§11. The Metric Manifold $\mathcal{M}^3$

We now continue the discussion begun in §6 of the metric and unimodular metric submanifolds $\mathcal{M}$ and $\bar{\mathcal{M}}$ of $GL(3,\mathbb{R})$. Suppose G is a 3-dimensional Lie group and $\{e_a\}$ a basis of its Lie algebra $\mathfrak{g}$ with dual basis $\{\omega^a\}$. With each left invariant Riemannian metric $g = g_{ab}\,\omega^a \otimes \omega^b$ on G we may associate a point $\underline{g} = g_{ab}\hat{e}^b{}_a$ of $\mathcal{M}$ and vice versa. In this way we may identify $\mathcal{M}$ with the set of all left invariant Riemannian metrics on G. The latter space is essentially the geometric configuration space associated with the dynamics of spatially homogeneous spacetimes, a fact which has led Misner to give $\mathcal{M}$ the name minisuperspace.[18] This is our motivation for studying $\mathcal{M}$.

Let $C = \{C^a{}_{bc}\} \in \mathbb{R}^{1,2}$ be the set of components of the structure constant tensor of $\mathfrak{g}$ in the basis $\{e_a\}$. We may enlarge the vector spaces $\rho_W^{r,s}(\mathcal{F}(\mathcal{M}))$ described in to spaces $\rho_W^{r,s}(\mathcal{F}(\mathcal{M}),C)$ by letting these classes of $\mathbb{R}^{r,s}$-valued functions on $\mathcal{M}$ depend on C in a "tensorial way" when expressed in terms of the component coordinate system. The usefulness of these spaces hinges upon the fact that if $T \in \rho_W^{r,s}(\mathcal{F}(\mathcal{M}),C)$, it will still satisfy (6.52) for all $\underline{A} \in I_C$, the matrix automorphism group of the Lie algebra $\mathfrak{g}$ with respect to the basis $\{e_a\}$:

(11.1) $$\tilde{f}_{\underline{A}}\,T = \rho_W^{r,s}(\underline{A})\,T, \qquad \underline{A} \in I_C = \mathrm{Aut}_e(\mathfrak{g}).$$

The elements of $\rho_W^{0,0}(\mathcal{F}(\mathcal{M}),C)$ are therefore invariant under the transpose action of the unimodular subgroup $SI_C = S\mathrm{Aut}_e(\mathfrak{g})$.

The components in the frame $\{e_a\}$ of left invariant fields associated with the left invariant metric $g_{ab}\,\omega^a \otimes \omega^b$ on G depend on g_{ab} and $C^a{}_{bc}$ in a tensorial way. By reinterpreting the components g_{ab} as the component functions on $\mathcal{M}$, we obtain elements of the spaces $\rho_W^{r,s}(\mathcal{F}(\mathcal{M}),C)$. The components of the metric connection and of the Riemann tensor and its contractions provide examples of this sort, given explicitly by (8.15), (8.16) and (8.17). In a similar way we may extend the spaces $\rho_W^{r,s}(T^{p,q}(\mathcal{M}))$ discussed after (6.59) to spaces $\rho_W^{r,s}(T^{p,q}(\mathcal{M}),C)$.

Define the gravitational potential function $U^* \in \rho^{0,0}_1(\mathcal{F}(\mathcal{M}), C)$ and the Einstein 1-form $G^* \in \rho^{0,0}_1(\mathcal{X}^*(\mathcal{M}), C)$ by:

$$(11.2)\quad U^* = -g^{\frac{1}{2}} R^* = \tfrac{1}{4}\mathcal{S}^{-1}{}_{abcd}\, n^{ab} n^{cd} + 6 g^{\frac{1}{2}} g^{ab} a_a a_b ,$$
$$G^* = G^{*ab} dg_{ab} .$$

The matrix $\underline{G}^* = G^*(\underline{E})$ of G^* is given by (8.17). An elementary computation shows that the exterior derivative of U^* is given by:

$$(11.3)\quad dU^* = g^{\frac{1}{2}} G^* + Q^* \quad \text{or} \quad -g^{\frac{1}{2}} G^* = -dU^* + Q^* ,$$

where $Q^* = Q^{*ab} dg_{ab} \in \rho^{0,0}_1(\mathcal{X}^*(\mathcal{M}), C)$ is an inexact 1-form whose traceless matrix is:

$$(11.4)\quad \underline{Q}^* = 2 g^{\frac{1}{2}} (a^c \underline{K}_c - 2\underline{a}\,\underline{a}^T) .$$

This is nonzero only for class B structure constant tensors. From the alternative calculation:

$$dU^* = d(-g^{1/2} g^{ab} R^*_{ab}) = g^{1/2} G^* - g^{1/2} g^{ab} dR^*_{ab} ,$$

we may read off another expression for Q^*:

$$Q^* = -g^{1/2} g^{ab} dR^*_{ab} .$$

If we interpret $-g^{1/2} G^*$ as a force field on $\mathcal{M}$, (11.3) shows that it has a conservative part $-dU^*$ and a nonconservative part Q^* which is nonzero only for class B structure constant tensors. Since $0 = \mathrm{Tr}\,\underline{Q}^* = \mathrm{Tr}(\underline{1}\,\underline{Q}^*) = Q^*(\mathrm{Tr}\,\underline{E})$, the nonconservative force Q^* has no component along the vector field $\mathrm{Tr}\,\underline{E}$.

Let $\mathcal{M}_D$ be the submanifold of $\mathcal{M}$ for which $\underline{g}$ is diagonal and let $\underline{g}' = \mathrm{diag}(g_{11}', g_{22}', g_{33}')$ be the restriction of $\underline{g}$ to $\mathcal{M}_D$. $\{g_{11}', g_{22}', g_{33}'\}$ are coordinates on this submanifold (where they assume only positive values) in terms of which the inverse matrix and determinant function are simply:

$$(11.5)\quad (\underline{g}')^{-1} = \mathrm{diag}\left(\frac{1}{g_{11}'}, \frac{1}{g_{22}'}, \frac{1}{g_{33}'}\right) , \qquad g' = \det \underline{g}' = g_{11}' g_{22}' g_{33}' .$$

Notice that $\mathcal{M}_D$ is just the abelian scale group $D(3,\mathbb{R})^+$. The bases $\{\hat{\underline{e}}_a\}$ and $\{\hat{\underline{e}}_A\}$ of (10.10) yield convenient exponential parametrizations $e^{\underline{\beta}}$ of the matrix-valued function on this group, where $\underline{\beta} = \beta^a \hat{\underline{e}}_a = \beta^A \hat{\underline{e}}_A$. It is useful to note explicitly the relations between the two scale parametrizations and the variables $\beta^{ab} = \beta^a - \beta^b$ for $a \neq b$:

(11.6) $\beta^1 = \beta^0 + \beta^+ + \sqrt{3}\,\beta^-$, $\beta^2 = \beta^0 + \beta^+ - \sqrt{3}\,\beta^-$, $\beta^3 = \beta^0 - 2\beta^+$,
$\beta^{23} = 3\beta^+ - \sqrt{3}\,\beta^-$, $\beta^{31} = -3\beta^+ - \sqrt{3}\,\beta^-$, $\beta^{12} = 2\sqrt{3}\,\beta^-$.

The zeros in the $\beta^+\beta^-$ plane of the latter three functions are the lines containing the rays with angular coordinates $\frac{4\pi}{3}$, $\frac{2\pi}{3}$ and 0 radians respectively.

The transpose action of $D(3,\mathbb{R})^+$ on the point $\underline{1} \in \mathcal{M}_D$ induces a parametrization of $\underline{g}'$ differing by a factor of two from the previous parametrization and which we have already used in (10.18):

(11.7) $\underline{g}'(\underline{\beta}) = \tilde{f}_{e^{\underline{\beta}}}(\underline{1}) = (e^{\underline{\beta}})^T \underline{1}\, e^{\underline{\beta}} = e^{2\underline{\beta}}$

$(g')^{1/2} = \exp \operatorname{Tr} \underline{\beta} = e^{3\beta^0}$.

The first of these relations may be viewed as a coordinate transformation to new coordinates $\{\beta^a\}$ or $\{\beta^A\}$ on $\mathcal{M}_D$. (If $\underline{g}'$ is interpreted as the component matrix of a left-invariant metric with respect to a basis $\{e_a\}$ of $\mathfrak{g}$, then $e^{-\underline{\beta}}$ is the scale matrix which normalizes this orthogonal basis.) $\mathcal{M}_D$ has three 2-dimensional "Taub submanifolds" $\mathcal{M}_{T(a)}$ characterized by $\beta^{bc} = 0$ or equivalently $g'_{bb} = g'_{cc}$, where (a,b,c) is a cyclic permutation of $(1,2,3)$. These submanifolds intersect at the "isotropy submanifold" $\mathcal{M}_I = \{r\underline{1} \mid r \in \mathbb{R}^+\}$, the submanifold of matrices which are positive multiples of the identity matrix. $\mathcal{M}_{T(3)}$ is the submanifold for which β^- vanishes. The Taub submanifold $\mathcal{M}_{T(a)}$, with (a,b,c) as above, is left fixed by the transpose action of the 1-dimensional orthogonal subgroup generated by $\underline{k}_a^{IX} = \hat{\underline{e}}^b{}_c - \hat{\underline{e}}^c{}_b$ since if $\underline{y} \in \mathcal{M}_{T(a)}$ then:

$$(\exp \theta \underline{k}_a^{IX})^T \underline{y}\ \exp \theta \underline{k}_a^{IX} = (\exp - \theta \underline{k}_a^{IX}) \exp \theta \underline{k}_a^{IX}\ \underline{y} = \underline{y} .$$

The intersection of the three Taub submanifolds is left fixed by the entire orthogonal group.

Let $\underline{A}^{\#\prime} = \frac{1}{2}(\underline{A} + \underline{g}^{-1\prime}\underline{A}^T\underline{g}') = \frac{1}{2}\tilde{\underline{\xi}}'(\underline{A})$ denote the restriction to $\mathcal{M}_D$ of the symmetrization (6.58) of $\underline{A} \in gl(3,\mathbb{R})$ with respect to $\underline{g}$; it is one-half the matrix $\tilde{\underline{\xi}}(\underline{A}) = \underline{W}(\tilde{\xi}(\underline{A}))$ of the transpose generating vector field $\tilde{\xi}(\underline{A}) = 2\operatorname{Tr} \underline{A}\underline{E} = 2A^{\#}$ at $\mathcal{M}_D$. Inner products between such vector fields at $\mathcal{M}_D$ are given by the following formula which holds only at $\mathcal{M}_D$:

(11.8) $$\mathscr{G}(A,B) = g'^{1/2} \langle \underline{A}^{\#\prime}, \underline{B}^{\#\prime} \rangle_{DW} .$$

Since diagonal matrices commute and are trivially symmetric,

$\underset{\sim}{A}^{\#'}$ coincides with $\underset{\sim}{A}$ if $\underset{\sim}{A}$ is diagonal, and since $\mathcal{M}_D$ is an orbit of the transpose action of $D(3,\mathbb{R})^+$, such an $\underset{\sim}{A}$ is the matrix on $\mathcal{M}_D$ of a vector field A tangent to $\mathcal{M}_D$. The scale generators $\{\hat{\underset{\sim}{e}}_A\}$ are therefore the matrices of orthogonal vector fields $\{\hat{e}_A\}$ on $\mathcal{M}_D$ since on that submanifold (using (10.10)):

(11.9) $\mathcal{G}(\hat{e}_A,\hat{e}_B) = \mathcal{G}'_{AB} = g'^{1/2} \langle \hat{\underset{\sim}{e}}_A, \hat{\underset{\sim}{e}}_B \rangle_{DW} = 6 e^{3\beta^0} \eta_{AB}$.

In fact since the transpose action of $D(3,\mathbb{R})^+$ on $\mathcal{M}_D$ amounts to translation in the coordinates $\{\beta^A\}$, $2\hat{e}_A = \tilde{\xi}(\hat{\underset{\sim}{e}}_A) = \partial/\partial\beta^A$ holds there and hence these coordinates are orthogonal. The induced metric $\mathcal{G}'$ on $\mathcal{M}_D$ is then:

$$\tfrac{1}{4}\mathcal{G}' = \mathcal{G}'_{AB}\, d\beta^A \otimes d\beta^B .$$

The map $\underset{\sim}{A} \mapsto \underset{\sim}{A}^{\#'} = \tfrac{1}{2}\underset{\sim}{\tilde{\xi}}'(\underset{\sim}{A})$ is nontrivial for off-diagonal matrices:

$$\underset{\sim}{\tilde{\xi}}'(\hat{\underset{\sim}{e}}^a{}_b) = \hat{\underset{\sim}{e}}^a{}_b + e^{\beta^{ba}} \hat{\underset{\sim}{e}}^b{}_a = e^{\beta^{ba}} \underset{\sim}{\tilde{\xi}}'(\hat{\underset{\sim}{e}}^b{}_a) .$$

Since $E^a{}_b = g'_{bc}\partial^{ac} = g'_{bb}\partial^{ab}$ at $\mathcal{M}_D$, the vector fields whose matrices are $\underset{\sim}{\tilde{\xi}}'(\hat{\underset{\sim}{e}}^a{}_b)$ and $\underset{\sim}{\tilde{\xi}}'(\hat{\underset{\sim}{e}}^b{}_a)$ at $\mathcal{M}_D$, namely $2E^a{}_b$ and $2E^b{}_a$, are both proportional to ∂^{ab} there. This suggests the decomposition of the vector space *offdiag*$(3,\mathbb{R})$ of off-diagonal matrices into three subspaces corresponding to the three possible pairs of unequal values $\{2,3\}$, $\{3,1\}$ and $\{1,2\}$ that the indices a and b of $\hat{\underset{\sim}{e}}^a{}_b$ may take. These three subspaces are orthogonal to each other and to the diagonal matrices with respect to $\langle\,,\rangle_{DW}$ and induce vector fields on $\mathcal{M}$ which at $\mathcal{M}_D$ are proportional to ∂^{23}, ∂^{31} and ∂^{12} respectively.

(10.18) displays the symmetrizations $\underset{\sim}{K}_a' = \underset{\sim}{K}_a^{\#'}$ of the adjoint matrix Lie algebra generators (10.1) of each Bianchi type. The inner products:

(11.10) $\mathcal{G}'_{ab} = e^{3\beta^0}\bar{\mathcal{G}}'_{ab}$, $\bar{\mathcal{G}}'_{ab} = \bar{\mathcal{G}}(K_a,K_b) = \langle \underset{\sim}{K}_a', \underset{\sim}{K}_b' \rangle_{DW}$

of the corresponding vector fields $K_a = K_a^{\#}$ at $\mathcal{M}_D$ are easily evaluated:

(11.11) $\bar{\mathcal{G}}'_{aa} = \tfrac{1}{2}(n^{(b)}e^{\beta^{bc}} - n^{(c)}e^{-\beta^{bc}})^2 + \tfrac{1}{2}a^2(e^{2\beta^{a3}} - 5\delta^3_a)$,

$\bar{\mathcal{G}}'_{12} = -\tfrac{a}{2}(n^{(1)}e^{2\beta^{13}} - n^{(2)}e^{2\beta^{23}})$, $\bar{\mathcal{G}}'_{23} = \bar{\mathcal{G}}'_{31} = 0$.

In the first formula, (a,b,c) is a cyclic permutation of $(1,2,3)$; the first term in this formula for the three cases $\{n^{(b)}, n^{(c)}\} = \{1,0\}, \{1,-1\}$ and $\{1,1\}$ may be respectively written:

$$(11.12) \qquad \tfrac{1}{2} e^{2\beta^{bc}} \;, \quad 2\cosh^2 \beta^{bc} \;, \quad 2\sinh^2 \beta^{bc} .$$

The inner products $\bar{\mathcal{S}}'_{aA} = \langle \underline{K}_a', \hat{\underline{e}}_A \rangle_{DW}$ are also easy to evaluate since $\mathrm{Tr}\, \underline{K}_a' = \mathrm{Tr}\, \underline{K}_a' \hat{\underline{e}}_0 = \mathrm{Tr}\, \underline{K}_a' \hat{\underline{e}}_+ = 2a_a$ and $\mathrm{Tr}\, \underline{K}_a' \hat{\underline{e}}_- = 0$:

$$\bar{\mathcal{S}}'_{a0} = -4a_a \;, \quad \bar{\mathcal{S}}'_{a+} = 2a_a \;, \quad \bar{\mathcal{S}}'_{a-} = 0 .$$

For class A Lie algebras, $\bar{\mathcal{S}}_{ab}$ is diagonal and $\bar{\mathcal{S}}_{aA} = 0$ so the transpose generators $\tilde{\xi}(\underline{k}_a) = 2K_a$ are orthogonal to each other and to $\mathcal{M}_D$ at $\mathcal{M}_D$. For class B Lie algebras, $\underline{k}_3 = \overset{\circ}{\underline{k}}_3 + \frac{a}{3}(2\hat{\underline{e}}_0 + \hat{\underline{e}}_+)$ where $\overset{\circ}{\underline{k}}_3$ is off-diagonal so K_3 loses its orthogonality to $\mathcal{M}_D$ and even becomes tangent to $\mathcal{M}_D$ for type V where $\overset{\circ}{\underline{k}}_3$ vanishes. K_1 and K_2 remain orthogonal to $\mathcal{M}_D$ but not to each other.

In the following three paragraphs let (a,b,c) be one of the three cyclic permutations of $(1,2,3)$. The 4-dimensional "symmetric case" submanifold $\mathcal{M}_{S(a)}$ consists of those points of $\mathcal{M}$ for which g_{ab} and g_{ac} vanish leaving g_{bc} as the only nonvanishing off-diagonal component of $\underline{g}$. The term "symmetric case" comes from spatially homogeneous cosmology in connection with a reflection symmetry. If the component matrix of a left invariant metric on a 1-dimensional Lie group G with respect to a basis $\{e_d\}$ of $\mathfrak{g}$ lies in $\mathcal{M}_{S(a)}$, then $\{e_a, -e_b, -e_c\}$ have the same inner products as $\{e_a, e_b, e_c\}$. When the components of the SCT in this basis are in diagonal form, this reflection of $\mathfrak{g}$ leaves $\underline{n}$ invariant, but only the reflection associated with $\mathcal{M}_{S(3)}$ leaves $a_d = a\,\delta^3_d$ invariant (if it is nonzero). When both are invariant, the reflection is an automorphism of $\mathfrak{g}$ and hence by exponentiation, one obtains an automorphism of the group G which is a discrete isometry of the metric.(29)

Let $\underline{g}'' = \mathrm{diag}(g''_{11}, g''_{22}, g''_{33}) + g''_{bc}(\hat{\underline{e}}^b{}_c + \hat{\underline{e}}^c{}_b)$ be the restriction of $\underline{g}$ to $\mathcal{M}_{S(a)}$. $\{g''_{11}, g''_{22}, g''_{33}, g''_{bc}\}$ are coordinates on this submanifold. Suppose $\underline{k}$ is any nonzero linear combination of $\hat{\underline{e}}^b{}_c$ and $\hat{\underline{e}}^c{}_b$; $\exp \underline{k}$ may be evaluated by comparison with

(10.1) and (10.2). If we consider the action on $\mathcal{M}$ of the 1-dimensional subgroup generated by $\underline{k}$, it is easy to see explicitly that any point of $\mathcal{M}_{S(a)}$ may be connected to $\mathcal{M}_D$ by an orbit, suggesting the following parametrization of $\underline{g}''$:

(11.13) $\underline{g}''(\underline{\beta},\theta) = (\exp\theta\underline{k})^T e^{2\underline{\beta}} \exp\theta\underline{k} = \tilde{f}_{e^{\underline{\beta}}\exp\theta\underline{k}}(\underline{1})$.

This extends the functions $\underline{g}' = e^{2\underline{\beta}}$ and $\underline{\beta}$ from $\mathcal{M}_D$ to $\mathcal{M}_{S(a)}$ and represents a coordinate transformation to new coordinates $\{\underline{\beta},\theta\}$ on $\mathcal{M}_{S(a)}$.

(Note that $\det\underline{g}'' = \det\underline{g}$ since $\underline{k}$ is traceless.)

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The nonzero class A adjoint generators $\overset{0}{\underline{k}}_a = n^{(c)}\hat{\underline{e}}^b{}_c - n^{(b)}\hat{\underline{e}}^c{}_b$ may be used to construct such a coordinate system. When $n^{(b)} = n^{(c)} = 1$, $\overset{0}{\underline{k}}_a$ generates a rotation and $e^{\theta\overset{0}{\underline{k}}_a}$ is an orthogonal matrix with the property that:

$$(e^{\theta\overset{0}{\underline{k}}_a})^T\,\underline{y}\;e^{\theta\overset{0}{\underline{k}}_a} = \underline{y}\;,\quad \underline{y}\in\mathcal{M}_{T(a)}\subset\mathcal{M}_{S(a)}\,,$$

θ is therefore undetermined at $\mathcal{M}_{T(a)}$ and the resulting coordinates which we will call spherical coordinates are singular there. Evaluating (11.13) explicitly for this case and using double angle trigonometric identities, one finds the result:

(11.14)
$$g_{11}'' = \tfrac{1}{2}(g_{11}' + g_{22}') - \tfrac{1}{2}(g_{22}' - g_{11}')\cos 2\theta$$
$$g_{22}'' = \tfrac{1}{2}(g_{11}' + g_{22}') + \tfrac{1}{2}(g_{22}' - g_{11}')\cos 2\theta$$
$$g_{12}'' = \tfrac{1}{2}(g_{22}' - g_{11}')\sin 2\theta \qquad g_{33}'' = g_{33}'.$$

If we consider this as a map from $\mathcal{M}_D\times S^1$ into $\mathcal{M}_{S(3)}$, where the circle S^1 is the manifold of the 1-dimensional subgroup of $SO(3,\mathbb{R})$ generated by $\underline{k}$, on which θ assumes the values $[0,2\pi)$, then for fixed $\underline{g}'$, S^1 is wrapped around $\mathcal{M}_{S(3)}$ twice. Restricting θ to the interval $[0,\pi)$, the map becomes one-to-one. What is happening is that $\exp\pi\underline{k} = \hat{\underline{e}}^a{}_a - \hat{\underline{e}}^b{}_b - \hat{\underline{e}}^c{}_c$ is diagonal and so commutes with $\underline{g}'$ and its square is $\underline{1}$, so $\tilde{f}_{\exp\pi\underline{k}}(\underline{g}') = \underline{g}'$. Thus the discrete reflection subgroup $\{\underline{1},\exp\pi\underline{k}\}$ acts as the identity on $\mathcal{M}_D$ so for fixed $\underline{g}'$, all elements of a given right coset of this discrete subgroup within the 1-dimensional subgroup are mapped onto the same point in $\mathcal{M}_{S(a)}$.

A similar but more complicated situation occurs when one attempts to extend these "spherical coordinates" to all of $\mathcal{M}$ by the following parametrization of $\underset{\sim}{g}$, motivated by the fact that any symmetric matrix may be diagonalized by an element of the special orthogonal group:

(11.15) $$\underset{\sim}{g}(\underset{\sim}{\beta},\underset{\sim}{S}) = \underset{\sim}{S}^T e^{2\underset{\sim}{\beta}} \underset{\sim}{S} = \tilde{f}_{\underset{\sim}{S}}(e^{2\underset{\sim}{\beta}}) = \tilde{f}_{e^{\underset{\sim}{\beta}}\underset{\sim}{S}}(\underset{\sim}{1}),$$
$$\underset{\sim}{\beta} \in d(3,\mathbb{R}) \ , \ \underset{\sim}{g}' = e^{2\underset{\sim}{\beta}} \in \mathcal{M}_D \ , \ \underset{\sim}{S} \in SO(3,\mathbb{R}) .$$

Unfortunately the eigenvalues $\{g_{11}', g_{22}', g_{33}'\}$ of $\underset{\sim}{g}$ are only determined up to permutations while the eigenvectors of $\underset{\sim}{g}$ (i.e. the matrix $\underset{\sim}{S}$) are only determined up to orientation preserving permutations and reflections if the eigenvalues are distinct, with an additional arbitrary rotation about the third eigenvector if two eigenvalues coincide and an arbitrary rotation if the three eigenvalues coincide.

Let P be the discrete subgroup of SO(3,$\mathbb{R}$) generated by the elements $\{\exp \frac{n\pi}{2} \overset{IX}{\underset{\sim}{k}_a} \mid a = 1,2,3 ; n = -1,0,1\}$ and R its subgroup consisting of the four elements $\{\underset{\sim}{1}, \exp\pi \overset{IX}{\underset{\sim}{k}_1}, \exp\pi \overset{IX}{\underset{\sim}{k}_2}, \exp\pi \overset{IX}{\underset{\sim}{k}_3}\}$, where $\overset{IX}{\underset{\sim}{k}_a} = \underset{\sim}{\hat{e}}^b{}_c - \underset{\sim}{\hat{e}}^c{}_b$ and (a,b,c) is a cyclic permutation of $(1,2,3)$. $\{\overset{IX}{\underset{\sim}{k}_a}\}$ is the canonical basis of the type IX adjoint matrix Lie algebra $so(3,\mathbb{R})$; from §10 one may derive the following expressions (which are special cases of (11.25)):

$$\exp \tfrac{\pi}{2} \overset{IX}{\underset{\sim}{k}_a} = \underset{\sim}{1} + (\overset{IX}{\underset{\sim}{k}_a})^2 + \overset{IX}{\underset{\sim}{k}_a} = \underset{\sim}{\hat{e}}^a{}_a + \underset{\sim}{\hat{e}}^b{}_c - \underset{\sim}{\hat{e}}^c{}_b ,$$
$$\exp \pi \overset{IX}{\underset{\sim}{k}_a} = \underset{\sim}{1} - (\overset{IX}{\underset{\sim}{k}_a})^2 = \underset{\sim}{\hat{e}}^a{}_a - \underset{\sim}{\hat{e}}^b{}_b - \underset{\sim}{\hat{e}}^c{}_c .$$

The elements in P not in R correspond to orientation preserving permutations and those in R to the orientation preserving reflections. P acts naturally on $\mathcal{M}_D \times SO(3,\mathbb{R})$ in the following way:

(11.16) $$(\underset{\sim}{g}', \underset{\sim}{S}) \mapsto (\underset{\sim}{S}_0 \underset{\sim}{g}' \underset{\sim}{S}_0^T, \underset{\sim}{S}_0 \underset{\sim}{S}) \ , \quad \underset{\sim}{S}_0 \in P$$
$$\underset{\sim}{S}^T \underset{\sim}{g}' \underset{\sim}{S} \mapsto \underset{\sim}{S}^T \underset{\sim}{S}_0^T (\underset{\sim}{S}_0 \underset{\sim}{g}' \underset{\sim}{S}_0^T) \underset{\sim}{S}_0 \underset{\sim}{S} = \underset{\sim}{S}^T \underset{\sim}{g}' \underset{\sim}{S} .$$

All elements of the orbit of this action containing the point $(\underset{\sim}{g}', \underset{\sim}{S})$ (a discrete set of points) are mapped onto the single point $\underset{\sim}{S}^T \underset{\sim}{g}' \underset{\sim}{S} \in \mathcal{M}$ by the parametrization (11.15) considered as a map from $\mathcal{M}_D \times SO(3,\mathbb{R})$ onto $\mathcal{M}$. Similarly for fixed $g' \in \mathcal{M}_{T(a)}$, all elements of a given right coset in $SO(3,\mathbb{R})$ of the subgroup generated by the matrix $\underset{\sim}{k}_a^{\mathrm{II}}$ are mapped onto the same point in $\mathcal{M}$ by (11.15). Presumably one could choose an open subset of $\mathcal{M}_D \times SO(3,\mathbb{R})$ which is in a one-to-one correspondence with the orbits of the action of P on this manifold and on which (11.15) is almost everywhere nonsingular (except when g' lies on a Taub submanifold) and therefore represents a transformation from component coordinates to "spherical coordinates" $\{\underset{\sim}{\beta}, \underset{\sim}{S}\}$ on $\mathcal{M}$, the latter of which are singular on the orbits of the Taub submanifolds of $\mathcal{M}$ under the transpose action of $SO(3,\mathbb{R})$. A parametrization of $\underset{\sim}{\beta}$ and $\underset{\sim}{S}$ would then lead to an explicit local coordinate system on $\mathcal{M}$; for example one might take $\{\beta^A, \theta^a\}$, where $\{\theta^a\}$ are exponential coordinates on $SO(3,\mathbb{R})$. However, it is unnecessary to choose such a subset of $\mathcal{M}_D \times SO(3,\mathbb{R})$ since our only interest will be in studying the solution curves of certain classical mechanical systems with $\mathcal{M}$ as the configuration space and the differential equations do not care about the lack of a global one-to-one correspondence between (almost all of) $\mathcal{M}_D \times SO(3,\mathbb{R})$ and $\mathcal{M}$ as long as it exists locally. Although what we will really be doing is using (11.15) to pull back these systems from $\mathcal{M}$ to $\mathcal{M}_D \times SO(3,\mathbb{R})$ where new solution curves will be obtained and then mapped back to $\mathcal{M}$ onto solution curves of the original systems, we will simply act as though we were employing new coordinates on $\mathcal{M}$.

Other 3-dimensional matrix groups $\hat{G} \subset GL(3,\mathbb{R})^+$ may be used in place of $SO(3,\mathbb{R})$ in constructing such a "coordinate system" on $\mathcal{M}$, provided that the orbits of the transpose action of $\hat{G}$ are (almost everywhere) transversal to $\mathcal{M}_D$ and that any point of $\mathcal{M}$ lies on an orbit through $\mathcal{M}_D$. (Two submanifolds N and Q of a manifold M are transversal if at each point $x \in M$ of their intersection, $TM_x = TN_x \oplus TQ_x$. The orbits of $SO(3,\mathbb{R})$ fail to be transversal to the 3-dimensional submanifold $\mathcal{M}_D$ at the Taub submanifolds since the orbit dimension degenerates from three to two there except at the isotropy submanifold $\mathcal{M}_I$ which is fixed under this action.) The corresponding parametrization:

$$(11.16) \qquad \underset{\sim}{g}(\underset{\sim}{\beta},\underset{\sim}{S}) = \underset{\sim}{S}^T e^{2\underset{\sim}{\beta}} \underset{\sim}{S} = \tilde{f}_{\underset{\sim}{S}}(\underset{\sim}{g}'),$$

$$\underset{\sim}{g}' = e^{2\underset{\sim}{\beta}} \in \mathcal{M}_D \,, \quad \underset{\sim}{S} \in G,$$

has similar global problems when $\hat{G}$ has a compact subgroup. An explicit parametrization of $\underset{\sim}{\beta}$ and $\underset{\sim}{S}$ converts this into a "coordinate transformation" (in the above generalized sense) from component coordinates to new coordinates on $\mathcal{M}$, say $\{\beta^A, \theta^a\}$ for example, where $\{\theta^a\}$ are exponential coordinates on $\hat{G}$. $\{\beta^A, \theta^a\}$ are really product coordinates on $\mathcal{M}_D \times \hat{G}$ which is (almost everywhere) in a local one-to-one correspondence with $\mathcal{M}$. Note that:

$$(11.17) \qquad g^{1/2}(\underset{\sim}{\beta},\underset{\sim}{S}) = \det \underset{\sim}{S}\; e^{3\beta^0}$$

so only when $\hat{G}$ is unimodular does β^0 equal $\frac{1}{6} \ln g$.

We will leave the choice of coordinate system on $\hat{G}$ free and use the matrix-valued function $\underset{\sim}{S}$ and the invariant fields $\{e_a, \tilde{e}_a, \omega^a, \tilde{\omega}^a\}$ generated by a basis $\{\underset{\sim}{k}_a\}$ of its matrix Lie algebra. These and the matrix-valued functions $\underset{\sim}{\beta}$ and $\underset{\sim}{g}' = e^{2\underset{\sim}{\beta}}$ on $\mathcal{M}_D$ naturally induce fields (denoted by the same symbols) on the product manifold $\mathcal{M}_D \times \hat{G}$ which we are locally identifying with $\mathcal{M}$ through (11.16). Since the transpose action of $\hat{G}$ on $\mathcal{M}$ corresponds to right translation on the factor $\hat{G}$ of $\mathcal{M}_D \times \hat{G}$, the transpose generators $\tilde{\xi}(\underset{\sim}{k}_a) = 2 \operatorname{Tr} \underset{\sim}{k}_a \underset{\sim}{E}$ correspond to the left invariant vector fields $\{e_a\}$ while $\{\tilde{e}_a, \tilde{\omega}^a\}$ are invariant under that action. The following relations also hold on $\mathcal{M}_D \times \hat{G}$:

(11.18) $\quad d\underline{S}\,\underline{S}^{-1} = \underline{\kappa}_a \tilde{\omega}^a \;, \quad \underline{g}'^{-1} d\underline{g}' = 2d\underline{\beta} = 2d\beta^A \hat{\underline{e}}_A \,.$

Since the collection of component functions $\{g_{ab}\}$ is an element of $\rho^{0,2}_0(\mathcal{F}(\mathcal{M}))$, (11.16) may be written as:

(11.19) $\quad \{g_{ab}\} = \rho^{0,2}_0(\underline{S}^{-1})\,\{g'_{ab}\},$

$\qquad\qquad \{g'_{ab}\} = \rho^{0,2}_0(\underline{S})\,\{g_{ab}\}.$

Similarly by reexpressing the elements $T \in \rho^{r,s}_W(\mathcal{F}(\mathcal{M}))$ in terms of $\underline{g}' = e^{2\underline{\beta}}$ and $\underline{S}$ using (11.16), one will obtain:

(11.20) $\quad T = \rho^{r,s}_W(\underline{S}^{-1})T' \quad , \; T' = \rho^{r,s}_W(\underline{S})T \,,$

where T' is the same function of g'_{ab} as T is of g_{ab}, so this use of the prime notation is consistent with its appearance in (10.21) and earlier in this section.

In these equations we are really interpreting T as its pullback from $\mathcal{M}$ to $\mathcal{M}_D \times \hat{G}$. Similarly differential forms and other covariant tensor fields on $\mathcal{M}$ may be pulled back to $\mathcal{M}_D \times \hat{G}$ simply by reexpressing them in terms of $\underline{\beta}$ and $\underline{S}$ using (11.16). In fact since the differential of the map (11.16) is invertible almost everywhere on $\mathcal{M}_D \times \hat{G}$, vector fields and therefore any tensor fields may be pulled back to $\mathcal{M}_D \times \hat{G}$, although pulled back tensor fields with contravariant valence will have singularities where the differential is singular. We will denote all pulled back fields on $\mathcal{M}_D \times \hat{G}$ by the same symbols used for the corresponding fields on $\mathcal{M}$. In particular the spaces $\rho^{r,s}_W(T^{p,q}(\mathcal{M}))$ have pullbacks, say $\rho^{r,s}_W(T^{p,q}(\mathcal{M}_D \times \hat{G}))$, but we will suppress the distinction between them. The notation (11.19) may then be extended to elements T of these new spaces, but now T' will involve derivatives of $\underline{S}$.

For example, since $W^a{}_b \in \rho^{1,1}_0(\mathcal{X}^*(\mathcal{M}))$, $\underline{W} = \underline{g}^{-1}d\underline{g}$ transforms according to the adjoint representation:

$$\underline{W} = \underline{S}^{-1}\underline{W}'\underline{S} \,.$$

Evaluating this in terms of $\underline{\beta}$ and $\underline{S}$ using (11.16) and the two formulas:

$$\underline{g}^{-1} = \underline{S}^{-1}\underline{g}'^{-1}\underline{S}^{-1T} \quad , \; \underline{g}'^{-1} = e^{-2\underline{\beta}} \,,$$

$$d\underline{g} = \underline{S}^T(\,d\underline{g}' + \underline{g}'(d\underline{S}\,\underline{S}^{-1}) + (d\underline{S}\,\underline{S}^{-1})^T\underline{g}'\,)\underline{S} \;,$$

one may read off the result:

(11.21) $$\underline{W}' = \underline{g}'^{-1} d\underline{g}' + d\underline{S}\underline{S}^{-1} + \underline{g}'^{-1} (d\underline{S}\underline{S}^{-1})^T \underline{g}'$$
$$= 2\,(d\beta^A \hat{\underline{e}}_A + \underline{\kappa}'_a \tilde{\omega}^a),$$

where we have introduced the notation:

(11.22) $$\mathcal{K}'_a = \tfrac{1}{2}\,(\underline{\kappa}_a + \underline{g}'^{-1} \underline{\kappa}_a{}^T \underline{g}'),$$

for the symmetrization of $\underline{\kappa}_a$ with respect to $\underline{g}'$.

Having evaluated $\underline{W}'$, the DeWitt metric $\mathcal{G}$ is easily evaluated (i.e. pulled back to $\mathcal{M}_D \times \hat{G}$) using (11.9), (11.17) and the invariance of the trace under conjugation:

(11.23) $$\begin{aligned}\mathcal{G} &= g^{1/2}\,(\,\mathrm{Tr}\,(\underline{W} \otimes \underline{W}) - \mathrm{Tr}\,\underline{W} \otimes \mathrm{Tr}\,\underline{W}\,) \\ &= \det \underline{S}\; g'^{1/2}\,(\,\mathrm{Tr}\,(\underline{W}' \otimes \underline{W}') - \mathrm{Tr}\,\underline{W}' \otimes \mathrm{Tr}\,\underline{W}'\,) \\ &= 4 \det \underline{S}\; e^{3\beta^0} (\, 6\,\eta_{AB}\, d\beta^A \otimes d\beta^B + \\ &\qquad \bar{\mathcal{G}}'_{Aa}\,(d\beta^A \otimes \tilde{\omega}^a + \tilde{\omega}^a \otimes d\beta^A) + \bar{\mathcal{G}}'_{ab}\, \tilde{\omega}^a \otimes \tilde{\omega}^b)\end{aligned}$$

where:

$$\bar{\mathcal{G}}'_{Aa} = \langle \hat{\underline{e}}_A, \mathcal{K}'_a \rangle_{DW}, \quad \bar{\mathcal{G}}'_{ab} = \langle \mathcal{K}'_a, \mathcal{K}'_b \rangle_{DW}.$$

This takes its simplest form when the group $\hat{G}$ is unimodular ($\det \underline{S} = 1$ or $\mathrm{Tr}\,\underline{\kappa}_a = 0$) and the basis $\{\underline{\kappa}_a\}$ of its matrix Lie algebra is such that the transpose generators $\{\tilde{\xi}(\underline{\kappa}_a)\}$ are orthogonal to $\mathcal{M}_D$ and to themselves at $\mathcal{M}_D$, i.e. $\bar{\mathcal{G}}'_{Aa} = 0$ and $\bar{\mathcal{G}}'_{ab}$ is diagonal. The frame $\{\partial/\partial\beta^A, \tilde{e}_a\}$ with dual frame $\{d\beta^A, \tilde{\omega}^a\}$ is then orthogonal and the expression for $\mathcal{G}^{-1}$ immediate:

(11.24) $$\begin{aligned}\tfrac{1}{4}\mathcal{G} &= e^{3\beta^0} (\,6\,\eta_{AB}\, d\beta^A \otimes d\beta^B + \bar{\mathcal{G}}'_{ab}\, \tilde{\omega}^a \otimes \tilde{\omega}^b) \\ 4\mathcal{G}^{-1} &= e^{-3\beta^0} (\tfrac{1}{6}\,\eta_{AB}\, \partial/\partial\beta^A \otimes \partial/\partial\beta^B + \bar{\mathcal{G}}'^{ab}\, \tilde{e}_a \otimes \tilde{e}_b) \\ \bar{\mathcal{G}}'^{ab} &= \delta_{ab} / \bar{\mathcal{G}}'_{aa}\,.\end{aligned}$$

The condition $0 = \langle \underline{\kappa}_a, \hat{\underline{e}}_A \rangle_{DW}$ obtained by evaluating $\bar{\mathcal{G}}'_{Aa} = 0$ at $\underline{g}' = \underline{1}$ and the condition $\mathrm{Tr}\,\underline{\kappa}_a = 0$ together require that $\underline{\kappa}_a$ be an off-diagonal matrix which guarantees that the first condition is satisfied on all of $\mathcal{M}_D$. The remaining condition is satisfied if each of the three matrices $\{\underline{\kappa}_a\}$ lies in a different one of the three orthogonal subspaces of *offdiag*$(3,\mathbb{R})$ corresponding to the three possible pairs of unequal values $\{2,3\}$, $\{3,1\}$ and $\{1,2\}$ that the indices a and b of $\hat{\underline{e}}^a{}_b$

may take. This also guarantees that the orbits of the transpose action of $\hat{G}$ which intersect $\mathcal{M}_0$ cover $\mathcal{M}$. All of the nondegenerate class A adjoint matrix Lie algebra bases $\{\underset{\sim}{k}{}^{\circ}_a\}$ of (10.1) satisfy these conditions; from that section and in its notation their exponentials are given by:

(11.25) $$\exp \theta \underset{\sim}{k}{}^{\circ}_a = \underset{\sim}{1} - (\underset{\sim}{k}{}^{\circ}_a)^2 + C_a (\underset{\sim}{k}{}^{\circ}_a)^2 + S_a \underset{\sim}{k}{}^{\circ}_a \,.$$

When $n^{(b)}$ and $n^{(c)}$ are nonzero and have the same sign and (a,b,c) is a cyclic permutation of $(1,2,3)$ then C_a and S_a are trigonometric functions so $\underset{\sim}{k}{}_a$ generates a compact subgroup.

We will only consider groups $\hat{G}$ and bases $\{\underset{\sim}{k}_a\}$ satisfying the above conditions and which we will describe as "nice". Now suppose that in addition, $\hat{G} \subset I_c$. The prime notation may be extended to $\rho_W^{r\,s}(\mathfrak{F}(\mathcal{M}), C)$ and then to $\rho_W^{r\,s}(T^{p,q}(\mathcal{M}), C)$ by the formulas (11.20). If $T \in \rho_W^{r\,s}(\mathfrak{F}(\mathcal{M}), C)$, then T' is the same function of g'_{ab} and $C^a{}_{bc}$ as T is of g_{ab} and $C^a{}_{bc}$ since $\hat{G} \subset I_c$. Note that the weight W is irrelevant since $\hat{G}$ is unimodular. In particular if $T \in \rho_W^{0\,0}(\mathfrak{F}(\mathcal{M}), C)$ then $T' = T$, so such functions are independent of $\underset{\sim}{S}$. For example the gravitational potential $U^* = U^{*\prime}$ is easily evaluated on $\mathcal{M}_0$ for C in the diagonal form (9.18) with the result:

(11.26) $$U^* = e^{\beta^0} V + 6a^2 e^{\beta^0 + 4\beta^+}$$

$$V = \tfrac{1}{2} e^{4\beta^+} (n^{(1)} e^{2\sqrt{3}\beta^-} - n^{(2)} e^{-2\sqrt{3}\beta^-})^2$$
$$- n^{(3)} e^{-2\beta^+} (n^{(1)} e^{2\sqrt{3}\beta^-} + n^{(2)} e^{-2\sqrt{3}\beta^-}) + \tfrac{1}{2} (n^{(3)})^2 e^{-8\beta^+} .$$

The terms enclosed by parentheses in V may be rewritten as in (11.12).

Similarly $\underset{\sim}{Q}^{*\prime} = \underset{\sim}{S} \underset{\sim}{Q}^* \underset{\sim}{S}^{-1}$ is easily evaluated since $\underset{\sim}{g}'$ is diagonal:

$$\underset{\sim}{Q}^{*\prime} = 2 g'^{1/2} (a'^3 \underset{\sim}{K}'_3 - 2 \underset{\sim}{g}'^T \underset{\sim}{a}') = 2a g'^{1/2} g'^{33} (\underset{\sim}{K}'_3 - 2a \underset{\sim}{\hat{e}}{}^3{}_3)$$
$$= 2a\, e^{\beta^0 + 4\beta^+} (\underset{\sim}{K}{}^{\circ\prime}_3 + a \underset{\sim}{\hat{e}}_+) ,$$

where $\underset{\sim}{K}{}^{\circ\prime}_a$ is the symmetrization of $\underset{\sim}{k}{}^{\circ}_3 = \underset{\sim}{k}_3 - a \underset{\sim}{I}^{(3)}$ with respect to $\underset{\sim}{g}'$. Then since the trace is invariant under conjugation and $\underset{\sim}{\mathcal{K}}{}'_a$ and $\underset{\sim}{K}{}^{\circ\prime}_3$ are off-diagonal:

Correction for page 11.12: After sentence following EQ (11.26), add:

Note that U^* has the property that its derivative with respect to β^- vanishes at $\beta^-=0$:

$$\partial/\partial\beta^- \big|_{\beta^-=0} \; U^* = 0 .$$

(11.27) $$Q^* = \mathrm{Tr}\,\underline{Q}^*\underline{W} = \mathrm{Tr}\,\underline{Q}^{*\prime}\underline{W}' = 4a\,e^{\beta^0+4\beta^+}\left(6a\,d\beta^+ + \mathrm{Tr}(\underline{\mathcal{K}}'_a\underline{K}^{0\prime}_3)\,\tilde{\omega}^a\right).$$

As another useful example define $M_a \in \rho^{0;1}_{1}(\mathcal{H}^*(\mathcal{M}),C)$ by:

(11.28) $$M_a = -\tfrac{1}{2}g^{1/2}(\mathrm{Tr}\,\underline{k}_a\underline{W} - 2a_b\underline{W}^{b}{}_a).$$

For C in the diagonal form (9.18), $M'_a = M_b S^{-1b}{}_a$ may be evaluated as follows using (1.17) with $\underline{A}$ replaced by $\underline{S}$:

$$M'_a = -\tfrac{1}{2}g'^{1/2}(\mathrm{Tr}\,\underline{k}_a\underline{W}' - 2a\,W'^3{}_a)$$
$$= -e^{3\beta^0}\left((\langle \underline{k}_a,\hat{\underline{e}}_A\rangle - 2a\,\hat{e}_A{}^3{}_a)\,d\beta^A + (\langle \underline{k}_a,\underline{\mathcal{K}}'_b\rangle - 2a\,\mathcal{K}'_b{}^3{}_a)\,\tilde{\omega}^b\right).$$

Since only $\underline{k}_3 = \underline{k}^0_3 + a\underline{I}^{(3)} = \underline{k}^0_3 + \frac{a}{3}(2\hat{\underline{e}}_0 + \hat{\underline{e}}_+)$ has diagonal components ($\underline{k}^0_3$ is off-diagonal), the first term inside the outer parentheses is $6a\,\delta^3_a\,d\beta^+$.

For a given C in the diagonal form (9.18), if the unimodular matrix group $\hat{G}$ is "nice" in the above sense and contained in the automorphism matrix group I_C and therefore in the special automorphism matrix group SI_C, then the resulting "special automorphism coordinates" $\{\underline{\beta},\underline{S}\}$ on $\mathcal{M}$ (actually $\mathcal{M}_0\times\hat{G}$) are conveniently adapted both to the DeWitt geometry and the symmetry properties of the function spaces $\rho^{r;s}_{W}(\mathfrak{F}(\mathcal{M}),C)$. We are only interested in the cases for which C assumes its canonical value for each Bianchi type. For all Bianchi types except I, II and V, the canonical special automorphism group SI_C has dimension three and is "nice" and is therefore the unique candidate for constructing "special automorphism coordinates." For these types SI_C is the adjoint group in the class A case (generated by $\{\underline{k}_a\}$) and the corresponding class A adjoint group in the class B case (generated by $\{\underline{k}^0_a\}$). For the remaining types I, II and V there is some freedom in choosing a "nice" 3-dimensional subgroup of SI_C. For types I and II we may simply replace the vanishing canonical adjoint generators $\{\underline{k}_a\}$ and $\underline{k}_3$ respectively by the corresponding canonical type IX adjoint generators $\{\underline{k}^{IX}_a\}$ and $\underline{k}^{IX}_3$ respectively, while for type V the vanishing generator $\underset{\sim}{k}{}^{\circ}_3$ may be replaced by $\underset{\sim}{k}{}_3^{IX}$. In fact according to the discussion of section ten; a "nice" basis $\{\underset{\sim}{\kappa}_a\}$ of generators of a "nice" special automorphism subgroup for all class B types is:

(11.29) $\quad \underset{\sim}{\kappa}_1 = \hat{\underset{\sim}{e}}{}^3{}_1 \quad , \underset{\sim}{\kappa}_2 = \hat{\underset{\sim}{e}}{}^3{}_2 \quad , \underset{\sim}{\kappa}_3 = \underset{\sim}{k}{}^{\circ}_3 + \delta^{V}_{Z}\, \underset{\sim}{k}{}_3^{IX}$,

where $\delta^{Z'}_{Z}$ is the "Bianchi type Kronecker delta". Similarly the suggested class A choice may be written:

(11.30) $\quad \underset{\sim}{\kappa}_a = \underset{\sim}{k}_a + \delta^{II}_{Z}\delta_a^3\, \underset{\sim}{k}{}_3^{IX} + \delta^{I}_{Z}\, \underset{\sim}{k}{}_a^{IX}$.

Let $\hat{C}^a{}_{bc}$ be the components of the SCT of the Lie algebra of $\hat{G}$ with respect to the basis $\{\underset{\sim}{\kappa}_a\}$. Then $\hat{C}^a{}_{bc} = C^{\circ a}{}_{bc} \equiv \epsilon_{bcd}\, n^{ad}$ holds for all Bianchi types except I, II and V where $\hat{C}^a{}_{bc}$ equals the canonical values $C^a{}_{bc}(IX)$, $C^a{}_{bc}(VII_0)$ and $C^a{}_{bc}(VII_0)$ respectively.

The inner products $\bar{g}'_{ab}$ and the matrices $\mathcal{K}_a'$ for these choices of the basis $\{\underset{\sim}{\kappa}_a\}$ may be read off from (11.12) and (10.18) when the appropriate correspondences are made. The latter formulas hold for the nondegenerate class A types while they must be supplemented by type IX formulas for types I and II. For the class B case they are explicitly:

$$(11.31)\quad \begin{aligned} \mathcal{K}_1' &= \tfrac{1}{2}(\hat{\underset{\sim}{e}}{}^3{}_1 + e^{2\beta^{13}}\hat{\underset{\sim}{e}}{}^1{}_3) & \bar{g}'_{11} &= \tfrac{1}{2} e^{2\beta^{13}} \\ \mathcal{K}_2' &= \tfrac{1}{2}(\hat{\underset{\sim}{e}}{}^2{}_1 + e^{2\beta^{23}}\hat{\underset{\sim}{e}}{}^2{}_3) & \bar{g}'_{22} &= \tfrac{1}{2} e^{2\beta^{23}} \\ \mathcal{K}_3' &= K_3^{\circ\prime} + \delta^{V}_{Z}\, K_3^{IX\prime} & \bar{g}_{33} &= \bar{g}^{\circ\prime}_{33} + \delta^{V}_{Z}\,\bar{g}^{IX\prime}_{33}, \end{aligned}$$

where $\bar{g}^{\circ\prime}_{33} = \langle \underset{\sim}{K}{}_3^{\circ\prime}, \underset{\sim}{K}{}_3^{\circ\prime}\rangle$ and $\bar{g}^{IX\prime}_{33} = \langle \underset{\sim}{K}{}_3^{IX\prime}, \underset{\sim}{K}{}_3^{IX\prime}\rangle$. The relation between $\{\underset{\sim}{k}_a\}$ and $\{\underset{\sim}{\kappa}_a\}$ for these types is:

(11.32) $\quad \underset{\sim}{k}_1 = -a\underset{\sim}{\kappa}_1 - n^{(2)}\underset{\sim}{\kappa}_2 \,, \quad \underset{\sim}{k}_2 = n^{(1)}\underset{\sim}{\kappa}_1 - a\,\underset{\sim}{\kappa}_2 \,, \quad \underset{\sim}{k}_3 = \underset{\sim}{\kappa}_3 + a\underset{\sim}{I}^{(3)}$,

and the inner product appearing in Q^* in (11.27) is:

(11.33) $\quad \mathrm{Tr}\,(\underset{\sim}{K}{}_3^{\circ\prime}\mathcal{K}_a') = \delta_a^3(1-\delta^{V}_{Z}) = \delta_a^3\,\bar{g}^{\circ\prime}_{33}$.

To complete the evaluation of M_a' it is convenient to consider the class A and class B cases separately. In the first case:

(11.34) $\quad M_a' = -\tfrac{1}{2} g'^{\,1/2}\, \mathrm{Tr}\, \underset{\sim}{k}_a \underset{\sim}{W}' = -e^{3\beta^{\circ}}\bar{g}'_{ab}\,\tilde{\omega}^b\,(1-\delta^{I}_{Z}-\delta^{II}_{Z}\delta_a^3)$.

$\{M_a'\}$ and M_3' vanish for types I and II respectively. In the second case using (11.31) and noting that $\mathcal{K}_b'{}^3{}_a$ is nonzero only when neither a or b equals 3, one finds after a little algebra:

$$M_1' = e^{3\beta^0}\left(3a\,\bar{g}'_{11}\,\tilde{\omega}^1 + n^{(2)}\bar{g}'_{22}\,\tilde{\omega}^2\right) \qquad (11.35)$$
$$M_2' = e^{3\beta^0}\left(-n^{(1)}\bar{g}'_{11}\,\tilde{\omega}^1 + 3a\,\bar{g}'_{22}\,\tilde{\omega}^2\right)$$
$$M_3' = e^{3\beta^0}\left(-6a\,d\beta^+ - \bar{g}^{0\prime}_{33}\,\tilde{\omega}^3\right).$$

Comparison of the last line with (11.27) and (11.33) shows that Q^* and M_3' are proportional. According to (9.6), $9a^2 + n^{(1)}n^{(2)} = n^{(1)}n^{(2)}(9h+1)$ when $n^{(1)}n^{(2)} \neq 0$. This can vanish in the class B case only when $h = -1/9$ which occurs at the canonical type $\mathrm{VI}_{-1/9}$ and hence for that type only, M_1' and M_2' are linearly dependent.

Ryan has evaluated Q^* in terms of spherical coordinates for all class B types, although not at our canonical values of C.[35] The results are unbelievably complicated since these coordinates bear no relation to the symmetry properties of Q^*. For the same reason "Hamiltonian cosmology" has been limited to "diagonal metrics" since the expression for the gravitational potential U^* is correspondingly monstrous. The only exception is type IX where spherical and special automorphism coordinates coincide, allowing Ryan to study the general case.[32]

We now want to consider $\mathcal{M}$ as the configuration space of a classical mechanical system and apply some of the results of §7, abandoning the notation $\{q_{ab}, \dot{q}_{ab}, p^{ab}\}$ for the coordinates induced on $T\mathcal{M}$ and $T^*\mathcal{M}$ by the component coordinates on $\mathcal{M}$ in favor of the more familiar symbols $\{g_{ab}, \dot{g}_{ab}, \pi^{ab}\}$. We extend our index raising and lowering conventions and matrix conventions to $T\mathcal{M}$ and $T^*\mathcal{M}$ so for example, $\pi^a{}_b = \pi^{ac}g_{cb}$ and $\underline{\pi} = \pi^a{}_b\,\underline{\hat{e}}^b{}_a$. Poisson brackets of functions on $T^*\mathcal{M}$ are defined in terms of the induced component coordinates as one would expect:

$$\{B, D\} = \partial B/\partial g_{ab}\,\partial D/\partial \pi^{ab} - \partial D/\partial g_{ab}\,\partial B/\partial \pi^{ab}.$$

The canonical generating map or moment $\tilde{n} = f \circ \tilde{\xi}$ of the lifted transpose action of $GL(3,\mathbb{R})$ on $T^*\mathcal{M}$ is easily evaluated since $\tilde{\xi}(\underline{A}) = 2A_{ab}\,\partial^{ab}$:

(11.36) $$\tilde{\pi}(\underline{A}) = f(\tilde{\xi}(\underline{A})) = 2A_{ab}\Pi^{ab} = 2\,\mathrm{Tr}\,\underline{A}\,\underline{\Pi},$$
$$\{\tilde{\pi}(\underline{A}), \tilde{\pi}(\underline{B})\} = -\tilde{\pi}([\underline{A},\underline{B}]).$$

The additional minus sign in the Poisson bracket relation relative to (7.17) occurs since the transpose action is a right action. According to that section the lifted transpose action on T is given in terms of the induced component coordinates by:
$$g_{ab}\circ \bar{f}^{*}_{\underline{A}} = A^{c}{}_{a}\, g_{cd}\, A^{d}{}_{b}\,, \quad \Pi^{ab}\circ \bar{f}^{*}_{\underline{A}} = A^{-1a}{}_{c}\,\Pi^{cd} A^{-1b}{}_{d}\,,$$
where we have suppressed a tilde in the symbol $\bar{f}^{*}_{A}$ for typographical reasons. The corresponding dragging action on $\mathfrak{F}(T^{*}\mathcal{M})$ is then:

(11.37) $$\bar{f}^{*}_{\exp t\underline{A}}\, T = (\exp t\,\mathrm{ad}(\tilde{\pi}(A)))T\,, \quad T\in \mathfrak{F}(T^{*}\mathcal{M}).$$

Suppose we now define spaces $\rho^{r,s}_{W}(\mathfrak{F}(T^{*}\mathcal{M}))$ and $\rho^{r,s}_{W}(\mathfrak{F}(T^{*}\mathcal{M}), C)$ whose elements T satisfy:

(11.38) $$\bar{f}^{*}_{\underline{A}}\, T = \rho^{r,s}_{W}(A)\,T,$$ provided that $\underline{A}\in I_C$ if T lies in one of the latter spaces.

These classes of $\mathbb{R}^{n^{s}}$-valued functions depend on g_{ab}, g^{ab}, Π^{ab}, powers of g and in the latter case on $C^{a}{}_{bc}$ "in a tensorial way" as before. (Note that $\Pi^{ab}\in \rho^{2,0}_{0}(\mathfrak{F}(T^{*}\mathcal{M}))$ has zero weight.) From (11.37), (11.38) and (4.7) we then find that for $T\in \rho^{r,s}_{W}(\mathfrak{F}(T^{*}\mathcal{M}))$:

(11.39) $$\mathrm{ad}(\tilde{\pi}(\underline{A}))T = d/dt|_{0}\, \bar{f}^{*}_{\exp t\underline{A}}\, T = \rho'^{r,s}_{W}(\underline{A})T = \{\tilde{\pi}(A), T\},$$
while this holds only for $\underline{A}$ in the matrix Lie algebra of I_C if $T\in \rho^{r,s}_{W}(\mathfrak{F}(T^{*}\mathcal{M}), C)$. The last equality enables us to compute many interesting Poisson brackets by inspection.

An important classical mechanical system with $\mathcal{M}$ as its configuration space has as its solution curves the geodesics of the DeWitt metric. According to section seven,

(11.40) $$L_0 = \tfrac{1}{4} f(G) = \tfrac{1}{4}\mathcal{G}^{abcd}\,\dot{g}_{ab}\,\dot{g}_{cd} = g^{1/2}\langle \underline{K}, \underline{K}\rangle_{DW}$$

may be taken as a Lagrangian for this classical mechanical system, where $\underline{K} = -\tfrac{1}{2} f(\underline{W}) = -\tfrac{1}{2}\underline{g}^{-1}\dot{\underline{g}}$ is the "extrinsic curvature matrix" on $T\mathcal{M}$. The correspondence between $T\mathcal{M}$ and $T^{*}\mathcal{M}$ determined by this Lagrangian may be stated in terms of the induced component coordinates as follows:

(11.41) $\quad \Pi^{ab} = \partial L_o / \partial \dot{g}_{ab} = \frac{1}{2} \mathcal{G}^{abcd} \dot{g}_{cd} \quad , \quad \dot{g}_{ab} = 2 \mathcal{G}^{-1}{}_{abcd} \Pi^{cd}$,

and reflects index raising and lowering associated with the DeWitt metric. Comparison with (6.65) leads to the equivalent relations:

(11.42) $\quad \underline{\Pi} = -g^{1/2}(\underline{K} - \underline{1}\,\mathrm{Tr}\,\underline{K}) \quad , \quad \underline{K} = -g^{-1/2}(\underline{\Pi} - \frac{1}{2}\underline{1}\,\mathrm{Tr}\,\underline{\Pi})$.

This correspondence between $T\mathcal{M}$ and $T^*\mathcal{M}$ extends to a correspondence between $\mathfrak{F}(T\mathcal{M})$ and $\mathfrak{F}(T^*\mathcal{M})$ by reexpressing functions of $\{g_{ab}, \dot{g}_{ab}\}$ in terms of $\{g_{ab}, \Pi^{ab}\}$ and vice versa using (11.41). If $B \in \mathfrak{F}(T\mathcal{M})$ and $D \in \mathfrak{F}(T^*\mathcal{M})$ let $^*B \in \mathfrak{F}(T^*\mathcal{M})$ and $^*D \in \mathfrak{F}(T\mathcal{M})$ be the functions so obtained. (Note that $^{**}B = B$.) To be consistent we really should have used asterisks in (11.41) and (11.42) since the equalities relate functions on two different manifolds.) The Hamiltonian associated with the Lagrangian L_o is obtained in this way:

(11.43)
$$H_o = {}^*L_o = \mathcal{G}^{-1}{}_{abcd} \Pi^{ab} \Pi^{cd} = {}^*f(\mathcal{G}^{-1})$$
$$= g^{-1/2}(\mathrm{Tr}\,\underline{\Pi}^2 - \tfrac{1}{2}\mathrm{Tr}^2\,\underline{\Pi}) .$$

H_o and L_o are the kinetic energy functions, $\dot{g}_{ab}$ the velocity and Π^{ab} the mechanical momentum for the system. Similarly:

(11.44)
$$f(M_a) = g^{1/2}(\mathrm{Tr}\,\underline{R_a}\,\underline{K} - 2a_b K^b{}_a)$$
$$^*f(M_a) = -(\mathrm{Tr}\,\underline{R_a}\underline{\Pi} - 2a_b \Pi^b{}_a)$$
$$^*\eta(\underline{A}) = 2\mathcal{G}^{abcd} A_{ab}\,\dot{g}_{cd} = \mathcal{G}^{abcd} (\tilde{\xi}(A))_{ab}\,\dot{g}_{cd} .$$

The classical mechanical time t is an affine parameter for the geodesics obtained as solutions of the Lagrange or Hamiltonian equations (beware of identifications):

$$-\delta L_o/\delta g_{ab} = (\partial L/\partial \dot{g}_{ab})^{\cdot} - \partial L/\partial g_{ab} = 0$$
$$\dot{g}_{ab} = \{g_{ab}, H_o\} \qquad \dot{\Pi}^{ab} = \{\Pi^{ab}, H_o\} .$$

The time rate of change along a solution curve of a possibly time dependent function B on $T^*\mathcal{M}$ is given by:

$$\dot{B} = \{B, H_o\} + \partial B/\partial t .$$

Since $\partial H_o/\partial t = 0$, the kinetic energy is conserved. By choosing the constant of energy E_o to be $\frac{1}{4}$ and $-\frac{1}{4}$ in the spacelike and timelike case respectively one makes t the arclength, since $4E_o$ represents the inner product of the tangent vector to a geodesic with itself. Since $H_0 \in \mathcal{P}^{0,0}_{-1}(\mathcal{D}(T^*\mathcal{M}))$, $\{\tilde{\pi}(\underline{A}), H_0\} = -(\mathrm{Tr}\,\underline{A})H_0$, so the canonical generators of the action of $SL(3,\mathbb{R})$ commute with H_0 and are therefore conserved. Since $*\pi(\underline{A}) = \mathcal{G}^{abcd}(\tilde{\xi}(\underline{A}))_{ab}\,\dot{g}_{cd}$ is the function on $T\mathcal{M}$ representing the inner product of the vector field $\tilde{\xi}(\underline{A})$ with the tangent to a curve, the conservation of $\tilde{\pi}(\underline{A})$ for $\underline{A} \in sl(3,\mathbb{R})$ just reflects the fact that the inner product of a Killing vector field with the tangent to a geodesic is constant.

Suppose we introduce the Einstein force field $-g^{1/2}G^* = -dU^* + Q^*$ of (11.3) to deflect the geodesics of $(\mathcal{M},\mathcal{G})$. We therefore introduce the gravitational potential U^* into the Lagrangian and Hamiltonian and the nonconservative force Q^* into the equations of motion:

(11.45) $$L = L_0 - U^*\,, \qquad -\delta L/\delta g_{ab} = Q^{*ab}$$
$$H = H_0 + U^*\,, \quad \dot{g}_{ab} = \{g_{ab}, H\}\,, \quad \dot{\pi}^{ab} = \{\pi^{ab}, H\} + Q^{*ab}.$$

Since U^* is an ordinary and not a velocity-dependent potential, the canonical momentum and mechanical momentum coincide. The time rate of change of a possibly time-dependent function B on $T^*\mathcal{M}$ is now given by:

(11.46) $$\dot{B} = \partial B/\partial g_{ab}\,\dot{g}_{ab} + \partial B/\partial\pi^{ab}\,\dot{\pi}^{ab} + \partial B/\partial t$$
$$= \{B,H\} + Q^{*ab}\,\partial B/\partial\pi^{ab} + \partial B/\partial t\,.$$

Indulging in several identifications, we have in particular:

(11.47) $$\dot{H} = Q^{*ab}\,\partial H/\partial\pi^{ab} = Q^{*ab}\,\dot{g}_{ab} = -2\mathrm{Tr}\,\underline{Q}^*\underline{K}$$
$$= 2g^{-1/2}\,\mathrm{Tr}\,\underline{Q}^*\underline{\pi} = 4a^a\,{}^*f(M_a).$$

The fourth equality follows from (11.42) and the fifth from (11.4) and comparison with (11.44). In the class A case the energy is conserved and ${}^*f(M_a) = -\tfrac{1}{2}\tilde{\pi}(\underline{k}_a)$. Since $\{\underline{k}_a\}$ lie in the Lie algebra of SI_C in this case, $\{\tilde{\pi}(\underline{k}_a)\}$ commute with both H_0 and U^* and are therefore conserved. In the class B case it turns out that $\{{}^*f(M_a)\}$ are also conserved and hence when they vanish, the energy is conserved.

The "special automorphism coordinates" $\{\underline{B},\underline{S}\}$ on $\mathcal{M}$ (really on $\mathcal{M}_0\times\hat{G}$) induce "coordinates" on $T\mathcal{M}$ and $T^*\mathcal{M}$ (really on $T(\mathcal{M}_0\times\hat{G})$ and $T^*(\mathcal{M}_0\times\hat{G})$) which are not quite canonical. Namely, introduce "coordinates" $\{\beta^A,\underline{S},\dot{\beta}^A,\dot{\tilde{\omega}}^a\}$ on $T\mathcal{M}$ and $\{\beta^A,\underline{S},p_A,\mathcal{P}_a\}$ on $T^*\mathcal{M}$ corresponding to taking components of tangent vectors and covectors in the frame $\{\partial/\partial\beta^A,\tilde{e}_a\}$ with dual frame $\{d\beta^A,\tilde{\omega}^a\}$ as in (7.3):

(11.48) $$\dot{\beta}^A=f(d\beta^A) \qquad p_A=f(\partial/\partial\beta^A)$$
$$\dot{\tilde{\omega}}^a=f(\tilde{\omega}^a) \qquad \mathcal{P}_a=f(\tilde{e}_a).$$

Since $-f:\mathcal{X}(M)\to\mathcal{F}(T^*M)$ is a Lie algebra homomorphism for any manifold M as one may easily check by a computation in induced coordinates, the brackets of the (noncanonical) momenta $\mathcal{P}_a$ are:

(11.49) $$\{\mathcal{P}_a,\mathcal{P}_b\}=-\hat{C}^c{}_{ab}\mathcal{P}_c\,.$$

$\longleftarrow$ Since $\mathcal{P}_a=f(\tilde{e}_a)$ is essentially the moment for the 1-parameter group of left translations generated by $\tilde{e}_a$ on the factor manifold $\hat{G}$ of $\mathcal{M}_0\times\hat{G}$:

(11.50) $$(\tilde{e}_a)_t\,\underline{S}=(\exp-t\underline{k}_a)\,\underline{S}$$
$$\{\mathcal{P}_a,\underline{S}\}=d/dt|_0\ (\exp t\,\mathrm{ad}(\mathcal{P}_a))\,\underline{S}=d/dt|_0(\tilde{e}_a^{\,*})_t\,\underline{S}$$
$$=d/dt|_0(\tilde{e}_a)_t\,\underline{S}=-\underline{k}_a\underline{S}\,.$$

This makes use of (7.16) and several identifications.

We may evaluate the extrinsic curvature matrix and kinetic energy functions on $T\mathcal{M}$ and $T^*\mathcal{M}$ by inspection using (11.21) and (11.24):

(11.51) $$\underline{K}'=-\tfrac{1}{2}f(\underline{W}')=-(\dot{\beta}^A\underline{e}_A+\underline{\mathcal{X}}'_a\dot{\tilde{\omega}}^a)$$
$$L_0=\tfrac{1}{4}f(\mathcal{G})=e^{3\beta^0}(6\eta_{AB}\dot{\beta}^A\dot{\beta}^B+\bar{\mathcal{G}}'_{ab}\dot{\tilde{\omega}}^a\dot{\tilde{\omega}}^b)$$
$$H_0=f(\mathcal{G}^{-1})=\tfrac{1}{4}e^{-3\beta^0}(\tfrac{1}{6}\eta^{AB}p_Ap_B+\bar{\mathcal{G}}'^{ab}\mathcal{P}_a\mathcal{P}_b).$$

The relations between the new velocities and momenta are:

(11.52) $$p_A=\partial L_0/\partial\dot{\beta}^A=12e^{3\beta^0}\eta_{AB}\dot{\beta}^B,\qquad \dot{\beta}^A=\tfrac{1}{12}e^{-3\beta^0}\eta^{AB}p_B,$$
$$\mathcal{P}_a=\partial L_0/\partial\dot{\tilde{\omega}}^a=2e^{3\beta^0}\bar{\mathcal{G}}'_{ab}\dot{\tilde{\omega}}^b,\qquad \dot{\tilde{\omega}}^a=\tfrac{1}{2}e^{-3\beta^0}\bar{\mathcal{G}}'^{ab}\mathcal{P}_b.$$

From these and the primed version of (11.42), one may evaluate the momentum matrix $\underline{\Pi}'=\underline{S}\,\underline{\Pi}\,\underline{S}^{-1}$:

(11.53) $$\underline{\Pi}' = \tfrac{1}{12}\,n^{AB} p_A \hat{\underline{e}}_B + \tfrac{1}{4} P_0 \hat{\underline{e}}_0 + \tfrac{1}{2}\,\bar{g}'^{ab} P_a \underline{\mathcal{X}}_b' \,.$$

The expression for $f(M_a')$ may be written down by inspection of (11.34) and (11.35), while ${}^*f(M_a')$ may be obtained from this result and (11.51):

(11.54) class A: $$f(M_a') = -e^{3\beta^0}\bar{g}'_{ab}\dot{\omega}^b(1-\delta^{I}_{z}-\delta^{II}_{z}\delta^3_a)$$
$${}^*f(M_a') = -\tfrac{1}{2}P_a\,(1-\delta^{I}_{z}-\delta^{II}_{z}\delta^3_a)$$

class B: $$f(M_1') = e^{3\beta^0}(3a\bar{g}_{11}'\dot{\omega}^1 + n^{(2)}\bar{g}_{22}'\dot{\omega}^2)$$
$$f(M_2') = e^{3\beta^0}(-n^{(1)}\bar{g}_{11}'\dot{\omega}^1 + 3a\bar{g}_{22}'\dot{\omega}^2)$$
$$f(M_3') = e^{3\beta^0}(-6a\dot{\beta}^+ - \bar{g}^{\circ\,\prime}_{33}\dot{\omega}^3)$$
$${}^*f(M_1') = \tfrac{1}{2}(3aP_1 + n^{(2)}P_2)$$
$${}^*f(M_2') = \tfrac{1}{2}(-n^{(1)}P_1 + 3aP_2)$$
$${}^*f(M_3') = -\tfrac{1}{2}(ap_+ + (1-\delta^{V}_{z})P_3)\,.$$

Since the gravitational potential (11.26) is independent of the automorphism variables, the equations of motion for those variables may be derived from the kinetic energy alone. Since only the automorphism momenta appear in the Hamiltonian, their equations of motion are easily evaluated using (11.49):

(11.55) $$\dot{P}_a = \{P_a, H\} + Q^*_a = P_b \hat{C}^b{}_{ac}\left(\tfrac{1}{2} e^{-3\beta^0}\bar{g}'^{cd} P_d\right) + Q^*_a$$

where:

(11.56) $$Q^*_a = \delta^3_a\, 4a\, e^{\beta^0+4\beta^+}\bar{g}'^{\,0}{}_{33} .$$

These may be rewritten as first order equations for the automorphism velocities:

(11.57) $$(\dot{\tilde{\omega}}^a)^{\cdot} = \bar{g}'_{bd}\hat{C}^d{}_{ec}\bar{g}'^{ea}\dot{\tilde{\omega}}^b\dot{\tilde{\omega}}^c + (e^{-3\beta^0}\bar{g}'^{ac})^{\cdot} e^{3\beta^0}\bar{g}'_{cb}\dot{\tilde{\omega}}^b + \tfrac{1}{2} e^{-3\beta^0}\bar{g}'^{ab}Q^*_b .$$

Notice that it is not necessary to choose coordinates on $\hat{G}$ nor evolve the matrix $\underline{S}$ using its equation of motion:

(11.58) $$\dot{\underline{S}}\underline{S}^{-1} = \underline{K}_a\dot{\tilde{\omega}}^a = \tfrac{1}{2} e^{-3\beta^0}\bar{g}'^{ab} P_b \underline{K}_a ,$$

until after one has solved the equations of motion for the remaining variables $\{\beta^A, \dot{\beta}^A, \dot{\tilde{\omega}}^a\}$ or $\{\beta^A, p_A, P_a\}$.

The class B nonpotential force in general has only two nonvanishing components, Q^*_+ and Q^*_3. The component Q^*_3, when nonvanishing, guarantees that at least one off-diagonal automorphism degree of freedom be nontrivial. The other component satisfies

(11.59) $$-\partial U^*/\partial\beta^+ + Q^*_+ = -\partial(e^{\beta^0}V)/\partial\beta^+ .$$

The term $e^{\beta^0}V$ is the potential of the corresponding class A Bianchi type obtained by setting the structure constant tensor component a to zero. Thus the class A part of the potential alone generates the correct driving force for the $\beta^\pm$ equations of motion.

§12. Group Kinematics

A spatially homogeneous spacetime is essentially a spacetime built from a 1-parameter family of homogeneous 3-geometries, so we are led to consider product manifolds of the form $M = \mathbb{R} \times G$ where $\mathbb{R}$ is the real line and G a 3-dimensional Lie group. Before introducing a spacetime metric on such a manifold, it is worth exploring an independent group structure.

Consider the real line $\mathbb{R}$ with natural coordinate $t = x^0$. This is a 1-dimensional Lie group under addition with multiplication function $\varphi(t_1, t_2) = t_1 + t_2$. The coordinate frame and dual frame are bi-invariant:

(12.1) $$e_0 = \tilde{e}_0 = d/dt \qquad \omega^0 = \tilde{\omega}^0 = dt.$$

Now consider the product manifold $M = \mathbb{R} \times G$. This is itself a 4-dimensional Lie group with the direct product group structure. Namely, if (t, x) denotes a point in $\mathbb{R} \times G$, then the multiplication is:

(12.2) $$(t_1, x_1)(t_2, x_2) = (t_1 + t_2, x_1 x_2).$$

Let $G_t = \{(t, x) \mid x \in G\}$. We may identify the normal subgroup G_0 with G. If $\{x^a\}$ are local coordinates on G, then $\{x^\alpha\}$ extended in the natural way to M are local product coordinates on that manifold and $\{\partial/\partial x^\alpha\}$, $\{dx^\alpha\}$ the associated frame and dual frame. (See (A.10).)

Tensor fields on either component manifold induce corresponding tensor fields on the product manifold. Such a field has the same expression in a local product coordinate system as in the corresponding local coordinates on the component manifold from which it arises. Vector fields arising in this way from different component manifolds therefore commute and are annihilated by 1-forms induced from the other component. We will sloppily denote an induced field on M by the same symbol given the original field, until a distinction is required. Note that by applying (3.10) in the product local coordinates $\{x^\alpha\}$, it follows that $\mathcal{L}_{e_0} T = 0$ for all tensor fields T on M induced by tensor fields on G, since by definition $e_0{}^\mu = \delta^\mu_0$ and:

$$\frac{\partial}{\partial t} T^{\alpha\cdots}{}_{\beta\cdots} = \frac{\partial}{\partial t}(\delta^{\alpha}_{a\cdots} T^{a\cdots}{}_{b\cdots}\, \delta^{b}_{\beta\cdots}) = 0 .$$

All such fields are therefore invariant under dragging along by the flow of e_0, which is simply $(e_0)_t\,(t_1, x) = (t_1+t, x)$.

By these remarks it follows that if $\{e_a\}$ is a basis of $\mathfrak{g}$, then $\{e_\alpha\}$ and $\{\tilde{e}_\alpha\}$ are two mutually commuting frames on M with respective dual frames $\{\omega^\alpha\}$ and $\{\tilde{\omega}^\alpha\}$. In fact $\{e_\alpha\}$ is a basis of the Lie algebra of M and $\{\tilde{e}_\alpha\}$ the corresponding right invariant basis. The components of the SCT in this basis are:

(12.3) $\qquad C^{\alpha}{}_{\beta\gamma} = \delta^{\alpha}{}_{a} C^{a}{}_{bc}\, \delta^{b}_{\beta}\, \delta^{c}_{\gamma}.$

The product structure of $M = \mathbb{R}\times G$ is completely determined by the vector field e_0 and the submanifold G_0 which we identify with G. The 1-parameter family of hypersurfaces $\{G_t \mid t\in\mathbb{R}\}$ or "slicing of M" is swept out by the action of the flow of e_0 on G_0:

(12.4) $\qquad G_t = (e_0)_t\, G_0 = \{ (e_0)_t\,(0,x) \mid x\in G\}.$

We call e_0 an ADM generator for this slicing. Given a basis $\{e_a\}$ of $\mathfrak{g}$, we obtain a frame for G_0 by its identification with G and a frame on each member of the slicing by dragging along by e_0, yielding the vector fields $\{e_a\}$ on M. [Namely, let $e_a(t,x) = d(e_0)_t\,(0,x)\; e_a(0,x)$ where $e_a(0,x)\in TG_{0\,(0,x)}$ is identified with $e_a(x)\in TG_x$. $e_a \in \mathfrak{X}(M)$ will then have the property $(e_0)_t\, e_a = e_a$.) The frame $\{e_\alpha\}$ on M is therefore called ~~a~~ comoving ADM frame for the slicing, comoving meaning that the "reduced frame" $\{e_a\}$ which is tangent to the slicing is also dragged along by e_0. Note that we have been using the symbol t for a variety of purposes, first as a coordinate function for both $\mathbb{R}$ and $\mathbb{R}\times G$ and second as the parameter describing the flow of e_0 and the product slicing. These are all related by the action of the flow of e_0 on G_0.

Let us call spatially homogeneous (SH) any tensor field on $\mathbb{R}\times G$ which is invariant under left translation by the subgroup

G_o. Since the vector fields $\{\tilde{e}_a\}$ are a basis of generators of the left action of G_o on $\mathbb{R}\times G$, the Lie derivatives of any SH field with respect to them vanish. The components of a SH field in the left invariant frame $\{e_\alpha\}$ must therefore be independent of position on each hypersurface G_t and thus depend only on the coordinate t. These components are therefore naturally interpreted as functions on the real line and their derivative by e_o might just as well be written d/dt with this understanding. The members of the family $\{G_t \mid t\in\mathbb{R}\}$ are called SH hypersurfaces. A tensor field (intrinsic) over a hypersurface G_t is called SH if it is invariant under the left action of G_o restricted to that submanifold. By means of the natural projection map from $\mathbb{R}\times G$ into G which identifies each SH hypersurface with G, a SH tensor field over the family $\{G_t \mid t\in\mathbb{R}\}$ becomes a time-dependent left invariant field on G, where "time-dependent" is used in the same sense as in the second section.

The frame $\{e_\alpha\}$ is an example of a SH comoving ADM frame for the SH slicing of M. Consider a SH vector field $\bar{e}_o = N e_o + N^a e_a$, where N is strictly positive or negative so that $\bar{e}_o$ is nowhere tangent to any G_t. Call N the lapse function and the vector field $\vec{N} = N^a e_a$ which is tangent to the SH slicing the shift vector field. Being SH, $\vec{N}$ may be interpreted as a time-dependent field on G and N and N^a as simply functions on $\mathbb{R}$. It also implies $[\tilde{e}_a, \bar{e}_o] = 0$ and since $\{\tilde{e}_a\}$ are tangent to the slicing, it follows that dragging along G_o by $\bar{e}_o$ also generates the SH slicing. We may complete $\bar{e}_o$ to a SH comoving ADM frame $\{\bar{e}_\alpha\}$ on M (relative to this slicing) by dragging along the reduced frame $\{e_a\}$ on G_o, while dragging $\{\tilde{e}_a\}$ off G_o reproduces the existing reduced frame $\{\tilde{e}_a\}$ on M since $[\bar{e}_o, \tilde{e}_a] = 0$.

Let $\bar{t}$ denote the parameter describing the flow of $\bar{e}_o$. Since $(\bar{e}_o)_{\bar{t}}\, G_o$ is a member of the SH slicing, a function T^{-1} on the real line with inverse T is defined by:

(12.5) $\quad G_{T^{-1}(\bar t)} = (\bar e_0)_{\bar t}\, G_0 = (T^{-1}(\bar t),\ \vec N_{T^{-1}(\bar t)}\, G_0)$,

where $\vec N_t$ indicates the flow of the time-dependent vector field $\vec N(t)$ on G as discussed in the second section. In other words the hypersurface G_t is obtained by letting G_0 flow a parameter interval $\bar t = T(t)$ along $\bar e_0$. In terms of the product structure this represents the combined action of the flow of the time-dependent vector field $N e_0 = N d/dt$ on the origin of $\mathbb{R}$ by an amount $T(t)$ and a flow of the time-dependent left invariant vector field $\vec N(t)$ on G by an amount t, leading to a time-dependent right translation of G. The equation $\bar t = T(t)$ defines a new coordinate $\bar t$ on $\mathbb{R}$ (and therefore a function on $\mathbb{R}\times G$) which by construction is the one for which $N e_0 = N\, d/dt = d/d\bar t$. Therefore $N d\bar t/dt = 1$ and the function T is defined by

$$\bar t = T(t) = \int_0^t d\tau / N(\tau) ,$$

where N is considered an explicit function $N(t)$. To summarize, the action of the flow of $\bar e_0$ on G_0 leads to a reparametrization of the SH slicing and a parameter-dependent right translation on its members G_t.

The flow of $\bar e_0$ acting on G_0 generates a new product structure for M which might be denoted by $M = \overline{\mathbb{R}\times G}$ with points $\overline{(\bar t, \bar x)}$:

(12.6) $\quad \overline{(\bar t, \bar x)} = (T^{-1}(\bar t),\ \vec N_{T^{-1}(\bar t)}\, \bar x)$.

We can now consider the direct product group structure associated with this new product. However, the left action of G_0 on M will be the same since its generators $\{\tilde e_a\}$ are unchanged. So if this left action of the subgroup G_0 is the only property of the 4-dimensional group structure of M that interests us, we can allow a product generated from G_0 by any such $\bar e_0$. (If $\bar e_0$ is sufficiently pathological, trouble can arise, as does in the type IX / VIII Taub-Nut spacetimes.[50, 53])

Now consider the relation between the frames $\{e_\alpha\}$, $\{\tilde e_\alpha\}$ and their dual frames. Being tangent to the SH slicing, $\bar e_a$ is a linear combination of $\{e_b\}$ only and being SH, the coefficients depend only on t and like N and N^a may be identified with functions on $\mathbb{R}$:

(12.7) $\bar{e}_0 = N e_0 + N^a e_a \qquad \bar{\omega}^0 = N^{-1}\omega^0 = d\bar{t}$

$\bar{e}_a = e_b \mathcal{R}^{-1b}{}_a \qquad \bar{\omega}^a = \mathcal{R}^a{}_b(\omega^b - \omega^0 N^b/N)$

$$\bar{N}^a = \mathcal{R}^a{}_b N^b.$$

Expressing the comoving assumption $[\bar{e}_0, \bar{e}_a] = 0$ in terms of the original frame leads to the following differential equation for the time-dependent matrix $\underline{\mathcal{R}}(t)$:

(12.8) $\underline{\mathcal{R}}^{-1}\, d\underline{\mathcal{R}}/dt = \underline{k}_a N^a/N \quad , \quad \underline{\mathcal{R}}(0) = \underline{1}$.

The initial condition follows from the coincidence of $\{e_a\}$ and $\{\bar{e}_a\}$ on G_0. The unique solution $\underline{\mathcal{R}}(t)$ represents a curve through the identity of the adjoint matrix group of $\mathfrak{g}$ with respect to the basis $\{e_a\}$. Furthermore the reduced frame $\{\bar{e}_a\}$ may be interpreted as a time-dependent basis of $\mathfrak{g}$ which satisfies:

(12.9) $[\bar{e}_a, \bar{e}_b] = C^c{}_{ab}\bar{e}_c$,

since the adjoint matrix group leaves the components of the SCT invariant under a change of basis. The time-dependent adjoint transformation $\underline{\mathcal{R}}(t)$ is just the result of dragging the left invariant vector fields $\{e_a\}$ by the time-dependent left invariant vector field $\vec{N}$. The structure functions of the frame $\{\bar{e}_\alpha\}$ are therefore the constants $C^\alpha{}_{\beta\gamma}$. In fact $\{\bar{e}_\alpha\}$ and $\{\bar{e}_0, \tilde{e}_a\}$ are bases of the Lie algebras of left and right invariant vector fields on M with the direct product group structure generated by $\bar{e}_0$.

Making use of the adjoint identity listed in (5.9) and using $N\, d/dt = d/d\bar{t}$, the equations for $\underline{\mathcal{R}}$ and $\det \underline{\mathcal{R}}$ (see (6.29)) may be rewritten in the form:

(12.10) $d\underline{\mathcal{R}}/d\bar{t}\ \underline{\mathcal{R}}^{-1} = \underline{k}_a \bar{N}^a \quad , \quad d/d\bar{t}\,(\ln \det \underline{\mathcal{R}}) = 2a_a \bar{N}^a$.

These will be needed in the following section. Notice that the curve $\underline{\mathcal{R}}(t)$ is completely determined by a choice of lapse and shift through the comoving assumption. The class of SH comoving ADM frames for the

SH slicing generated from a fixed SH frame $\{e_a\}$ on G_0 therefore corresponds to the freedom of choice of the SH lapse and shift relative to the original product structure. This in turn is intimately connected with the adjoint matrix group associated with the basis $\{e_a\}$ of $\mathfrak{g}$.

However, suppose we relax the condition that the shift vector field $\vec{N}$ lie in $\mathfrak{g}$ and instead only require that it lie in

$$(12.11) \qquad \mathfrak{X}(\mathfrak{g}) = aut(G) \oplus \mathfrak{g} = aut(G) \oplus \tilde{\mathfrak{g}} \, .$$

Instead of generating only a time-dependent right translation of the family $\{G_t \mid t \in \mathbb{R}\}$ which leaves the elements of $\tilde{\mathfrak{g}}$ fixed and transforms the basis $\{e_a\}$ by a time-dependent inner automorphism $\underline{\mathcal{R}}$, the more general shift vector field will generate a time-dependent automorphism / translation of this family which now transforms both $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ by time-dependent automorphisms. These two time-dependent automorphisms correspond to the two ways of projecting $\vec{N}$ into $aut(G)$ using the direct sums (12.11).

The relation between the original comoving SH ADM frame and the comoving ADM frame $\{\bar{e}_\alpha\}$ generated by dragging along the reduced frame $\{e_a\}$ from G_0 by $\bar{e}_0$ is now given by:

$$(12.12) \qquad \begin{aligned} \bar{e}_0 &= N e_0 + N^a e_a & \bar{\omega}^0 &= N^{-1}\omega^0 = d\bar{t} \\ \bar{e}_a &= e_b \, \mathcal{S}^{-1\,b}{}_a & \bar{\omega}^a &= \mathcal{S}^a{}_b\,(\omega^b - \omega^0 N^b/N) \end{aligned}$$

$$\bar{N}^a = \mathcal{S}^a{}_b N^b \, ,$$

where $\underline{\mathcal{S}} \in Aut_e(\mathfrak{g})$ now satisfies either of the following differential equations derived from the comoving condition:

$$(12.13) \qquad \begin{aligned} \underline{\mathcal{S}}^{-1}\, d\underline{\mathcal{S}}/d\bar{t} &= ad_e(\vec{N}) = \hat{\underline{e}}_A \dot{\omega}^A & \qquad \underline{\mathcal{S}}(0) = \underline{1} \\ d\underline{\mathcal{S}}/d\bar{t}\, \underline{\mathcal{S}}^{-1} &= ad_{\tilde{e}}(\vec{N}) = \hat{\underline{e}}_A \dot{\tilde{\omega}}^A \end{aligned}$$

with the consequence:

$$(12.14) \qquad d/d\bar{t}\,(\ln \det \underline{\mathcal{S}}) = (\mathrm{Tr}\,\hat{\underline{e}}_A)\,\dot{\omega}^A = (\mathrm{Tr}\,\hat{\underline{e}}_A)\,\dot{\tilde{\omega}}^A \, .$$

Here $\{\hat{\underline{e}}_A\}$ is a basis of $aut_e(\mathfrak{g})$ generating invariant bases $\{e_A, \omega^A, \tilde{e}_A, \tilde{\omega}^A\}$ on $Aut_e(\mathfrak{g})$ and $\dot{\omega}^A$ and $\dot{\tilde{\omega}}^A$ are the components of the tangent to the curve $\underline{\mathcal{S}}(\bar{t})$ with respect to $\{e_A\}$ and $\{\tilde{e}_A\}$ respectively. The frame $\{\bar{e}_\alpha\}$ is the most general comoving ADM frame in which the spatial components of SH tensor fields are functions only of time, apart from the action on $\{\bar{e}_a\}$ of constant elements of $Aut_e(g)$ which corresponds to relaxing the initial condition $\underline{\mathcal{S}}(0) = \underline{1}$.

Let us extend the adjoint notation to $\mathfrak{X}(g)$ acting on $\tilde{g}$:

(12.14) $$\begin{aligned} &ad : \mathfrak{X}(g) \to aut(\tilde{g}) \\ &ad(\xi)\tilde{X} = \pounds_\xi \tilde{X} \qquad \xi \in \mathfrak{X}(g),\ \tilde{X} \in \tilde{g} \\ &\underline{ad}_{\tilde{e}}(\xi) = \tilde{\omega}^a(ad(\xi)\tilde{e}_b)\, \underline{\hat{e}}^b{}_a \ . \end{aligned}$$

Since $\{e_a\}$ and $\{\tilde{e}_a\}$ have the same SCT components apart from an overall sign, it follows that $aut_e(g) = aut_e(\tilde{g})$ and $Aut_e(g) = Aut_{\tilde{e}}(\tilde{g})$. Let $\{\bar{\tilde{e}}_a\}$ be the reduced frame generated by dragging along $\{\tilde{e}_a\}$ from G_0 by $\bar{e}_0$:

(12.15) $$\bar{\tilde{e}}_a = \Upsilon^{-1\,b}{}_a \tilde{e}_a \ .$$

Then the condition $[e_0, \tilde{e}_a] = 0$ leads to equations similar to (12.13) for the time-dependent matrix $\underline{\Upsilon} \in Aut_e(g)$. For example :

(12.16) $$\underline{\Upsilon}^{-1}\, d\underline{\Upsilon}/dt = \underline{ad}_{\tilde{e}}(\vec{N}) \qquad \underline{\Upsilon}(0) = \underline{1} \ .$$

When $Aut(G) \cong Aut(g)$, as for example occurs when G is simply connected, then given any smooth curve $\underline{\mathcal{S}}(t)$ in $Aut_e(g)$, one can generate the transformation (12.12), i.e. determine a compatible shift vector field $\vec{N}(t)$ and hence an ADM generator $\bar{e}_0$ which completes the reduced frame $\{\bar{e}_a\}$ to a comoving ADM frame on $\mathbb{R} \times G$. The freedom remaining in the choice of $\vec{N}(t) \in \mathfrak{X}(g)$ for a given $\underline{\mathcal{S}}(t)$ leads to different curves $\underline{\Upsilon}(t)$.

When the SCT components $C^a{}_{bc}$ are canonical, we call $\{\bar{e}_\alpha\}$ a "canonical" comoving ADM frame. Once the shift vector field $\vec{N}(t) \in \mathfrak{X}(g)$ is chosen, there remains only the freedom to transform the reduced frame by time-independent elements of the canonical automorphism matrix group $Aut_e(g)$.