§13. Spatially Homogeneous Spacetime Models

A spatially homogeneous space-time model (M, g) is a 4-dimensional manifold M with a Lorentz metric g and a 3-dimensional Lie group G acting simply transitively on a family of spacelike hypersurfaces of M as an isometry group. We assume the initial construction of the previous section with $M = \mathbb{R} \times G$ and the subgroup G_0 acting on the left as the isometry group. For t in an open interval about zero the SH hypersurface G_t is assumed to be spacelike and therefore a homogeneous 3-geometry.

The spacetime metric g is assumed to be a solution of the Einstein equations with some SH energy-momentum tensor T acting as a source of the field ($T=0$ being the vacuum case). Expressing these equations in a SH frame leads to a set of ordinary differential equations for the component functions $g_{\alpha\beta}$ which depend only on the coordinate t and describe the dynamics of a SH spacetime or "cosmological model." Many approaches are available to deal with these equations (for example, see the work of Ellis and coworkers) but the problem is ideally suited to the ADM formulation which is concisely reviewed in chapter 21 of MTW. Familiarity with this chapter and its notation will be assumed in what follows.

The family of SH hypersurfaces is a distinguished slicing of the spacetime since its elements are orbits of an isometry group. As discussed in the previous section, a choice of a basis $\{e_a\}$ of its Lie algebra $\mathfrak{g}$ leads to a SH comoving ADM frame $\{e_\alpha\}$ for this family, for which the vector field $e_0 = \partial/\partial t$ is an ADM generator in the sense that the slicing arises by dragging G_0 along its flow; the reduced frame $\{e_a\}$ is also dragged along.

The components of the spacetime metric in this frame may be interpreted in terms of a lapse function N, a shift vector field $\vec{N} = N^a e_a$ and the induced metric $g^* = g_{ab}\,\omega^{a*} \otimes \omega^{b*}$ (here an asterisk denotes restriction of a form or covariant tensor field to the SH hypersurfaces), which are essentially time-dependent left invariant fields on G:

$$(13.1)\quad g = g_{\alpha\beta}\,\omega^\alpha \otimes \omega^\beta = -N^2\,\omega^0 \otimes \omega^0 + g_{ab}\,(\omega^a + N^a \omega^0) \otimes (\omega^b + N^b \omega^0)$$
$$e_0 = N e_\perp + \vec{N}$$

The last relation connects the lapse, shift and ADM generator with the field $e_\perp$ of unit normals to the SH hypersurfaces. A normal comoving ADM frame for the slicing is one in which $e_\perp$ itself is taken to be the ADM generator e_0 so that the corresponding lapse is unity and the shift zero. Let us assume this choice initially. The transformation to a general SH comoving ADM frame for the SH slicing determined by a general SH ADM generator $\bar{e}_0 = N e_0 + N^a e_a$ is precisely the topic of the previous section, the fundamental equation being (12.7). Relative to the original (normal) product structure of M, the new reduced frame $\{\bar{e}_a\}$ may be interpreted as a time-dependent basis of $\mathfrak{g}$ related to the fixed basis $\{e_a\}$ by the time-dependent adjoint matrix $\underline{\mathcal{R}}$. The time-dependent metric $\bar{g}^*$ induced on G by the product structure associated with $\bar{e}_0$ is related to g^* by this transformation. In terms of the matrices of components in their respective frames, we have:

(13.2) $\quad \underline{\bar{g}} = \underline{\mathcal{R}}^{-1T} \underline{g}\, \underline{\mathcal{R}}^{-1} \qquad \underline{g} = \underline{\mathcal{R}}^{T} \underline{\bar{g}}\, \underline{\mathcal{R}}$.

The formula for the components of the extrinsic curvature of the SH hypersurfaces (an intrinsic tensor field on the hypersurfaces) may be evaluated using the results of §8, in particular (8.26) and (8.18). A dot denotes differentiation by t:

(13.3) $\quad K_{ab} = -(1/2N)(\dot{g}_{ab} - N_{(a|b)}) = -(1/2N)(\dot{g}_{ab} + 2N^c K_{cab})$,

With our matrix notation convention this may be written with its immediate consequence:

(13.4) $\quad 2N\underline{K} = -\underline{g}^{-1}\underline{\dot{g}} \qquad -2\underline{K}_c N^c$

$\qquad\qquad N\,\mathrm{Tr}\,\underline{K} = -(\log g^{1/2})^{\cdot} - 2a_c N^c$

When we specialize to a normal frame these are very simple:

(13.5) $\quad \underline{K} = -\tfrac{1}{2}\underline{g}^{-1}\underline{\dot{g}} \qquad \mathrm{Tr}\,\underline{K} = -(\log g^{1/2})^{\cdot}$

The extrinsic curvature matrix $\underline{\bar{K}}$ in a general frame, given by the barred version of (13.4), is related to the matrix (13.5) by the adjoint transformation:

(13.6) $\quad \underline{K} = \underline{\mathcal{R}}^{-1}\underline{\bar{K}}\,\underline{\mathcal{R}}\,. \qquad \underline{\bar{K}} = \underline{\mathcal{R}}\,\underline{K}\,\underline{\mathcal{R}}^{-1}$

Inserting the second equation of (13.2) into the first of (13.5) and computing its derivative using (12.10) and the relation $\dot{f} = df/dt = N^{-1} df/d\bar{t}$, one may read off from (13.6) the barred version of (13.4) in which the dot refers to differentiation by $\bar{t}$: $\quad (-\tfrac{1}{2})$

$$K = -\tfrac{1}{2}\underline{g}^{-1}\underline{\dot{g}} = (\underline{\mathcal{R}}^{-1}\underline{\bar{g}}^{-1}\underline{\mathcal{R}}^{-1T})\,\underline{\mathcal{R}}^{T}(\underline{\dot{\bar{g}}} + \underline{\bar{g}}\,\underline{\dot{\mathcal{R}}}\,\underline{\mathcal{R}}^{-1} + \underline{\mathcal{R}}^{-1T}\underline{\dot{\mathcal{R}}}^{T}\underline{\bar{g}})\,\underline{\mathcal{R}}$$
$$= (-\tfrac{1}{2}N)\,\underline{\mathcal{R}}^{-1}(\underline{\bar{g}}^{-1} d\underline{\bar{g}}/d\bar{t} + 2\bar{N}^a \underline{\bar{K}}_a)\,\underline{\mathcal{R}}\,.$$

This is typical of the way results obtained easily in a normal frame may be transformed to a general frame using the ideas of §12, showing how the reparametrization freedom and action of the adjoint group introduces a general lapse and shift. Similarly one may transform to a comoving ADM frame characterized only by (12.9), the relevant group now being the automorphism group, leading to the introduction of the automorphism velocities rather than the lapse components through (12.13).

We now restrict ourselves to a normal frame for the considerable simplification achieved, resorting to the above tricks to transform to other interesting frames. All Latin indices are lowered and raised with g_{ab}, g^{ab} while the zero index (we occasionally use the symbol $\perp$ to emphasize normal components) is lowered and raised with a minus sign:

$$g_{00} = -1 = g^{00} \qquad\qquad g_{0a} = 0 = g^{0a}.$$

The normal frame is convenient in the respect that the Latin indexed components of space-time fields are simply related to the components in the reduced frame of corresponding 3-dimensional fields on the SH hypersurfaces, without the lapse and shift mixing things up. The latter components of various geometric fields were evaluated in §8 where an asterix signaled their 3-dimensional nature and a slash denoted covariant differentiation. The space-time SH components may be evaluated using the standard formulas of the appendix, the structure constants $C^\alpha{}_{\beta\gamma}$ of the frame given by (12.3) and the fact that such components depend only on the coordinate t which in the normal product is the proper time along the integral curves of $n = e_0$:

$$e_\gamma T^{\alpha\cdots}{}_{\beta\cdots} = \partial_\gamma T^{\alpha\cdots}{}_{\beta\cdots} = \delta^0_\gamma \dot{T}^{\alpha\cdots}{}_{\beta\cdots}.$$

Making use of (A.13) one finds the only nonvanishing connection coefficients to be:

$$(13.7) \qquad \Gamma^0{}_{ab} = -K_{ab} \qquad\qquad \Gamma^a{}_{0b} = -K^a{}_b = \Gamma^a{}_{b0}$$

$$\Gamma^c{}_{ab} = \Gamma^{*c}{}_{ab} = \tfrac{1}{2} C^c{}_{ab} + K^c{}_{ab}.$$

These may be summarized by:

$$(13.8) \qquad \nabla_{e_0} e_0 = 0 \qquad\qquad \nabla_{e_a} e_0 = -K^b{}_a e_b = \nabla_{e_0} e_a$$

$$\nabla_{e_a} e_b = -K_{ab} e_0 + \Gamma^c{}_{ab} e_c.$$

These state that the acceleration of the field of unit normals vanishes (so its integral curves are geodesics), that K is the extrinsic curvature

Correction for page 13.3. Insert the following at the end of the first paragraph:

For example, equations (13.4) become:

$$2N\underline{K} = -\underline{g}^{-1}\dot{\underline{g}} - 2\,\mathrm{ad}_e(\vec{N})^{\#} = -\underline{g}^{-1}\dot{\underline{g}} - 2\,\underline{e}_A^{\#}\,\dot{\tilde{\omega}}^A$$

$$N\,\mathrm{Tr}\,\underline{K} = -(\log g^{1/2})^{\cdot} - (\mathrm{Tr}\,\underline{e}_A)\,\dot{\tilde{\omega}}^A \quad .$$

of the homogeneous hypersurfaces (MTW conventions) and that $\Gamma^c{}_{ab}$ are the components of the intrinsic connection induced on the hypersurfaces. From (13.7), (13.5) and the appendix, useful expressions may be derived for the space-time divergences of SH vector fields and symmetric and antisymmetric second rank tensors respectively (conveniently defining $\underline{K}_0 = \underline{K}$):

$$(13.9)\qquad \begin{aligned} X^\beta{}_{;\beta} &= g^{-1/2}(g^{1/2}X^0)^{\cdot} - 2a_c X^c \\ T_\alpha{}^\beta{}_{;\beta} &= g^{-1/2}(g^{1/2}T^0{}_\alpha)^{\cdot} + \mathrm{Tr}\,\underline{K}_\alpha \underline{T} - 2a_f T^f{}_\alpha \\ F^{a\beta}{}_{;\beta} &= -g^{-1/2}(g^{1/2}F^{0a})^{\cdot} - \tfrac{1}{2}\epsilon_{bfg}F^{fg}C^{ba} \\ F^{0\beta}{}_{;\beta} &= -2a_c F^{0c}\,. \end{aligned}$$

Except for $T_0{}^\beta{}_{;\beta}$, the terms other than time-derivatives are 3-divergences; compare with (8.25).

Evaluating the Ricci tensor components according to (A.16) yields:

$$(13.10)\qquad R_{00} = \mathrm{Tr}\,\dot{\underline{K}} - \mathrm{Tr}\,\underline{K}^2 \qquad R_{0a} = -\mathrm{Tr}\,\underline{k}_a\underline{K} + 2a_f K^f{}_a$$

$$R_{ab} = -\dot{K}_{ab} + \mathrm{Tr}\,\underline{K}\,K_{ab} - 2K_{af}K^f{}_b + R^*{}_{ab}\,.$$

Raising the index and rearranging the derivative using (13.5) gives:

$$(13.11)\qquad \underline{R} = -\dot{\underline{K}} + \underline{K}\,\mathrm{Tr}\,\underline{K} + \underline{R}^* = -g^{-1/2}(g^{1/2}\underline{K})^{\cdot} + \underline{R}^*\,.$$

The scalar curvature is therefore:

$$(13.12)\qquad R = -R_{00} + \mathrm{Tr}\,\underline{R} = -2\mathrm{Tr}\,\dot{\underline{K}} + \mathrm{Tr}^2\underline{K} + \mathrm{Tr}\,\underline{K}^2 + \underline{R}^*\,.$$

Again making use of (13.5) the scalar curvature density may be written:

$$(13.13)\qquad g^{1/2}R = -2(g^{1/2}\,\mathrm{Tr}\,\underline{K})^{\cdot} + g^{1/2}(\mathrm{Tr}\,\underline{K}^2 - \mathrm{Tr}^2\underline{K} + \underline{R}^*)\,.$$

Similarly the Einstein tensor and Einstein field equations

$$M^\alpha{}_\beta = g^{1/2}(G^\alpha{}_\beta - kT^\alpha{}_\beta) = 0$$

may be evaluated, leading to the well known result:

$$(13.14)\qquad \begin{aligned} g^{1/2}\underline{G} &= \dot{\underline{\Pi}} - \tfrac{1}{2}\mathcal{T}\underline{1} + g^{1/2}\underline{G}^* = kg^{1/2}\underline{T} \\ g^{1/2}G^0{}_a &= g^{1/2}(\mathrm{Tr}\,\underline{k}_a\underline{K} - 2a_f K^f{}_a) = -\mathrm{Tr}\,\underline{k}_a\underline{\Pi} + 2a_f\Pi^f{}_a = kg^{1/2}T^0{}_a \\ 2g^{1/2}G^0{}_0 &= \mathcal{T} - g^{1/2}R^* = 2kg^{1/2}T^0{}_0 \\ \mathcal{T} &= \mathcal{G}^{abcd}K_{ab}K_{cd} = \mathcal{G}^{-1}{}_{abcd}\Pi^{ab}\Pi^{cd} \\ \Pi^{ab} &= -\mathcal{G}^{abcd}K_{cd}\,. \end{aligned}$$

We have introduced the field momentum Π^{ab} and the kinetic energy function $\mathcal{T}$, both involving the DeWitt metric. $T_{\alpha\beta}$ are the components of a SH energy-momentum tensor assumed to drive the geometry. It depends on certain variables describing the source which may or may not be SH and which satisfy field equations not yet concerning us but such that the energy-momentum tensor is divergenceless. The tensor with components $g^{-1/2} M^{\alpha}{}_{\beta}$ is then divergenceless by the contracted Bianchi identities regardless of the Einstein field equations which require it to vanish. Using the divergence formula (13.9), this condition takes the form:

$$\text{(13.15)} \qquad \dot{M}^{0}{}_{0} = -\mathrm{Tr}\,\underline{K}\underline{M} + 2a_f M^{f}{}_{0}$$
$$\dot{M}^{0}{}_{a} = -\mathrm{Tr}\,\underline{K_a}\underline{M} + 2a_f M^{f}{}_{a}.$$

The quantities $M^{0}{}_{a}$ are therefore first integrals of the dynamical equations $\underline{M}=0$. This is true also of $M^{0}{}_{0}$ except in the class B models when in addition the "momentum constraints" $M^{0}{}_{a}=0$ must be satisfied. The equation $M^{0}{}_{0}=0$ is called the energy constraint.

These ordinary differential equations together with the source equations define a classical mechanical system. The gravitational configuration space is the metric manifold $\mathcal{M}$ on which g_{ab} are coordinates. The solution to the full equations with given initial conditions will involve a curve $c(t)$ on this manifold specified in the component coordinate system by functions $g_{ab}(t) = g_{ab} \circ c(t)$. t plays the role of the classical mechanical time. The tangent $c'(t)$ is called the velocity; its components in the metric coordinate frame are the functions $\dot{g}_{ab}(t)$. $\mathcal{T}$ is a homogeneous quadratic form kinetic energy which is just a fourth of the squared length of the velocity in the DeWitt geometry, given by (11.53) and (11.54):

$$\text{(13.16)} \qquad \mathcal{T}(\underline{g}, \underline{\dot{g}}) = \tfrac{1}{4}\, \mathcal{G}^{abcd}\, \dot{g}_{ab}\, \dot{g}_{cd}.$$

By interpreting $\dot{g}_{ab}$ as coordinate functions on $T\mathcal{M}$ this becomes a function on the tangent bundle or velocity phase space. It serves as a Lagrangian for the geodesics of the DeWitt metric. Apart from the extra factor of $\frac{1}{2}$ in the kinetic energy which is inserted to conform to convention, π^{ab} are the covariant components of the velocity in the DeWitt geometry and the components of the mechanical momentum determined by the kinetic energy:

$$\text{(13.17)} \qquad \pi^{ab} = \partial\mathcal{T}/\partial \dot{g}_{ab} = \tfrac{1}{2}\, \mathcal{G}^{abcd}\, \dot{g}_{cd}.$$
$$\mathcal{T} = \mathcal{G}^{-1}{}_{abcd}\, \pi^{ab}\pi^{cd}.$$

By identifying $\{g_{ab}, \pi^{ab}\}$ with the coordinates induced on $T^{*}\mathcal{M}$ by the metric coordinate system we obtain a kinetic energy function on the cotangent bundle or momentum phase space which serves as a

Hamiltonian for the geodesics on $\mathcal{M}$.

The other partial derivatives of the kinetic energy function on $T\mathcal{M}$ may be evaluated by applying (11.5) to the explicit expression (11.16) for $\mathcal{G}^{abcd}$:

(13.18) $\partial\mathcal{T}/\partial g_{ab} = \frac{1}{2}\mathcal{T} g^{ab} + 2(\underline{\Pi}\,\underline{K})^{ab}$.

The dynamical equations give us an expression for $\dot{\underline{\Pi}}$. By using the identity:

(13.19) $(\underline{\Pi}\,\underline{g}^{-1})^{\cdot} = \dot{\underline{\Pi}}\,\underline{g}^{-1} + 2\underline{\Pi}\,\underline{K}\,\underline{g}^{-1}$,

an expression for $\dot{\Pi}^{ab}$ is obtained. Comparing with (13.18), this expression may be recognized as implying the result:

(13.20) $-M^{*ab} = \dot{\Pi}^{ab} - \partial\mathcal{T}/\partial g_{ab} = (\partial\mathcal{T}/\partial\dot{g}_{ab})^{\cdot} - \partial\mathcal{T}/\partial g_{ab} = -\delta\mathcal{T}/\delta g_{ab}$

$$\underline{M}^* = g^{1/2}(\underline{G}^* - k\underline{T}).$$

The dynamical equations therefore equate the negative of the Lagrange derivative of the kinetic energy to a force with components $-M^{*ab}$ in the component coordinate frame. A force field or 1-form $M^* = M^{*ab}\,dg_{ab}$ is therefore defined on $\mathcal{M}$ involving the source variables as time-dependent parameters which deflects the solution curves from geodesics of the DeWitt metric. The gravitational part of this force has already been discussed in §11, where it was seen to consist of a conservative part associated with the scalar curvature potential $U^* = -g^{1/2}R^*$ and a nonconservative part:

(13.21) $-g^{1/2}G^* = -dU^* + Q^*$.

This allows us to write the dynamical equations in the form of driven Lagrange equations:

(13.22) $-\delta L_G/\delta g_{ab} = Q^{*ab} + k g^{1/2} T^{ab}$

$$L_G = \mathcal{T} - U^* = g^{1/2}(\mathrm{Tr}\,\underline{K}^2 - \mathrm{Tr}^2\underline{K} + R^*).$$

The gravitational Lagrangian L_G, by comparison with (13.13), is seen to be the SH space-time scalar curvature density apart from a total time derivative. It is just the ADM gravitational Lagrangian density evaluated on a spatially homogeneous field in a SH frame. Notice that even for vacuum fields the dynamical equations are not the Lagrange equations following from the SH ADM Lagrangian unless the nonconservative force Q^* vanishes (class A models). We explore this in great detail in the next section.

The gravitational energy of the classical mechanical system is just twice the normal-normal component of the Einstein tensor density:

(13.23) $E_G = \mathcal{T} + U^* = g^{1/2}(\mathrm{Tr}\,\underline{K}^2 - \mathrm{Tr}^2\underline{K} - R^*) = 2g^{1/2}\, G^{\perp}{}_{\perp}$.

By expressing $\mathcal{T}$ in terms of the momentum as in (13.17), we obtain by our usual identification the gravitational Hamiltonian function on $T^*\mathcal{M}$:

(13.24) $H_G = \mathcal{S}^{-1}{}_{abcd}\, \pi^{ab}\pi^{cd} + U^*$.

The Hamiltonian form of the dynamical equations is as one would expect except for the addition of the nonconservative force to the momentum equation:

(13.25) $\dot{g}_{ab} = \partial H_G / \partial \pi^{ab}$

$$\dot{\pi}^{ab} = -\partial H_G / \partial g_{ab} + Q^{*ab} + kg^{1/2} T^{ab}.$$

Unfortunately Lagrangian and Hamiltonian dynamics with nonconservative forces have been largely ignored in physics but the modifications introduced by such forces are rather straightforward even if unfamiliar.

The full configuration space of our dynamical system is a product manifold $\mathcal{M} \times \mathcal{S}$ where $\mathcal{S}$ describes the source associated with the energy-momentum tensor. This decomposition is not unique since different choices of independent source variables may be related by the space-time metric (such as covariant or contravariant components of tensor fields describing the source). For an appropriate choice of source variables the quantity $U_S = -2kg^{1/2}\, T^{\perp}{}_{\perp}$, a function on $\mathcal{M} \times \mathcal{S}$, may serve as a potential or partial potential for the source contribution to the force $-M^*$ on $\mathcal{M}$ (considering the source variables in M^* and U_S as external parameters):

(13.26) $kg^{1/2} T^{ab} = -\partial U_S / \partial g_{ab} + Q_S{}^{ab}$.

The slopover defines a source component of the total nonconservative force $Q = Q^* + Q_S$; for familiar sources this is not required. The dynamical equations may then be written in the form:

(13.27) $-\delta L / \delta g_{ab} = Q^{ab}$ $\qquad L = L_G - U_S$.

We assume nonderivative coupling to exclude velocity-dependent potentials. The mechanical and canonical momentum therefore coincide.

Defining $U = U^* + U_S$, the total energy function is just:

(13.28) $E = \mathcal{T} + U = 2M^{\perp}{}_{\perp}$.

The energy constraint requires it to vanish. Consider the vacuum case. When the scalar curvature of the SH hypersurfaces of the spacetime is positive, zero or negative, the energy constraint causes the velocity to be space-like, null or time-like respectively in the DeWitt geometry. In particular the solution curves for vacuum type I models are null geodesics on $\mathcal{M}$. The geodesics of $\mathcal{M}$ have been solved for explicitly by DeWitt, as described in appendix B.

The Hamiltonian H corresponding to the Lagrangian L is $H_G + U_S$ and the driven Hamiltonian equations are:

$$\text{(13.29)}\qquad \begin{aligned} \dot{g}_{ab} &= \partial H/\partial \pi^{ab} &&= \{g_{ab}, H\} \\ \dot{\pi}^{ab} &= -\partial H/\partial g_{ab} + Q^{ab} &&= \{\pi^{ab}, H\} + Q^{ab} \end{aligned}$$

By (13.14) the momentum constraints are:

$$-\operatorname{Tr} k_a \pi + 2a_f \pi^f{}_a = R g^{1/2} T^{\perp}{}_a .$$

For class A models these equate the adjoint generators to the components of the flux of source energy-momentum relative to the SH hypersurfaces. In vacuum these generators must vanish along solution curves. When written in terms of the velocity as in (13.14) the vacuum class A constraints require the velocity to be orthogonal to the vector fields $\xi(k_a)$ generating the adjoint action on $\mathcal{M}$. In class B models the interpretation is not quite so neat.

We have explored the dynamical system induced on $\mathcal{M}\times\mathcal{S}$ by the Einstein equations expressed in a normal comoving ADM frame, using the metric coordinate system on $\mathcal{M}$. By choosing a new ADM generator $\bar{e}_0$ for the SH slicing we may induce a new system on $\mathcal{M}\times\mathcal{S}$ describing the evolution of the new components $\bar{g}_{ab}$ and explicitly involving the specified lapse and shift components. Alternatively we may reinterpret the original system in terms of a reparametrization of the classical mechanical time due to nontrivial SH lapse and a time-dependent change of component coordinates on $\mathcal{M}$ generated by nonzero shift. (38)

Choosing $\bar{t} = T(t)$ as a new time for the classical mechanical system, with $d/d\bar{t} = N\, d/dt$, means that we reparametrize the solution curves:

$$\text{(13.30)}\qquad \bar{c}(\bar{t}) = c(t) \qquad \bar{c}'(\bar{t}) = N c'(t).$$

The lapse function relates the new and old velocities; other time derivatives and forces (since they are equated to accelerations) transform similarly. Reparametrization of the action integral shows that a Lagrangian picks up the lapse as a multiplier too, so NL is the

Lagrangian appropriate to the new time and NQ the corresponding force field on $\mathcal{M}$.

Equations (13.2) may be interpreted as a transformation to time-dependent component coordinates $\bar{g}_{ab}$ on $\mathcal{M}$, exactly as one may consider a rotating cartesian coordinate system on $\mathbb{R}^3$. In reexpressing the Lagrangian in the new coordinates, it is useful to note that its value at a fixed point is the component of a weight one scalar density on G. Substitution of (13.2) into the Lagrangian is therefore equivalent to transforming the corresponding scalar density which has a component $\bar{L}$ in the new frame, the gravitational part of which is given by the barred version of (13.22):

(13.31) $$L = \det \underline{R}\, \bar{L}\,.$$

The SH lapse components appear through the time-derivatives of $\underline{R}$ exactly as in the discussion following (13.6) and play the role of the components of the angular velocity in rotating cartesian coordinates. The components in the component coordinate system of the nonconservative force Q, a 1-form on $\mathcal{M}$, correspond to the components of a weight-one tensor density on G and transform similarly, picking up a factor of $\det \underline{R}$.

The combined effect of the time reparametrization and the coordinate transformation leads to a time-dependent Lagrangian $N\bar{L} \det \underline{R}$ and a nonconservative force with components $N\bar{Q}^{ab} \det \underline{R}$ in the time-dependent metric coordinate system. The corresponding Lagrange equations are:

(13.32) $$-\bar{\delta}/\delta \bar{g}_{ab}\,(N\bar{L}\det \underline{R}) = N\bar{Q}^{ab}\det \underline{R}$$
$$-\bar{\delta}/\delta\bar{g}_{ab} \equiv \; d/d\bar{t}\,(\,\partial/\partial(d\bar{g}_{ab}/d\bar{t})) - \partial/\partial \bar{g}_{ab}\,.$$

However, it is $N\bar{L}$ which corresponds to the ADM Lagrangian expressed in the new frame on space-time as in the barred forms of (13.22) and (13.4). (See MTW 21.86). The dynamical equations may be written in terms of its Lagrange derivative by using the expression (12.10) for the logarithmic derivative of $\det \underline{R}$:

(13.33) $$-\bar{\delta}/\delta\bar{g}_{ab}\,(N\bar{L}) = N\bar{Q}^{ab} - 2a_c\bar{N}^c\,\bar{\pi}^{ab}$$
$$\bar{\pi}^{ab} = \partial(N\bar{L})/\partial(d\bar{g}_{ab}/d\bar{t}) = -\mathscr{G}^{abcd}\bar{K}_{cd}$$

A new force proportional to the momentum thus appears when expressing the dynamical equations in a general SH comoving ADM frame. This force must also enter the Hamiltonian form of these equations treating $N\bar{L}$ as the Lagrangian, since the "true Hamiltonian" is derived from the Lagrangian $N\bar{L}\det\underline{R}$. The true momentum components in the time-dependent component coordinates also differ from $\bar{\Pi}^{ab}$ by a factor of $\det\underline{R}$ for the same reason.

The new velocity may be expressed in terms of the "momentum" by inverting the second of equations (13.33) supplemented by the barred version of (13.4):

(13.34) $$d\bar{g}_{ab}/d\bar{t} = N\bar{\mathcal{G}}^{-1}{}_{abcd}\bar{\Pi}^{cd} - 2\bar{K}_{cab}\bar{N}^c.$$

The Hamiltonian corresponding to the Lagrangian $N\bar{L}$ is therefore:

(13.35) $$\bar{H} = \bar{\Pi}^{ab}\, d\bar{g}_{ab}/d\bar{t} - N\bar{L} = N\bar{\mathcal{H}} + \bar{N}^c(-2\operatorname{Tr}\underline{\bar{K}}_c\underline{\bar{\Pi}})$$
$$\bar{\mathcal{H}} = \bar{\mathcal{G}}^{-1}{}_{abcd}\bar{\Pi}^{ab}\bar{\Pi}^{cd} + \bar{U} = 2\bar{g}^{1/2}(\bar{G}^{\perp}{}_{\perp} - k\bar{T}^{\perp}{}_{\perp}) = 2\bar{M}^{\perp}{}_{\perp},$$

and the Hamiltonian equations are:

(13.36) $$d\bar{g}_{ab}/d\bar{t} = \partial\bar{H}/\partial\bar{\Pi}^{ab}$$
$$d\bar{\Pi}^{ab}/d\bar{t} = -\partial\bar{H}/\partial\bar{g}_{ab} + N\bar{Q}^{ab} - 2a_c\bar{N}^c\bar{\Pi}^{ab}.$$

The momentum dependent force enters since $\bar{H}$ is the true Hamiltonian H expressed in the new component coordinates.

$\bar{\mathcal{H}}$ is the SH super-Hamiltonian but the coefficients of the shift components in $\bar{H}$ are not the components of the SH geometrical super momentum: (see (8.25)):

(13.37) $$\bar{\mathcal{H}}^*_a = -2\bar{\Pi}_a{}^b{}_{|b} = -2\operatorname{Tr}\underline{\bar{K}}_a\underline{\bar{\Pi}} + 4a_b\bar{\Pi}^b{}_a = \bar{g}^{1/2}\bar{G}^{\perp}{}_a.$$

However, as the canonical generators of the adjoint action they perform the same role as the geometrical super-momentum components in the full theory, namely, causing the solution curves $\bar{g}_{ab}(t)$ to evolve with the additional inverse adjoint transformation (13.2) relative to the normal solution curves $g_{ab}(t)$.

For the logical separation of the full configuration space into the cross product $\mathcal{M}\times\mathcal{S}$ it will turn out that the source component of the supermomentum will be independent of the metric components, at least for familiar sources. We assume this is so. The full ADM "Hamiltonian" $\bar{H}_{ADM}$ is:

(13.38) $$\bar{H}_{ADM} = N\bar{\mathcal{H}} + \bar{N}^a\bar{\mathcal{H}}_a$$
$$\bar{\mathcal{H}}_a = 2\bar{g}^{1/2}(\bar{G}^{\perp}{}_a - k\bar{T}^{\perp}{}_a) = 2\bar{M}^{\perp}{}_a.$$

If we now replace the canonical adjoint generators in $\bar{H}$ by the components of the SH supermomentum, we obtain the ADM Hamiltonian. In order that the corresponding driven Hamiltonian equations with $\bar{H}_{ADM}$ as the Hamiltonian agree with (13.36), we must subtract off the additional terms generated by partial derivatives of the extra term $4a_b\bar{\Pi}^b{}_a\bar{N}^a$:

$$\bar{A}_{ab} = -\partial/\partial\bar{\Pi}^{ab}\,(4a_c\bar{\Pi}^c{}_d\bar{N}^d) = -4a_{(a}\bar{N}_{b)}$$

$$\bar{B}^{ab} = \partial/\partial\bar{g}_{ab}\,(4a_c\bar{\Pi}^c{}_d\bar{N}^d) = 4a_c\bar{\Pi}^{c(a}\bar{N}^{b)}.$$ (36)

The dynamical equations may therefore be written in the final form:

(13.39) $$d\bar{g}_{ab}/d\bar{t} = \partial\bar{H}_{ADM}/\partial\bar{\Pi}^{ab} + \bar{A}_{ab}$$

$$d\bar{\Pi}^{ab}/d\bar{t} = -\partial\bar{H}_{ADM}/\partial\bar{g}_{ab} + N\bar{Q}^{ab} - 2a_c\bar{N}^c\bar{\Pi}^{ab} + \bar{B}^{ab}.$$

In this form the constraint functions are the coefficients of the lapse and shift components in the Hamiltonian, but the price one must pay is the most complicated form of the dynamical equations. Since the constraints are known and the dynamical equations are of primary interest, (13.35) and (13.36) are the most convenient arrangements of these equations.

Since the coefficient $\bar{\mathcal{H}}$ of the lapse N must vanish in (13.35) or in $\bar{H}_{ADM}$, one may choose N to be a function of the dynamic variables without affecting the equations of motion (the partial derivatives this generates in the equations of motion have vanishing coefficients). The same is true of the shift in the class A system if we employ (13.35); for the class B system (13.38) must be used to exploit this freedom. The shift is usually set to zero except when a comoving fluid frame is used (26). A judicious choice of the lapse is the key to obtaining many of the known exact solutions.(25,52)

Ryan chooses $\bar{N}^a = 0$

(so that the bars may be dropped except on $\bar{t}$) and $N = 6g^{1/2}/\mathrm{Tr}\,\Pi$. (15)

Let $\Omega = -\beta^0 = -1/6\ln g$. This choice of lapse has the effect of making Ω the new time variable:

$$d\Omega/d\bar{t} = -1/6\,g^{ab}\,dg_{ab}/d\bar{t} = -1/6\,N g_{ab}\,\partial\mathcal{H}/\partial\Pi^{ab} = (N/6)\,g^{-1/2}\,\mathrm{Tr}\,\Pi = 1.$$

Correction for page 13.11. Add to the 2nd paragraph (beginning with "In this form...)

It is instructive to repeat this discussion for the more general class of shift vector fields. From the more general point of view, the situation becomes even more transparent. This has been done in [103].

The components of the spacetime Riemann tensor in a normal frame are easily evaluated using (A.16) and (13.7). The results are:

$$R^{ab}{}_{cd} = 2K^a{}_{[c}K^b{}_{d]} + R^{*ab}{}_{cd} \tag{13.40}$$
$$R^{0b}{}_{0d} = -\dot{K}^b{}_d + K^b{}_f K^f{}_d$$
$$R^{0b}{}_{cd} = K^b{}_e C^e{}_{cd} - 2K_{e[c}\Gamma^e{}_{d]}{}^b .$$

These naturally divide into three parts (in a decomposition adapted to the normal vector field) as in MTW Exercise 14.14:

$$E^a{}_b = R^{0a}{}_{0b}\ , \quad F^a{}_b = \tfrac{1}{4}\epsilon^{acd}\epsilon_{bfg}R^{fg}{}_{cd}\ , \quad H^{ba} = \tfrac{1}{2}R^{0b}{}_{cd}\epsilon^{acd} . \tag{13.41}$$

Evaluating these with the help of (13.40), (8.18), (13.10), (13.11), (13.14):

$$\underline{E} = -\dot{\underline{K}} + \underline{K}^2 = \underline{K}^2 - \underline{K}\,\mathrm{TR}\,\underline{K} - \underline{R}^* + \underline{R} \tag{13.42}$$
$$\underline{F} = \underline{E} - \underline{R} - \underline{1}\,G^0{}_0$$
$$H^{ba} = \tfrac{1}{2}(K^b{}_f C^{fa} - \epsilon^{acd}K_{fc}\Gamma^f{}_d{}^b) = H^{(ba)} + \tfrac{1}{4}\epsilon^{baf}R^0{}_f .$$

E_{ab} and F_{ab} are symmetric and $H^b{}_b = 0$.

For vacuum spacetimes, $R^\alpha{}_\beta = 0$ so that the Weyl and Riemann tensors coincide and furthermore $\underline{E} = \underline{F}$ and $H^{[ab]} = 0 = \mathrm{TR}\,\underline{E}$. Consider the vacuum class A models for which $\underline{g}$ is always diagonal and so $\underline{K}$, $\dot{\underline{K}}$ and $\underline{K}^2$ are diagonal. When $\underline{n}$ is also diagonal it follows from (8.17) that $\underline{R}^*$ is diagonal and hence $\underline{E} = \underline{F}$ is diagonal. It is easily seen that $\underline{H}$ is also diagonal in this case. Since $\underline{E}$ and $\underline{H}$ are both diagonal these spacetimes are of Petrov type I [17], while if both $\underline{g}$ and $\underline{n}$ have corresponding pairs of coincident diagonal values, so do $\underline{E}$ and $\underline{H}$ and the spacetime is then of Petrov type IA (D). The Petrov type of other vacuum and nonvacuum models is not obvious from inspection. Joseph has shown that the diagonal type V vacuum spacetimes are of Petrov type I. [58]

§14. Spatial Homogenization of Lagrangian and Routhian Formulations

We quickly review the naive Lagrangian and so called Hamiltonian formulations of the vacuum Einstein equations. Let $\{e_\alpha\}$ be some fixed frame on a region C of spacetime with dual frame $\{\omega^\alpha\}$ and structure functions $C^\alpha{}_{\beta\gamma}$. The Einstein Lagrangian, field equations and action are:

(14.1) $$\mathcal{L}(g_{\alpha\beta}, \partial_\gamma g_{\alpha\beta}, \partial_\gamma \partial_\delta g_{\alpha\beta}, C^\alpha{}_{\beta\gamma}, \partial_\delta C^\alpha{}_{\beta\gamma}) = ({}^4 g)^{1/2} R$$
$$0 = \delta\mathcal{L}/\delta g_{\alpha\beta} = -({}^4 g)^{1/2} G^{\alpha\beta}$$
$$S[g, C] = \int_C \mathcal{L}[g]\, \omega^{0123}.$$

The field equations follow from the extremization of the action as discussed in the third section and hold for all choices of frame. Let us specialize to a comoving ADM frame with respect to a family of spacelike hypersurfaces $\{M_t \mid t \in \mathbb{R}\}$ representing a slicing of spacetime (assuming one exists). We are free to choose new variables to describe the component functions $g_{\alpha\beta}$; the optimal choice is the set $\{g_{ab}, N, N^a\}$ where:

$$N = (-g^{00})^{-1/2} \qquad N^a = -g^{0a}/g^{00}.$$

We are also free to drop divergences in $\mathcal{L}$ since they have no effect on the field equations. By eliminating a divergence containing "bad derivatives" (see MTW), one obtains the following form of the Lagrangian density:

(14.2) $$\mathcal{L}_{ADM}(g_{ab}, \partial_c g_{ab}, \partial_c \partial_d g_{ab}, \dot{g}_{ab}, N, N^a, \partial_b N^a, C^a{}_{bc}, \partial_d C^a{}_{bc})$$
$$= N \mathcal{G}^{abcd} K_{ab} K_{cd} + N g^{1/2} R^* \equiv T - U^*$$
$$K_{ab} = -(2N)^{-1}(\dot{g}_{ab} - 2 N_{(a|b)}).$$

Here we use the notation $\dot{g}_{ab} = \partial_0 g_{ab} = e_0 g_{ab}$ (which is just $\partial g_{ab}/\partial t$ in an coordinate system comoving with e_0) and a slash to refer to the connection associated with the induced metric on the hypersurfaces M_t which has components g_{ab} in the frame $\{e_a\}$ on M_t. The Lagrange equations may now be be written:

(14.3) $$\delta\mathcal{L}_{ADM}/\delta g_{ab} - (\partial\mathcal{L}_{ADM}/\partial \dot{g}_{ab})^{\cdot} = 0$$
$$\delta\mathcal{L}_{ADM}/\delta N = 0 = \delta\mathcal{L}_{ADM}/\delta N^a,$$

where here the Lagrange derivative refers to the 3-dimensional one on M_t having no zero component derivatives and with respect to the frame $\{e_a\}$. The Lagrangian formulation is at this point completely decomposed into a three-plus-one split associated with our given slicing of spacetime. Note the obvious decomposition of $\mathcal{L}_{ADM}$ into kinetic and potential energy terms.

In a Lagrangian system with a finite number of degrees of freedom we are free to go to the Routhian formulation by a Legendre transformation with respect to a subset of the coordinates describing the system rather than all of them as is done when going to the Hamiltonian formulation. We then obtain a new function from the Lagrangian called the Routhian whose natural variables are the coordinates and momenta associated with the chosen subset and the coordinates and velocities of the remaining set. As far as the equations of motion are concerned, it acts like a Hamiltonian for the first set and a Lagrangian for the second set. Poisson brackets may be introduced for the Hamiltonian variables and a rich structure may be explored. This formulation is especially useful when the momenta associated with the second set vanish; this happens when the Lagrangian is independent of the velocities of that set. The Routhian equations for that set then yield contraints.

Unfortunately classical field theory has been somewhat neglected as an extension of classical mechanics, but one may generalize Routhian dynamics to this case. The so called Hamiltonian formulation of general relativity is in fact a Routhian formulation. The momenta conjugate to N and N^a vanish, so we divide the gravitational variables up into a dynamical set $\{g_{ab}\}$ and a constraint set $\{N, N^a\}$, where "spatial" derivatives are understood to be included in these sets. We then define the Routhian appropriate to this decomposition. The momenta conjugate to g_{ab} are:

(14.4) $\quad \pi^{ab} = \partial L_{ADM}/\partial \dot{g}_{ab} = -\mathcal{G}^{abcd} K_{cd}$.

This relation is easily inverted to give the velocities in terms the momenta:

(14.5) $$K_{ab} = -\mathcal{G}^{-1}{}_{abcd}\,\pi^{cd}$$
$$\dot{g}_{ab} = N\mathcal{G}^{-1}{}_{abcd}\,\pi^{cd} + 2N_{(a|b)}.$$

The Routhian density $\mathcal{R}$ is defined by:

(14.6) $$\mathcal{R}(g_{ab}, \partial_c g_{ab}, \partial_c\partial_d g_{ab}, \pi^{ab}, \partial_c \pi^{ab}, N, N^a, \partial_b N^a, C^a{}_{bc}, \partial_d C^a{}_{bc})$$
$$= \pi^{ab}\dot{g}_{ab} - \mathcal{L}_{ADM} = N\mathcal{G}^{-1}{}_{abcd}\pi^{ab}\pi^{cd} - Ng^{1/2}R^* + 2\pi^{ab}N_{a|b}$$
$$= N\mathcal{H} + N^a\mathcal{H}_a + (2\pi^{ab}N_b)_{|a}.$$

We have used the definition (3.17) of $\mathcal{H}_a$ and the relation preceding it (just a integration by parts). Dropping the 3-divergence, we obtain $\mathcal{R}_{ADM}$ which is a function of the variables $\{g_{ab}, \pi^{ab}, N, N^a\}$. The lapse N and shift components N^a appear only as coefficients of the super-Hamiltonian $\mathcal{H}$ and super-momentum $\mathcal{H}_a$.

If we couple gravity to electromagnetism, then starting with the variables $\{g_{\alpha\beta}, A_\alpha\}$, we obtain the three-plus-one ADM Lagrangian and add A_0 to the constraint set since its momentum vanishes. The corresponding Routhian, in which we again integrate by parts to remove all "spatial" derivatives from the constraint variables as above is essentially the ADM Hamiltonian. The same is true for other 1-form theories (massive vector meson and Yang-Mills).

In our present case, the Routhian equations are:

(14.7) $$\dot{g}_{ab} = \delta\mathcal{R}_{ADM}/\delta\pi^{ab}, \quad \dot{\pi}^{ab} = -\delta\mathcal{R}_{ADM}/\delta g_{ab},$$
$$0 = \delta\mathcal{R}_{ADM}/\delta N = \mathcal{H}, \quad 0 = \delta\mathcal{R}_{ADM}/\delta N^a = \mathcal{H}_a.$$

By (A.17) and the definition of the Lagrange derivative:

(14.8) $$\delta(Ng^{1/2}R^*)/\delta g_{ab} = \mathcal{G}^{abcd} N_{;cd}.$$

(3.35) provides the only other nontrivial Lagrange derivatives that must be evaluated, leaving only terms which are ordinary partial derivatives with respect to g_{ab} and π^{ab}. The explicit expressions are given by MTW (21.114) and (21.115). It is useful to note that $\mathcal{R}_{ADM}, \mathcal{H}, \mathcal{H}_a$ are respectively $2g^{1/2}G^{\perp}{}_0$, $2g^{1/2}G^{\perp}{}_{\perp}$, $2g^{1/2}G^{\perp}{}_a$, where $\perp$ indicates components perpendicular to M_t.

In order to consider spatially homogeneous spacetimes, we limit our attention to spacetime manifolds $\mathbb{R}\times G$ and assume a SH comoving ADM frame for the SH slicing as discussed in section twelve (so $\partial_\delta C^\alpha{}_{\beta\gamma}=0$). Next we introduce an operation Hom called homogenization. Let ϕ symbolize the frame components of a set of fields and let $F(\phi, \partial_\alpha\phi, \partial_\alpha\partial_\beta\phi, C^\alpha{}_{\beta\gamma})$ be a function of the indicated arguments. Then a new function $(\mathrm{Hom}F)(\phi, \dot\phi, \ddot\phi, C^\alpha{}_{\beta\gamma})$ is defined by inserting zeros in all arguments of F containing "spatial" derivatives. Hom F agrees with F when evaluated on spatially homogeneous fields. From the definition of the Lagrange derivative:

(14.9) $$\delta F/\delta\phi = \partial F/\partial\phi - \gamma_\alpha(\partial F/\partial\partial_\alpha\phi) + \gamma_\beta\gamma_\alpha(\partial F/\partial\partial_\alpha\partial_\beta\phi)$$
$$\gamma_\alpha = \partial_\alpha - C^\beta{}_{\alpha\beta} = \partial_\alpha - 2a_a\delta^a_\alpha,$$

and the assumption that F contains no second zero component derivatives, we easily derive the important relation:

(14.10) $$\mathrm{Hom}\,\delta F/\delta\phi = \partial\,\mathrm{Hom}F/\partial\phi - (\partial\,\mathrm{Hom}F/\partial\dot\phi)^{\cdot} + 2a_a\,\mathrm{Hom}\,\partial F/\partial\partial_a\phi + 4a_a a_b\,\mathrm{Hom}\,\partial F/\partial\partial_a\partial_b\phi .$$

If F is a Lagrangian density for a field theory, then HomF is a Lagrangian for an ordinary classical mechanical system in the finite number of variables ϕ with velocities $\dot\phi$. The first term in Hom $\delta F/\delta\phi$ is just the finite-dimensional Lagrange derivative:

$$\delta\,\mathrm{Hom}F/\delta\phi = \partial\,\mathrm{Hom}F/\partial\phi - (\partial\,\mathrm{Hom}F/\partial\dot\phi)^{\cdot}.$$

$\delta\,\mathrm{Hom}F/\delta\phi = 0$ are the equations of motion for the finite dimensional system, while Hom $\delta F/\delta\phi = 0$ are the homogenized field equations satisfied by spatially homogeneous fields, whose components ϕ depend only on t. These are ordinary differential equations which agree with $\delta\,\mathrm{Hom}F/\delta\phi = 0$ when $a_a = 0$. However, when $a_a \neq 0$ the correspondence between the classical mechanical system and the homogenized field theory is broken.[26,36] The additional terms in the homogenized Lagrange derivative play the role of additional forces which must be introduced into the classical mechanical system to restore the correspondence.

Suppose $\mathcal{H}(\phi, \partial_a\phi, \partial_a\partial_b\phi, \Pi, \partial_a\Pi, C^a{}_{bc})$ is a Hamiltonian density derived from a Lagrangian density $\mathcal{L}$. Then Hom $\mathcal{H}$ is a Hamiltonian for a classical mechanical system but the homogenized Hamiltonian equations are:

(14.11) $\dot{\phi} = \partial\, \mathrm{Hom}\mathcal{H}/\partial\pi + 2a_a\, \mathrm{Hom}\, \partial\mathcal{H}/\partial\partial_a\pi$

$\dot{\pi} = -\partial\, \mathrm{Hom}\mathcal{H}/\partial\phi - 2a_a\, \mathrm{Hom}\, \partial\mathcal{H}/\partial\partial_a\pi - 4a_a a_b\, \mathrm{Hom}\, \partial\mathcal{H}/\partial\partial_a\partial_b\phi.$

When $a_a \neq 0$, the additional terms break the correspondence between the canonical structure of the field theory and the finite-dimensional homogenized system. Moreover, if one drops spatial divergences in going from $\mathcal{L}$ to $\mathcal{H}$, Hom $\mathcal{H}$ will not be the Hamiltonian one obtains from Hom $\mathcal{L}$ by the usual methods. Similar statements hold for a Routhian formulation.

Consider the gravitational Lagrangian density $\mathcal{L}_{ADM} = N(T-U^*)$. The kinetic energy T only involves first derivatives of g_{ab} so we need only evaluate:

$$2a_c\, \mathrm{Hom}\,\partial(NT)/\partial\partial_c g_{ab} = 2a_c\, \mathrm{Hom}\,(-2\pi^{fg}\,\partial/\partial\partial_c g_{ab}\, N_{(f|g)})$$
$$= -2a_c N^c \pi^{ab},$$

since by (A.13):

$$(\partial/\partial\partial_c g_{ab})\, N_{(f|g)} = \partial/\partial\partial_c g_{ab}\,(N^e g_{e(f,g)} - \tfrac{1}{2} N^e \Gamma_{e(f,g)})$$
$$= \partial/\partial\partial_c g_{ab}\,(\tfrac{1}{2} N^e g_{fg,e}) = \tfrac{1}{2} N^c \delta^a_{(f}\delta^b_{g)}.$$

The homogenized Lagrange derivative of $NU^* = -Ng^{1/2}R^*$ is just the homogenized Einstein tensor density $Ng^{1/2}G^{*ab}$ so by (13.21):

$$\mathrm{Hom}\ \delta(NU^*)/\delta g_{ab} = N\partial\, \mathrm{Hom}\, U^*/\partial g_{ab} - NQ^{*ab}.$$

The homogenized dynamical Lagrangian equations are therefore:

$$0 = \mathrm{Hom}\ \delta\mathcal{L}/\delta g_{ab} = \delta\, \mathrm{Hom}\,\mathcal{L}/\delta g_{ab} + NQ^{*ab} - 2a_c N^c \pi^{ab},$$

which agrees with (13.32). Similarly (13.37) may be obtained in this way.[36] When a source for the gravitational field is present, there is no problem unless spatial derivatives are involved.

The important observation to make is that in the class B case, the ideas of the general theory don't quite apply to the finite-dimensional spatially homogenized theory. In particular naive quantization on the class B minisuperspaces has no relevance because of the failure of the Poisson bracket algebra of the classical mechanical system to agree with the homogenized field theoretic Poisson brackets.

§15. A Spatially Homogeneous Perfect Fluid Source.

We now consider a perfect fluid source, assuming familiarity with the relevant sections of chapter 22 of MTW whose notation and results we take for granted. As in §13, we at first use a normal SH comoving ADM frame $\{e_\alpha\}$ to discuss the fluid equations and then relate these to more general frames using the transformation equations of §12. The components of the SH energy-momentum tensor are given by:

(15.1) $$T^{\alpha\beta} = (\rho + p) u^\alpha u^\beta + p\, g^{\alpha\beta}$$
$$-1 = u^\alpha u_\alpha = -(u^0)^2 + g^{ab} u_a u_b \;, \quad u^\perp = u^0 = (1 + g^{ab} u_a u_b)^{1/2} .$$

The index $\perp$ is equivalent to 0 in a normal frame and indicates a component along the unit normal.

If one is given the metric g and the energy-momentum tensor T of a perfect fluid, then ρ, p and the 4-velocity u are uniquely determined, so it should be no surprise that given a SH metric, T is SH if and only if ρ, p and u are. This is easily checked by writing out $\pounds_\xi T = 0$ in terms of the Lie derivatives of ρ, p and u and showing that each must vanish by taking various contractions of the equation with u and g, using $\pounds_\xi g = 0$ and $(\pounds_\xi u)^\alpha u_\alpha = u^\alpha (\pounds_\xi u)_\alpha = 0$, a consequence of the unit nature of u. In the electromagnetic case the energy-momentum tensor T determines the electromagnetic field F only up to a duality rotation so spatial homogeneity of T does not necessarily require that F be SH. (55)

The 4-velocity and thermodynamic variables are therefore SH. Choosing n and s to be the primary thermodynamic variables, an equation of state $\rho(n, s)$ enables one to determine p and $\mu = (\rho + p)/n$ in terms of them. The spatial homogeneity of s together with its constancy along the flow lines implies that it is a constant, leaving n and u as the only dynamic variables:

(15.2) $$0 = \nabla_u s = s_{,\alpha} u^\alpha = \dot{s} u^0 .$$

In this case the following equations hold:

(15.3) $(\rho+p)^{-1}dp = d\ln n \qquad (\rho+p)^{-1}dp = d\ln\mu$.

The current vector V (21) and a function ℓ are defined by:

(15.4) $V=\mu u, \; V^{\perp}=\mu u^{\perp}=(\mu^2+g^{ab}V_aV_b)^{1/2}, \quad \ell = g^{1/2}nu^{\perp}$.

The various components of the energy-momentum tensor (15.1) may be written in terms of these new quantities as follows, where a factor $-kg^{1/2}$ has been included for later purposes:

(15.5)
$$-kg^{1/2}T^{\perp}{}_{\perp} = k\ell V^{\perp} - kpg^{1/2}$$
$$-kg^{1/2}T^{\perp}{}_{a} = -k\ell V_a$$
$$kg^{1/2}T^{ab} = k\ell V^aV^b/V^{\perp} + kpg^{1/2}g^{ab}.$$

Notice that the momentum constraints (13.14) for models of type I and II (where $\underline{k}_a=0$ and $\underline{k}_3=0$ respectively) require that $V_a=0$ and $V_3=0$ respectively.

In a general spacetime the divergence condition $T_\alpha{}^\beta{}_{;\beta}=0$ may easily be decomposed into the following set of equations:

(15.6)
$$(nu^\beta)_{;\beta}=0$$
$$a_\alpha = u_{\alpha;\beta}u^\beta = -(\ln\mu)_{,\beta}(\delta^\beta_\alpha+u^\beta u_\alpha).$$

In terms of the current vector the second equation becomes:

(15.7) $V_{\alpha;\beta}u^\beta = -\mu_{,\alpha}$.

These equations may be evaluated for SH fluid variables using (13.9) and the following formula which holds for a SH vector field X:

(15.8) $X_{a;\beta}X^\beta = \dot{X}_aX^0 + X_fC^f{}_{ag}X^g$.

Rewritten in terms of ℓ and V with the help of (15.8), (15.6) and (15.7) become for SH fluid variables:

(15.9) $(\ln\ell)^{\cdot} = 2a_aV^a/V^{\perp} \quad , \quad \dot{V}_a = -V_fC^f{}_{ag}V^g/V^{\perp}$.

ℓ is therefore a constant of the motion for class A models. In type II for which $n^{(3)}=C^3{}_{12}=-C^3{}_{21}=1$ is the only nonvanishing canonical component of the SCT, $V_fC^f{}_{ag}=V_3C^3{}_{ag}=0$ since V_3 must vanish. V_1 and V_2 are therefore constants for this type.

Under the transformation (12.7), ℓ and V_a transform as follows:

(15.10) $\bar{\ell} = (\det\underline{R}^{-1})\ell \quad , \quad \bar{V}_a = V_bR^{-1\,b}{}_a$.

Applying $d/d\bar{t} = N\,d/dt$ to these equations and using (12.10) and (15.9) one finds:

(15.11) $$d/d\bar{t}\,(\ln\bar{\ell}) = 2Na_a(\bar{V}^a/V^\perp - \bar{N}^a/N)$$
$$d/d\bar{t}\ \bar{V}_a = -N\bar{V}_f C^f{}_{ag}(\bar{V}^g/V^\perp - \bar{N}^g/N).$$

If we choose a SH ADM generator proportional to the fluid 4-velocity, the right hand sides vanish and $\bar{\ell}, \bar{V}_a$ are constants.[26] On the other hand suppose we transform to the more general comoving ADM frame (12.12). Then using (12.13) and (12.14), a similar calculation yields the more general equations:

(15.12) $$(\ln\bar{\ell})^{\cdot} = 2Na_a\bar{V}^a/V^\perp - (\mathrm{Tr}\,\hat{e}_B)\,\dot{\tilde{\omega}}^B$$
$$(\bar{V}_a)^{\cdot} = N\bar{V}_f C^f{}_{ag}\bar{V}^g/V^\perp - \bar{V}_f\,\hat{e}_B{}^f{}_a\,\dot{\tilde{\omega}}^B.$$

Define a function V^2 by:

(15.13) $$V^2 = |n^{ab}V_aV_b| = \epsilon\, n^{ab}V_aV_b,$$

where ϵ equals $1, 0, -1$ when $n^{ab}V_aV_b$ is positive, zero or negative. If $V \neq 0$, we may "normalize" the spatial current vector:

(15.14) $$V_a = V\,\hat{V}_a \quad , \quad n^{ab}\hat{V}_a\hat{V}_b = \epsilon.$$

From (15.9) one may compute the derivative of V^2 finding:

(15.15) $$(\ln \ell^{-1} V^2)^{\cdot} = 0 \quad .$$

$\ell^{-1}V^2$ is therefore a constant of the motion for class B models while V^2 is itself one for class A models.

For types I and II this is trivial but for types VI_0, VII_0, VIII, IX it reduces the number of dynamically independent components of V_a to two, suggesting the reparametrization of $\hat{V}_a$ by two new independent variables $\{\lambda^1, \lambda^2\}$ adapted to the symmetry of the various quadratic surfaces (15.14). For canonical values of $\underline{n}$ these surfaces are:

$$n^{(1)}(\hat{V}_1)^2 + n^{(2)}(\hat{V}_2)^2 + n^{(3)}(\hat{V}_3)^2 = \epsilon,$$

where $\epsilon = \pm 1$, and the appropriate coordinates on these surfaces are:

(15.16) VI_0, $\epsilon = 1$: $\hat{V}_1 = \cosh\lambda_1$ $\hat{V}_2 = \sinh\lambda_1$ $\hat{V}_3 = \lambda_2$

$\epsilon = -1$: $\hat{V}_1 = \sinh\lambda_1$ $\hat{V}_2 = \cosh\lambda_1$ $\hat{V}_3 = \lambda_2$

VII_0: $\hat{V}_1 = \cos\lambda_1$ $\hat{V}_2 = \sin\lambda_1$ $\hat{V}_3 = \lambda_2$

$\mathrm{VIII}_0,\ \epsilon=1:\quad \hat{V}_1=\cosh\lambda_1\cos\lambda_2 \quad \hat{V}_2=\cosh\lambda_1\sin\lambda_2 \quad \hat{V}_3=\sinh\lambda_1$

$\epsilon=-1:\quad \hat{V}_1=\sinh\lambda_1\cos\lambda_2 \quad \hat{V}_2=\sinh\lambda_1\sin\lambda_2 \quad \hat{V}_3=\cosh\lambda_1$

$\mathrm{IX}:\quad \hat{V}_1=\sin\lambda_1\cos\lambda_2 \quad \hat{V}_2=\sin\lambda_1 \quad \hat{V}_3=\cos\lambda_1$

One may derive somewhat complicated evolution equations for λ_1 and λ_2 from (15.7). Ryan has done this for type IX in a slightly different context.(14,15) One could also use the variables $\{\lambda_1,\lambda_2\}$ defined for types VI_0 and VII_0 in the type VI_h and VII_h cases, but V has to be determined from ℓ through the constant of the motion ℓV^a.

In class A models ℓ is a constant while it evolves according to (15.9) for class B models. The equation defining ℓ in terms of n, V_a and g_{ab} is an equation implicitly determining n in terms of ℓ, V_a and g_{ab} since $u^\perp=(1+\mu^{-2}g^{ab}V_aV_b)$ involves μ which is a function of n. This in turn allows the determination of functions of n like μ and p.(30) In the dust case, defined by vanishing pressure p, then $\mu=1$ and $n=\rho=\ell g^{-1/2}$ so this is unnecessary; the same is true when V_a remains zero and one has explicitly $n=\ell g^{-1/2}$, enabling one to determine μ and p as well.

Let the fluid configuration space $\mathcal{S}$ be the manifold on which $\{V_a, \ell, n\}$ are global coordinates, the latter two restricted to positive values. Although ℓ really depends on g_{ab}, n and V_a, treating it as an "independent variable" saves us the trouble of having to deal with a fluid component of the nonconservative force Q on $\mathcal{M}$. Define the fluid potential on $\mathcal{M}\times\mathcal{S}$ according to §13 by:

(15.17) $U_S(g_{ab},V_a,\ell,n) = -2kg^{1/2}T^\perp{}_\perp = 2k\ell V^\perp - 2kpg^{1/2}$.

$-\partial U_S/\partial g_{ab} = kg^{1/2}T^{ab}, \quad Q_S=0.$

The defining relation (15.4) for ℓ is then considered as an implicit evolution equation for n.

From (15.9) and the equations of motion follow immediately the relations:

(15.18) $(\partial U_S/\partial V_a)\dot{V}_a=0 \qquad (\partial U_S/\partial\ell)\dot{\ell}=4k\ell a^bV_b.$

To evaluate $\partial U_S/\partial n$ when the equations of motion hold one uses $d/dn\,(\ln\mu) = (\rho+p)^{-1}dp/dn = (\mu n)^{-1}dp/dn$ and $\ell = g^{1/2} n u^{\perp}$:

(15.19) $\partial U_S/\partial n = (2k\ell\mu^2/V^{\perp})\, d/dn(\ln\mu) - 2kg^{1/2}\, dp/dn = 0.$

As far as the geometric equations are concerned, U_S is a potential function on $\mathcal{M}$ in which V_a, ℓ, n are treated as explicit functions of the classical mechanical time t and as such are operated on by $\partial/\partial t$; when the equations of motion hold one has:

(15.20) $\partial U_S/\partial t = \dot{V}_a\, \partial U_S/\partial V_a + \dot{\ell}\, \partial U_S/\partial \ell + \dot{n}\, \partial U_S/\partial n = 2k\ell a^b V_b.$

The time rate of change of the energy $E = \mathcal{T} + U$ of a Langrangian system driven by a nonconservative force equals $\partial U/\partial t$ plus the scalar product of the nonconservative force with the velocity; in our case:

(15.21) $\dot{E} = Q^{ab}\dot{g}_{ab} + \partial U/\partial t = -2\,\mathrm{Tr}\,\underline{Q}\underline{K} + 4k\ell a^b V_b = 2a^b M_{b0}.$

Because of (13.28), this reproduces (13.15) with $\mathbb{M} = 0$.

In a general SH comoving ADM frame, the fluid contributions to the super-Hamiltonian (13.35) and supermomentum (13.38) are:

(15.22) $\bar{H}_S(\bar{g}_{ab}, \bar{V}_a, \bar{\ell}, n) = -2k\bar{g}^{1/2}T^{\perp}{}_{\perp} = U_S(\bar{g}_{ab}, \bar{V}_a, \bar{\ell}, n),$

$\bar{H}_{Sa}(\bar{V}_b, \bar{\ell}) = -2k\bar{g}^{1/2}\bar{T}^{\perp}{}_a = -2k\bar{\ell}\,\bar{V}_a.$

The fluid kinematical quantities may be evaluated using the connection component expressions, the equations of motion and the standard formulas for a timelike congruence.(40) Only the rotation and expansion are worth stating here:

(15.23) $\Theta = u^{\alpha}{}_{;\alpha} = g^{-1/2}(g^{1/2}u^{\perp})^{\cdot} - 2a_b V^b/\mu$

$\omega^{\alpha} = -\tfrac{1}{2}\eta^{\alpha\beta\gamma\delta}u_{\beta}u_{\gamma;\delta} = (2\mu^2 V^{\perp} g^{1/2})^{-1}(\mathcal{V}^2 V^{\alpha} + \mu^2 V_f C^{fa}\delta^{\alpha}_a)$

$\omega = (\omega^{\alpha}\omega_{\alpha})^{1/2} = (\mathcal{V}^4 + \mu^2 V_f V_g C^{fa}C^{gb}g_{ab})^{1/2}/(2\mu g^{1/2}V^{\perp})$

$C^{ab} = \tfrac{1}{2}C^a{}_{fg}\epsilon^{bfg} = n^{ab} + \epsilon^{abc}a_c.$

The sign of ω^{α} differs from (40) since we observe the conventions:

(15.24) $\eta_{\alpha\beta\gamma\delta} = ({}^4g)^{1/2}\epsilon_{\alpha\beta\gamma\delta}, \quad \eta^{\alpha\beta\gamma\delta} = -({}^4g)^{-1/2}\epsilon^{\alpha\beta\gamma\delta},$

$\epsilon_{0123} = 1 = \epsilon^{0123},$

while Ellis uses the opposite sign convention. Notice from (15.23) that if $V_f C^{fa} = 0$ then $\mathcal{V}^2 = |n^{ab}V_aV_b| = |C^{ab}V_aV_b| = 0$, so the rotation vanishes. Thus type I and II models cannot support rotating fluids.

The evolution equations for V_a may be written:

(15.25) $$V^{\perp}\dot{V}_a = \underline{?}\,\epsilon_{afd}V^f(V_g C^{gd}).$$

It is therefore clear that $\dot{V}_a$ vanishes if and only if V_a is an eigenvector of C^{ab}:

(15.26) $$V_a C^{ab} = \lambda V^b.$$

If at any time (15.26) holds then $\dot{V}_a = 0$ and V_a remains constant for all time. For class A models V_a must be an eigenvector of $C^{ab} = n^{ab}$. Since we assume $\underline{n}$ to be diagonal, it is sufficient to choose (28) V_a to have only one nonvanishing component initially, say V_3:

$$0 = \lambda g^{13}V_3 = \lambda g^{23}V_3 \quad , \quad V_3 n^{(3)} = \lambda g^{33}V_3 .$$

If $n^{(3)} \neq 0$ then $\lambda = n^{(3)}/g^{33}$ and $g^{13} = 0 = g^{23}$ which implies $g_{13} = g_{23} = 0$ and $g_{33} = 1/g^{33}$ so $\lambda = g_{33}n^{(3)}$. Since these conditions are maintained by the evolution, the metric matrix will only have one pair of nonvanishing off-diagonal components, a situation called the symmetric case in which $\underline{g}$ is confined to $\mathcal{M}_{S(3)}$. For this symmetric case the rotation scalar (15.23) reduces to:

(15.27) $$\omega = g_{33}{}^{1/2}\,|n^{(3)}V_3|\,/(2\mu g^{1/2}).$$

In type VIII two inequivalent cases of this kind occur, namely when the eigenvector direction is chosen so that the corresponding value $n^{(a)}$ is either positive or negative.

If $n^{(3)} = 0$, then $V_a = V_3\delta^3{}_a$ will be an eigenvector with zero eigenvalue and therefore the fluid rotation vanishes. Similarly in class B models examination of (15.26) shows that V_a must be proportional to $a_a = a\delta^3{}_a$ to be an eigenvector in which case the eigenvalue must also be zero and again the rotation vanishes. In §16 we will see that here too $\underline{g}$ is in general confined to a 4-dimensional submanifold of $\mathcal{M}$.

From (15.25) a short calculation yields the result:

(15.28) $$V^{\perp}\,(n^{ab}V_b)^{\cdot} = n^{ab}\,\epsilon_{bcd}\,V^c(n^{de}V_e) - a_c V^c\,(n^{ab}V_b)\,.$$ gn ?

Thus if $n^{ab}V_b$ vanishes initially, it will remain zero. The only interesting case not already discussed occurs for type IV where $\underline{n} = \mathrm{diag}(n^{(1)},0,0)$ so that V_1 remains zero if it vanishes initially. This also follows from the vanishing of the constant of the motion $\ell^{-1}V^2 = \ell^{-1}(n^{(1)}V_1)^2$. Ellis and King[45] call $n^{ab}V_b$ an "invariant relation quantity" following Estabrook and Wahlquist[24] since it has the property that it remains zero if it is initially so and hence remains nonzero if it is initially nonzero. Other examples are $\epsilon^{abc}a_bV_c$, $V^2 = |n^{ab}V_aV_b|$ and the tilt parameter $\beta = \cosh^{-1}u^{\perp}$ used by Ellis and King. The vanishing of the first of these when $a_b \neq 0$ corresponds to the class B symmetric case.

When $C^a{}_{bc}$ is in the standard diagonal form (9.6), the fluid equations of motion (15.9) are explicitly:

$$\text{(15.29)}\quad V^\perp \dot{\ell} = 2a\ell V^3$$

$$V^\perp \dot{V}_1 = -(aV_1 + n^{(2)}V_2)V^3 + n^{(3)}V_3V^2$$

$$V^\perp \dot{V}_2 = -(aV_2 - n^{(1)}V_1)V^3 - n^{(3)}V_3V^1$$

$$V^\perp \dot{V}_3 = a(V_1V^1 + V_2V^2) + n^{(1)}V_1V^2 - n^{(2)}V_2V^1 \quad .$$

For type VI_h, the noncanonical structure constant tensor components (10.22) are useful. In this case, letting $\bar{V}_a = V_b A^{-1b}{}_a$, one has:

$$\text{(15.30)}\quad V^\perp \dot{\bar{V}}_1 = -(a-q)\bar{V}_1\bar{V}^3$$

$$V^\perp \dot{\bar{V}}_2 = -(a+q)\bar{V}_2\bar{V}^3$$

$$V^\perp \dot{\bar{V}}_3 = a(\bar{V}_1\bar{V}^1 + \bar{V}_2\bar{V}^2) + q(\bar{V}_1\bar{V}^2 + \bar{V}_2\bar{V}^1) \quad .$$

Notice that both $\bar{V}_1$ and $\bar{V}_2$ are invariant relation quantities, while $\bar{V}_1$ is a constant of the motion for Bianchi type $\mathrm{VI}_{-1} = \mathrm{III}$ where $a = q = 1$. If we define m by the equation of motion:

$$\text{(15.31)}\quad V^\perp (\ln m)^{\cdot} = \bar{V}^3 \quad ,$$

then ℓ, $\bar{V}_1$ and $\bar{V}_2$ are expressable in the form:

$$\text{(15.32)}\quad \ell/\ell_0 = m^{2a} \quad , \quad \bar{V}_1/\bar{V}_{10} = m^{-(a-q)} \quad , \quad \bar{V}_2/\bar{V}_{20} = m^{-(a+q)} \, .$$