§8. Isometry Groups and Homogeneous Geometries

Let (M,g) be an n-dimensional (pseudo-) Riemannian manifold M with a nondegenerate metric tensor field g of signature $s=2q-n$. Let $\underline{\eta}=\mathrm{diag}(1,\dots,-1,\dots)$ be the diagonal matrix whose first q diagonal values are 1 and the remaining ones -1. An orthonormal frame $\{e_a\}$ is one satisfying $g(e_a,e_b)=\eta_{ab}$. Let $O(q,n-q)=\{\underline{O}\in GL(n,\mathbb{R}) \mid \underline{O}^T\underline{\eta}\,\underline{O}=\underline{\eta}\}$ be the orthogonal group for the signature s and $SO(q,n-q)$ the corresponding special orthogonal group. (For $q=n$ these are $O(n,\mathbb{R})$ and $SO(n,\mathbb{R})$ respectively.) Then any two orthonormal frames at a point are related by an element of $O(q,n-q)$ or if they have the same orientation, by an element of $SO(q,n-q)$.

An isometry of (M,g) into itself is an element $h\in\mathcal{D}(M)$ which leaves the metric invariant under dragging: $hg=g$. It then follows from (3.2) that:

$$g(X,Y)(x) = hg(hX,hY)(h(x)) = g(hX,hY)(h(x)).$$

Thus a frame at $h(x)$ dragged along from an orthonormal frame at x is also orthonormal. We may also consider isometries between manifolds; $h:(M,g)\to(\bar{M},\bar{g})$ is an isometry if in addition to being a diffeomorphism it drags g into $\bar{g}$.

An r-parameter group of isometries of (M,g) is an action of an r-dimensional group G on M such that $\{f_a \mid a\in G\}$ are isometries. The Lie derivative of g with respect to the Lie algebra of generators of that action must vanish:

(8.1) $$\mathcal{L}_{\xi(X)}\, g = 0 \quad , \quad X\in\mathfrak{g} \quad .$$

This is called Killing's equation and the elements of $\xi(\mathfrak{g})$ Killing vector fields.

Suppose the isometric action of G on (M,g) is simply transitive. This means that the action is transitive but the isotropy group at every point of M is trivial, so M is diffeomorphic to $G/\{a_0\}=G$. If x_0 is any fixed point of M, then $F_{x_0}: G\to M$ is a diffeomorphism which maps the right invariant vector fields onto the generators of the action of G on M according to (1.23):

(8.2) $F_{X_0}\tilde{X} = \xi(X)$, $F_{X_0}^{-1}\xi(X) = \tilde{X}$.

Pulling the metric g back to G yields a left invariant metric $F_{X_0}^{-1}g$ on G since by (8.1) , (8.2) and (3.8):

$$\mathcal{L}_{\tilde{X}}(F_{X_0}^{-1}g) = 0 \ , \qquad \tilde{X} \in \tilde{\mathfrak{g}},$$

and the elements of $\tilde{\mathfrak{g}}$ generate the left translations. Thus (M,g) is isometric to G with a left invariant metric. Since any two points of M may be mapped into each other by an isometry, we are led to call (M,g) a homogeneous geometry. Lie groups with left invariant metrics are homogeneous geometries and any homogeneous geometry is isometric to one of these. The elements of $\tilde{\mathfrak{g}}$ are Killing vector fields of any left invariant metric on G; such metrics have constant components in any left invariant frame.

The space of left invariant Riemannian metrics on G is naturally identifiable with the space $\mathcal{M}(\mathfrak{g})$ of positive-definite inner products on $\mathfrak{g}$. Let $\{e_a\}$ be a basis of $\mathfrak{g}$ and $\{\omega^a\}$ its dual basis. An inner product $<,>$ on $\mathfrak{g}$ is essentially a tensor $g = g_{ab}\,\omega^a \otimes \omega^b$ over $\mathfrak{g}$, where $g_{ab} = \langle e_a, e_b \rangle = g(e_a, e_b)$. Since we have identified the dual basis $\{\omega^a\}$ with left invariant 1-forms on G, g itself is identified with a left invariant metric on G.

A choice of basis $\{e_a\}$ of $\mathfrak{g}$ enables $\mathcal{M}(\mathfrak{g})$ to be identified in a natural way with $\mathcal{M}^r$, the metric submanifold of $GL(r,\mathbb{R})$:

(8.3) $g = g_{ab}\,\omega^a \otimes \omega^b \in \mathcal{M}(\mathfrak{g}) \quad \leftrightarrow \quad \underline{g} = g_{ab}\,\hat{\underline{e}}^b{}_a \in \mathcal{M}^r$.

Depending on the choice of the matrix $\underline{g}$, the complete group of isometries of (G,g) may be larger than the left translations alone. The existence of Killing vector fields linearly independent of $\tilde{\mathfrak{g}}$ may or may not impose restrictions on $\underline{g}$. When such fields exist G will be a subgroup of a larger group of isometries of (G,g). (A manifold on which a group acts transitively is called a homogeneous space. Our homogeneous geometries are therefore examples of homogeneous spaces which are themselves groups.)

Suppose an r-dimensional Riemannian homogeneous geometry (G,g) admits a complete group of isometries of dimension $m > r$

so that the isotropy group I_x at each point x and the linear isotropy group DI_x acting on TM_x are each of dimension $p \doteq m-r$. (I_x and DI_x are isomorphic for an isometry group since fixing a point x and its tangent space fixes all geodesics emanating from x and thus immobilizes the geometry. This implies that the kernel of the homomorphism from I_x onto DI_x is trivial and hence $I_x \cong DI_x$.) The action of DI_x on TM_x is dragging along by isometries, which leaves orthonormal frames orthonormal. The matrix representation of DI_x with respect to an orthonormal basis of TM_x is therefore a subgroup of $O(r,\mathbb{R})$ and so its dimension p must equal the dimension of a possible subgroup of $O(r,\mathbb{R})$. When this subgroup is $O(r,\mathbb{R})$ itself, (G, g) is not only homogeneous but isotropic about every point; it is in fact a space of constant curvature. Since the subgroups of $O(3,\mathbb{R})$ have dimensions $0, 1$ and 3, the complete isometry group of a homogeneous 3-geometry can only have dimension $3, 4$ or 6.

For the special dimension 3 the SCT of a Lie algebra admits a very useful decomposition into a covector and a symmetric second rank tensor density.(24) In component language, this arises from the equivalence in this dimension of an antisymmetric pair of indices with one index via the duality operation. Taking the dual on the lower indices of $C^a{}_{bc}$ yields a weight-one density C^{ab} which has a symmetric part $n^{ab} = C^{(ab)}$ and an antisymmetric part $C^{[ab]} = \epsilon^{abc} a_c$ which is the dual of a covector which is not surprisingly the trace $C^b{}_{ab} = 2a_a$:

$$(8.4)\qquad \begin{aligned} C^{ab} &= \tfrac{1}{2} C^a{}_{fg} \epsilon^{bfg} = n^{ab} + \epsilon^{abc} a_c \\ C^a{}_{bc} &= C^{ad}\epsilon_{dbc} = \epsilon_{bcd} n^{ad} + a_f \delta^{fa}_{bc} \\ 2a_c &= C^{ab}\epsilon_{abc} = \tfrac{1}{2} C^a{}_{fg} \epsilon^{bfg} \epsilon_{abc} = C^a{}_{ca}. \end{aligned}$$

The contracted Jacobi identity (1.15) implies that n^{ab} annihilates a_b:

$$(8.5)\qquad 0 = \tfrac{1}{2}\epsilon^{abc} C^d{}_{ab} C^e{}_{ce} = a_c C^{cd} = a_c n^{cd}.$$

When a_b is nonzero this implies that the matrix of cofactors $\underline{\Delta}(\underline{n})$ of the matrix $\underline{n} = n^{ab}\,\hat{\underline{e}}^b{}_a$ is proportional to $a_a a_b \hat{\underline{e}}^b{}_a$

so a proportionality factor h (of weight zero) is defined by: (49)

(8.6) $a_a a_b = h\,\Delta(\underline{n})_{ab} = \frac{1}{2} h\, \epsilon_{acd}\, \epsilon_{bfg}\, n^{cf} n^{dg}$.

Evaluating the components of the Killing form defined by (1.14) in terms of this decomposition quickly yields:

(8.7) $\gamma_{ab} = -2(1+h)\,\Delta(\underline{n})_{ab}$.

Let $\{e_a\}$ be a basis of a 3-dimensional Lie algebra in which the SCT has components $C^a{}_{bc}$ and consider a new basis $\bar{e}_a = e_b A^{-1b}{}_a$, $\underline{A} \in GL(3,\mathbb{R})$. The components of the SCT transform as in (1.16). The component transformation laws for the other quantities just defined are:

(8.8) $\bar{n}^{ab} = (\det \underline{A}^{-1})\, A^a{}_f\, n^{fg} A^b{}_g$,

$\bar{a}_b = a_c A^{-1c}{}_b$, $\quad \bar{h} = h$.

The only invariants of the set of components of the SCT under basis transformation are the rank and the absolute value of the signature of the symmetric matrix $\underline{n}$ and the parameter h (defined to be zero when a_b vanishes).

Let (G, g) be a homogeneous 3-geometry with g given by (8.3) in a left invariant frame $\{e_a\}$ with dual frame $\{\omega^a\}$ and SCT components $C^a{}_{bc}$. This frame will be used to take components of the various geometric fields generated by the metric. All indices will be lowered and raised using g_{ab} and its inverse g^{ab}, except for those on ϵ_{abc} and ϵ^{abc} which are the components of the basis 3-form $\omega^{123} = \omega^1\wedge\omega^2\wedge\omega^3$ and 3-vector $e_{123} = e_1\wedge e_2\wedge e_3$ respectively:

$\epsilon_{abc} = \omega^{123}(e_a, e_b, e_c) \qquad \epsilon^{abc} = e_{123}(\omega^a, \omega^b, \omega^c)$.

The volume 3-form η provides G with a left invariant measure. Its components are defined by:

(8.9) $\eta_{abc} = g^{1/2}\,\epsilon_{abc}$, $\quad g = \det \underline{g}$.

It is convenient to introduce η_{abc} into the structure constant tensor decomposition so that indices may be raised and lowered indiscriminantly in calculations:

(8.10) $C^a{}_{bc} = \eta_{bcd}\, m^{ad} + a_d\, \delta^{da}_{bc}$, $\quad m^{ab} = g^{-1/2} n^{ab}$.

A matrix notation is adopted wherever possible. The entries of all square matrices except $\underline{g}$, $\underline{g}^{-1}$ and $\underline{\eta}$ are understood to be the components of the mixed forms of the corresponding second rank fields, the upper left index labeling the rows and the lower right index the columns. For example:

(8.11) $\underline{m} = m^a{}_b \underline{\hat{e}}^b{}_a$, $\underline{a}\underline{a}^T = a^a a_b \underline{\hat{e}}^b{}_a$.

A very useful set of matrices are the symmetrizations with respect to $\underline{g}$ of the adjoint matrix group generators:

(8.12) $\underline{K}_a = \frac{1}{2}(\underline{k}_a + \underline{g}^{-1}\underline{k}_a^T\underline{g})$ $\quad$ $\mathrm{Tr}\,\underline{K}_a = 2a_a$

$K_a{}^{bc} = C^{(b}{}_a{}^{c)}$

$(\mathcal{L}_{e_a} g)_{bc} = -2K_{abc}$.

The last relation is an immediate consequence of (A.14) or alternatively, follows by directly taking the Lie derivative of (8.3) using (3.23). Note that if $\underline{T}$ is a matrix arising from a symmetric tensor field:

(8.13) $\mathrm{Tr}\,\underline{K}_a\underline{T} = \mathrm{Tr}\,\underline{k}_a\underline{T}$.

We now evaluate the formulas of the appendix for the components of the metric connection and curvature fields and covariant derivatives and divergences. We break with the conventions of MTW only in defining the connection components:

(8.14) $\nabla_{e_a} e_b = \Gamma^c{}_{ab}\, e_c$.

With $\partial_a g_{bc} = 0$, the connection formulas become:

(8.15) $\Gamma^c{}_{ab} = \frac{1}{2}C^c{}_{ab} + K^c{}_{ab}$.

$\Gamma_{[ca]b} = \frac{1}{2}C_{bca}$ $\qquad$ $\Gamma^c{}_{cb} = -2a_b = -\Gamma_b{}^c{}_c$ $\qquad$ $\Gamma^c{}_{bc} = 0$

With $\partial_d\Gamma^a{}_{bc} = 0$ the curvature formulas become:

(8.16) $R^{*a}{}_{bcd} = \Gamma^a{}_{cf}\Gamma^f{}_{db} - \Gamma^a{}_{df}\Gamma^f{}_{cb} - \Gamma^a{}_{fb}C^f{}_{cd}$

$R^*{}_{bd} = R^{*g}{}_{bgd} = -2a_f\Gamma^f{}_{db} - \Gamma^g{}_{fb}\Gamma^f{}_{gd}$

$R^* = g^{bd}R^*{}_{bd} = -4a_f a^f - \Gamma^g{}_{fb}\Gamma^f{}_{gd}$.

The asterix will help us later distinguish these from spacetime fields when (G, g) is a spacelike hypersurface in spacetime. In order to be useful these formulas must be evaluated in terms of the SCT decomposition. After some rather lengthy algebra one finds for the matrix of the Ricci tensor:

Correction for page 8.6. Insert after 1st sentence:

Occasionally it is convenient to use an alternative expression for the last two terms in the formulas for $\underline{R}^*$ and $\underline{G}^*$:

$$-2\underline{a}\,\underline{a}^T - 2a^c\underline{K}_c = [\underline{m}, \underline{A}] - 2a_c a^c \underline{1}$$

$$\underline{A} = a_c \eta^{ca}{}_b \,\underline{\hat{e}}^b{}_a \,.$$

§9. Bianchi Classification

We may classify homogeneous 3-geometries by the type of 3-dimensional Lie group G defining the homogeneity. Since the Lie algebra $\mathfrak{g}$ determines G up to global structure, our problem is reduced to the classification of 3-dimensional Lie algebras. Bianchi was the first to classify both homogeneous 3-geometries (23) and 3-dimensional Lie algebras (22), hence the classification bears his name. However, a more modern approach due to Behr (24) will be followed here.

Let $\{e_a\}$ be the standard basis of $\mathbb{R}^3 = \mathfrak{g}$ and $\{\omega^a\}$ its dual basis. Consider the 9-dimensional real vector space $\mathfrak{g} \otimes (\mathfrak{g}^* \wedge \mathfrak{g}^*)$ of $\binom{1}{2}$-tensors over $\mathfrak{g}$ which are antisymmetric in their covariant arguments:

$$T = \tfrac{1}{2} T^a{}_{bc}\, e_a \otimes (\omega^b \wedge \omega^c).$$

The three relations $T^d{}_{[ab} T^e{}_{c]d} = 0$ define a 6-dimensional submanifold $\mathcal{C}$ in this space whose elements we denote by:

(9.1) $$C = \tfrac{1}{2} C^a{}_{bc}\, e_a \otimes (\omega^b \wedge \omega^c).$$

Each $C \in \mathcal{C}$ may be used to convert $\mathfrak{g}$ into a Lie algebra by defining:

(9.2) $$[e_a, e_b] = C^c{}_{ab} e_c.$$

$GL(3,\mathbb{R})$ has a natural left action on $\mathcal{C}$ described in component form by:

(9.3) $$C \mapsto f_{\underline{A}}(C), \quad (f_{\underline{A}}(C))^a{}_{bc} = A^a{}_d\, C^d{}_{fg}\, A^{-1f}{}_b\, A^{-1g}{}_c.$$

By (1.16), $f_{\underline{A}}(C)$ is a new structure constant tensor on $\mathfrak{g}$ which induces a new Lie algebra structure isomorphic to the one induced by C. [Alternatively a passive viewpoint may be taken by interpreting the components of the new tensor $f_{\underline{A}}(C)$ in the natural basis as the components of the same tensor C in a new basis $\bar{e}_a = e_b A^{-1b}{}_a$.] Let us call two points of $\mathcal{C}$ equivalent if they correspond to isomorphic Lie algebras on $\mathfrak{g}$. The equivalence classes are just the orbits of $\mathcal{C}$ under this action. Our problem is to characterize these orbits or equivalence classes and then set up a classification scheme.

The decomposition (8.4) of the structure constant tensor makes this easy:

(9.4) $$C^a{}_{bc} = \epsilon_{bcd}\, n^{ad} + a_d\, \delta^{da}_{bc}$$

$$n^{ad} a_d = 0 \qquad a_a a_b = \tfrac{1}{2} h\, \epsilon_{acd}\, \epsilon_{bfg}\, n^{cf} n^{dg}\,.$$

The only invariants of $C^a{}_{bc}$ under the action of $GL(3,\mathbb{R})$ are the constant h and the rank r and absolute value of the signature s of the matrix $\underline{n}$. These quantities therefore completely characterize the equivalence classes.

The equivalence classes divide into two general categories, class A and class B, according to whether h (or equivalently a_b) vanishes or not. For class A, h vanishes and there are only six different combinations of r and $|s|$. For class B, a_b is nonvanishing and so the rank of $\underline{n}$ is at most two, limiting the number of combinations to four, but with the additional freedom in the value of h which is infinite for $r = 0, 1$ and finite for $r = 2$. Rather than presenting a table of these combinations, it is more useful to choose a canonical element as a representative of each equivalence class. This is done by using the action of $GL(3,\mathbb{R})$ to map a general element of the class to a fixed one with a canonical set of components.

Since $\underline{n}$ is symmetric it may be transformed to "Sylvester form":

(9.5) $$\bar{n} = (\det A^{-1})\, A\, n\, A^T = \mathrm{diag}(1,\ldots,-1,\ldots,0,\ldots)\,.$$

We accomplish this by first diagonalizing $\underline{n}$ with an orthogonal matrix, then scaling the nonzero diagonal values to absolute value unity and changing their overall sign if desired (which we may do using f_{-1} since n^{ab} transforms like a weight-one density), and finally by permuting the diagonal values if necessary. The end results $\bar{n}^{ab}$ and $\bar{a}_b = a_c A^{-1c}{}_b$ may be interpreted as components in a new basis $\bar{e}_a = e_b A^{-1b}{}_a$. For class B orbits the relation $\bar{n}^{ab} \bar{a}_b = 0$ implies that the new basis may be chosen so that only one of the components a_b

is nonvanishing, in which case the corresponding diagonal value of $\bar{n}$ must vanish. We make the choice $\bar{a}_b = \bar{a}\,\delta^3{}_b$.

When C has been "diagonalized" in this way we have (dropping bars):

$$(9.6)\qquad \begin{array}{lll} n = \mathrm{diag}\,(n^{(1)}, n^{(2)}, n^{(3)}) & & [e_2, e_3] = n^{(1)} e_1 - a\, e_2 \\ a_b = a\,\delta^3{}_b & a n^{(3)} = 0 & [e_3, e_1] = n^{(2)} e_2 + a\, e_1 \\ h = a^2/(n^{(1)} n^{(2)}) & & [e_1, e_2] = n^{(3)} e_3 \,. \end{array}$$

For general values of $n^{(a)}$, a we call this the diagonal form for the basis of the Lie algebra $\mathfrak{g}$, here adapted to the third element of the basis. When $a \neq 0$ and $r < 2$, then $h = \infty$ and a may be normalized to unity, while if $r = 2$ the sign of a is unimportant and may be assumed positive. The following table lists our choice of canonical components for each equivalence class. (For later convenience we permute the Sylvester form for type II.)

(9.7)	CLASS	BIANCHI TYPE	a	$n^{(1)}$	$n^{(2)}$	$n^{(3)}$	h	dim(orbit)	dim(Ad(G))	dim(isometry group)	dim(Aut($\mathfrak{g}$))
		I	0	0	0	0	0	0	0	6 ($k=0$)	9
		II	0	0	0	1	0	3	2	4	6
	A	VI_0	0	1	-1	0	0	5	3	3	4
		VII_0	0	1	1	0	0	5	3	4, 6 ($k=0$)	4
		VIII	0	1	1	-1	0	6	3	3	3
		IX	0	1	1	1	0	6	3	4, 6 ($k>0$)	3
		V	1	0	0	0	∞	3	3	4, 6 ($k<0$)	6
		IV	1	1	0	0	∞	5	3	3	4
	B	$\mathrm{III} = \mathrm{VI}_{-1}$	1	1	-1	0	-1	5	2	4	4
		$\mathrm{VI}_{h\neq 0,-1}$	$a>0$	1	-1	0	$-a^2$	5+1	3	3	4
		$\mathrm{VII}_{h\neq 0}$	$a>0$	1	1	0	a^2	5+1	3	4, 6 ($k<0$)	4

interchange last two columns!!

The Roman numerals (Bianchi type) were assigned by Bianchi who used a different classification scheme involving the structure of the derived Lie algebra $[\mathfrak{g}, \mathfrak{g}]$ which may have dimension zero (I), one (II, III), two (IV–VII) or three (VIII, IX). (These four cases

correspond to the possible values of the rank of the matrix $C^{ab}\hat{\underline{e}}^{b}{}_{a}$.]
This leads to the
slightly different choices of canonical components which often appear in the literature.[25] Three columns give the dimension of each equivalence class and of its corresponding adjoint and automorphism groups. The last column gives the possible dimensions of larger isometry groups of homogeneous 3-geometries of each Bianchi type, with the sign of the constant curvature indicated when that dimension is 6. These will be discussed in the next section.

Consider again the action of the 9-dimensional group $GL(3,\mathbb{R})$ on the 6-dimensional manifold $\mathcal{C}$. The isotropy group I_C at a point $C \in \mathcal{C}$ is at least of dimension $9-6=3$. If $\underline{A} \in I_C$, then (1.17) holds; therefore I_C is the matrix representation in the basis $\{e_a\}$ of the automorphism group $Aut(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ with SCT C. The orbit or equivalence class containing C is diffeomorphic to $GL(3,\mathbb{R})/I_C$ so its dimension is $9-\dim I_C$. Let SI_C denote the special isotropy group at C and hence the matrix representation in the basis $\{e_a\}$ of the special automorphism group $SAut(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ with SCT C. This group is important in the dynamics of spatially homogeneous universe models.

A basis $\{e_a\}$ of a 3-dimensional Lie algebra $\mathfrak{g}$ for which the SCT components $C^{a}{}_{bc}$ assume the canonical values will be called a canonical basis. Since the matrix representations of the adjoint group (or group of inner automorphisms) and automorphism group of $\mathfrak{g}$ are determined completely by the values of the SCT components, they are independent of the particular canonical basis chosen and hence one can speak of the canonical adjoint and automorphism matrix group for each Bianchi type, namely $IAut_e(\mathfrak{g})$ and $Aut_e(\mathfrak{g})$ for any canonical basis $\{e_a\}$ of $\mathfrak{g}$.

§10. Bianchi Groups, Adjoint and Automorphism Groups, 3-geometries and Maximal Isometry Groups

In this section we imbed in $GL(3,\mathbb{R})^+$ a Lie group of each of the Bianchi types by manipulation of the adjoint Lie algebra. When the structure constant tensor components are in the diagonal form (9.6), the matrices of the adjoint matrix Lie algebra corresponding to the diagonalizing basis are given by:

$$\begin{aligned}
(10.1)\quad \underline{k}_1 &= -n^{(2)}\hat{\underline{e}}^3{}_2 + n^{(3)}\hat{\underline{e}}^2{}_3 - a\hat{\underline{e}}^3{}_1 \\
\underline{k}_2 &= -n^{(3)}\hat{\underline{e}}^1{}_3 + n^{(1)}\hat{\underline{e}}^3{}_1 - a\hat{\underline{e}}^3{}_2 \\
\underline{k}_3 &= -n^{(1)}\hat{\underline{e}}^2{}_1 + n^{(2)}\hat{\underline{e}}^1{}_2 + a(\hat{\underline{e}}^1{}_1 + \hat{\underline{e}}^2{}_2) \\
[\underline{k}_2, \underline{k}_3] &= n^{(1)}\underline{k}_1 - a\underline{k}_2 \qquad\qquad a n^{(3)} = 0 \\
[\underline{k}_3, \underline{k}_1] &= n^{(2)}\underline{k}_2 + a\underline{k}_1 \qquad\qquad \mathrm{Tr}\,\underline{k}_a = 2a_a = 2a\delta^3_a \\
[\underline{k}_1, \underline{k}_2] &= n^{(3)}\underline{k}_3
\end{aligned}$$

The matrix $\underline{k}_3$ is responsible for the nonunimodularity of the class B adjoint group.

Canonical coordinates of the second kind lead to the following parametrization of the adjoint matrix $\underline{R}$; the exponentiation can be done explicitly:

$$(10.2)\qquad \underline{R}(x) = e^{x^1\underline{k}_1} e^{x^2\underline{k}_2} e^{x^3\underline{k}_3} =$$

$$\begin{bmatrix} 1 & 0 & -ax^1 \\ 0 & c_1 & -n^{(2)}s_1 \\ 0 & n^{(3)}s_1 & c_1 \end{bmatrix}
\begin{bmatrix} c_2 & 0 & n^{(1)}s_2 \\ 0 & 1 & -ax^2 \\ -n^{(3)}s_2 & 0 & c_2 \end{bmatrix}
\begin{bmatrix} c_3 e^{ax^3} & -n^{(1)}s_3 e^{ax^3} & 0 \\ n^{(2)}s_3 e^{ax^3} & c_3 e^{ax^3} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The following abbreviations and identities are used:

$$\begin{aligned}
(10.3)\quad m^{(a)} &= (-n^{(b)}n^{(c)})^{1/2}, \quad \tilde{m}^{(a)} = (n^{(b)}n^{(c)})^{1/2}, \quad (a,b,c)\ \text{cyclic} \\
c_a &= \cosh m^{(a)}x^a = \cos\tilde{m}^{(a)}x^a \\
s_a &= (m^{(a)})^{-1}\sinh m^{(a)}x^a = (\tilde{m}^{(a)})^{-1}\sin\tilde{m}^{(a)}x^a \\
(c_a, s_a) &\longrightarrow (1, x^a) \quad \text{as} \quad m^{(a)} \to 0 \\
(c_a)^2 - (m^{(a)}s_a)^2 &= (c_a)^2 + (\tilde{m}^{(a)}s_a)^2 = 1 \\
dc_a &= (m^{(a)})^2 s_a\, dx^a \qquad ds_a = c_a\, dx^a .
\end{aligned}$$

Let $\underline{I}^{(a)} = \underline{1} - \hat{\underline{e}}^a{}_a = \hat{\underline{e}}^b{}_b + \hat{\underline{e}}^c{}_c$ (no summations and (a,b,c) cyclic) and introduce the notation $\overset{\circ}{\underline{k}}_a$ for the matrices obtained by setting $a=0$ in (10.1). One may easily verify the relations:

$$\begin{aligned}
(\underline{k}_a)^2 &= (m^{(a)})^2\,\underline{I}^{(a)}, \quad (\underline{k}_a)^3 = (m^{(a)})^2\,\underline{k}_a \qquad a \neq 3 \\
(\overset{\circ}{\underline{k}}_3)^2 &= (m^{(3)})^2\,\underline{I}^{(3)}, \quad (\overset{\circ}{\underline{k}}_3)^3 = (m^{(a)})^2\,\overset{\circ}{\underline{k}}_3
\end{aligned}$$

These allow us to gather even and odd powers of the above matrix exponential series into trigonometric, hyperbolic and truncated series for the coefficients of $\underline{1}$, $\underline{I}^{(a)}$, $\underline{k}_a$ for $a \neq 3$. Since $\underline{k}_3 = \underline{k}_3^\circ + a\underline{I}^{(3)}$ and $\underline{k}_3$ and $\underline{I}^{(3)}$ commute, we may exponentiate them separately and multiply the results, exponentiating $\underline{k}_3^\circ$ as above. This quickly yields (10.2), (10.3).

The left and right invariant dual frames determined by the "basis" $\{\underline{k}_a\}$ are computable from the relations:

(10.4) $$\underline{R}^{-1} d\underline{R} = \underline{k}_a \omega^a \qquad d\underline{R}\,\underline{R}^{-1} = \underline{k}_a \tilde{\omega}^a .$$

The invariant frames themselves are easily constructed by the duality relations. The calculation is made less tedious by the method of (6.33). The result is:

(10.5)
$$\begin{aligned}
\omega^1 &= e^{-ax^3}(c_2 c_3\, dx^1 + n^{(1)} s_3\, dx^2) & \tilde{\omega}^1 &= dx^1 + (n^{(1)} s_2 - a x^1)\, dx^3 \\
\omega^2 &= e^{-ax^3}(-n^{(2)} c_2 s_3\, dx^1 + c_3\, dx^2) & \tilde{\omega}^2 &= c_1 dx^2 - (n^{(2)} s_1 c_2 + a x^2)\, dx^3 \\
\omega^3 &= n^{(3)} s_2\, dx^1 + dx^3 & \tilde{\omega}^3 &= n^{(3)} s_1 dx^2 + c_1 c_2\, dx^3 \\
e_1 &= e^{ax^3}(n^{(2)} s_3 \partial_2 + c_3 c_2^{-1}(\partial_1 - n^{(3)} s_2 \partial_3)) & \tilde{e}_1 &= \partial_1 \\
e_2 &= e^{ax^3}(c_3 \partial_2 - n^{(1)} s_3 c_2^{-1}(\partial_1 - n^{(3)} s_2 \partial_3)) & \tilde{e}_2 &= c_1 \partial_2 - n^{(3)} s_1 c_2^{-1}(\partial_3 - n^{(1)} s_2 \partial_1) \\
e_3 &= \partial_3 & \tilde{e}_3 &= n^{(2)} s_1 \partial_2 + c_1 c_2^{-1}(\partial_3 - n^{(1)} s_2 \partial_1) \\
& & &\quad + a(x^1 \partial_1 + x^2 \partial_2)
\end{aligned}$$

$$\omega^{123} = e^{-2ax^3} c_2\, dx^{123}, \quad \tilde{\omega}^{123} = c_2\, dx^{123}, \quad \det \underline{R} = e^{2ax^3}$$

In the above we assumed that $\{\underline{k}_a\}$ are linearly independent so that the adjoint group is of full dimension three. However, for the degenerate types I, II, III this dimension is 0,2,2 respectively since $\underline{k}_a, \underline{k}_3$ vanish for types I, II while $\underline{k}_1 = -\underline{k}_2$ for type III. The type I adjoint matrix group is trivial, namely $\{\underline{1}\}$; canonical parametrizations on the type II, III adjoint matrix groups may be obtained by setting x^3 and x^2 respectively to zero and in the latter case relabling x^3 by x^2. Since everything is analytic in the parameters $\{n^{(a)}, a\}$ however, (10.5) are the invariant fields expressed in canonical coordinates of the second kind on 3-dimensional groups of the degenerate types as well. To recover these expressions legitimately as in (6.27) for actual 3-dimensional subgroups of $GL(3,\mathbb{R})^+$ we must "exponentiate and differentiate" the following Lie algebra bases $\{\tilde{e}_a\}$:

(10.6) I. $\{\hat{\underline{e}}^1{}_1, \hat{\underline{e}}^2{}_2, \hat{\underline{e}}^3{}_3\}$

II. $\{\hat{\underline{e}}^3{}_2, -\hat{\underline{e}}^2{}_1, \hat{\underline{e}}^3{}_1\}$

III. $\{(\hat{\underline{e}}^3{}_1 + \hat{\underline{e}}^2{}_2)/\sqrt{2}, (-\hat{\underline{e}}^3{}_1 + \hat{\underline{e}}^2{}_2)/\sqrt{2}, 2\hat{\underline{e}}^1{}_1\}$ or $\{\hat{\underline{e}}^3{}_1, \hat{\underline{e}}^2{}_2, \hat{\underline{e}}^1{}_1\}$.

The invariant fields in canonical coordinates of the second kind obtained from these bases agree with (10.5) evaluated at the canonical values of $\{n^{(a)}, a\}$. The alternative basis for type III is somewhat simpler but with different structure constant components:

(10.7) $S(x) = e^{x^1\hat{\underline{e}}_1}\, e^{x^2\hat{\underline{e}}_2} e^{x^3\hat{\underline{e}}_3} = \mathrm{diag}\,[e^{x^3}, e^{x^2}, 1] + x^1\hat{\underline{e}}^3{}_1$

$[\hat{e}_2, \hat{e}_3] = 0 = [\hat{e}_1, \hat{e}_2] \quad [\hat{e}_3, \hat{e}_1] = \hat{e}_1$

$\omega^2 = dx^2 = \tilde{\omega}^2, \; \omega^3 = dx^3 = \tilde{\omega}^3, \; \omega^1 = e^{-x^3}dx^1, \; \tilde{\omega}^1 = dx^1 - x^1 dx^3$

$e_1 = e^{x^3}\partial_1, \; e_2 = \partial_2 = \tilde{e}_2, \; e_3 = \partial_3, \; \tilde{e}_1 = \partial_1, \; \tilde{e}_3 = \partial_3 + x^1\partial_1$.

Evaluating (10.1) − (10.5) at the canonical values of $\{n^{(a)}, a\}$ given by (9.7) leads to the canonical adjoint matrix group of each type. It is instructive to examine each of these separately in order not to be overwhelmed by our simultaneous notation.

The canonical adjoint matrix groups of types VI_0, VII_0, VIII and IX are more familiar as the group of isometries of the Lorentz and Euclidean planes, the 3-dimensional (proper) Lorentz group $SO(2,1)$ and the rotation group $SO(3,\mathbb{R})$ respectfully. In the first two cases $y^1 = n^{(1)}x^2$ and $y^2 = -n^{(2)}x^1$ describe the translations of the plane in cartesian coordinates and x^3 the angle of hyperbolic or ordinary rotation. The type VI_h, VII_h canonical adjoint matrix groups have the same interpretation as their $h=0$ counterparts except that a boost or rotation by x^3 is accompanied by a dilation of the plane by a factor e^{ax^3}, while type V involves only the dilation (as well as the translations). The type I group is better known as the group of translations of $\mathbb{R}^3$, while the type II representative is the subgroup of $GL(3,\mathbb{R})^+$ consisting of unit upper-diagonal matrices (zeros below the diagonal with ones on the diagonal) (the Heisenberg group). Representations of the class A groups as well as their global structure and 2x2 matrix realizations are discussed by Vilenkin [11] but the class B groups have been somewhat neglected by mathematicians. Representation theory is important for perturbation theory and particle creation in spatially homogeneous cosmology.

In the literature Euler angle coordinates are used almost exclusively for type IX. These are a special case of the following coordinates valid for types VI_0, VII_0, VIII, IX :

$$(10.8)\qquad \underline{R}(x) = e^{x^2 \underline{k}_3}\, e^{x^1 \underline{k}_1}\, e^{x^3 \underline{k}_3}.$$

This exponentiation can be done as above. The left invariant frame and dual frame are similarly found to be:

$$(10.9)\qquad \begin{aligned} \omega^1 &= c_3\,dx^1 + n^{(1)}n^{(2)} s_1 s_3\,dx^2 & e_1 &= c_3\partial_1 + s_3(s_1)^{-1}(\partial_2 - c_1\partial_3) \\ \omega^2 &= n^{(2)}(-s_3\,dx^1 + s_1 c_3\,dx^2) & e_2 &= (n^{(2)})^{-1}(-n^{(1)}n^{(2)} s_3\partial_1 + c_3(s_1)^{-1}(\partial_2 - c_1\partial_3)) \\ \omega^3 &= c_1\,dx^2 + dx^3 & e_3 &= \partial_3 \end{aligned}$$

These coordinates fail at the identity.

The above $GL(3,\mathbb{R})$ realization of the type I group is just the scale group $D(3,\mathbb{R})^+$ with Lie algebra $d(3,\mathbb{R}) = \mathrm{span}\{\underline{1}_3\} \oplus sd(3,\mathbb{R})$. In addition to the basis $\{\underline{\hat{e}}_a = \cancel{\sum_a} \underline{\hat{e}}^a{}_a\}$ of (10.7), the following basis is useful:[18]

$$(10.10)\qquad \{\hat{e}_A\}_{A=0,+,-} = \{\underline{1}, \mathrm{diag}(1,1,-2), \mathrm{diag}(\sqrt{3},-\sqrt{3},0)\}$$

$$\langle \underline{\hat{e}}_A, \underline{\hat{e}}_B\rangle_{DW} = 6\,\eta_{AB}\,,\quad \underline{\eta} = \mathrm{diag}(-1,1,1).$$

Adapted to the above direct sum decomposition corresponding to the dilation and shear subgroups, this basis is orthogonal with respect to the DeWitt inner product. The traceless basis $\{\hat{e}_\pm\}$ is adapted to the third axis. Rotations of this basis by 120° permute the diagonal values cyclically forward; rotations by any angle leave the basis orthogonal. The entire basis $\{\hat{e}_A\}$ remains orthogonal under the action of $SO(2,1)$; boosts have found application in spatially homogeneous dynamics.[18]

We now investigate the canonical automorphism and special automorphism matrix groups $\mathrm{Aut}_e(\mathfrak{g})$ and $\mathrm{SAut}_e(\mathfrak{g})$ for each Bianchi type. For types IX and VIII the weighted character of $\underline{n} = \mathrm{diag}(1,1,\pm 1)$ adds no freedom and the corresponding canonical automorphism-adjoint matrix groups are $SO(3,\mathbb{R})$ and $SO(2,1)$ which are unimodular. When $r=2$ we may scale the two basis vectors $\{e_1, e_2\}$ of the Lie algebra uniformly without affecting $a_a = a\delta^3_a$ or $\underline{n} = \mathrm{diag}(1,\pm 1,0)$ due to its weighted nature. This scaling is generated by the matrix $\underline{I}^{(3)} = \mathrm{diag}(1,1,0)$, in terms of which $\underline{k}_3 = \underline{\mathring{k}}_3 + a\underline{I}^{(3)}$. Except for the degenerate type III, the 4-dimensional automorphism group has one extra dimension than the adjoint group corresponding to this scaling. In fact for all of these types it is generated by the basis $\{\underline{\hat{e}}^3{}_1, \underline{\hat{e}}^3{}_2, \underline{\mathring{k}}_3, \underline{I}^{(3)}\}$, the first three elements of which generate the special automorphism group.

Except for type III, $\{\hat{e}^3{}_1, \hat{e}^3{}_2\}$ may be replaced by the linearly independent combinations $k_1 = -a\,\hat{e}^3{}_1 - n^{(2)}\hat{e}^3{}_2$, $k_2 = n^{(1)}\hat{e}^3{}_1 - a\,\hat{e}^3{}_2$, while $\{k^0_3, I^{(3)}\}$ may be replaced by $\{k_3, I^{(3)}\}$. For the two cases $r=1$ we assume $n^{(a)} = \delta^a_1$ for uniformity of discussion. Invariance of $\underline{n}$ imposes the condition $A^a{}_1 = \delta^a_1 \,|\det \underline{A}|^{\frac{1}{2}}$ on elements of $GL(3,\mathbb{R})$ yielding for the class A type II a 6-dimensional group whose unimodular subgroup is the 5-dimensional subgroup of $SL(3,\mathbb{R})$ for which $A^a{}_1 = \delta^a_1$. (In fact $\mathrm{aut}_e(g) = \mathrm{ad}_e(g) \oplus \mathrm{span}\{\underline{1} + \hat{e}^3{}_3\} \oplus \mathcal{D}_e$ where $\mathcal{D}_e$ is the Lie algebra of the $SL(2,\mathbb{R})$ subgroup of $SL(3,\mathbb{R})$ which leaves the third axis of $\mathbb{R}^3$ fixed when acting on that space in the natural way.) When $a_a = a\,\delta^3_a$ is nonzero (type IV), its invariance requires in addition that $A^3{}_a = \delta^3{}_a$ which forces $A^1{}_1 = A^2{}_2 > 0$ by the previous condition, resulting in a 4-dimensional group (again generated by $\{k_1, k_2, k^0_3, I^{(3)}\}$) whose unimodular subgroup has $A^1{}_1 = A^2{}_2 = 1$ and is therefore the above matrix realization of the type II group given in (10.3). When $r=0$ but $a \neq 0$ (type V), invariance of a_a requires only $A^3{}_a = \delta^3{}_a$ resulting in a 6-dimensional group with a 5-dimensional unimodular subgroup. (Here $\mathrm{aut}_e(g) = \mathrm{ad}_e(g) \oplus \mathcal{D}_e$ where $\mathcal{D}_e$ has the same significance as above.) In the trivial type I case, $GL(3,\mathbb{R})$ and $SL(3,\mathbb{R})$ are the automorphism and special automorphism matrix groups. Finally, subtracting the automorphism group dimension from 9 (the dimension of $GL(3,\mathbb{R})$) yields the equivalence class dimensions quoted in the table (9.7).

We now return our attention to the homogeneous 3-geometries (G, g) of §8. Regardless of the global structure of G, it will have the same local structure as one of the above matrix groups. For example, a local isomorphism is provided by identifying canonical coordinates of the second kind taken with respect to bases in which the structure constant tensor components agree. Our choice of coordinates and the expressions (10.5) are therefore always valid near the identity and we will assume them in discussing possible additional Killing vector fields. Here the original work of Bianchi will be quoted with some modification. One must see for what specializations of the constant matrix $\underline{g}$ of components taken in the canonical basis $\{e_a\}$ of (10.5) do Killing's equations admit additional solutions linearly independent of the three right invariant vector fields $\xi_a = \tilde{e}_a$. We already know from §8 that only one or three additional linearly independent Killing vectors are allowed.

Types I, II, III, V are special in the respect that any left invariant metric necessarily has a complete isometry group of dimension 6, 4, 4 and 6 respectively. Consider first types I and V whose homogeneous 3-geometries are necessarily spaces of constant zero and negative curvature respectively. The left invariant vector fields and 1-forms (10.5) are explicitly:

(10.11) $$\omega^a = \exp[-\epsilon(\delta^a_1+\delta^a_2)x^3]\,dx^a, \quad e_a = \exp[\epsilon(\delta^a_1+\delta^a_2)x^3]\,\partial_a ,$$

where $\epsilon = 0$ and 1 respectively. By choosing new coordinates $\{x^1, x^2\}$ linear in the old coordinates so that $\{\partial_a\}$ are orthogonal and $g(\partial_1,\partial_1) = g(\partial_2,\partial_2) = g_{33}$, a general metric $g_{ab}\,\omega^a\otimes\omega^b$ may be reduced to the form $g_{33}\,\delta_{ab}\,\omega^a\otimes\omega^b$ where $\{\omega^a\}$ are the same expressions (10.11) but in the new coordinates. (In type I the factor g_{33} may be eliminated as well.) In other words, apart from a constant conformal factor in type V, all type I and V metrics are isometric to the metric:

(10.12) $$\delta_{ab}\,\omega^a\otimes\omega^b = e^{-2\epsilon x^3}(dx^1\otimes dx^1 + dx^2\otimes dx^2) + dx^3\otimes dx^3 .$$

For type I (flat space), the three Killing vectors $\xi_{a+3} = \epsilon_{abc}\,x^b\partial_c$ generate the action of the rotation group which is the isotropy group at the identity. For type V the space has constant negative curvature -1 and the extra Killing vectors are:

(10.13) $$\begin{aligned} \xi_4 &= x^1x^2\partial_1 - 1/2\,(e^{2x^3} + (x^1)^2 - (x^2)^2)\,\partial_2 + x^2\partial_3 \\ \xi_5 &= 1/2\,(e^{2x^3} + (x^2)^2 - (x^1)^2)\,\partial_1 - x^1x^2\partial_2 - x^1\partial_3 \\ \xi_6 &= x^1\partial_2 - x^2\partial_1 . \end{aligned}$$

The isotropy group at the identity is generated by the ordered basis $\{\xi_4 + \tfrac{1}{2}\tilde e_2,\ \xi_5 - \tfrac{1}{2}\tilde e_1,\ \xi_6\}$ which has the canonical type IX bracket algebra apart from an overall sign. In the type I and type V cases the Killing vector fields $\{\tilde e_1, \tilde e_2, \tilde e_3 + \xi_6\}$ and $\{\tilde e_1, \tilde e_2, a\tilde e_3 + \xi_6\}$ respectively agree with the expressions (10.5) for the canonical fields $\{\tilde e_a\}$ of types VII_0 and VII_h respectively. The metrics (10.12) therefore admit simply transitive isometry subgroups of these respective types.

In fact consider the type VII_h metric with $g = \mathrm{diag}(g_{11}, g_{11}, g_{33})$:

(10.14) $$\begin{aligned} g_{ab}\,\omega^a\otimes\omega^b &= g_{11}\,e^{-2ax^3}(dx^1\otimes dx^1 + dx^2\otimes dx^2) + g_{33}\,dx^3\otimes dx^3 \\ &= g_{33}\left(e^{-2y^3}(dy^1\otimes dy^1 + dy^2\otimes dy^2) + dy^3\otimes dy^3\right)/a^2 \end{aligned}$$

where $y^1 = a(g_{11}/g_{33})^{1/2}x^1$, $y^2 = a(g_{11}/g_{33})^{1/2}x^2$, $y^3 = ax^3$. Apart from a

Correction for page 10.6.

Add to the end of the paragraph following (10.13):

For type I the three Killing vector fields $\{\xi_a\}$ belong to $aut(G)$ and satisfy $\quad \mathrm{ad}_e(\xi_{a+3}) = \epsilon_{abc}\,\underset{\sim}{\hat{e}}{}^c{}_b$, provided that G has the topology of $\mathbb{R}^3$, i.e. is simply connected. In the type V case only ξ_6 belongs to $aut(G)$ and satisfies $\quad \mathrm{ad}_e(\xi_6) = \underset{\sim}{\hat{e}}{}^2{}_1 - \underset{\sim}{\hat{e}}{}^1{}_2$.

Corrections for page 10.7.

Insert at end of paragraph (after $\bar{x}^3(xt) = \cdots$ etc) containing (10.15) the following:

ξ_4 is in fact an element of $aut(G)$ since by explicit calculation one finds that $\mathrm{ad}_e(\xi_4) = \underset{\sim}{\hat{e}}{}^1{}_2 - \underset{\sim}{\hat{e}}{}^2{}_1$. For a metric with $\underset{\sim}{g} = \mathrm{diag}(g_{11}, g_{22}, g_{33})$, a simple scaling of the coordinates shows that the additional Killing vector field ξ_4 may be written:

(10.17) $\quad \xi_4 = \frac{1}{2}\left[(g_{22}/g_{11})(x^2)^2 - (g_{11}/g_{22})(x^1)^2\right]\partial_3 + x^1\partial_2 - x^2\partial_1$.

Insert at the end of paragraph containing (10.18) the following:

The parameter n labels the 1-parameter family of conformal isometry classes of left invariant type III Riemannian metrics. It is easy to see by a simple scaling of the coordinates that any metric g with $\underset{\sim}{\bar{g}} = \mathrm{diag}(\bar{g}_{11}, \bar{g}_{22}, \bar{g}_{33})$ in the noncanonical basis belongs to the class with $n=0$. In this case the additional Killing vector field ξ_4 can be written:

(10.19) $\quad \xi_4 = \frac{1}{2}\left[(\bar{g}_{33}/\bar{g}_{11})e^{2\bar{x}^3} - (\bar{x}^1)^2\right]\bar{\partial}_1 - \bar{x}^1\bar{\partial}_3$.

constant scale factor this is just (10.12) with the replacement $x \to y$. Replacing x^a by $y^a(x)$ in the expressions (10.13) yields the extra Killing vectors $\{\xi_4, \xi_5, \xi_6\}$ of (10.14). (Note that ξ_6 is invariant under this substitution.) All such metrics are therefore isometric to within a conformal constant and admit a simply transitive type V isometry group generated by $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3 - \xi_6\}$. However, any type VII_h not reducible to this form admits no additional Killing fields.

By similar arguments one may show that all type II metrics are isometric to within a constant conformal factor and admit one additional Killing field. For $\underline{g} = \underline{1}$ the canonical metric and corresponding Killing field ξ_4 are:

(10.15) $$\delta_{ab}\, dx^a \otimes dx^b + x^2 (dx^1 \otimes dx^3 + dx^3 \otimes dx^1) + (x^2)^2\, dx^1 \otimes dx^1$$
$$\xi_4 = \tfrac{1}{2}((x^2)^2 - (x^1)^2)\,\partial_3 + x^1 \partial_2 - x^2 \partial_1 .$$

ξ_4 generates a rotation about the x^3 coordinate axis which can be written down explicitly:

(10.16) $$\bar{x}^1(x,t) = x^1 \cos t - x^2 \sin t$$
$$\bar{x}^2(x,t) = x^1 \sin t + x^2 \cos t$$
$$\bar{x}^3(x,t) = x^3 + \tfrac{1}{4}((x^2)^2 - (x^1)^2) \sin 2t - \tfrac{1}{2} x^1 x^2 (1 + \cos 2t).$$

Bianchi shows that apart from an overall scale factor any type III metric is isometric to a metric with components $\bar{\underline{g}} = \underline{1} + n(\underline{\hat{e}}^2{}_1 + \underline{\hat{e}}^1{}_2)$ in the alternative basis (10.6), (10.7). n may be assumed nonnegative; positive-definiteness then requires $n < 1$. Let $m = (1-n^2)^{-1}$. The metric and additional Killing vector are explicitly:

(10.18) $$e^{-2\bar{x}^3} d\bar{x}^1 \otimes d\bar{x}^1 + d\bar{x}^2 \otimes d\bar{x}^2 + n e^{-\bar{x}^3} (d\bar{x}^1 \otimes d\bar{x}^2 + d\bar{x}^2 \otimes d\bar{x}^1) + d\bar{x}^3 \otimes d\bar{x}^3$$
$$\xi_4 = \tfrac{1}{2}(m e^{2\bar{x}^3} - (\bar{x}^1)^2)\bar{\partial}_1 - mn e^{\bar{x}^3} \bar{\partial}_2 - \bar{x}^1 \bar{\partial}_3 .$$

The vector fields $\{\tilde{e}_3, \xi_4, \tilde{e}_1\}$ generate a Lie subalgebra; for $n \neq 0$ they are also everywhere linearly independent and hence generate a simply transitive type VIII subgroup of isometries (the ordered basis $\{\tilde{e}_3, (\tilde{e}_1 + \xi_4)/\sqrt{2}, (\tilde{e}_1 - \xi_4)/\sqrt{2}\}$ has the canonical type VIII bracket algebra). The metric component matrix in the canonical basis is $\underline{g} = \mathrm{diag}(1+n, 1-n, 4)$.

We next consider the nondegenerate class A types, for which it is convenient to recall the relation (8.18):

(10.20) $\quad \pounds_{e_c}\, \underset{\sim}{g} = -2\, K_{cab}\, \omega^a \otimes \omega^b, \quad \underline{K}_c = \tfrac{1}{2}(\underline{k}_c + \underline{g}^{-1}\underline{k}_c^T \underline{g})$

When the matrix $\underline{K}_c$ for fixed c vanishes, e_c becomes an additional Killing field. If the entire set $\{\underline{K}_c\}$ vanishes, the metric will also be right invariant and hence bi-invariant. For a diagonal metric matrix $\underline{g}' = \mathrm{diag}(g_{11}, g_{22}, g_{33}) = e^{2\underline{\beta}}$ in the canonical basis, using the scale group parametrization and the notation $\beta^{bc} = \beta^b - \beta^c$, the matrices $\{\underline{K}'_a\}$ are given by:

(10.21)
$$2\underline{K}_1' = (n^{(3)} - n^{(2)} e^{2\beta^{23}})\,\hat{\underline{e}}^2{}_3 + (n^{(3)} e^{2\beta^{32}} - n^{(2)})\,\hat{\underline{e}}^3{}_2 - a\,(\hat{\underline{e}}^3{}_1 + e^{2\beta^{13}}\hat{\underline{e}}^1{}_3)$$
$$2\underline{K}_2' = (n^{(1)} - n^{(3)} e^{2\beta^{31}})\,\hat{\underline{e}}^3{}_1 + (n^{(1)} e^{2\beta^{13}} - n^{(3)})\,\hat{\underline{e}}^1{}_3 - a\,(\hat{\underline{e}}^3{}_2 + e^{2\beta^{23}}\hat{\underline{e}}^2{}_3)$$
$$2\underline{K}_2' = (n^{(2)} - n^{(1)} e^{2\beta^{12}})\,\hat{\underline{e}}^1{}_2 + (n^{(2)} e^{2\beta^{21}} - n^{(1)})\,\hat{\underline{e}}^2{}_1 + 2a\,(\hat{\underline{e}}^1{}_1 + \hat{\underline{e}}^2{}_2).$$

The prime reminds us of the restriction to diagonal metric matrices.

For types VII_0, VIII, IX when $\beta^{12} = 0$ $(g_{11} = g_{22})$, $\underline{K}_3'$ vanishes making $\xi_6 = e_3$ an additional Killing vector, while in type IX $\beta^{12} = \beta^{23} = \beta^{31} = 0$ $(g_{11} = g_{22} = g_{33})$ corresponds to a bi-invariant metric of a geometry having constant positive curvature $\tfrac{1}{4}$ when $\underline{g} = \underline{1}$. The $\beta^{12} = 0$ metrics are explicitly:

(10.22) $\quad \mathrm{VII}_0: \quad g_{11}\,(dx^1 \otimes dx^1 + dx^2 \otimes dx^2) + g_{33}\, dx^3 \otimes dx^3$

$\mathrm{VIII}, \mathrm{IX}: \quad g_{11}\,(C_2^2\, dx^1 \otimes dx^1 + dx^2 \otimes dx^2) +$
$\qquad g_{33}\,(S_2^2\, dx^1 \otimes dx^1 + S_2\,(dx^1 \otimes dx^3 + dx^3 \otimes dx^1) + dx^3 \otimes dx^3)$

The first shows that such type VII_0 metrics correspond to flat space and hence admit two additional Killing fields:

(10.23) $\quad \xi_4 = \gamma x^2 \partial_3 - \gamma^{-1} x^3 \partial_2, \quad \xi_5 = \gamma^{-1} x^3 \partial_1 - \gamma x^1 \partial_3, \quad \gamma = (g_{11}/g_{33})^{\frac{1}{2}}.$

Type VIII requires an indefinite metric $\underline{g} = \mathrm{diag}(1, 1, -1)$ to admit further Killing fields. The remaining Bianchi types VI_0, VI_h, IV admit no additional Killing fields. (Type VI_0 requires an indefinite metric $\underline{g} = \mathrm{diag}(g_{11}, -g_{11}, g_{33})$ for e_3 to become a Killing field as in the other nondegenerate class A types.)

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Only part of this additional Killing structure has any implication for spatially homogeneous cosmology. Consider the following metric on $\mathbb{R} \times G$ with g_{ab} functions only of t (this will be developed in more detail in §12):

(10.23) $\quad \underset{\sim}{g} = -dt \otimes dt + g_{ab}\, \omega^a \otimes \omega^b.$

Suppose $\xi = \xi^a e_a$ (with ξ^a possibly t-dependent in addition to its

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x-dependence) is a Killing vector field of the induced metric $g_{ab}\,\omega^a \otimes \omega^b$ on constant t hypersurfaces. Only the $(0,a)$ components of the spacetime Killing equation (A.14) in the basis $\{e_\alpha\} = \{\partial/\partial t, e_a\}$ remain to be satisfied. These are equivalent to $\partial \xi^a/\partial t = 0$. Therefore essentially only those additional Killing vector fields discussed above which are independent of the metric components remain Killing vector fields of the spacetime metric. Whether or not it is possible to have metrics of the special form for which this is true depends on the dynamics. It will turn out that only three situations exist corresponding to flat spacetime ($\mathrm{I} \cap \mathrm{VII}_0$), Taublike models with an additional Killing vector field ($\mathrm{I} \cap \mathrm{VII}_0, \mathrm{II}, \mathrm{III}, \mathrm{V} \cap \mathrm{VII}_h, \mathrm{VIII}, \mathrm{IX}$) and Friedmann models with three additional Killing vector fields ($\mathrm{I} \cap \mathrm{VII}_0$, $\mathrm{V} \cap \mathrm{VII}_h$, IX).

When $\underline{n}$ is diagonal and has a pair of diagonal values of equal magnitude but opposite sign, a rotation by $\frac{\pi}{4}$ about the eigenvector corresponding to the remaining diagonal value brings $\underline{n}$ into a new canonical form $\bar{\underline{n}}$. Consider type VI_h with $\underline{n} = q(\hat{\underline{e}}^1{}_1 - \hat{\underline{e}}^2{}_2)$ and $a_a = a\,\delta^3_a$:

$$\text{(10.24)} \qquad \underline{A} = \exp \tfrac{\pi}{4} \underline{k}_3^{\mathrm{IX}} = 2^{-1/2} \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2^{1/2} \end{pmatrix}, \quad \underline{A}^{-1} = \underline{A}^T,$$

$$\bar{\underline{n}} = \underline{A}\,\underline{n}\,\underline{A}^T = q\,(\hat{\underline{e}}^2{}_1 + \hat{\underline{e}}^1{}_2), \qquad \bar{a}_a = a_a,$$

$$\bar{\underline{k}}_1 = -(q+a)\,\hat{\underline{e}}^3{}_1, \quad \bar{\underline{k}}_2 = (q-a)\,\hat{\underline{e}}^3{}_2, \quad \bar{\underline{k}}_3 = (a+q)\,\hat{\underline{e}}^1{}_1 + (a-q)\,\hat{\underline{e}}^2{}_2 .$$

Notice that the special automorphism generator $\bar{\underline{k}}_3^0 = q(\hat{\underline{e}}^1{}_1 - \hat{\underline{e}}^2{}_2)$ is now diagonal and that now $\{\hat{\underline{e}}^1{}_1, \hat{\underline{e}}^2{}_2, \hat{\underline{e}}^3{}_1, \hat{\underline{e}}^3{}_2\}$ is a basis of $\mathrm{aut}_e(g)$. q is related to a and h by the relation $a^2 = -h q^2$; define $\lambda = q/a$. Then $q = 1$, $\lambda = a^{-1}$ corresponds to our canonical type VI_h components, $q = a = \lambda = 1$ to type III, $q = 0 = \lambda$ to type V and $q = 0 = a$ to type I. If $\{e_a\}$ is a basis of the Lie algebra of a Lie group with the unbarred SCT components, then $\{\bar{e}_a = e_b A^{-1b}{}_a\}$ is a basis with the following brackets:

$$\text{(10.25)} \qquad [\bar{e}_2, \bar{e}_3] = (q-a)\,\bar{e}_2, \quad [\bar{e}_3, \bar{e}_1] = (q+a)\,\bar{e}_1, \quad [\bar{e}_1, \bar{e}_2] = 0 .$$

Suppose we perform the corresponding rotation about the first basis vector in type VIII where $\underline{n} = \hat{\underline{e}}^1{}_1 + \hat{\underline{e}}^2{}_2 - \hat{\underline{e}}^3{}_3$:

(10.28) $A = \exp \frac{\pi}{2} \underline{k}_1^{\text{II}} = 2^{-1/2} \begin{pmatrix} 2^{1/2} & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}$, $\bar{e}_a = e_b A^{-1b}{}_a$,

$\bar{\underline{n}} = \underline{A}\,\underline{n}\,\underline{A}^T = \hat{\underline{e}}^1{}_1 + \hat{\underline{e}}^2{}_3 + \hat{\underline{e}}^3{}_2$,

$[\bar{e}_2, \bar{e}_3] = \bar{e}_1$, $[\bar{e}_3, \bar{e}_1] = \bar{e}_3$, $[\bar{e}_1, \bar{e}_2] = \bar{e}_2$.

The basis $\{\bar{e}_a\}$ here is closely related to Bianchi's choice. For the "Taublike" type VIII metrics, $\underline{g} = \text{diag}(g_{11}, g_{11}, g_{33})$ and e_3 is an additional Killing field which commutes with all the right invariant vector fields $\{\tilde{e}_a\}$ and hence with $\bar{\tilde{e}}_a = \tilde{e}_b A^{-1b}{}_a$. Since $[\bar{\tilde{e}}_3, \bar{\tilde{e}}_1] = -\bar{\tilde{e}}_3$, the ordered set $\{\bar{\tilde{e}}_1 = \tilde{e}_1, e_3, \bar{\tilde{e}}_3 = 2^{-1/2}(\tilde{e}_2 + \tilde{e}_3)\}$ has the same bracket algebra as (10.7) and generates a simply transitive isometry subgroup of type III.

This second canonical form for types VI_h and VIII is halfway between our diagonal form and one used by Ellis and MacCallum.[42] For type VI_h, it is closely related to Bianchi's choice of canonical SCT components. His parameter h_B is related to h by:

(10.29) $h = -(1+h_B)^2/(1-h_B)^2$.

A single-valued relation is obtained by restricting h_B to the interval $[-1, 1)$, on which:

(10.30) $h_B = (h - 1 + 2(-h)^{\frac{1}{2}})/(1+h)$, $a = (1+h_B)/(1-h_B)$.

If we let $\underline{B} = \underline{A}\,((n_B - 1)/2)^{-1}$, where $\underline{A}$ is as in (10.22), then $\underline{n}_B$ and $(a_B)_a = a_B \delta^3_a$ for Bianchi's SCT components are related to our canonical $\underline{n}$ and a_a by the formula (8.8) with $\underline{B}$ in place of $\underline{A}$ there:

(10.31) $\underline{n}_B = \frac{1}{2}(h-1)(\hat{\underline{e}}^2{}_1 + \hat{\underline{e}}^1{}_2)$ $\qquad a_B = \frac{1}{2}(h_B - 1)a = -\frac{1}{2}(1+h_B)$.

Similarly Bianchi's choice of type VII_h SCT components is related to our canonical choice by the same transformation (8.8) but with $\underline{B}$ now a combined dilation and "boost" rather than a dilation and rotation:

(10.32) $\underline{B} = -\cosh 2\theta \exp(-\theta(\hat{\underline{e}}^1{}_2 + \hat{\underline{e}}^2{}_1))$, $\tanh 2\theta = (h/(1+h))^{1/2} = \frac{1}{2} h_B$,

$\underline{n}_B = -\underline{I}^{(3)} + \frac{1}{2} h_B (\hat{\underline{e}}^1{}_2 + \hat{\underline{e}}^2{}_1)$, $a_B = -\frac{1}{2} h_B$.

Correction for page (10.9). At bottom of page add (same paragraph):

For the explicit type VI_h basis $\{e_a\}$ given in (10.5), the coordinate expressions for $\{\bar{e}_a\}$ and $\{\bar{\omega}^a\}$ in terms of new coordinates $\bar{x}^a = A^a{}_b x^b$ (canonical coordinates of the second kind with respect to the new basis) are given by:

$$\text{(10.26)}\quad \bar{e}_1 = e^{(q+a)\bar{x}^3}\bar{\partial}_1 \qquad \bar{\omega}^1 = e^{-(q+a)\bar{x}^3}d\bar{x}^1$$
$$\bar{e}_2 = e^{-(q-a)\bar{x}^3}\bar{\partial}_2 \qquad \bar{\omega}^2 = e^{(q-a)\bar{x}^3}d\bar{x}^2$$
$$\bar{e}_3 = \bar{\partial}_3 \qquad \bar{\omega}^3 = d\bar{x}^3 \quad .$$

The corresponding right invariant basis is:

$$\text{(10.27)}\quad \tilde{\bar{e}}_1 = \bar{\partial}_1 \qquad \tilde{\bar{e}}_2 = \bar{\partial}_2 \qquad \tilde{\bar{e}}_3 = \bar{\partial}_3 + (q+a)\bar{x}^1\bar{\partial}_1 - (q-a)\bar{x}^2\bar{\partial}_2 \quad .$$

The type $\mathrm{III} = \mathrm{VI}_{-1}$ bases of (10.7) are exactly of this form with $q = a = \frac{1}{2}$.

After the first paragraph of page (10.9) (which is to be put at the end of the section) add (same paragraph):

As an example consider type III metrics with $\bar{g}$ diagonal in the noncanonical basis. The extra Killing vector discussed above depends only on the ratio $(\bar{g}_{11}/\bar{g}_{33})$ and hence is time-independent if and only if $(\bar{g}_{11}/\bar{g}_{33})^{\cdot} = 0$: Since $\hat{\underline{e}}'_1$ generates an automorphism, one may pick a new basis with the same structure constant tensor components in which this ratio is unity when it is time-independent. Thus for type III spacetime metrics with an additional Killing vector field of this sort one may assume $\underline{\bar{g}} = \mathrm{diag}\,(\bar{g}_{33}, \bar{g}_{22}, \bar{g}_{33})$.