§16. Exploration of the Dynamical System

In this section we will discuss the dynamics of SH perfect fluid spacetimes using the tools already developed. In the general situation the solution of the three momentum constraints and the energy constraint will be described as a means of reducing the number of gravitational degrees of freedom. This will be followed by an investigation, roughly speaking, of which degrees of freedom may be suppressed to obtain more specialized and less complicated subsystems, a feature which strongly depends on the Bianchi type. The Bianchi types I, II, V and $VI_{-1/9}$ require special treatment, due to certain degeneracies, and will be referred to as degenerate types. The remaining types will be called nondegenerate or general. Examples of tractable subsystems will be given to illustrate various techniques and point out problems beyond the scope of this paper and a few remarks about linearization stability and other matters will be made. The non-Hamiltonian analogue of our approach will also be summarized.

Now that we have discussed SH perfect fluids (with configuration space $\mathcal{S}$), our dynamical system on $\mathcal{M}\times\mathcal{S}$ determined by a normal canonical comoving ADM frame is completely defined, describing the evolution of a SH perfect fluid spacetime model. However, we must remember that such a spacetime frame is determined only up to the (time-independent) action of the canonical automorphism matrix group on the reduced frame, under which the components of all the fields involved transform as follows:

$$(16.1)\qquad \{\underline{g}, V_a, \ell, n\} \mapsto \{\underline{A}^{-1T}\underline{g}\,\underline{A}^{-1}, V_b A^{-1b}{}_a, |\det \underline{A}^{-1}|\ell, n\}$$
$$\underline{A}\in \mathrm{Aut}_e(\mathfrak{g}).$$

This freedom may be used to eliminate unimportant constants in the solutions of the equations of motion.

The structure of the dynamical system becomes much more transparent once we abandon the component coordinates in favor of the special automorphism coordinates developed in § 11, since the latter are adapted to the symmetry of the system. All the work required to transform the geometric variables to the new coordinates was done in that section. We need only extend the transformation to the fluid velocity:

(16.2) $$V_a' = V_b S^{-1b}{}_a \quad .$$

Note that $\ell' = \ell$ since $\det \underline{S} = 1$. The fluid equations of motion in the new coordinates $\{g'_{ab}, S^a{}_b, V'_a, \ell, n\}$ on $\mathcal{M} \times \mathcal{S}$ then follow from (15.12):

(16.3) $$(\ln \ell)^{\cdot} = 2a_a V'^a / V^\perp = 2a V'^3 / V^\perp = 2a\, e^{-2\beta^0 + 4\beta^+} V_3' / V^\perp$$

$$(V_a')^{\cdot} = V_f' C^f{}_{ag} V'^g / V^\perp - V_f' \kappa_b{}^f{}_a \tilde{\omega}^b$$

$$V^\perp = (\mu^2 + g'^{ab} V_a' V_b')^{1/2}$$

$$\mathbb{V}^2 = \epsilon n^{ab} V_a V_b = \epsilon n^{ab} V_a' V_b' \,.$$

When $\mathbb{V}^2 \neq 0$, one may apply the change of variables (15.14), (15.16) to $\{V_a'\}$ and thus use the constant of the motion $\mathbb{V}^2$ (class A) or $\ell \mathbb{V}^2$ (class B) to eliminate one fluid degree of freedom. Due to (15.31) and (15.32), only the variables $\{m, \bar{V}_3 = \bar{V}_3'\}$ are needed for type VI_h fluids.

The fluid potential energy function (15.17) is given by:

(16.4) $$U_S(\underline{g}', V_a', \ell, n) = 2k\ell V^\perp - 2kp\, e^{3\beta^0} \,.$$

The expressions for the the kinetic energy of the gravitational Lagrangian (13.22) and Hamiltonian (13.24), when compared with (11.40) and (11.43), are seen to be given by (11.51), while the gravitational potential energy function U^* is given by (11.26). The nonpotential force Q^* is given by (11.27). Let $U = U^* + U_S$ be the total potential energy.

By introducing all the identifications we have made between dots and time derivatives and coordinates on the configuration and phase spaces, we have essentially reduced our notation to the sloppy but convenient language of physics rather than of precise mathematics. We exploit this sloppiness, interpreting quantities as functions on $T\mathcal{M}$ or $T^*\mathcal{M}$ depending on whether we "express them in terms of velocities or momenta." For example, the kinetic energy $\mathcal{T}$ of (13.14) "equals" L_0 of (11.40), (11.51) and H_0 of (11.43), (11.51):

(16.5) $$\mathcal{T} = e^{3\beta^0}(6\eta_{AB}\dot{\beta}^A\dot{\beta}^B + \mathcal{G}'_{ab}\tilde{\omega}^a\tilde{\omega}^b) = \tfrac{1}{4}e^{-3\beta^0}(\tfrac{1}{6}\eta^{AB}p_A p_B + \mathcal{G}'^{ab}P_a P_b),$$

while the functions $f(M_a')$ "=" $^*f(M'_a)$ of (11.54) are the gravitational contributions to the momentum constraint functions:

(16.6) $\mathcal{M}^{\perp}{}_a{}' = f(M'_a) - k\ell V'_a$.

The complete Lagrangian and Hamiltonian are just $L = \mathcal{T} - U$ and $H = \mathcal{T} + U$ respectively.

Note that of the automorphism variables, only the velocities or momenta appear in the system except in the combination V'_a where the matrix $\underset{\sim}{S}$ appears explicitly. Since the fluid potential depends on V'_a, one must add an extra term to the equations of motion (11.55) and (11.57) for P_a and $\dot{\omega}^a$ respectively. For example, from the Hamiltonian point of view, one must add to (11.55) the term:

(16.7) $\{P_a, U_S\} = 2k\ell(V^{\perp})^{-1} g'^{bc} V'_b \{P_a, V'_c\} = 2k\ell(V^{\perp})^{-1} V'_d \mathcal{K}_a{}^d{}_c V'^c$,

in which we have used the relation:

(16.8) $\{P_a, V'_b\} = V'_d \mathcal{K}_a{}^d{}_c$

which follows immediately from (11.50). As long as these new brackets (16.8) are introduced and the primed fluid variables are used, the matrix $\underset{\sim}{S}$ decouples from the remaining variables, in the sense that one can can always obtain it after a solution of the remaining equations of motion are solved by using (11.59).

It is not surprising that the matrix $\underset{\sim}{S}$ can be eliminated from the system leaving behind only the automorphism velocities because of the possibility of interpreting the primed variables as components with respect to a new comoving ADM frame generated by a nonzero shift vector field which is itself determined by those velocities. However, the approach we have taken using adapted coordinates on $\mathcal{M}$ and zero shift dynamics automatically incorporates the condition that this equivalent shift be chosen so that $\underset{\sim}{g}$ be diagonalized using the subgroup $\hat{G} \subset SAut_e(\mathfrak{g})$. If $\{\underset{\sim}{g}(t), v_a(t), \ell(t), n(t)\}$ is a solution curve of the dynamical system, representing components taken in a normal canonical comoving ADM frame $\{e_\alpha\}$, then $\{\underset{\sim}{g}'(t), V'_a(t), \ell(t), n(t)\}$ represent components taken in a new canonical comoving ADM frame $\{e'_\alpha\}$ whose reduced frame is orthogonal, as described in § 12 and assuming the topological conditions mentioned there. The (nonunique) shift vector field which generates this frame is determined by the automorphism velocities $\{\dot{\tilde{\omega}}^a\}$ as follows. Suppose $\{E'_a\} \subset \mathcal{X}(G)$ are such that $ad_{e'}(E'_a) = \underset{\sim}{k}_a$. Then $\vec{N} = \dot{\tilde{\omega}}^a E'_a$ is a shift compatible with the given primed frame, although it is defined only modulo time-dependent elements of the kernel of the map $ad : \mathcal{X}(g) \to aut(g)$. The spacetime metric is therefore known without integrating the equation for $\underline{S}$ which only serves to transform the primed frame back to the normal frame.

The scale variables $\{\beta^0, \beta^\pm\}$ are closely related to a parametrization of the isometry classes of left invariant Riemannian metrics on G, namely the orbits of the action of $Aut(g)$ on $\mathcal{M}(g)$ or equivalently of $Aut_e(g)$ on $\mathcal{M}$, assuming $Aut(G) \cong Aut(g)$.[100] Since every such orbit intersects $\mathcal{M}_D$ (in particular since it is true with respect to the subgroup $\hat{G} \subset Aut_e(g)$), it is sufficient to consider isomorphism classes on $\mathcal{M}_D$. The variable β^0 parametrizes the overall scale of the 3-metric, while $\{\beta^\pm\}$ parametrize the conformal 3-metric (see references in (101)). For types VIII and IX, both β^+ and β^- serve to parametrize the conformal isomorphism classes of $\mathcal{M}_D$ and hence of $\mathcal{M}$. For types IV, VI_0, VII_0, VI_h and VII_h, there is one extra automorphism generator $\underline{I}^{(3)} = \frac{1}{3}(\underline{e}_+ + 2\underline{e}_0) \in aut_e(g)$ not contained in the Lie algebra of $\hat{G}$. It generates translations in the β^0, β^+ plane and hence only β^- parametrizes the conformal isometry classes for these types. For the remaining types I, II and V, all left invariant metrics on G are conformally isometric (i.e., isometric modulo a constant scale factor). For type I, they are all isometric.

Since $\underline{g}' = (e^{\underline{\beta}})^T e^{\underline{\beta}}$, one may perform another frame transformation

$$e''_a = e'_b \, (e^{-\underline{\beta}})^b{}_a \tag{16.9}$$

which orthonormalizes the reduced frame. The orthonormal frame $\{e_\perp, e''_a\}$ is convenient for certain calculations. Suppose B is a SH mixed second rank spatial tensor, with matrices of components $\underline{B}'$ and $\underline{B}''$ with respect to the reduced frames $\{e'_a\}$ and $\{e''_a\}$ respectively. Since the

Correction for page 16.4. Second paragraph. 6th line, after "... on $\mathcal{M}_{D_0}$" ∧ insert

This fact was used in §10 to discuss the additional Killing vector fields possible for left invariant metrics by considering only "diagonal metrics".

diagonal part diag $\underline{B}'$ is invariant under the scaling (16.9):

$$B''^{a}{}_{b} = e^{\beta^{ab}} B'^{a}{}_{b} , \qquad (16.10)$$

it directly represents the diagonal orthonormal components, leaving only the off-diagonal components to be scaled. This is a very useful observation.

Our dynamical system is still subject to the constraints $\mathcal{M}^{\perp}{}_{a'} = 0 = H$. In the nondegenerate case and type V, there are three linearly independent momentum constraints, while for types I, II and $\mathrm{VI}_{-1/9}$ there are only 0, 2 and 2 linearly independent such constraints respectively. In the nondegenerate case we will be able to reduce the number of gravitational degrees of freedom by four using the constraints, arriving at an unconstrained (driven) Hamiltonian system with two gravitational degrees of freedom, namely those associated with $\{\beta^{\pm}\}$. In the degenerate case, fewer degrees of freedom can be eliminated.

Correction for page 16.8. Insert in line 7 after "β^+ is derived."

Alternatively, one may add a term to the component Q^+ of the nonpotential force:

$$(16.14) \qquad \dot{\beta}^+ = -(\partial H/\partial p_+)^S + Q^+ = -(\partial H^S/\partial p_+) + Q^+_S$$

$$Q^+_S = Q^+ + (\partial H/\partial \mathcal{P}_3)^S (\partial \mathcal{P}_3^S/\partial p_+) = Q^+ - \tfrac{1}{2} a e^{-3\beta^0} \bar{\mathcal{g}}'^{33} \mathcal{P}_3^S .$$

The S superscript indicates the replacement of $\mathcal{P}_3$ by $\mathcal{P}_3^S = 2k\ell V_3 - a p_+$.

As one can see from (11.53), the momentum constraints involve only the special automorphism velocities $\dot{\tilde{\omega}}^a$ or momenta P_a except in the class B case when $\dot{\beta}^+$ or p_+ enters the third constraint. The nontrivial momentum constraints in the class A case involve exactly the adjoint degrees of freedom; the type IX or "rotational" degrees of freedom supplementing these in types I and II are unrestricted and require special consideration. For class B the first two constraints involve only the adjoint degrees of freedom corresponding to $\underline{k}_1$ and $\underline{k}_2$; the last constraint links the third automorphism velocity or momentum to $\dot{\beta}^+$ or p_+ except in the type V case in which only the latter is involved. In the general case the automorphism degrees of freedom, which appear in the system only through their velocities or momenta, may therefore be eliminated from the system by solving the momentum constraints for these velocities or momenta in terms of the fluid variables and possibly $\dot{\beta}^+$ or p_+ (except in type V):

(16.11) class A.
$$P_a = -2k\ell V_a'$$
$$\dot{\tilde{\omega}}^a = -k\ell e^{-3\beta^\circ} \bar{g}'^{ab} V_b' .$$

These do not hold for type I or for a=3 in type II.

(16.12) class B.
$$P_1 = k\ell\,(3aV_1' - n^{(2)}V_2')/((9+h^{-1})a^2)$$
$$P_2 = k\ell\,(3aV_2' + n^{(1)}V_1')/((9+h^{-1})a^2)$$
$$P_3 = -2k\ell V_3' - a\,p_+$$
$$\dot{\tilde{\omega}}^a = \tfrac{1}{2} e^{-3\beta^\circ} \bar{g}'^{ab} P_b .$$

These hold only for P_3 and $\dot{\tilde{\omega}}^3$ in type $VI_{-1/9}$, while the third constraint becomes for type V:

(16.13) $P_+ = -2k\ell V_3'\,, \quad \dot{\beta}^+ = -\frac{1}{6}k\ell e^{-3\beta^0} V_3'.$

Except for P_3 which involves P_+ in the class B case, these expressions for the momenta may be substituted directly into the Hamiltonian without affecting the equations of motion for the scale variables. The expression for P_3 in the class B case may be used only after the equation of motion for β^+ is derived. In the Lagrangian approach, the expressions for the velocities may be used only after the equations of motion are derived, while they may be inserted directly into the fluid equations (16.3). The expressions for the momenta may also be inserted in (16.3) if one is employing the Hamiltonian approach.

In the nondegenerate case , let U_{eff} be the function $\frac{1}{4}e^{-3\beta^0}\bar{G}'^{ab}P_aP_b$ with the momenta replaced using (16.11) and (16.12) except for P_3 in the class B case where it is left independent until the scale equations of motion are derived. The net result from the Hamiltonian point of view is that the special automorphism degrees of freedom have dropped out of the system leaving behind an effective potential U_{eff} in the Hamiltonian and a "noninertial" force in the fluid velocity equations. We are left with a system involving only the scale degrees of freedom in addition to those of the fluid and having only one remaining constraint. This procedure can be carried out for any SH source including the actual substitution of the "solved momentum constraints" into the Hamiltonian provided that $\mathcal{J}_a = g^{1/2}T^{\perp}{}_a$ does not depend explicitly on the metric (as is true for electromagnetism and other 1-form theories).

The picture changes for the degenerate types I, II, V and $\text{VI}_{-1/9}$. For types I and II the rotational degrees of freedom are unrestricted and the corresponding components of the fluid velocity are required to vanish. The equations of motion for P_a in the first case are given by (11.55) with $\hat{C}^a{}_{bc} = \epsilon_{abc}$:

(16.15) $$\dot{P}_a = \tfrac{1}{2} e^{-3\beta^0} \mathcal{G}'^{cd} \epsilon_{bac} P_b P_d \,.$$

However, these rotational degrees of freedom are fake, reflecting a poor choice of coordinates on the configuration space. The type I vacuum and dust solutions are simply null and timelike geodesics respectively of $(\mathcal{M}, \mathcal{G})$. In the latter case, the Hamiltonian differs from the kinetic energy only by the constant term $2k\ell$ whose only effect is to determine the proportionality factor between the affine parameter t and the arclength s on the timelike geodesics. Appendix B shows that the geodesics of $(\mathcal{M}, \mathcal{G})$ are diagonal apart from the transpose action of fixed elements $\underline{O}\underline{A}$ of right cosets of $SO(3,\mathbb{R})$ in $GL(3,\mathbb{R})$. (By an appropriate choice of basis $\{e_a\}$ of $\mathfrak{g}$, we may therefore confine our attention to diagonal solutions.) The initial data $\underline{S} = \underline{1}$ and $P_a \neq 0$ corresponds to $\underline{A}$ diagonal and $\underline{N}$ not diagonal in the notation of that appendix so that $\underline{O} \neq \underline{1}$. However, the submanifold "$(\underline{O}\underline{A})^T \mathcal{M}_D (\underline{O}\underline{A})$" of $\mathcal{M}$ for fixed $\underline{O}\underline{A}$ does not coincide with a submanifold "$\underline{S}_0{}^T \mathcal{M}_D \underline{S}_0$" for any fixed $\underline{S}_0 \in SO(3,\mathbb{R})$, causing a complicated behavior of $\underline{S}$ to accomplish an irrelevant linear transformation.

The same feature is present in the general type I perfect fluid case, for which the Lagrangian in DeWitt coordinates differs from (B.5) only by terms trivial geometrically:

(16.16) $$L = \tfrac{1}{4}\left(-\dot{\zeta}^2 + (\kappa\zeta)^2 \mathrm{I}\right) - 2k\ell\mu + 2kp(\kappa\zeta)^2 .$$

In particular, (B.5) and (B.7) remain valid so again the solution is diagonal apart from an inessential constant linear transformation which may be eliminated by the choice of a new SH frame. Only the two geometric degrees of freedom associated with $\mathcal{J}$ and $\bar{S}$ are nontrivial as in (B.6):

(16.17) $4L = -\dot{\zeta}^2 + (K\zeta)^2(\dot{\xi}^2 + 8kp) - 8k\ell\mu$.

Since $(K\zeta)^2 = g^{1/2}$ and $\ell = g^{1/2}n$ when $V_a = 0$:

(16.18) $n = \ell(K\zeta)^{-2}$.

To proceed further an equation of state is needed. Consider the choice $p = (\gamma - 1)\rho$. From (15.3) one finds that p and μ are related to n by:

(16.19) $p = p_0(n/n_0)^\gamma$, $\mu = \mu_0(n/n_0)^{\gamma-1}$, $\mu_0 = \frac{\gamma}{\gamma-1}(p_0/n_0)$.

Letting $n_0 = \ell$, we then have explicitly:

(16.20) $p = p_0(K\zeta)^{-2\gamma}$, $\mu = \frac{\gamma}{\gamma-1}p_0(K\zeta)^{-2(\gamma-1)}$,

and hence after one step of algebra on the energy constraint:

(16.21) $\dot{\zeta}^2 = (K\alpha/\zeta)^2 + 8kp_0(\gamma-1)^{-1}(K\zeta)^{-2(\gamma-1)}$.

The problem is therefore reduced to a single quadrature.

It is instructive to obtain the diagonal type I vacuum and dust solutions from the Hamiltonian. The diagonal vacuum Hamiltonian is:

(16.22) $H = \frac{1}{24}e^{-3\beta^0}(-p_0^2 + p_+^2 + p_-^2)$.

Assuming $p_0 < 0$, one easily obtains the general solution:

(16.23) $\underline{g}(t) = (-p_0/4)^{2/3}\exp\left(\frac{2}{3}\ln t\,(\hat{\underline{e}}_0 + \hat{\underline{e}}_+ p_+/|p_0| + \hat{\underline{e}}_- p_-/|p_0|)\right)$
$= (-p_0/4)^{2/3}\,\mathrm{diag}(t^{2s_1}, t^{2s_2}, t^{2s_3})$,

where the momenta p_0, p_+ and p_- are constants satisfying $p_-^2 + p_+^2 = p_0^2$. With no loss of generality one may assume $-p_0/4 = 1$ (i.e. by a new choice of a SH frame). $\{s_a\}$ are the Kasner parameters

and satisfy $\Sigma s_a^2 = \Sigma s_a = 1$. If $p_0 = 0$, then the energy constraint forces $p_+ = p_- = 0$ and hence $\underline{\beta}$ and therefore $\underline{g}$ are constants, a situation corresponding to flat spacetime.

The diagonal type I dust Hamiltonian is:

(16.24) $H = \frac{1}{24}e^{-3\beta^0}(-p_0^2 + p_+^2 + p_-^2) + 2k\ell$.

Letting $\alpha = \frac{3}{4}k\ell$ and using the energy constraint, one easily solves the equations of motion for $\{\beta^0, p_0\}$ to find $e^{3\beta^0} = t(\alpha t + \gamma)$, suppressing an additive constant. The energy constraint reduces to $p_+^2 + p_-^2 = 16\gamma^2$. If $\gamma = 0$, then $p_\pm = 0$ and $\beta^\pm$ are constants which may be assumed to vanish with no loss of generality, yielding the flat Friedmann solution:

(16.25) $\quad \underline{g}(t) = \underline{1}(\alpha t^2)^{2/3}$.

When $\gamma \neq 0$, define $\cos\theta = p_+/(4\gamma)$ and $\sin\theta = p_-/(4\gamma)$; the other equations of motion are easily integrated yielding the final result:

(16.26) $\quad \underline{g}(t) = t^{2/3}(\alpha t+\gamma)^{2/3}\,(t/(\alpha t+\gamma))^{\frac{2}{3}(\cos\theta\,\hat{\underline{e}}_+ + \sin\theta\,\hat{\underline{e}}_-)}$.

This solution was first obtained by Heckman and Schucking.[19] (If $k = 8\pi$ and one defines $4\pi\rho/3 = m$, then $\alpha = 9m/2$.)

For type II, $V_3 = 0$ and V_1 and V_2 are constants. It is convenient to introduce an explicit parametrization of $\underline{S}$ and alter our scheme slightly since the momentum constraints may be used to eliminate only two of the automorphism degrees of freedom, namely, the adjoint degrees of freedom associated with $\underline{k}_1$ and $\underline{k}_2$:

(16.27) $\quad \underline{S}(\theta) = \exp\theta^3\underline{k}_3\;\exp(\theta^1\underline{k}_1 + \theta^2\underline{k}_2)$.

Note that $\underline{k}_a^T = \underline{k}_a(\underline{n})$ and $\underline{S}^T = \underline{R}(\underline{n})$ with $\underline{n} = \mathrm{diag}(-1,-1,0)$ as is revealed by comparison of (11.26), (16.27) with (10.1), (10.2). $\hat{G}$ is the transpose of a type VII$_0$ adjoint matrix group here. From this correspondence and the relation:

(16.28) $\quad d\underline{S}\,\underline{S}^{-1} = (\underline{R}^{-1}d\underline{R})^T = \omega^a(\underline{n})\,\underline{k}_a(\underline{n})^T = \tilde{\omega}^a\underline{k}_a$,

one may easily read off from (10.5) the expressions:

(16.29) $\quad \dot{\tilde{\omega}}^1 = c_3\dot\theta^1 - s_3\dot\theta^2\,,\ \dot{\tilde{\omega}}^2 = s_3\dot\theta^1 + c_3\dot\theta^2\,,\ \dot{\tilde{\omega}}^3 = \dot\theta^3$.

Similarly the primed components of the fluid velocity are:

(16.30)
$$V_1' = c_3V_1 - s_3V_2 = P_1/(2k\rho) = e^{3\beta^0}\bar{g}'_{11}\,\dot{\tilde{\omega}}^1/k\rho$$
$$V_2' = s_3V_1 + c_3V_2 = P_2/(2k\rho) = e^{3\beta^0}\bar{g}'_{22}\,\dot{\tilde{\omega}}^2/k\rho$$
$$V_3' = V_3 = 0.$$

c_3 and s_3 are ordinary trigonometric functions of θ^3 here.

It is also convenient to work directly with the variables θ^a and their velocities $\dot\theta^a$ and conjugate momenta p_a. Let:

(16.31)
$$\underline{g}'' = (\exp\theta^3\underline{k}_3)^T\,\underline{g}'\,(\exp\theta^3\underline{k}_3)$$
$$\bar{g}''_{ab} = (\exp\theta^3\underline{k}_3)^c{}_a\,\bar{g}'_{cd}\,(\exp\theta^3\underline{k}_3)^d{}_b$$
$$\bar{g}''^{ab} = (\exp-\theta^3\underline{k}_3)^a{}_c\,\bar{g}'^{cd}\,(\exp-\theta^3\underline{k}_3)^b{}_d\,.$$

Then:

(16.32) $\bar{\mathscr{g}}'_{ab}\dot{\tilde{\omega}}^a\dot{\tilde{\omega}}^b = \bar{\mathscr{g}}''_{ab}\dot{\theta}^a\dot{\theta}^b = \frac{1}{4}\bar{\mathscr{g}}''^{ab}p_a p_b,$

$p_a = 2e^{3\beta^0}\bar{\mathscr{g}}''_{ab}\dot{\theta}^b$

From (16.20) one finds the two nontrivial momentum constraints to be simply:

(16.33) $p_1 = -2k\ell v_1 \quad , \quad p_2 = 2k\ell v_2 .$

(p_1 and p_2 are constants since θ^1 and θ^2 are cyclic variables.) θ^3 enters the kinetic energy through $\bar{\mathscr{g}}''_{ab}$ or $\bar{\mathscr{g}}''^{ab}$ and the fluid potential through the combination $g^{ab}v_a v_b = g''^{ab}v_a v_b$ (valid since $v_3 = 0$). The degenerate type II constraints therefore enable us to reduce the gravitational degrees of freedom to $\mathcal{M}_{S(3)}$ rather than $\mathcal{M}_D$ as in the general case.

The type V case is similar. Again, since its rotational degree of freedom cannot be eliminated, it is useful to work with the variables $\{\theta^a, \dot{\theta}^a, p_a\}$ associated with the parametrization (16.27). The matrix group G chosen for this case coincides with the type II choice and although the basis $\{\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3\}$ for type V corresponds to the basis $\{-\mathbf{k}_2, \mathbf{k}_1, \mathbf{k}_3\}$ for type II, (16.29) continues to hold. In addition to (16.31) and (16.32), introduce the notation:

(16.36) $v_a'' = v_b(\exp(-\theta^1\mathbf{k}_1 - \theta^2\mathbf{k}_2))^b{}_a \quad , \quad v_3'' = v_3 = v_3' ,$

$g^{ab}v_a v_b = g''^{ab}v''_a v''_b \quad , \quad v''^a = g''^{ab}v''_b .$

The momentum contraints are:

(16.37) $v_1' = c_3 v_1'' - s_3 v_2'' = 3p_1/(k\ell) = 6\bar{\mathscr{g}}'_{11}\dot{\tilde{\omega}}^1/(k\ell)$

$v_2' = s_3 v_1'' + c_3 v_2'' = 3p_2/(k\ell) = 6\bar{\mathscr{g}}'_{22}\dot{\tilde{\omega}}^2/(k\ell)$

$v_3'' = v_3' = -p_+/(2k\ell)$

As above these simplify to:

(16.38) $p_1 = \frac{1}{3}k\ell v_1'' \; , \; p_2 = \frac{1}{3}k\ell v_2'' \; , \; p_+ = -2k\ell v_3'' .$

The first two expressions may be substituted directly into

Correction for page 16.12. Continue first paragraph with:

The only remnant of the adjoint degrees of freedom $\{\theta^1, \theta^2\}$ is an effective potential in the Hamiltonian, obtained by substituting (16.23) into (16.22):

(16.34) $$U_{eff} = k^2 \ell^2 e^{-3\beta^0} (\bar{g}''^{11} V_1^2 + \bar{g}''^{22} V_2^2 + 2\bar{g}''^{12} V_1 V_2).$$

If we had used the parametrization

(16.35) $$e^{\theta^1 \underline{K}_1 + \theta^2 \underline{K}_2} e^{\theta^3 \underline{K}_3}$$

instead of (16.18), the fluid potential would still be independent of θ^1 and θ^2 (due to the condition $V_3 = 0$) but the kinetic energy would be dependent on these variables and the reduction could not have been done.

the Hamiltonian resulting in a effective potential and into the equations for $\{V''_a\}$ which are obtained by a reapplication of (15.12) for the transformation (16.36):

$$(16.39)\qquad (V''_a)^{\cdot} = V''_f\, C^f{}_{ag} V''^g / V^\perp - V''_f \left(\kappa_1{}^f{}_a \dot{\theta}^1 + \kappa_2{}^f{}_a \dot{\theta}^2\right)$$
$$= V''_f\, C^f{}_{ag} V''^g / V^\perp + \tfrac{1}{6} k \ell e^{-3\beta^0} V''_f \left(\kappa_1{}^f{}_a \bar{\mathcal{G}}'^{1b} V''_b + \kappa_2{}^f{}_a \bar{\mathcal{G}}'^{2b} V''_b\right)$$
$$(\ln \ell)^{\cdot} = 2a V''^3 / V^\perp$$

Here the gravitational degrees of freedom have been reduced to $\mathcal{M}_{S(3)}$ but with the β^+ degree of freedom driven by V_3.

However, when $V_1 = V_2 = 0$ in the type II and V cases, the rotational degree of freedom is fake in the same sense as described for the type I case where necessarily $V_a = 0$. To understand why, it is useful to present an alternative argument for the type I case which can be specialized to the other two cases. Consider initial data $\{\underline{g}, \dot{\underline{g}}\}$ for the type I case. This can always be diagonalized by transforming $\underline{g}$ to $\mathcal{M}_I$ using $SL(3,\mathbb{R}) = SAut_e(\mathfrak{g})$ and then using an element of $SO(3,\mathbb{R}) \subset SL(3,\mathbb{R})$ to diagonalize the resulting $\dot{\underline{g}}$. Since diagonal initial data remains diagonal when evolved, it is sufficient to consider only diagonal solution curves. Notice that this argument hinges on the fact that the source potential is $SL(3,\mathbb{R})$ invariant, although this condition is not necessary, only sufficient. For types II and V, we need only check whether or not arbitrary initial data tangent to $\mathcal{M}_{S(3)}$ can be reduced to initial data tangent to $\mathcal{M}_D$ under the action of $SAut_e(\mathfrak{g})$. But this is just the 2-dimensional analogue of the type I discussion considering the submatrices obtained by eliminating the 3rd rows and columns from $\{\underline{g}, \dot{\underline{g}}\}$, since the corresponding $SL(2,\mathbb{R})$ subgroup of $SL(3,\mathbb{R})$ is contained in $SAut_e(\mathfrak{g})$ for these types. That diagonal data remains diagonal when evolved in these two cases will be seen later. This argument breaks down when V_1 and V_2 do not vanish because the source potential is no longer "$SL(2,\mathbb{R})$" invariant. The same feature may occur in the type I case when the source potential is not $SL(3,\mathbb{R})$ invariant. Electromagnetic fields, for example, can excite the rotational degrees of freedom.

For type $\mathrm{VI}_{-1/9}$ the first two momentum constraints are:

(16.40) $$0 = \tfrac{1}{2}(P_1 - P_2) - k\ell v_1' = -\tfrac{1}{2}(P_1 - P_2) - k\ell v_2' .$$

These imply $v_1' + v_2' = 0$, and hence they represent a single constraint on the difference $P_1 - P_2$. The third constraint is the same as in the general type VI_h case.

To examine the situation here, it is convenient to introduce the noncanonical basis $\bar{e}_a = e_b A^{-1b}{}_a$ of (10.24), (10.26) as well as a new basis $\{\bar{\underline{k}}_1, \bar{\underline{k}}_2, \underline{k}_3\}$ of $aut_e(g)$ with the image basis $\{\underline{k}_1, \underline{k}_2, \bar{\underline{k}}_3\}$ of $aut_{\bar{e}}(g)$:

(16.41) $$\bar{\underline{k}}_a = \underline{A}^{-1}\underline{k}_a\underline{A} = \underline{k}_b A^{-1b}{}_a \ , \quad \underline{k}_a = \underline{A}\bar{\underline{k}}_a\underline{A}^{-1} = \bar{\underline{k}}_b A^b{}_a \qquad a,b\neq 3$$ only for canonical points

$$\underline{A}\,\underline{k}_3\underline{A}^{-1} = \bar{\underline{k}}_3 = \hat{\underline{e}}^1{}_1 - \hat{\underline{e}}^2{}_2 = \hat{\underline{e}}_-/\sqrt{3}$$

$$[\underline{k}_3, \begin{pmatrix}\bar{\underline{k}}_1\\ \bar{\underline{k}}_2\end{pmatrix}] = \begin{pmatrix}\bar{\underline{k}}_1\\ -\bar{\underline{k}}_2\end{pmatrix} \qquad [\bar{\underline{k}}_3, \begin{pmatrix}\underline{k}_1\\ \underline{k}_2\end{pmatrix}] = \begin{pmatrix}\underline{k}_1\\ -\underline{k}_2\end{pmatrix} \ .$$

Under the change of special automorphism basis, the automorphism velocities and momenta transform as follows:

(16.42) $$\dot{\bar{\omega}}^a = A^a{}_b\dot{\omega}^b \qquad \bar{P}_a = P_b A^{-1b}{}_a \ ,$$

while the fluid current components transform under the change of basis of g as follows:

(16.43) $$\bar{V}_a' = V_b' A^{-1b}{}_a = \bar{V}_b \bar{S}^{-1b}{}_a \ ,$$

where $\bar{\underline{S}} = \underline{A}\,\underline{S}\,\underline{A}^{-1}$. The two constraints (16.40) then become:

(16.44) $$\bar{V}_2' = 0$$

$$-k\ell\bar{V}_1' = \bar{P}_1 = e^{3\beta^0}[(\bar{g}'_{11}+\bar{g}'_{22})\dot{\bar{\omega}}^1 + (\bar{g}'_{11}-\bar{g}'_{22})\dot{\bar{\omega}}^2]$$

$$= e^{3(\beta^0+2\beta^+)}[\cosh 2\sqrt{3}\beta^-\,\dot{\bar{\omega}}^1 + \sinh 2\sqrt{3}\beta^-\,\dot{\bar{\omega}}^2] \ .$$

Since the latter constraint involves a β^--dependent linear combination of the automorphism velocities, one cannot parametrize $\bar{\underline{S}}$ in any way that would enable one to use it to eliminate one of the automorphism variables.

However, consider the explicit parametrization:

(16.45) $$\bar{\underline{S}} = \exp\theta^3\bar{\underline{k}}_3 \exp(\bar{\theta}^1\underline{k}_1+\bar{\theta}^2\underline{k}_2) = \mathrm{diag}(e^{\theta^3}, e^{-\theta^3}, 1) + e^{\theta^3}\bar{\theta}^1\hat{\underline{e}}^3{}_1 + e^{-\theta^3}\bar{\theta}^2\hat{\underline{e}}^3{}_2$$

$$\{\dot{\bar{\omega}}^a\} = \{e^{\theta^3}\dot{\bar{\theta}}^1, e^{-\theta^3}\dot{\bar{\theta}}^2, \dot{\theta}^3\} \qquad \{\bar{P}_a\} = \{e^{-\theta^3}\bar{p}_1, e^{\theta^3}\bar{p}_2, \bar{p}_3\}$$

$$\{\bar{V}_a'\} = \{e^{-\theta^3}\bar{V}_1, e^{\theta^3}\bar{V}_2, \bar{V}_3 - \bar{\theta}^1\bar{V}_1 - \bar{\theta}^2\bar{V}_2\}$$

$$\bar{g}'^{ab}\bar{V}_a'\bar{V}_b' = \tfrac{1}{2}(g'^{11}+g'^{22})(\bar{V}_1'^2+\bar{V}_2'^2) + (g'^{11}-g'^{22})\bar{V}_1'\bar{V}_2' + g'^{33}\bar{V}_3'^2$$

$$= e^{-2(\beta^0+\beta^+)}[\cosh 2\sqrt{3}\beta^-(\bar{V}_1'^2+\bar{V}_2'^2) + 2\sinh 2\sqrt{3}\beta^-\,\bar{V}_1'\bar{V}_2']$$

$$+ e^{-2(\beta^0-2\beta^+)}\,\bar{V}_3'^2 \ .$$

The fluid constraint $\bar{V}_2'=0$ forces $\bar{V}_2=0$ and hence $\{\bar{V}_a'\}$ is independent of $\bar{\theta}^2$. Thus the fluid potential energy is $\bar{\theta}^2$-independent, as is the kinetic energy since the automorphism velocities are $\bar{\theta}^2$-independent. Since

(16.46) $$Q^*(\partial/\partial\theta^1) = Q^*(\partial/\partial\theta^2) = 0 \ ,$$

the nonpotential force has no component along $\bar{\theta}^2$ and therefore the momentum conjugate to the cyclic variable $\bar{\theta}^2$ is a constant of the motion:

(16.47) $$\bar{p}_2 = e^{-\theta^3}\bar{P}_2 = e^{-\theta^3+3\beta^0}[(\mathscr{g}'_{11}-\mathscr{g}'_{22})\bar{\tilde{\omega}}^1 + (\mathscr{g}'_{11}+\mathscr{g}'_{22})\bar{\tilde{\omega}}^2]$$
$$= e^{-\theta^3+3(\beta^0+2\beta^+)}[\sinh 2\sqrt{3}\beta^- \,\bar{\tilde{\omega}}^1 + \cosh 2\sqrt{3}\beta^- \,\bar{\tilde{\omega}}^2].$$

The contribution of the automorphism variables to the kinetic energy is:

(16.48) $$\tfrac{1}{4}e^{-3\beta^0}\bar{G}'^{ab}P_aP_b = \tfrac{1}{4}e^{-3\beta^0}[(\mathscr{g}'^{11}+\mathscr{g}'^{22})(\bar{P}_1^2+\bar{P}_2^2) + 2(\mathscr{g}'^{11}-\mathscr{g}'^{22})\bar{P}_1\bar{P}_2 + \mathscr{g}'^{33}P_3^2]$$
$$= e^{-3(\beta^0+2\beta^+)}[\cosh 2\sqrt{3}\beta^-(e^{-2\theta^3}\bar{p}_1^2 + e^{2\theta^3}\bar{p}_2^2) + 2\sinh 2\sqrt{3}\beta^- \,\bar{p}_1\bar{p}_2]$$
$$+ \tfrac{1}{8}e^{-3\beta^0}\cosh^{-2} 2\sqrt{3}\beta^- \, p_3^2$$

If we choose new fluid variables:

(16.49) $$\{\bar{V}_a''\} = \{\bar{V}_b[\exp(-\bar{\theta}^1\underline{K}_1 - \bar{\theta}^2\underline{K}_2)]^b{}_a\} = \{\bar{V}_1, 0, \bar{V}_3 - \bar{\theta}^1\bar{V}_1\}$$

with equations of motion similar to (16.39), then we may use the constraint (16.44) to determine $\bar{p}_1$:

(16.50) $$\bar{p}_1 = e^{\theta^3}\bar{P}_1 = k\ell\bar{V}_1'',$$

while $\bar{p}_2$ is a constant. The third constraint may be used to determine p_3 and evolve θ^3 which still appears in the kinetic energy and in the fluid potential and equations of motion for $\{\bar{V}_a''\}$:

(16.51) $$p_3 = -2k\ell\bar{V}_3'' - a\,p_+$$
$$\dot{\theta}^3 = -\tfrac{1}{4}e^{-3\beta^0}\cosh^{-2}2\sqrt{3}\beta^-(2k\ell\bar{V}_3'' + a\,p_+)$$

The only remnant of the $\{\theta^1, \theta^2\}$ variables is an effective potential in the kinetic energy and a "noninertial force" in the fluid equations of motion involving the momenta $\{\bar{p}_1, \bar{p}_2\}$. The metric matrix

(16.52) $$\underline{\bar{g}}'' = (e^{\theta^3\underline{K}_3})^T e^{2\underline{A}\underline{B}\underline{A}^{-1}} e^{\theta^3\underline{K}_3}$$

lies in $\mathcal{M}_{S(3)}$.

Having considered the degenerate types in detail, let us return to the general case which has so far been reduced to a system in the three scale degrees of freedom with a conserved Hamiltonian:

(16.53) $$H = \tfrac{1}{24} e^{-3\beta^0}(-p_0^2 + p_+^2 + p_-^2) + U_{eff} + U .$$

Let $-h$ be the constant of energy. The energy contraint is $h=0$. A standard technique in classical mechanics (12) easily generalized to partially canonical systems with nonconservative forces may be used to reduce our system by a further degree of freedom using the energy integral. The technique involves solving for the solution curves directly rather than parametrized by time, as in the Kepler problem where one solves for the radius as a function of the angular variable. From the frame transformation point of view, this corresponds to choosing the lapse $N(t)$ so that on the spacetime generated by a given solution curve, the new time coordinate t' equals $\beta^0(t)$. The nonconservative force appropriate to a description of the dynamics with the new time variable therefore picks up a factor of the lapse as discussed in §13.

Let $-I$ be the function obtained by solving $H+h=0$ for p_0 in terms of h and the remaining variables $\{\beta^0, \beta^\pm, P_\pm, V_a', \ell, n\}$:

(16.53) $$-I = \mp \left(P_+^2 + P_-^2 + 24 e^{3\beta^0}(U_{eff} + U + h)\right)^{1/2}$$
$$I_0 = I\big|_{h=0}.$$

I_0 acts as a Hamiltonian for the system with β^0 as the new integration variable, while $t(\beta^0)$ for a given solution is determined by integrating an equation which serves to define the lapse:

(16.54) $$d\beta^\pm/d\beta^0 = \partial I_0/\partial P_\pm \ , \quad dP_\pm/d\beta^0 = -\partial I_0/\partial\beta^\pm + NQ_\pm$$
$$N = dt/d\beta^0 = \partial I/\partial h\big|_{h=0} = 12 e^{3\beta^0}/I_0.$$

Recall that $Q_- = 0$ so only the component Q_+ of the nonconservative force is relevant to the reduced system. By (11.53), $TR\,\Pi = \frac{1}{2}P_0 = -\frac{1}{2}I_0$; as remarked in §13, Ryan chooses a lapse equal to $6g^{1/2}/TR\,\Pi = -12e^{3\beta^0}/I_0$ making $\Omega = -\beta^0$ the new integration variable. This agrees with (16.54).

The choice of the negative root in (16.29) corresponds to integrating in the direction of increasing β^0 as t increases (expansion) while the positive root corresponds to β^0 decreasing with increasing t (contraction). This reduction technique is only valid on a segment of a solution curve for which β^0 is either strictly increasing or decreasing with t. In particular, it breaks down at points of maximum expansion. Note that nothing would have prevented us from performing this energy reduction before the momentum reduction. The same technique may also be applied to the degenerate cases and type V, the only difference being the number of variables involved. To complete the scheme one must reparametrize the fluid equations:

(16.55) $dV_a'/d\beta^0 = N(V_a')^{\cdot}$ $\quad d\ln\ell/d\beta^0 = N(\ln\ell)^{\cdot}$.

The "cosmic time" t may be obtained as a function of β^0 by integrating the lapse function given as a function of the remaining essential variables by (16.54). However, this step is not necessary provided one is willing to accept a nontrivial lapse function. The lapse integration of (16.54) is completely analogous to the shift integration (11.58) which determines the transformation matrix $\underline{S}$.

We have therefore reduced the number of gravitational degrees of freedom to two in the general case and three or less in the remaining cases. For the class A case, ℓ and $V^2 = |\eta^{ab}V_a'V_b'|$ are constants leaving two independent fluid degrees of freedom except for types I and II where they are trivial. In the class B case, only ℓV^2 is conserved so three fluid degrees of freedom remain. However, for Bianchi types $\text{VI}_{h\neq 0}$, $\text{III} = \text{VI}_{-1}$ and V there are effectively only two fluid degrees of freedom due to equations (15.31) and (15.32).

The sense in which the number of gravitational degrees of freedom have been reduced can be made more precise by the following explanation, using the general case as an example. Given a solution curve

(16.56) $\{\beta^\pm(\beta^0), V_a'(\beta^0), \ell(\beta^0), n(\beta^0)\}$

of the reduced system of differential equations, the components of the spacetime metric and fluid energy-momentum tensor in a certain canonical comoving ADM frame are completely specified without any further integrations. The transformation to a normal canonical comoving ADM frame may then be accomplished by integration of ordinary differential equations for the proper time t and the special automorphism matrix $\underline{S}$.

For type IX, spherical coordinates coincide with our special automorphism coordinates. This has permitted Ryan to carry through the above program in this case and apply qualitative techniques to the reduced system which he has called "qualitative cosmology."[31,32] These same techniques may be extended to the remaining Bianchi types using our approach.

We now survey the submanifolds of $\mathcal{M}\times S$ to which a solution curve will be confined if it is initially tangent to one of them. To put it differently, we will determine which of the automorphism and scale degrees of freedom can be naturally suppressed, resulting in a partially canonical or completely canonical system in the remaining degrees of freedom. This is best accomplished before the energy reduction. Consider first the nondegenerate class A types, for which the generic or general case is characterized by the existence of at least two nonvanishing components of either V_a or V_a', i.e. neither is an eigenvector of the matrix $\underline{n}$. When this is true initially it remains so and the direction of the fluid current vector will always be changing although its "group magnitude" $\underline{V}^2$ is conserved. This has led to the description of the general case as a "tumbling case."[68]

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On the other hand since $\underline{g}'$, $\underline{n}$ and $\mathscr{S}'_{ab}$ are diagonal, it is easy to show from (16.1) with the constraints inserted that if two of the components V_a' vanish, the third is a constant, i.e. $(V_a')^{\cdot} = 0$ if (V_1', V_2', V_3') is an eigenvector of $\underline{n}$ of this special form. Suppose for definiteness that V_3' is the only nonvanishing component. Then by the momentum constraints $\tilde{\omega}^3$ is the only nonvanishing automorphism velocity. If we let $\theta^3(t) = \int_0^t \tilde{\omega}^3(\tau)d\tau$ where $\tilde{\omega}^3(t)$ is considered an explicit function of t, then we may integrate the equation for S:

(16.57) $\dot{\underset{\sim}{S}}\underset{\sim}{S}^{-1} = \dot{\tilde{\omega}}^3 \underset{\sim}{k}_3$, $\underset{\sim}{S}(t) = (\exp \theta^3(t)\underset{\sim}{k}_3)\, \underset{\sim}{S}_0$.

Without loss of generality we may assume the initial condition $\underset{\sim}{S}_0 = \underset{\sim}{1}$. Alternatively we may use the following explicit parametrization:

(16.58) $$\underset{\sim}{S}(\theta) = e^{\theta^3 \underset{\sim}{k}_3}\, e^{\theta^2 \underset{\sim}{k}_2}\, e^{\theta^1 \underset{\sim}{k}_1} .$$

With the index permutation $(123) \to (321)$ of (10.5) one finds that $\dot{\tilde{\omega}}^1$ and $\dot{\tilde{\omega}}^2$ are linear in $\dot{\theta}^1$ and $\dot{\theta}^2$ so the momentum constraints imply that θ^1 and θ^2 are constants which may be assumed to vanish. The geometric part of a solution curve on $\mathcal{M}\times S$ of this type is therefore confined to $\mathcal{M}_{\$(3)}$:

(16.59) $$g(t) = (\exp \theta^3(t)\underset{\sim}{k}_3)^T\, g'(t)\, (\exp \theta^3(t)\underset{\sim}{k}_3) .$$

This is called a symmetric case or "nontumbling" case, first considered for type IX by Gödel and more recently by Ryan.(31) Types VI$_0$, VII and VIII have two kinds of symmetric cases $\mathcal{M}_{\$(a)}$ depending on whether the corresponding component $n^{(a)}$ is vanishing or nonvanishing in the first two cases or positive or negative in the last. Ozsvath has examined a symmetric case $\mathcal{M}_{\$(a)}$ with $n^{(a)}$ nonvanishing for each of the nondegenerate class A dust cases, stimulating the present investigation.(28) Reducing these systems by the one nontrivial momentum constraint using a canonical transformation led him to the restriction of our special automorphism coordinates to $\mathcal{M}_{\$(a)}$. (Apart from permutations of the canonical SCT components, signs, factors of two and a possible additive constant of $\frac{\pi}{2}$, his **w** corresponds to the one nonzero automorphism variable θ^a for the $\mathcal{M}_{\$(a)}$ case, while $\{x, y, z\}$ merely relable $\{g'_{11}, g'_{22}, g'_{33}\}$.)

At this point the presence of the fluid source is felt by the geometric variables only through the appearance in the Lagrangian and Hamiltonian of the two constants ℓ and V_a' and the thermodynamic variables μ and p which are functions of n whose evolution is determined implicitly by the defining equation for ℓ. For dust, $p=0$ and $\mu=1$ and the latter complication is avoided, leaving a completely Lagrangian or Hamiltonian system on the velocity or momentum phase space associated with the configuration space $\mathcal{M}_{S(a)}$. The single variable θ^a is cyclic and its constant momentum may be replaced by $k\ell V_a'$ in the Hamiltonian, leaving a Hamiltonian system involving only the scale variables.

The type II case is similar in a way but with two nonzero constant fluid current components and two cyclic coordinates θ^1 and θ^2. Their elimination leads to a system involving only θ^3 and the scale variables. Only when p_1, p_2, V_1 and V_2 vanish does θ^3 become cyclic. In this case θ^1 and θ^2 are constants which may be assumed to vanish and we obtain a symmetric case associated with $\mathcal{M}_{S(3)}$.

When $V_a'=0$ in the nondegenerate class A case, the momentum constraints require that $\dot{\tilde{\omega}}^a=0$ and hence $\underline{S}$ is a constant matrix which might as well be $\underline{1}$, again bringing us to the diagonal case in which the solution curves lie in $\mathcal{M}_D$. For type I we saw that the diagonal case was equivalent to the general case.

Now consider subcases of the class A diagonal case. The first possible specialization occurs if initially $\beta^-=0=P_-$. Since β^- enters the Hamiltonian only in U^* at this point and since $\partial U^*/\partial\beta^-$ vanishes at $\beta^-=0$ for all class A types as examination of (11.21) reveals, these initial conditions remain satisfied along the solution curve which is therefore confined to the Taub submanifold $\mathcal{M}_{T(3)}$. (For type II, β^- is cyclic.) The vacuum equations may be integrated exactly for types IX, VIII, VI_0 and II. This was done by Taub for types IX and II;[25]

Correction for page 16.22. Add to second paragraph:

However, as discussed above, one may always assume that p_3 vanishes when $v_a = 0$ and hence this symmetric case is equivalent to the diagonal case, where the solution curves are confined to $\mathcal{M}_D$. The general solution of the type II vacuum diagonal case was obtained by Taub.[25] Note that when only v_1 or v_2 is nonvanishing, one gravitational degree of freedom is suppressed. If $v_1 \neq 0$ but $v_2 = 0$, for example, then $\dot{\theta}^2 = 0$ and one can assume $\theta^2 = 0$ which leads to solution curves which satisfy $g_{13} = 0$. Since one can easily see that the effective potential and fluid potential are then both annihilated by the operator $\partial/\partial\theta^3|_{\theta^3=0}$, the initial data $\theta^3 = 0 = p_3$ is a solution of the equations of motion for θ^3, leading to a symmetric case associated with $\mathcal{M}_{S(1)}$.

Since U^* is independent of β^-, as is the fluid potential when $v_a = 0$, β^- is a cyclic variable for the diagonal case and hence its conjugate momentum p_- is a constant of the motion.

the type VIII analogue of the Taublike type IX solution is discussed by Miller.[53] Types I and VII_0 coincide on $\mathcal{M}_{T(3)}$ and have an additional spacetime Killing vector field tangent to the SH slicing, namely ξ_6 in the type I discussion of section 10 or e_3 in the type VII_0 discussion. The only vacuum solutions of this type have Kasner parameters $(0,0,1)$ and $(\frac{2}{3},\frac{2}{3},-\frac{1}{3})$.[16] The first corresponds to flat spacetime. The dust solution for this case is obtained from (16.17) by setting $\theta=0$. Taublike dust models of type I, II, VIII and IX were considered by Ozsvath who regularized the Hamiltonian equations by choosing a lapse $N=(g_{33})^{1/2}$.[27] This choice of lapse in fact decouples g_{33} from the variables $\{g_{11}, g_{22}, g_{12}\}$ in the symmetric case associated with $\mathcal{M}_{S(3)}$ for all Bianchi types except VIII and IX.[102] For types VII_0, VIII and IX, e_3 becomes a fourth linearly independent spacetime Killing vector field when solution curves lie in $\mathcal{M}_{T(3)}$ as discussed in §10. For types I and II the Killing vectors $\xi_6 \in aut(G)$ and $\xi_4 \in aut(G)$ respectively given in §10 are spacetime Killing vector fields for this Taublike case.

The last specialization that can occur corresponds to the initial conditions $\beta^+=0=P_+$. These solution curves are confined to the 1-dimensional isotropy submanifold $\mathcal{M}_I$. However, only the type $I \cap VII_0$ and type IX potentials are such that $\partial U^*/\partial\beta^+$ vanishes when $\beta^\pm=0$ so there are only two such cases, the zero and positive curvature Friedmann models respectively. These models have three additional spacetime Killing vector fields as discussed in §10. Both are exactly integrable for dust and other simple equations of state. Since the scalar curvature is positive and U^* therefore negative for type IX on $\mathcal{M}_I$, the gravitational Hamiltonian is negative-definite for the Friedmann models and hence no vacuum solution exists since the energy constraint cannot be satisfied. The zero curvature vacuum Friedmann model is flat spacetime, at which e_1 becomes a Killing vector field.

The substructure of the class B system is quite different. We saw in §15 that only when V_a is proportional to $a_a = a\delta^3_a$ (and hence $V_a=V_a'$) can we obtain a symmetric case. $V_3 = V_3'$ is then the only nonvanishing constant component and the rotation of the fluid vanishes. The momentum constraints and $V_1' = V_2' = 0$ require that $\bar{P}_1 = \bar{P}_2 = \dot{\tilde{\omega}}^1 = \dot{\tilde{\omega}}^2 = 0$ and so confine $\underline{S}(t)$ to a right coset of the subgroup generated by $\underline{\kappa}_3$; without loss of generality we may assume it to be the identity coset and thus confine the solution curves to $\mathcal{M}_{S(3)}$. Alternatively we may use the parametrization (16.27), so that $\dot{\theta}^1$ and $\dot{\theta}^2$ are constants which may be assumed to vanish. The remaining variables are then $\{\beta^0, \beta^\pm, \theta^3\}$. The only exception to this situation is the degenerate type $\text{VI}_{-1/9}$, for which one must also set the constant $\bar{p}_2$ to zero to lead to this symmetric case.

To examine the type VI_h case, it is useful to introduce the noncanonical frame and alternative basis for $aut_e(g)$ as in the type $\mathrm{VI}_{-1/9}$ discussion above beginning with equations (16.41). Instead of (16.44), the first two momentum constraints then become:

(16.57) $$\tfrac{1}{2}(3a+q)\,\bar{P}_1 - k\ell\bar{V}_1{}' = 0$$
$$\tfrac{1}{2}(3a-q)\,\bar{P}_2 - k\ell\bar{V}_2{}' = 0\,,$$

where $q = n^{(1)} = -n^{(2)} = 1$. The remaining equations (16.41) through (16.48), as well as the relation between $\bar{P}_1$ and the velocities expressed in (16.44), are also valid for the general type VI_h case.

Recall that $\bar{V}_1 = e^{\theta^3}\bar{V}_1{}'$ and $\bar{V}_2 = e^{-\theta^3}\bar{V}_2{}'$ are invariant relation quantities and hence so are $\bar{V}_1{}'$ and $\bar{V}_2{}'$. In the general case both $\bar{V}_1{}'$ and $\bar{V}_2{}'$ are always nonzero. Suppose we set $\bar{V}_2{}' = 0 = \bar{V}_2$ as is in fact required for $h=-1/9$ where $a = 1/3$. Then $\bar{\theta}^2$ is cyclic and $\bar{p}_2$ is a constant of the motion but the constraints force $\bar{p}_2 = 0$ except when $h = -1/9$. Only when both $\bar{V}_1{}'$ and $\bar{V}_2{}'$ vanish do the constraints lead to $\bar{p}_1 = 0 = \bar{p}_2$ and hence $\dot{\bar{\theta}}^1 = 0 = \dot{\bar{\theta}}^2$ so that the solution curves may be confined to $\mathcal{M}_{S(3)}$. This is a symmetric case with respect to the noncanonical frame and also with respect to the canonical frame since according to (16.52):

(16.58) $$\underline{A}^{-1}\underline{g}''\underline{A} = (e^{\theta^3\underline{K}_3})^T e^{2\underline{\beta}}\, e^{\theta^3\underline{K}_3} \in \mathcal{M}_{S(3)}.$$

For the exceptional type $h=-1/9$, one must set the constant of the motion $\bar{P}_2{}'$ to zero when $\bar{V}_1{}' = \bar{V}_2{}' = 0$ to obtain this case. Otherwise both $\dot{\bar{\theta}}^1$ and $\dot{\bar{\theta}}^2$ remain nonzero.

However, at $\beta^- = 0$, every term in the kinetic energy and the fluid potential energy is annihilated by $\partial/\partial\beta^-$ if either $\bar{p}_1$ or $\bar{p}_2$ vanishes. Since the gravitational potential (11.26) has this property, the initial data $\beta^- = 0 = p_+$ is a solution of the equations of motion so one may set $\beta^- = 0$ to obtain a simpler system. Since $\underline{A}\,\underline{\beta}\,\underline{A}^{-1} = \underline{\beta}$ when $\beta^- = 0$, the metric matrix $\underline{g}''$ of (16.52) is diagonal with $\theta^3/\sqrt{3}$ playing the role of $\bar{\beta}^-$. Suppose $\bar{p}_1 = 0$ but $\bar{p}_2 \neq 0$; the constraints then force $\dot{\bar{\theta}}^1 = 0$. Assuming $\bar{\theta}^1 = 0$ leads to a symmetric case associated with $\mathcal{M}_{S(2)}$ with only $\bar{V}_1{}' = 0$. (Note that with respect to the canonical frame the solution curves for this case satisfy

$g_{13}=0$ as in the similar discussion for type II given above.) An equivalent situation exists when $\bar{p}_2=0$ but $\bar{p}_1\neq 0$, with obvious qualifications for both when $h=-1/9$. When both $\bar{p}_1$ and $\bar{p}_2$ vanish, we are left with a diagonal case with respect to the noncanonical frame (a $\beta^-=0$ symmetric case associated with $\mathcal{M}_{S(3)}$ with respect to the canonical frame). Note that this entire discussion also holds for type VI_0 where $a=0$.

As noted in § 15 for the type IV perfect fluid models, we may set $V_1=V_1'$ to zero and obtain a case intermediate between the symmetric and general cases. (16.12) shows that this corresponds to $P_1=0=\dot{\tilde{\omega}}^1$. The special automorphism group $\hat{G}$ in this case is just the type II subgroup of $GL(3,\mathbb{R})^+$ described in section 10, with $\{\underline{k}_1,\underline{k}_2,\underline{k}_3\}$ equal to $\{\hat{\underline{e}}_3,\hat{\underline{e}}_1,\hat{\underline{e}}_2\}$. However, since $\tilde{\omega}^1\wedge d\tilde{\omega}^1=-\tilde{\omega}^1\wedge\tilde{\omega}^2\wedge\tilde{\omega}^3$ for the basis $\{\underline{k}_a\}$, the constraint $\dot{\tilde{\omega}}^1=0$ is not holonomic, i.e. cannot be represented as $\theta^1=$ constant for some choice of coordinates on $\hat{G}$.

At this point for all class B types, consider the symmetric case associated with $\mathcal{M}_{S(3)}$ when $V_a=V_3\delta^3_a=V_a'$, $P_a=P_3\delta^3_a$ and $\dot{\tilde{\omega}}^a=\dot{\tilde{\omega}}^3\delta^3_a$. With the parametrization (16.27) we can assume $\theta^1=\theta^2=0$. The equation for P_3 at this point is simply:

(16.59) $$\dot{P}_3=Q_3=4a\,e^{\beta^0+4\beta^+}\,\bar{G}^{\circ}_{33}{}'=2a\,e^{\beta^0+4\beta^+}\left(n^{(1)}e^{2\sqrt{3}\beta^-}-n^{(2)}e^{-2\sqrt{3}\beta^-}\right)^2.$$

For type V, Q_3 vanishes identically and P_3 is a constant of the motion but as in the type II case, we can always assume that $P_3=0$ under these conditions, leading to $\bar{\theta}^3=0$. Thus the symmetric case is equivalent to the diagonal case here. The remaining constraint equates p_+ to $2k\ell V_3$ while ℓ has the equation of motion:

(16.60) $$(\ln\ell)^{\cdot}=2e^{-3\beta^0+4\beta^+}\,V_3\,.$$

⟵ β^- is a cyclic variable for this system and its conjugate momentum p_- a constant of the motion. If $p_-=0$, β^- is a constant which may be assumed to vanish since $\hat{e}_-$ generates a subgroup of the type V canonical automorphism matrix group. The solution curves are then confined to $\mathcal{M}_{T(3)}$, a Taublike case. Farnsworth first examined this case for dust using the fluid velocity vector field as an ADM generator[59], while Shepley has made some remarks about it (using a lightlike ADM generator) in connection with cosmological singularities.[60] A detailed analysis for the equation of state $p=(\gamma-1)\rho$ was given by Collins.[72] On the other hand suppose $V_3'=0$ (so that ℓ is a constant) but $p_-\neq 0$. Then p_+ vanishes and β^+ is a constant. We might as well set $\beta^+=0$ since a scale transformation by the automorphism matrix $\exp -3\beta^+ \underline{I}^{(3)} = \exp -\beta^+(\hat{e}_+ + 2\hat{e}_0)$ will eliminate β^+ without changing anything but the initial value of β^0. Only the variables $\{\beta^0, \beta^-\}$ remain at this point and β^- is cyclic. In the dust case the energy constraint may be solved for $\dot{\beta}^0$ in terms of β^0 and the constants ℓ and p_- and hence $\beta^0(t)$ may be obtained by integration of this expression. $\beta^-(t)$ may then be obtained by another integration. This system was mentioned by Heckmann and Schucking.[19] When $p_-=0$, we may assume $\beta^-=0$ and arrive at the negative curvature Friedmann models which are exactly integrable for simple equations of state.

Consider the type V dust $\beta^0\beta^-$ case, for which the Hamiltonian and equations of motion are:

(16.61) $$H = \tfrac{1}{24} e^{-3\beta^0}(-p_0^2 + p_-^2) + 6 e^{\beta^0} + 2k\ell = 0$$

$$e^{\beta^0}(e^{2\beta^0})^{\cdot} = 2e^{3\beta^0}\dot{\beta}^0 = -p_0/6 = 2\left(e^{4\beta^0} + (p_-/12)^2 + 12k\ell e^{3\beta^0}\right)^{1/2}$$

$$e^{\beta^0}\dot{\beta}^- = e^{-2\beta^0}(p_-/12).$$

Let $\alpha^2 = |p_-/12|$ and when $\alpha^2\neq 0$ let $\text{sign}\, p_- = p_-/|p_-|$. In the vacuum case $\ell = 0$ these equations are easily integrated. When $\alpha^2 \neq 0$, the choice of lapse $N = g_{33}{}^{1/2} = e^{\beta^0}$ such that $d/d\bar{t} = e^{\beta^0} d/dt$ reduces them to standard integrals:

(16.62) $$e^{2\beta^0} = \alpha^2 \sinh 2\bar{t} \quad , \quad e^{2\beta^-} = (\tanh \bar{t})^{\text{sign } P_-}$$
$$g_{11} = \alpha^2 \sinh 2\bar{t} \, (\tanh^{\sqrt{3}} \bar{t})^{\text{sign } P_-}$$
$$g_{22} = \alpha^2 \sinh 2\bar{t} \, (\tanh^{-\sqrt{3}} \bar{t})^{\text{sign } P_-}$$
$$g_{33} = \alpha^2 \sinh 2\bar{t} \, .$$

Since one may interchange e_1 and e_2 without affecting the type V canonical SCT components, the sign of P_- is unimportant and may be assumed positive. This is the Joseph solution.(58) If $P_- = 0$, one may assume $\beta^- = 0$ and the remaining equation produces the vacuum Friedmann solution:

(16.63) $$(e^{\beta^0})^{\cdot} = e^{\beta^0} \dot{\beta}^0 = -\tfrac{1}{12} e^{-2\beta^0} P_0 = 1 \quad , \quad e^{\beta^0} = t \, ,$$
$$\underline{g} = t^2 \underline{1} \, .$$

Both solution curves (16.36) and (16.37) hit the frontier of $\mathcal{M}$ at $t = 0$, exhibiting "cosmological singularities." However, the latter example is a "fictitious singularity" since $(\mathbb{R}^+ \times G, \underline{g})$ is isometric to the interior of the forward light cone of Minkowski space for this solution.(13)

For the remaining class B types IV, III = VI$_{-1}$, VI$_h$ and VII$_h$, the Hamiltonian, nonconservative force, gravitational potential and $\bar{G}_{3'3}$ are explicitly (for the symmetric case):

(16.64) $$H = \tfrac{1}{4} e^{-3\beta^0} \left(\tfrac{1}{6} \eta^{AB} P_A P_B + \bar{\mathscr{G}}'^{33} P_3{}^2 \right) + U^* + U_g$$
$$Q^* = 4a e^{\beta^0 + 4\beta^+} \left(6a d\beta^+ + \bar{\mathscr{G}}_{3'3} \, d\theta^3 \right)$$
$$U^* = e^{\beta^0 + 4\beta^+} \left(\bar{\mathscr{G}}_{3'3} + 6a^2 \right)$$
$$\bar{\mathscr{G}}_{3'3} = \tfrac{1}{2} \left(n^{(1)} e^{2\sqrt{3}\beta^-} - n^{(2)} e^{-2\sqrt{3}\beta^-} \right)^2 .$$

The component Q^*_+ cancels the term $-\partial/\partial\beta^+ (6a^2 e^{\beta^0 + 4\beta^+})$ occurring in the Hamiltonian equation for P_+. The Hamiltonian equations may be used to evolve the scale variables while the momentum constraint $P_3 = -aP_+ - 2k\ell V_3$ may be used to evaluate P_3; ℓ has the equation of motion:

(16.65) $$(\ln \ell)^{\cdot} = 2a e^{-2\beta^0 + 4\beta^+} V_3 \, .$$

Setting $V_3{}' = 0$ makes ℓ a constant but does not succeed in acheiving a diagonal case as for the nondegenerate class A types. The variable θ^3 may be set to zero only if it is possible for P_3 and $\dot{P}_3 = Q_3$ to vanish. This occurs only for type VII_h on the Taub submanifold $\mathcal{M}_{T(3)}$ where the type VII_h and type V systems coincide and the above discussion holds. No further specializations are possible for type IV, i.e. no diagonal type IV solution curves exist.

For type VI_h, as discussed above, one can set $\beta^- = 0$ even when $v_3 \neq 0$. The vacuum case is exactly integrable.[102] In fact when $v_3 = 0$, the single remaining momentum constraint becomes integrable (i.e. holonomic) and when the system is restricted to a submanifold of the corresponding distribution, the nonpotential force vanishes and the reduced system is Hamiltonian in the ordinary sense. In terms of the noncanonical frame of (10.24) and (10.26), the $\beta^- = 0$ symmetric case is a diagonal case. Using the relations:

$$(16.66) \qquad 2\mathscr{G}'_{33} = q \qquad 6\dot{\beta}^+ = (\ln (\bar{g}_{11}\bar{g}_{22})^{1/2}/\bar{g}_{33})^{\cdot} \qquad \hat{\omega}^3 = \hat{\theta}^3 = (\ln (\bar{g}_{11}/\bar{g}_{22})^{1/2})^{\cdot}$$

valid for this diagonal case, one easily finds that the momentum constraint function and nonpotential force are:

$$(16.67) \qquad \begin{aligned} f(\bar{M}_3)|_{T\mathcal{M}_D} &= q\,\bar{g}^{1/2}(\ln u)^{\cdot} \\ \bar{Q}^*|_{\mathcal{M}_D} &= 4\bar{g}^{1/2}\bar{g}^{33}\, d \ln u \\ u &= \bar{g}_{33}\,(\bar{g}_{11}\bar{g}_{22})^{-1/2}\,(\bar{g}_{11}/\bar{g}_{22})^{-\lambda/2} \,. \end{aligned}$$

Recall that $q = n^{(1)} = -n^{(2)}$ and $\lambda = q a^{-1}$. The momentum constraint requires $u = u_0 =$ constant when $v_3 = 0$ and hence $\bar{Q}^*$ vanishes on the submanifolds of $\mathcal{M}_D$ to which the solution curves must be restricted. For Bianchi type $\mathrm{III} = \mathrm{VI}_{-1}$, $\lambda = 1$ and $u = g_{33}/g_{11}$. Since $\hat{e}^1{}_1$ and $\hat{e}^2{}_2$ belong to $aut_e(g)$, one can scale $\bar{g}_{11}$ and $\bar{g}_{22}$ arbitrarily. One may therefore assume $u = 1$ and restrict the solution curves to $\mathcal{M}_{T(2)}$, a Taublike case for which the additional Killing vector field (10.19) is a spacetime Killing vector field. This case is just the negative curvature Kantowski-Sachs case which will be discussed later.[87]

Although a vacuum type IV $\beta^-=0$ symmetric case does not exist for arbitrary initial values of the remaining variables $\{\beta^0, P_0, \beta^+, P_+, \theta^3, \mathbb{P}_3\}$ (subject to the energy constraint and the third momentum constraint $\mathbb{P}_3 = -P_+$), one can ask under what conditions on these variables will such a class of solutions exist. In order for β^- and P_- to remain zero, one must have:

(16.68) $$0 = \dot{P}_- = -\partial H/\partial\beta^-|_{\beta^-=0=P_-} = 3^{1/2} e^{-3\beta^0}(P_+{}^2 - e^{4(\beta^0+\beta^+)}),$$
$$0 = C_1 = P_+{}^2 - e^{4(\beta^0+\beta^+)}.$$

In order that C_1 remain zero, $\dot{C}_1$ must vanish when β^-, P_- and C_1 vanish. Evaluating $\dot{C}_1$ under these conditions using the Hamiltonian equations, one obtains the additional condition:

(16.69) $$0 = C_2 = P_0 - 13P_+.$$

One finds that both $\dot{C}_2$ and the Hamiltonian H vanish when $\beta^- = P_- = C_1 = C_2 = 0$ so these four conditions define a submanifold of the constraint submanifold of the symmetric case momentum phase space which contains any solution curve which intersects it. It consists of two disjoint pieces corresponding to the two signs in the following equation:

(16.70) $$P_+ = \mp e^{2(\beta^0+\beta^+)}.$$

The Hamiltonian equations for β^0 and β^+ then become:

(16.71) $$\dot{\beta}^0 = -\tfrac{1}{12} e^{-3\beta^0} P_0 = \pm \tfrac{13}{12} e^{-\beta^0+2\beta^+},$$
$$\dot{\beta}^+ = \tfrac{1}{12} e^{-3\beta^0} P_+ = \mp \tfrac{1}{12} e^{-\beta^0+2\beta^+}.$$

From these one obtains $e^{\beta^0+13\beta^+} = \gamma$ with γ a positive constant, and by introducing $z = (g'_{33})^{1/2} = e^{\beta^0-2\beta^+}$:

(16.72) $$\dot{z} = z(\dot{\beta}^0 - 2\dot{\beta}^+) = \pm\tfrac{5}{4}, \qquad z = \pm\tfrac{5}{4}(t-t_0).$$

Setting $t_0 = 0$ one therefore has:

(16.73) $$e^{\beta^0} = \gamma^{2/5}(\pm\tfrac{5}{4}t)^{13/15} \qquad e^{\beta^+} = \gamma^{1/5}(\pm\tfrac{5}{4}t)^{-1/15}$$
$$g'_{11} = g'_{22} = \gamma^{2/5}(\pm\tfrac{5}{4}t)^{8/5} \qquad g'_{33} = (\tfrac{5}{4}t)^2.$$

Since $I^{(3)}$ generates an automorphism, we may set $\gamma = 1$. If we choose the negative root for P_+, t is confined to the interval $(0,\infty)$, while the positive root corresponds to the time-reversed solution with t confined to $(-\infty, 0)$.

In the former case:

(16.74) $\dot{\theta}^3 = e^{-3\beta^0} P_3 = -e^{-3\beta^0} P_+ = e^{-(\beta^0 - 2\beta^-)} = \left(\frac{5}{4}t\right)^{-1}$

$\theta^3 = \frac{4}{5} \ln\left(\frac{5}{4}t\right)$

This is singular when the solution curve hits the frontier of $\mathcal{M}$ at $t=0$. This solution was recently found by Harvey and Tsoubelis who in addition showed that the perfect fluid system does not admit a corresponding solution.(70)

Although we have emphasized the Hamiltonian viewpoint which in general ultimately leads to the Misner-Ryan picture [15] of the dynamics in terms of a point particle moving on β^+,β^- space under the influence of moving potentials (using "Ω-time"), an alternative approach favored by the Soviet school should not be overlooked. This amounts to working directly with the Einstein equations expressed in the time-dependent orthogonal frame $\{Ne_\perp, e_a' = S^{-1\,b}{}_a e_b\}$. The choice of lapse $N = g'^{1/2}$ is suggested immediately by the form of the dynamical equations but we will leave the lapse general. All time derivatives in previous formulas for which the lapse was unity must be multiplied by the factor N^{-1} to represent the new time derivatives.

The dynamical equations are chosen to be:

(16.75) $$\underline{R}' = k\,(\underline{T}' - \tfrac{1}{2}T'\underline{1})$$

rather than:

(16.76) $$\underline{G}' = k\,\underline{T}'$$

which are equivalent to the Hamiltonian equations. [102] The two sets of equations differ by a term involving the super-Hamiltonian constraint function. By calculations similar to the one following (13.6), starting with the relations

(16.77) $$\underline{g}' = (\underline{S}^{-1})^T \underline{g}\,\underline{S} \qquad \underline{K}' = \underline{S}\underline{K}\underline{S}^{-1} \qquad \underline{R}' = \underline{S}\underline{R}\underline{S}^{-1}$$
$$\dot{S}S^{-1} = \underline{k}_a \tilde{\omega}^a \quad ,$$

one finds that the primed components of (13.11) are given by:

(16.78) $$2g'\underline{R}' = -2N^{-1}g'^{1/2}(g'^{1/2}\underline{K}')^{\cdot} - 2N^{-1}g'[\underline{K}', \underline{k}_a\dot{\tilde{\omega}}^a] + 2g'\underline{R}^{*\prime}$$
$$2N\underline{K}' = -(\ln \underline{g}')^{\cdot} - 2\underline{\mathcal{K}}_a'\dot{\tilde{\omega}}^a \, .$$

If we limit ourselves to the cases for which $\text{rank}\,\underline{n} \geq 2$ where $\hat{G} = SAut_e(\mathfrak{g})$ is unique and if we choose the basis $\{\underline{k}_a = \underline{k}_a^\circ\}$ for its Lie algebra, then we may write equations describing all such Bianchi types simultaneously. (Recall that $\{\underline{k}_a^\circ\}$ are the matrices (10.1) with the structure constant component a set to zero in the class B case.) The matrices $\{\underline{\mathcal{K}}_a'\}$ are given by (10.18) except that we are abandoning the notation $e^{2\beta^{bc}}$ in favor of g_{bb}'/g_{cc}'. Similarly we have the formulas:

(16.79) $2\bar{\mathscr{B}}'_{aa} = (\frac{1}{2}\bar{\mathscr{B}}'^{aa})^{-1} = (g'_{bb}g'_{cc})^{-1}\,(n^{(b)}g'_{bb} - n^{(c)}g'_{cc})^2$

$P_a = 2N^{-1}g'^{1/2}\bar{\mathscr{B}}'_{ab}\,\dot{\tilde{\omega}}^b$.

Here and in what follows, (a,b,c) will be assumed to by a cyclic permutation of (1,2,3).

The off-diagonal part of (16.75) evolves the automorphism variables, reproducing (11.55) with a source driving term:

(16.80) $2Ng'^{1/2}\,\mathrm{Tr}\,\underline{k}_a\underline{R}' = (P_a)^{\cdot} - P_b\hat{C}^b{}_{ac}\dot{\tilde{\omega}}^c - NQ^*_a = 2N k g'^{1/2}\,\mathrm{Tr}\,\underline{k}_a\underline{T}'$

$Q^*{}_a = -2g'^{1/2}\,\mathrm{Tr}\,\underline{k}_a\underline{R}^{*\prime} = \delta^3{}_a Q^*_3$.

The diagonal part of (16.75) evolves the scale variables:

(16.81) $2g'\,\mathrm{diag}\,\underline{R}' = N^{-1}g'^{1/2}(N^{-1}g'^{1/2}(\ln \underline{g}')^{\cdot})^{\cdot} + N^{-2}g'\underline{F}' + 2g'\,\mathrm{diag}\,\underline{R}^{*\prime}$

$= 2kg'\,\mathrm{diag}(\underline{T} - \frac{1}{2}T\underline{1})$.

$\underline{F}' = \sum_{a=1}^{3}\mathrm{diag}\,[\underline{g}'^{-1}\underline{k}_a^T\underline{g},\underline{k}_a](\dot{\tilde{\omega}}^a)^2$

$= -\sum_{a=1}^{3} F_a'(\dot{\tilde{\omega}}^a)^2(\hat{\underline{e}}^b{}_b - \hat{\underline{e}}^c{}_c) = -\frac{1}{4}N^2g'^{-1}\sum_{a=1}^{3} F^{a\prime}P_a^2(\hat{\underline{e}}^b{}_b - \hat{\underline{e}}^c{}_c)$

$F_a' = (n^{(b)}e^{\beta^{bc}} - n^{(c)}e^{\beta^{cb}})^2 = (g'_{bb}g'_{cc})^{-1}((n^{(b)}g'_{bb})^2 - (n^{(c)}g'_{cc})^2)$

$F^{a\prime} = F_a'(\frac{1}{2}\bar{\mathscr{B}}'^{aa})^2 = (n^{(b)}e^{\beta^{bc}} + n^{(c)}e^{\beta^{cb}})/(n^{(b)}e^{\beta^{bc}} - n^{(c)}e^{\beta^{cb}})$

$= (n^{(b)}g'_{bb} - n^{(c)}g'_{cc})/(n^{(b)}g'_{bb} + n^{(c)}g'_{cc})$.

$2g'R^{*a}{}_a' = (n^{(a)}g'_{aa})^2 - (n^{(b)}g'_{bb} - n^{(c)}g'_{cc})^2 - 4a^2g'_{11}g'_{22}$.

The first equation can also be written in the form:

(16.82) $N^{-1}g'^{1/2}(N^{-1}g'^{1/2}(\ln g'_{aa})^{\cdot})^{\cdot} + \frac{1}{4}(F'_bP_b^2 - F'_cP_c^2) + 2g'R^{*\prime a}{}_a$

$= 2kg'(T'^a{}_a - \frac{1}{2}T)$.

The super-Hamiltonian constraint is:

(16.83) $4g'G^{\perp}{}_{\perp} = -N^{-2}g'[(\ln g'_{22})^{\cdot}(\ln g'_{33})^{\cdot} + (\ln g'_{33})^{\cdot}(\ln g'_{11})^{\cdot} + (\ln g'_{11})^{\cdot}(\ln g'_{22})^{\cdot}]$

$+\frac{1}{4}\sum_{a=1}^{3}\bar{\mathscr{B}}'^{aa}(P_a)^2 + \sum_{a=1}^{3}(n^{(a)}g'_{aa})^2$

$-2[n^{(2)}n^{(3)}g'_{22}g'_{33} + n^{(3)}n^{(1)}g'_{33}g'_{11} + n^{(1)}n^{(2)}g'_{11}g'_{22}] + 12a^2g'_{11}g'_{22}$.

Since we have chosen a different basis $\{\underline{k}_a\}$ in the class B case, the momentum constraint functions (11.53) must be reevaluated:

(16.84)
$$N g'^{-1/2} f(M_1)' = -\bar{\mathcal{g}}'_{11}\dot{\tilde{\omega}}^1 + (3a/n^{(1)})\bar{\mathcal{g}}'_{22}\dot{\tilde{\omega}}^2 = N g'^{-1/2} k\ell V'_1$$
$$N g'^{-1/2} f(M_2)' = -(3a/n^{(2)})\bar{\mathcal{g}}'_{11}\dot{\tilde{\omega}}^1 - \bar{\mathcal{g}}'_{22}\dot{\tilde{\omega}}^2 = N g'^{-1/2} k\ell V'_2$$
$$N g'^{-1/2} f(M_3)' = -\bar{\mathcal{g}}_{33}\dot{\tilde{\omega}}^3 + a(\ln g'_{33}(g'_{11}g'_{22})^{-1/2})^{\cdot} = N g'^{-1/2} k\ell V'_3$$

These are easily inverted to determine the automorphism velocities (or momenta) except for type $VI_{-1/9}$:

(16.85)
$$\dot{\tilde{\omega}}^1 = -N g'^{-1/2}\bar{\mathcal{g}}'^{11} k\ell (V'_1 + (3a/n^{(1)})V'_2)/(1+9h)$$
$$\dot{\tilde{\omega}}^2 = -N g'^{-1/2}\bar{\mathcal{g}}'^{22} k\ell (V'_2 - (3a/n^{(2)})V'_2)/(1+9h)$$
$$\dot{\tilde{\omega}}^3 = -\bar{\mathcal{g}}'^{33}(N g'^{-1/2} k\ell V'_3 + (\ln g'_{33}(g'_{11}g'_{22})^{-1/2})^{\cdot}) .$$

Finally one must evaluate the fluid equations:

(16.86)
$$(V'_a)^{\cdot} = N V'_f C^f{}_{ag} V'^g/V^\perp - V'_f \underline{K}_b{}^f{}_a \dot{\tilde{\omega}}^b$$
$$(\ln \ell)^{\cdot} = 2aN V'^3/V^\perp = 2aN g'^{33}V'_3/V^\perp .$$

By substituting the expressions (16.85) in the first of these equations one finds:

(16.87)
$$(V'_a)^{\cdot} = N V'_b V'_c\{[n^{(c)} g'^{bb} - n^{(b)} g'^{cc}]/V^\perp + N g'^{-1/2} k\ell [n^{(c)}\bar{\mathcal{g}}'^{bb} - n^{(b)}\bar{\mathcal{g}}'^{cc}]\} + aN W'_a ,$$

$$\begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = -g'^{33}V'_3/V^\perp \begin{bmatrix} V'_1 \\ V'_2 \end{bmatrix} - (\ln g'_{33}(g'_{11}g'_{22})^{-1/2})^{\cdot}\,\bar{\mathcal{g}}'^{33}\begin{bmatrix} n^{(2)}V'_2 \\ n^{(1)}V'_1 \end{bmatrix}$$
$$W_3 = (g'^{11}(V'_1)^2 + g'^{22}(V'_2)^2)/V^\perp + 3N g'^{-1/2} k\ell (n^{(1)}n^{(2)})^{-1}((n^{(1)}V'_1)^2\bar{\mathcal{g}}'^{22} + (n^{(2)}V'_2)^2\bar{\mathcal{g}}'^{33}) .$$

To complete the energy constraint and dynamical equations, one needs the following energy-momentum components (no sum on a):

(16.88)
$$-4g' k T^\perp{}_\perp = 4k\ell g'^{1/2}V^\perp - 4kpg'$$
$$2g'k(T'^a{}_a - \tfrac{1}{2}T) = 2k\ell g'^{1/2} g'^{aa}(V'_a)^2/V^\perp + 2kg'(\rho - p)$$

One may derive similar equations for the remaining Bianchi types I, II, V and IV.

One may find essentially these equations for type IX in [82] and for the symmetric type VII_h case with $V'_a = 0$ in [84]. Ryan summarizes their utility in studying the type IX singularity in [33]. Note that we have eliminated the special automorphism variables in favor of the fluid variables. Alternatively one may use all four constraints to eliminate the fluid variables in favor of the geometrical variables and obtain completely geometric equations. This technique together with a partial Lagrangian/Hamiltonian approach was used by Bogoyalenskiy to study certain properties of the perfect fluid type IX system.[93]

To illustrate this general approach, we examine the $\beta^- = \bar\theta^1 = \theta^3 = 0$ vacuum type $VI_{-1/9}$ case using the noncanonical frame. The barred metric matrix is of the form:

(16.89) $\underline{\bar g} = (e^{\bar\theta^2 \underline{K}_2})^T \underline{\bar g}' e^{\bar\theta^2 \underline{K}_2} = \underline{\bar g}' + \bar g'_{22} \bar\theta^2 (\underline{\hat e}^2{}_3 + \underline{\hat e}^3{}_2) + \bar g'_{22} (\bar\theta^2)^2 \underline{\hat e}^3{}_3$

where $\underline{\bar g}' \in \mathcal{M}_D$. The primed spatial Ricci tensor component matrix for any h is found by evaluating (8.17) for diagonal $\underline{g}'$:

(16.90) $2\bar g'_{33} \underline{R}^{*\prime} = -4a^2 \,\mathrm{diag}\,(1+\lambda, 1-\lambda, 1+\lambda^2)$.

With the lapse choice $N = (\bar g'_{33})^{1/2}$ and $\lambda = 3$, one may derive the following equations in a way similar to that used above:

(16.91) $2\bar g'_{33} \underline{\bar R}' = x^{-1}(x(\ln \underline{\bar g}')^{\cdot})^{\cdot} - \beta^2(\underline{\hat e}^2{}_2 - \underline{\hat e}^3{}_3) - (8/9)\,\mathrm{diag}\,(2,-1,5)$
$+ (zx)^{-1}(\bar p_2)^{\cdot} z^{-1}(\underline{\hat e}^3{}_2 + \underline{\hat e}^2{}_3)$

$x = (\bar g'_{11} \bar g'_{22})^{1/2}$ $\quad z = (\bar g'_{22}/\bar g'_{33})^{1/2}$ $\quad \bar p_2 = xz^2 \dot{\bar\theta}^2$

$\beta = z\dot{\bar\theta}^2 = (\bar g'_{33}/\bar g'_{11})^{1/2} (\bar g'_{22})^{-1} \bar p_2$

$3(\bar g'_{33})^{1/2} \bar G^{\perp}{}_{3'} = (\ln u)^{\cdot}$ $\quad u = \bar g'_{33} \bar g'_{22} (\bar g'_{11})^{-2}$

$4\bar g'_{33} \bar G^{\perp}{}_{\perp} = -[(\ln \bar g'_{22})^{\cdot}(\ln \bar g'_{33})^{\cdot} + (\ln \bar g'_{33})^{\cdot}(\ln \bar g'_{11})^{\cdot} + (\ln \bar g'_{11})^{\cdot}(\ln \bar g'_{22})^{\cdot} + \beta^2 + 16/3$

$-2N\underline{K}' = (\ln \underline{\bar g}')^{\cdot} + \beta(z^{-1}\underline{\hat e}^3{}_2 + z\underline{\hat e}^2{}_3)$

$-2N\underline{K}'' = (\ln \underline{\bar g}')^{\cdot} + \beta(\underline{\hat e}^3{}_2 + \underline{\hat e}^2{}_3)$.

Here the double prime refers to the notation (16.1) for orthonormal components.

If $(\ln \underline{\bar g}')^{\cdot} = \underline{\gamma}$, a constant diagonal matrix, and if $\dot\beta = 0$ (i.e. $2N\underline{K}''$ is a constant matrix), then $2\bar g'_{33} \underline{R}'$ is a constant matrix and $4\bar g'_{33} \bar G^{\perp}{}_{\perp}$ and $3(\bar g'_{33})^{1/2} \bar G^{\perp}{}_{3'}$ are constants, thus reducing the vacuum Einstein equations to an overdetermined system of algebraic equations for the constants $\{\gamma_{11}, \gamma_{22}, \gamma_{33}, \beta\}$ which are already subject to the constraint $\dot\beta = 0$. From the two relations:

(16.92) $2(\ln \beta)^{\cdot} = \gamma_{33} - \gamma_{11} - 2\gamma_{22} = 0$ $\qquad (\ln u)^{\cdot} = \gamma_{33} + \gamma_{22} - 2\gamma_{11} = 0$

and the two nontrivial dynamical equations (of the three, one is just $(\ln u)^{\cdot\cdot} = 0$ due to the compatibility of the momentum constraint with the dynamical equations), one finds:

(16.93) $\underline{\gamma} = \mathrm{diag}\,(1, 1/3, 5/3)\,\gamma_{11}$ $\qquad \gamma_{11}{}^2 = 8/3$ $\qquad \beta^2 = 5/9\,\gamma_{11}{}^2$.

Fortunately the super-Hamiltonian constraint vanishes identically for these values of the constants and hence we have a particular exact solution of the full Einstein equations.

Integrating the conditions on $\underline{\bar{g}}'$ and β, one finds

(16.94) $\underline{\bar{g}}' = \text{diag}(Ae^{\gamma_{11}t}, Be^{\gamma_{11}t/3}, Ce^{5\gamma_{11}t/3})$

$\bar{\theta}^2 = (3\beta/2\gamma_{11})(\bar{g}'_{33}/\bar{g}'_{22})^{1/2}$.

The two possible signs for γ_{11} correspond to an expanding "big bang" solution ($\gamma_{11}=\sqrt{8/3}$) and the time-reversed "final crunch" solution ($\gamma_{11}=-\sqrt{8/3}$). The lapse is easily integrated to yield the proper time τ:

(16.95) $\tau = \int (\bar{g}'_{33})^{1/2}dt = (6/5\gamma_{11})(\bar{g}'_{33})^{1/2}$

$\bar{g}'_{33} = (50/27)\,\tau^2$.

Since both $\hat{\underline{e}}^1_1$ and $\hat{\underline{e}}^2_2$ belong to $aut_{\underline{e}}(g)$, one is free to scale $\bar{g}'_{11}$ and $\bar{g}'_{22}$ arbitrarily. Doing this and choosing $\gamma_{11}>0$ one obtains:

(16.96) $\underline{\bar{g}}' = \text{diag}(\tau^{6/5}, \tau^{2/5}, (50/27)\tau^2)$

$\bar{\theta}^2 = (\text{sign}\,\beta)\,(5\sqrt{10}/18)\;\tau^{4/5}$.

However, since $\underline{S}_0 = \text{diag}(1,-1,1) \in Aut_{\underline{e}}(g)$ satisfies $\underline{S}_0\underline{k}_2\underline{S}_0^{-1} = -\underline{k}_2$, one is also free to change the sign of $\bar{\theta}^2$ and hence one may assume sign $\beta = 1$. The final form of the spatial metric matrix is therefore:

(16.97) $\underline{\bar{g}} = \text{diag}(\tau^{6/5}, \tau^{2/5}, (25/6)\tau^2) + (5\sqrt{10}/18)\,\tau^{6/5}(\hat{\underline{e}}^2_3 + \hat{\underline{e}}^3_2)$.

Essentially this form was written down by Siklos in unpublished work.[(104)] It appears in an unrecognizable form in other papers and is claimed to be the only spatially homogeneous vacuum solution of Petrov type {3,1}.[(105,)] For the diagonal class A case with $N = g^{1/2}$, the following formulas hold:

(16.98)
$$
\begin{aligned}
2g\underline{R} &= (\ln \underline{g})^{\cdot\cdot} + 2g\underline{R}^{*} \\
2g R^{*a}{}_{a} &= (n^{(a)} g_{aa})^{2} - (n^{(b)} g_{bb} - n^{(c)} g_{cc})^{2} \\
4g G^{\perp}{}_{\perp} &= -[(\ln g_{22})^{\cdot}(\ln g_{33})^{\cdot} + (\ln g_{33})^{\cdot}(\ln g_{11})^{\cdot} + (\ln g_{11})^{\cdot}(\ln g_{22})^{\cdot}] - 2gR^{*} \\
2gR^{*} &= -[(n^{(1)}g_{11})^{2} + (n^{(2)}g_{22})^{2} + (n^{(3)}g_{33}^{2})] \\
&\quad + 2[n^{(2)}n^{(3)}g_{22}g_{33} + n^{(3)}n^{(1)}g_{33}g_{11} + n^{(1)}n^{(2)}g_{11}g_{22}] .
\end{aligned}
$$

← For type II the dynamical equations and their immediate consequences are:

(16.99) $(\ln g_{11})^{\cdot\cdot} = (g_{33})^2 = (\ln g_{22})^{\cdot\cdot}\,,\quad (\ln g_{33})^{\cdot\cdot} = -(g_{33})^2$

$(\ln g_{11}g_{33})^{\cdot\cdot} = 0 = (\ln g_{22}g_{33})^{\cdot\cdot}$

The general solution of these equations is:

(16.100) $g_{11}g_{33} = e^{c_1t+d_1}\,,\quad g_{22}g_{33} = e^{c_2t+d_2}\,,\quad g_{33} = \gamma\,\mathrm{sech}(\gamma t+\alpha)\,,$

where $\gamma^2 = c_1c_2$ is required by the energy constraint. The first two may be written:

(16.101) $g_{11} = AB\,e^{c_1t}/g_{33}\,,\qquad g_{22} = AB^{-1}e^{c_2t}/g_{33}\,.$

However, since $\hat{\underline{e}}_-$ generates a subgroup of the type II canonical automorphism matrix group we may scale away the constant B, i.e. we may set $B=1$ with no loss in generality. This solution was obtained by Taub.[25] g_{33} satisfies the same equation in the Taublike type IX case for which $g_{11}=g_{22}$. By defining the variable $z = g_{11}g_{33} = g_{22}g_{33}$ and using the explicit expression for g_{33}, the energy constraint may be written $((\ln z)^{\cdot})^2 + 4z = \gamma^2$. Solving for $z^{\cdot}$ and integrating yields Taub's result $z = \frac{\gamma^2}{4}\mathrm{sech}^2(\frac{\gamma}{2}t+\delta)$.[25]

The electromagnetic type IX/VIII Taublike (Taub-Nut) spacetimes are Taublike type IX/VIII geometries driven by the energy-momentum tensor of a SH electromagnetic field for which $\mathcal{E}^a = \mathcal{E}^3\delta^a_3$ and $\mathcal{B}^a = \mathcal{B}^3\delta^a_3$ (see appendix D). Evaluation of (D.4) for this case yields:

(16.102) $-T^{\perp}{}_{\perp} = -T^3{}_3 = T^1{}_1 = T^2{}_2\,,\quad -8\pi A^2 T^{\perp}{}_{\perp} = 2\mathcal{U} = e^2,$

where $\underline{g} = \mathrm{diag}(A,A,B)$ and $\mathcal{U} = e^2/2$ is a constant according to the discussion following (D.9). Since the energy momentum tensor of an electromagnetic field is traceless, $R^\alpha{}_\beta = G^\alpha{}_\beta$ holds. The relations $T^1{}_1 + T^3{}_3 = 0 = T^1{}_1 + T^{\perp}{}_{\perp}$ then imply that the vacuum equations $R^1{}_1 + R^3{}_3 = 0 = R^1{}_1 + G^{\perp}{}_{\perp}$ hold. Evaluating $R^1{}_1$, $R^3{}_3$ and $G^{\perp}{}_{\perp}$ using the formulas (16.40), (16.41) and (16.44) with $g_{11}=g_{22}=A$ and $g_{33}=B$, introducing a lapse $N = 2\ell B^{-1/2}$ where ℓ is some positive constant, using a prime to denote $d/d\bar{t}$

and defining $x = n^{(3)}AB$, one finds:

(16.103) $0 = 2N^2(R^1{}_1 + G^\perp{}_\perp) = A''/A - \frac{1}{2}(A'/A)^2 - 2\ell^2/A^2,$

$0 = 2N^2(R^1{}_1 + R^3{}_3) = (x'' + 8\ell^2)/x.$

These have the solutions:

(16.104) $A = \bar{t}^2 + \ell^2, \quad x = -4\ell^2\Delta, \quad \Delta = \bar{t}^2 - 2m\bar{t} + \gamma,$

where m and γ are constants. γ is determined by the energy constraint. Letting $k = 8\pi$ this becomes:

(16.105) $0 = 2N^2(G^\perp{}_\perp - 8\pi T^\perp{}_\perp) = \frac{1}{2}(A'/A)^2 - A'x'/(Ax) + 2\ell^2/A^2 + 8\ell^2 \frac{n^{(3)}}{x}\left(\frac{e^2}{A} - 1\right)$

$= -\frac{8\ell^2}{Ax}(\gamma + \ell^2 - n^{(3)}e^2),$

so $\gamma = -\ell^2 + n^{(3)}e^2$ and $\Delta = \bar{t}^2 - 2m\bar{t} - \ell^2 + n^{(3)}e^2$. The vacuum Taub-Nut solutions are obtained by setting $e = 0$. $n^{(3)} = 1$ corresponds to type IX and $n^{(3)} = -1$ to type VIII.

Notice that the only effect of the electromagnetic field on the Taublike geometry is to shift the zeros of Δ (assume $e^2 < m^2 + \ell^2$ in the type IX case so that these are real):

(16.106) $\bar{t}_\pm = m \pm (m^2 + \ell^2 - n^{(3)}e^2)^{1/2}.$

When $\Delta < 0$, $\Delta = 0$ and $\Delta > 0$ the SH hypersurfaces are spacelike, null and timelike respectively for type IX and the reverse for type VIII, since $B = -4\ell^2 n^{(3)}\Delta/A$. However, since $N^2 = 4\ell^2/B = -n^{(3)}A/\Delta$ is infinite at the zeros of Δ, the metric:

(16.107) $g = -N^2 d\bar{t} \otimes d\bar{t} + B\,\omega^3 \otimes \omega^3 + A(\omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2)$

is singular at the null SH hypersurfaces. Alternatively, we may attribute the singular behavior to the comoving frame $\{\bar{e}_0 = \partial/\partial\bar{t}, e_a\}$, caused by the fact that $\bar{e}_0$ is normal to a family of hypersurfaces which are changing from spacelike to timelike or vice versa there. A nonsingular frame will therefore be related to this frame by a singular transformation.

Two natural nonsingular canonical SH comoving ADM frames $\{\bar{e}^+_\alpha\}$ and $\{\bar{e}^-_\alpha\}$ do exist for which the ADM generators e^+_0 and e^-_0 are null. Define:

(16.108) $\lambda^\pm = \pm n^{(3)}N^2/(2\ell) = \pm A/(-2\ell\Delta)$

$$\bar{e}_0^\pm = \bar{e}_0 + \lambda^\pm e_3 \,, \qquad \bar{\omega}^0_\pm = d\bar{t}$$
$$\bar{e}_3^\pm = e_3 \,, \qquad \bar{\omega}^3_\pm = \omega^3 - \lambda^\pm d\bar{t} \,,$$

and let $\{\bar{e}_\alpha^\pm\}$ be a canonical SH comoving ADM frame for the SH slicing such that at $\bar{t}=0$ the reduced frame $\{e_a^\pm\}$ coincides with $\{e_a\}$. Since $\bar{e}_0^\pm$ differs from $\bar{e}_0$ only by a shift along e_3, $e_1^\pm$ and $e_2^\pm$ are related to e_1 and e_2 by a time dependent rotation as they are dragged along from the $\bar{t}=0$ hypersurface by $\bar{e}_0^\pm$. According to section twelve this rotation is explicitly:

(16.109) $$e_a^\pm = e_b (R^\pm)^{-1\,b}{}_a \,, \quad (\underline{R}^\pm)^{-1}(\underline{R}^\pm)' = \lambda^\pm \underline{k}_3 \,,$$
$$\underline{R}^\pm(\bar{t}) = \exp\left[\underline{k}_3\left(\textstyle\int_0^{\bar{t}} \lambda^\pm(\tau)\, d\tau\right)\right],$$

where $\lambda^\pm(\bar{t})$ indicates the functional dependence on $\bar{t}$. This rotation is singular at $\bar{t}_-$ and $\bar{t}_+$ and cannot be extended outside the open interval $(\bar{t}_-, \bar{t}_+)$. Reexpressing the metric in the new frames using the fact that:

(16.110) $$\omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2 = \omega^1_\pm \otimes \omega^1_\pm + \omega^2_\pm \otimes \omega^2_\pm \,,$$

one finds:

(16.111) $$g_\pm = \pm 2 \ell\, n^{(3)} (\bar{\omega}^0_\pm \otimes \bar{\omega}^3_\pm + \bar{\omega}^3_\pm \otimes \bar{\omega}^0_\pm) + B\, \bar{\omega}^3_\pm \otimes \bar{\omega}^3_\pm + A(\omega^1_\pm \otimes \omega^1_\pm + \omega^2_\pm \otimes \omega^2_\pm).$$

The above three frames are analytically related to each other on the region for which $\Delta < 0$ so $g = g_+ = g_-$ there but the transformations between them all become singular at $\Delta = 0$. By assuming that $\{e_\alpha^+\}$ is a nonsingular frame for all $\bar{t}$, one changes the manifold structure of $M = \mathbb{R} \times G$ on which $\{\bar{e}_0, e_a\}$ is analytic, so denote by M_+ the resulting manifold with the natural product structure induced by $\bar{e}_0^+$. $\{\bar{e}_\alpha^-\}$ is also singular on M_+ because it is still related to $\{\bar{e}_\alpha^+\}$ by a singular transformation. Similarly define M_-. (M_+, g_+) and (M_-, g_-) are then two analytic spacetimes which are inequivalent extensions of the region of (M,g) for which $\Delta<0$. In the first $\Delta \bar{e}_0^-$ is tangent to the null SH hypersurfaces while $\Delta \bar{e}_0^+$ is tangent to these hypersurfaces in the second. (The map $(\bar{e}_0^+)_{\bar{t}}(0,x) \in M_+ \mapsto (\bar{e}_0^-)_{-\bar{t}}(0,x) \in M_-$ is an isometry between the two spacetimes as was brought to the author's attention by J.G. Miller.) More explicit discussion of these spacetimes may be found in references [50, 51, 53]. The interesting feature they display is the (nonunique) evolution of the spacelike SH hypersurfaces into null and then timelike hypersurfaces. Similar strange behavior occurs in perfect fluid SH spacetimes of certain Bianchi types.[46]

← It is this problem which led us to include in our definition of a spatially homogeneous spacetime $(\mathbb{R}\times G, g)$ the qualification that the SH hypersurfaces be spacelike only for t in some open interval of the origin of $\mathbb{R}$. The type VIII Taub-Nut spacetime indicates that we should drop the requirement that the origin be in this open interval.

The vacuum class A models offer several interesting examples of "minisuperspace linearization instabilities." All of these models have the three linearly independent Killing vector fields $\{\tilde{e}_a\}$ in common, but there are additional Killing vector fields for special models. At flat spacetime occurring in the type I ∩ VII_0 minisuperspace, e_0 becomes an additional Killing vector field, while e_3 becomes one at the Taublike models of type VII_0, VIII and IX. These SH Killing vector fields induce linearization instabilities in the SH perturbations of these models. It is sufficient to consider the linearization stability of the constraint equations:

$$\mathcal{H}=0 \quad , \quad \mathcal{H}_a=0 .$$

The diagonal type I energy constraint is a cone (the momentum variables) crossed with a plane (the position variables) in the momentum phase space $T^*\mathcal{M}_D$, with flat space at the vertex $P_A=0$ where $e_0=\partial/\partial t$ becomes a Killing vector field:

(6.112) $$\mathcal{H}(\beta,P) = \tfrac{1}{24} e^{-3\beta^0}(-P_0{}^2+P_+{}^2+P_-{}^2) = 0 .$$

The linearized constraint vanishes identically at $P_A=0$:

(6.113) $$D\mathcal{H}(\beta,P)(\beta',P') = \tfrac{1}{12} e^{-3\beta^0}(-P_0P_0'+P_+P_+'+P_-P_-') - 3\beta^{0\prime}\mathcal{H}(\beta,P)$$

$$D\mathcal{H}(\beta,0)(\beta',P') = 0 .$$

The integrand of Moncrief's second order constraint integral associated with e_0 is just the second derivative of the energy constraint which at $p_A=0$ is:

(16.114) $$D^2\mathcal{H}(\beta,0)((\beta',p'),(\beta',p')) = \tfrac{1}{6}e^{-3\beta^0}(-p_0'^2+p_+'^2+p_-'^2)$$

which reimposes the correct constraint when set to zero. For the perfect fluid system the addition of the term $2k\ell\mu$ to the super-Hamiltonian removes the vertex from the constraint set.

For types VII_0, VIII and IX, e_3 is an additional Killing vector field ... for the Taublike models, causing the linearization of the momentum constraint $\mathcal{H}_3$ to vanish identically:

(16.115) $$\begin{aligned}\mathcal{H}_3(g,\pi) &= -2\,\mathrm{Tr}\,\underline{k_3\pi} = 2(\pi^2{}_1-\pi^1{}_2)\\ &= 2\pi^{12}(g_{11}-g_{22})-2g_{12}(\pi^{11}-\pi^{22})+2(\pi^{23}g_{31}-\pi^{31}g_{23}).\end{aligned}$$

Both factors in each term vanish for $(g,\pi)\in T^*(\mathcal{M}_{T(3)})\subset T^*\mathcal{M}$ and cause the linearized constraint to vanish identically on this submanifold:

(16.116) $$\begin{aligned}\tfrac{1}{2}D\mathcal{H}_3(g,\pi)(g',\pi') = {}&\pi'^{12}(g_{11}-g_{22})-g_{12}(\pi'^{11}-\pi'^{22})+\pi'^{23}g_{31}\\ &-\pi'^{31}g_{23}+\pi^{12}(g'_{11}-g'_{22})-g'_{12}(\pi^{11}-\pi^{22})+\pi^{23}g'_{31}-\pi^{31}g'_{23}.\end{aligned}$$

Again the Moncrief integrand associated with e_3 is the second derivative which reestablishes the correct constraint:

(16.117) $$\begin{aligned}\tfrac{1}{4}D^2\mathcal{H}_3(g,\pi)((g',\pi'),(g',\pi')) = {}&\pi'^{12}(g'_{11}-g'_{22})-g'_{12}(\pi'^{11}-\pi'^{22})\\ &+\pi'^{23}g'_{31}-\pi'^{31}g'_{23}.\end{aligned}$$

This linearization instability extends to the electromagnetic Taublike solutions for which A_3 is the only nonzero component of the vector potential, but the situation is different for the perfect fluid models. In the latter case the linearization of the constraint $\mathcal{H}_3=0$ forces $V_3'=0$ and the second order constraint links the first order gravitational perturbations to the second order fluid perturbation V_3''. The positive curvature Friedmann models, which are known to be linearization stable[70], exhibit this feature in all three momentum constraints.

A paper by Demianski and Grishchuck [61] deserves comment here. They give up an explicit equation of state connecting μ and p to n and impose a constraint on the geometry to determine such a relation. A symmetric type VII_0 system is considered with $\underline{g}$ confined to $\mathcal{M}_{S(1)}$ and β^- is set to zero causing the spatial curvature to vanish and the geometry to be flat. This requires that $0 = \dot{P}_- = -\partial H/\partial\beta^-|_{\beta^-=0}$ which serves to determine μ explicitly as a function of the geometric variables and the constants ℓ and V_1. This in turn determines n explicitly through the definition of ℓ and hence $\rho + p = \mu n$ is determined. The pressure alone may be obtained from the evolution equation $\dot{p} = n\dot{\mu}$. By this trick they obtain a system representing a class of spacetimes with flat SH hypersurfaces but containing a rotating fluid. If we assume the parametrization:

$$\underline{g} = (\exp\theta^1 \underline{k}_1)^T \underline{g}' (\exp\theta^1 \underline{k}_1) \in \mathcal{M}_{S(1)} , \quad (16.118)$$

then $\underline{g}' \in \mathcal{M}_{T(3)}$ for these models. The introduction of a shift whose only nonzero component is $N^1 = \dot{\theta}^1$ therefore confines the solution curves to $\mathcal{M}_{T(3)}$ (a Taublike system). Note that on any given SH hypersurface the induced metric has a higher symmetry than the restriction of the fluid current 1-form. Lukash also considers this symmetric type VII_0 system and the symmetric type VII_h system and has an interesting interpretation of the corresponding spacetimes as gravitational waves on Friedmann spacetimes. [62] In [94] he exhibits two special vacuum solutions for the symmetric type VII_h case. The gravitational wave interpretation for type IX models is discussed in [95].

An apparently more drastic anzsatz is made by Dunn and Tupper in [56]. They consider a class of perfect fluid and combined perfect fluid and electromagnetic Taublike type VI spacetimes in which the metric matrix is fixed up to two arbitrary constants. In the first case the Einstein equations are then used to define ρ and p

which turn out to satisfy an equation of state $p=(\gamma-1)\rho$ with γ a constant. These models therefore do fit into our scheme. Another example which does not is provided by [63]. Here a type III fluid spacetime is considered with all degrees of freedom except β^0 suppressed and the trace of the diagonal energy-momentum tensor required to vanish (a noncanonical frame is used). The latter condition enables one to integrate β^0 explicitly. The Einstein equations then define ρ and three unequal pressures P_1, P_2 and P_3 so the resulting fluid is not a perfect fluid.