}_a \underline{M}_b = \gamma_{ab} X^a X^b \underline{1}$ which follows from (C.5):

(C.10) $\quad \underline{A}(x) = e^{\frac{1}{2}X^0} \left(\underline{1} \cosh u(x)/2 + X^a \underline{M}_a\, u(x)^{-1} \sinh u(x)/2 \right)$

$\qquad u(x) = (\gamma_{ab} X^a X^b)^{1/2} = i\, (\overset{*}{\gamma}_{ab} X^a X^b)^{1/2} = i\, v(x).$

The hyperbolic combinations of $u(x)$ are equivalent to trigonometric combinations of $v(x)$ as in (10.3). The functions a^α expressed in canonical coordinates of the first kind may be read off from (C.10).

Let $\not\Sigma\, X^a \underline{M}_a$ indicate no summation over a; specializing (C.10) yields:

(C.11) $\quad \exp(\not\Sigma\, X^a \underline{M}_a/2) = \underline{1}\, C_a + \underline{M}_a S_a \,,$

where C_a and S_a are as in (10.3) except for the replacement of the argument X^a by $X^a/2$. This enables us to evaluate the functions a^α in canonical coordinates of the second kind or in Euler angle coordinates when valid. For example, using (C.11) to expand the product:

$$a^\alpha \underline{M}_\alpha = e^{\frac{1}{2}X^0}\, e^{\frac{1}{2}X^1 \underline{M}_1}\, e^{\frac{1}{2}X^2 \underline{M}_2}\, e^{\frac{1}{2}X^3 \underline{M}_3} \,,$$

and using the multiplication (C.2) one finds:

(C.12) $$\begin{aligned} a^0 &= e^{\frac{1}{2}x^0}\,(c_1c_2c_3 - n^{(1)}n^{(2)}n^{(3)} s_1s_2s_3) \\ a^1 &= e^{\frac{1}{2}x^0}\,(s_1c_2c_3 + n^{(1)} c_1 s_2 s_3) \\ a^2 &= e^{\frac{1}{2}x^0}\,(c_1 s_2 c_3 - n^{(2)} s_1 c_2 s_3) \\ a^3 &= e^{\frac{1}{2}x^0}\,(c_1 c_2 s_3 + n^{(3)} s_1 s_2 c_3), \end{aligned}$$

while expanding the product:

$$a^\alpha M_\alpha = e^{\frac{1}{2}x^0} e^{x^2 \underline{M}_3/2}\, e^{x^1 \underline{M}_1/2}\, e^{x^3 \underline{M}_3/2},$$

and using hyperbolic identities one obtains:

(C.13) $$a^1 = e^{\frac{1}{2}x^0} s_1(\tfrac{x^1}{2}) c_3(\tfrac{x^2-x^3}{2}) \qquad a^3 = e^{\frac{1}{2}x^0} c_1(\tfrac{x^1}{2}) s_3(\tfrac{x^2+x^3}{2})$$
$$a^2 = e^{\frac{1}{2}x^0} s_1(\tfrac{x^1}{2}) s_3(\tfrac{x^2-x^3}{2}) n^{(2)} \qquad a^0 = e^{\frac{1}{2}x^0} c_1(\tfrac{x^1}{2}) c_3(\tfrac{x^2+x^3}{2}),$$

where we have had to indicate the coordinate arguments explicitly.

Now let $\{\hat{e}_\alpha\}$ be the natural basis of $\mathbb{R}^4$ and define an associative algebra $q_{\underline{n}}$ on $\mathbb{R}^4$ by:

(C.14) $$\hat{e}_\alpha \hat{e}_\beta = M^\gamma{}_{\alpha\beta}\, \hat{e}_\gamma$$
$$a\,b = \hat{e}_\alpha M^\alpha{}_{\beta\gamma} a^\beta b^\gamma \;, \quad a = a^\alpha \hat{e}_\alpha \;, \; b = b^\alpha \hat{e}_\alpha \,.$$

e_0 is the unit element of $q_{\underline{n}}$ and

(C.15) $$a = a^\alpha \hat{e}_\alpha \in q_{\underline{n}} \mapsto \underline{A}[a] = a^\alpha \underline{M}_\alpha$$

is an isomorphism. By introducing the corresponding notation $\tilde{\underline{A}}[a] = a^\alpha \tilde{\underline{M}}_\alpha$, we may write:

(C.16) $$a\,b = \hat{e}_\alpha\, A[a]^\alpha{}_\beta\, b^\beta = \hat{e}_\alpha \tilde{A}[b]^\alpha{}_\beta\, a^\beta .$$

We call the elements of the algebra $q_{\underline{n}}$ Bianchi quaternions of type $\underline{n}$, and when $\underline{n}$ assumes its canonical value, canonical Bianchi quaternions of the given Bianchi type. Canonical type IX and type VIII Bianchi quaternions are known as ordinary and Gödel quaternions respectively.[66,67]

Define quaternion conjugation $*$ by:

(C.17) $$a = a^\alpha \hat{e}_\alpha \mapsto a^* = a^0 \hat{e}_0 - a^a \hat{e}_a \;, \quad (ab)^* = b^* a^*$$
$$a^* a = a a^* = \hat{e}_0\, \gamma^*_{\alpha\beta} a^\alpha a^\beta = \hat{e}_0 |a|^2 , \quad |ab|^2 = |a|^2 |b|^2 .$$

The second line defines the quaternion norm which is explicitly:

(C.18) $$|a|^2 = (a^0)^2 + n^{(2)}n^{(3)}(a^1)^2 + n^{(3)}n^{(1)}(a^2)^2 + n^{(1)}n^{(2)}(a^3)^2 .$$

A unit quaternion is one for which $|a|^2 = 1$. The unit quaternions of the canonical types IX, VIII, VII$_0$ and VI$_0$ are confined to the 3-sphere S^3, the hyperboloid H^3, a cylinder and a hyperbolic cylinder respectively, while the unit quaternions of types I and II correspond to $q^0 = \pm 1$.

Define the general and special linear groups in one Bianchi quaternion (of type $\underline{n}$) dimension and their respective quaternion Lie algebras by:

$$\text{(C.19)}\quad gl(1,q_{\underline{n}}) = q_{\underline{n}} \qquad GL(1,q_{\underline{n}}) = \{a \in q_{\underline{n}} \mid |a|^2 \neq 0\}$$
$$sl(1,q_{\underline{n}}) = \{a \in q_n \mid a^0 = 0\} \qquad SL(1,q_{\underline{n}}) = \{a \in q_{\underline{n}} \mid |a|^2 = 1\}.$$

$\hat{e}_0$ is the identity of $GL(1,q_{\underline{n}})$ and the inverse of any $a \in GL(1,q_{\underline{n}})$ is $a^{-1} = |a|^{-2} a^*$. Our nomenclature arises from the fact that the natural action of $GL(1,q_n)$ on $q_{\underline{n}}$ by left and right multiplication is linear while by (C.16), $SL(1,q_{\underline{n}})$ leaves the quaternion norm invariant under this action. The matrix groups describing these left and right linear actions in the natural basis $\{\hat{e}_\alpha\}$ are just those generated by $\{M_\alpha\}$ and $\{\tilde{M}_\alpha\}$ respectively:

$$L_a(b) = ab = A[a]^\alpha{}_\beta b^\beta \hat{e}_\alpha, \quad R_a(b) = ba = \tilde{A}[a]^\alpha{}_\beta b^\beta \hat{e}_\alpha.$$

Let the superscript + denote the component of $GL(1,q_{\underline{n}})$ connected to the identity.

Because of the isomorphism (C.15), the exponential formulas (C.9) through (C.13) may be taken over exactly to the quaternion case upon replacement of $\{\underline{M}_\alpha\}$ by $\{\underline{\hat{e}}_\alpha\}$, where the quaternion exponential from $q_{\underline{n}}$ into $GL(1,q_n)^+$ is defined by the usual power series. The functions $a^\alpha(x)$ then parametrize the nonsingular quaternions in terms of canonical coordinates of the first or second kinds or Euler angle coordinates. $GL(1,q_{\underline{n}})^+$ is just the direct product group $\mathbb{R} \times SL(1,q_{\underline{n}})^+$, where x^0 corresponds to the canonical coordinate on $\mathbb{R}$. $|a| = (|a|^2)^{1/2}$ is real and positive on $GL(1,q_{\underline{n}})^+$; in fact one may verify that $|a| = e^{\frac{1}{2}x^0}$ by evaluating (C.18) on (C.10). $SL(1,q_n)^+$ is therefore the hypersurface $x^0 = 0$, which explains why $sl(1,q_n)$ is its Lie algebra.

Let $\{a^\alpha\}$ be cartesian coordinates on $q_{\underline{n}} = \mathbb{R}^4$ with coordinate frame $\{\partial_\alpha\}$, and let $\varphi^\alpha(a_1,a_2) = M^\alpha{}_{\beta\gamma} a_1{}^\beta a_2{}^\gamma$ be the multiplication function in these coordinates. By (1.31) the left and right invariant frames on $GL(1,q_n)$ which reduce to $\{\tfrac{1}{2}\partial_\alpha\}$ at the identity are:

(C.20) $$2e_\alpha = a^\beta M^\gamma{}_{\beta\alpha}\,\partial_\gamma \qquad 2\tilde{e}_\alpha = a^\beta M^\gamma{}_{\alpha\beta}\,\partial_\gamma$$
$$[e_\alpha, e_\beta] = C^\gamma{}_{\alpha\beta}\, e_\gamma \qquad [\tilde{e}_\alpha, \tilde{e}_\beta] = -\,C^\gamma{}_{\alpha\beta}\,\tilde{e}_\gamma\,.$$

We have already taken the liberty of identifying the Lie algebra of $GL(1,q_n)$ with q_n itself. Under this identification, e_α corresponds to $\hat{e}_\alpha/2$. The dual frames are easily seen to be:

(C.21) $$\omega^\alpha = 2|a|^{-2} M^\alpha{}_{\beta\gamma}\, a^{*\beta} da^\gamma, \quad \tilde{\omega}^\alpha = 2|a|^{-2} M^\alpha{}_{\gamma\beta}\, a^{*\beta} da^\gamma,$$
$$\omega^0 = \tilde{\omega}^0 = 2|a|^{-2}\,\gamma^*{}_{\beta\gamma}\, a^\beta da^\gamma = d\ln|a|^2 = dx^0.$$

By introducing the quaternion valued forms $\omega = \omega^\alpha \hat{e}_\alpha$ and $\tilde{\omega} = \tilde{\omega}^\alpha \hat{e}_\alpha$, these relations may be written:

(C.22) $$\omega = 2a^{-1}da \qquad \tilde{\omega} = 2\,da\,a^{-1}.$$

If one evaluates $\{\omega^a\}$ and $\{\tilde{\omega}^a\}$ in the canonical coordinates of the second kind (C.12) or $\{\omega^a\}$ in the Euler angle coordinates (C.13) when valid, one will obtain exactly (10.5) and (10.9) respectively because of the local isomorphism provided by identifying canonical or noncanonical coordinates defined with respect to Lie algebra bases whose SCT components agree.

If we compare (10.2) and the quaternion analogue of (C.12) for the canonical type VII_0, VIII and IX cases, $(x^1, x^2, x^3) = (0,0,2\pi)$ corresponds to the identity of the $GL(3,\mathbb{R})^+$ subgroup and $-\hat{e}_0$ in $GL(1,q_n)^+$. In fact, the kernel of the homomorphism which sends the point of $GL(1,q_n)$ with coordinates (x^1,x^2,x^3) to the corresponding point of the $GL(3,\mathbb{R})^+$ subgroup is the discrete subgroup $\{\hat{e}_0, -\hat{e}_0\}$, so the $GL(3,\mathbb{R})^+$ subgroup is isomorphic to $GL(1,q_n)^+/\{\hat{e}_0,-\hat{e}_0\}$ in these cases. $GL(1,q_n)$ is said to be a double covering of the other group. The type VII_0 and VIII Bianchi quaternion groups are closed in one direction, with the range of the canonical coordinate x^3 of the second kind being $[0,4\pi)$. By extending the range of this coordinate to the real line, one obtains the "simply connected covering group" of each type.

Note that $GL(1,q_n)^+ = \mathbb{R}\times SL(1,q_n)^+$ is exactly the kind of product manifold we assumed ~~for the spacetime~~ manifold of a SH spacetime of Bianchi type II. This quaternion realization is used by Misner and Taub [50] to discuss the type IX Taub-Nut vacuum solution.

D. A Spatially Homogeneous Electromagnetic Field.

Consider a SH electromagnetic field F with components $F_{\alpha\beta}$ in a normal SH comoving ADM frame. Its energy-momentum tensor has components:

(D.1) $-4\pi T^{\alpha\beta} = F^{\alpha\gamma} F_\gamma{}^\beta + \frac{1}{4} g^{\alpha\beta} F_{\gamma\delta} F^{\gamma\delta}$.

Make the following definitions:

(D.2) $E^a = F^{0a}$ $\quad B^a = {}^*F^{0a} = \frac{1}{2}\eta^{abc} F_{bc}$

$\mathcal{E}^a = g^{1/2} E^a$ $\quad \mathcal{B}^a = g^{1/2} B^a$.

$4\pi S_a = \eta_{abc} E^b B^c$ $\quad 4\pi \mathcal{S}_a = 4\pi g^{1/2} S_a = \epsilon_{abc} \mathcal{E}^b \mathcal{B}^c$.

In a normal frame the following relations are also valid:

(D.3) $F_{a0} = E_a$ $\quad F^{bc} = \eta^{bca} B_a$

${}^*F_{a0} = B_a$ $\quad {}^*F^{ab} = -\eta^{abc} E_c$.

Evaluating (D.1) in terms of these variables yields:

(D.4) $4\pi T^{ab} = -(E^a E^b + B^a B^b) + \frac{1}{2} g^{ab}(E_c E^c + B_c B^c)$

$-4\pi k g^{1/2} T^\perp{}_\perp = \frac{1}{2} k g^{-1/2} g_{ab} (\mathcal{E}^a \mathcal{E}^b + \mathcal{B}^a \mathcal{B}^b)$

$-4\pi k g^{1/2} T^\perp{}_a = -4\pi k \mathcal{S}_a = -k \epsilon_{abc} \mathcal{E}^b \mathcal{B}^c$.

To study the dynamics of a SH gravitational field coupled to a SH electromagnetic field we take $\mathcal{E}^a$ and $\mathcal{B}^a$ as the independent source variables. No source component of the nonconservative force is then required. The electromagnetic potential energy is:

(D.5) $U_s(g_{ab}, \mathcal{E}^a, \mathcal{B}^a) = k(8\pi g^{1/2})^{-1} g_{ab}(\mathcal{E}^a \mathcal{E}^b + \mathcal{B}^a \mathcal{B}^b)$

$\partial U_s / \partial g_{ab} = -k g^{1/2} T^{ab}$.

By (13.9) the sourcefree Maxwell equations $F^{\alpha\beta}{}_{;\beta} = 0 = {}^*F^{\alpha\beta}{}_{;\beta}$ take the form:

(D.6) $\dot{\mathcal{E}}^a + g^{-1/2} \mathcal{B}_b C^{ba} = 0$ $\quad a_c \mathcal{E}^c = 0$

$-\dot{\mathcal{B}}^a + g^{-1/2} \mathcal{E}_b C^{ba} = 0$ $\quad a_c \mathcal{B}^c = 0$.

In a general SH ADM frame $\bar{\mathcal{E}}^a$ and $\bar{\mathcal{B}}^a$ are defined by:

(D.7) $\bar{\mathcal{E}}^a = ({}^4\bar{g})^{1/2} \bar{F}^{0a} = N \bar{g}^{1/2} \bar{F}^{0a} = \bar{g}^{1/2} \bar{F}^{\perp a} = \det \mathcal{R}^{-1} \mathcal{R}^a{}_b \mathcal{E}^b$

$\bar{\mathcal{B}}^a = ({}^4\bar{g})^{1/2} {}^*\bar{F}^{0a} = \bar{g}^{1/2} {}^*\bar{F}^{\perp a} = \det \mathcal{R}^{-1} \mathcal{R}^a{}_b \mathcal{B}^b$.

By transforming Maxwell's equations using (12.10) one obtains the result:

(D.8) $d\bar{\mathcal{E}}^a / d\bar{t} = -2 a_b \bar{N}^b \bar{\mathcal{E}}^a + \bar{N}^c C^a{}_{cb} \bar{\mathcal{E}}^b - N \bar{g}^{-1/2} \bar{\mathcal{B}}_b C^{ba}$

$d\bar{\mathcal{B}}^a / d\bar{t} = -2 a_b \bar{N}^b \bar{\mathcal{B}}^a + \bar{N}^c C^a{}_{cb} \bar{\mathcal{B}}^b + N \bar{g}^{-1/2} \bar{\mathcal{E}}_b C^{ba}$

$$a_a \bar{E}^a = 0 = a_a \bar{\mathcal{B}}^a.$$

As in section fifteen one may examine the source equations to see under what conditions $\mathcal{S}_a$ and $n^{ab}\mathcal{S}_a\mathcal{S}_b$ are constants of the motion.

Suppose we introduce a vector potential $A = A_\alpha \omega^\alpha$ and work in a normal frame. Then E_a, $\mathcal{B}^a$ and the nontrivial sourcefree vacuum Maxwell equations are given in terms of A by:

(D.9) $$F = d(A_\alpha \omega^\alpha) = d(A_a \omega^a) = \dot{A}_a \omega^0 \wedge \omega^a - \tfrac{1}{2} A_a C^a{}_{bc} \omega^b \wedge \omega^c$$

$$E_a = -\dot{A}_a \qquad \mathcal{B}^a = - A_d C^{da}$$

$$g^{1/2} (g^{1/2} g^{ab} \dot{A}_b)^{\cdot} + A_b C^{bc} g_{cd} C^{da} = 0 .$$

Define $\mathcal{U} = -\frac{1}{4} \gamma_{ab} (\mathcal{E}^a \mathcal{E}^b + \mathcal{B}^a \mathcal{B}^b)$. (Since $\gamma_{ab} = -2(1+h^{-1}) a_a a_b$ in the class B case, the sourcefree divergence constraints force $\mathcal{U}=0$.) For class A models the sourcefree Maxwell equations imply $\dot{\mathcal{U}}=0$. This is nontrivial only for types IX/VIII where $-\frac{1}{2} \gamma_{ab} \underline{\hat{e}}^b{}_a = \mathrm{diag}(\pm 1, \pm 1, 1)$ in a canonical frame. In types $\mathrm{VII}_0/\mathrm{VI}_0$, $-\frac{1}{2} \gamma_{ab} \underline{\hat{e}}^b{}_a = \mathrm{diag}(0,0,\pm 1)$ but $\dot{\mathcal{E}}^3 = 0 = \dot{\mathcal{B}}^3$ separately since $n^{(3)}=0$, while in types I and II, $\gamma_{ab}=0$ and $\mathcal{E}^a$ and $\mathcal{B}^a$ are constant anyway (except for $a=3$ in type II).

For the Taublike type IX/VIII electromagnetic spacetimes,[51, 53] the constant $\mathcal{U} = \frac{1}{2} e^2$ enables one to solve the Einstein equations independent of the Maxwell equations. Conversely, by a judicious choice of lapse the Maxwell equations may be solved independent of the Einstein equations. For these spacetimes, $\underline{g} = \mathrm{diag}(A, A, B)$ and only A_3 is nonvanishing. By defining $dt = (A/B^{1/2})\, du$, the one nontrivial Maxwell equation becomes:

$$(d^2/du^2 + 1) A_3 = 0$$

$$A_3 = e(\sin\alpha \cos u + \cos\alpha \sin u).$$

The duality rotation parameter α is an arbitrary constant and the overall amplitude e is chosen so that $\mathcal{U} = \frac{1}{2} e^2$. From the Einstein equations one finds $A = \bar{t}^2 + \ell^2$ where $dt = (2\ell / B^{1/2})\, d\bar{t}$ so $du = (2\ell/A)\, d\bar{t}$ and:

$$u = \int 2\ell\, d\bar{t} / (\bar{t}^2 + \ell^2) = 2 \tan^{-1}(\bar{t}/\ell)$$

$$\tan u/2 = \bar{t}/\ell \qquad \sin u = 2\ell\bar{t}/(\bar{t}^2+\ell^2) \qquad \cos u = (\ell^2 - \bar{t}^2)/(\bar{t}^2+\ell^2).$$