§6. Matrix Groups

We now apply our theory to matrix groups, after the introduction of a good deal of notation. $g\ell(n,\mathbb{R})$ is the vector space of all real $n\times n$ matrices with the natural basis $\{\underline{\hat{e}}^b{}_a\}$ of matrices having 1 in the b^{th} column, a^{th} row and zeros elsewhere. Let $\{\hat{\omega}^a{}_b\}$ be the dual basis, satisfying the duality relations $\hat{\omega}^a{}_b(\underline{\hat{e}}^c{}_d) = \delta^a_d\delta^c_b$. Any matrix $\underline{A}$ may be expressed as $A^a{}_b\underline{\hat{e}}^b{}_a$ with $A^a{}_b = \hat{\omega}^a{}_b(\underline{A})$ so that the upper left index in the component symbol refers to the row and the lower right one to columns. Note that $g\ell(n,\mathbb{R})$ and its dual space may be identified through the correspondence $A^a{}_b\underline{\hat{e}}^b{}_a \leftrightarrow A^a{}_b\hat{\omega}^b{}_a$.

$g\ell(n,\mathbb{R})$ may also be viewed as an n^2-dimensional Euclidean space $\mathbb{R}^{n^2}$ with cartesian coordinates $\{a^a{}_b\}$ defined by $a^a{}_b(\underline{A}) = \hat{\omega}^a{}_b(\underline{A}) = A^a{}_b$ and with the natural manifold structure. All the indices of ordinary tensor analysis are doubled in these coordinates. The coordinate frame $\{\partial^b{}_a = \partial/\partial a^a{}_b\}$ and dual frame $\{da^a{}_b\}$ satisfy duality relations of the form:

(6.1) $\quad da^a{}_b(\partial^c{}_d) = \partial^c{}_d a^a{}_b = \delta^a_d\delta^c_b$.

Vector fields X and 1-forms σ, for example, have the expansions $X^a{}_b\partial^b{}_a$ and $\sigma^a{}_b da^b{}_a$ respectively, with $X^a{}_b = X a^a{}_b$ and $\sigma^a{}_b = \sigma(\partial^a{}_b)$. The identity map on $g\ell(n,\mathbb{R})$ may be written as a matrix-valued function $\underline{a} = a^a{}_b\underline{\hat{e}}^b{}_a$ satisfying $\underline{a}(\underline{A}) = \underline{A}$.

$g\ell(n,\mathbb{R})$ has two interesting scalar functions, the trace Tr and the determinant det:

(6.2) $\quad \mathrm{Tr}\,\underline{A} = A^b{}_b$

$$\det\underline{A} = \Delta(\underline{A}) = (1/n!)\,\delta^{c_1\cdots c_n}_{d_1\cdots d_n}A^{d_1}{}_{c_1}\cdots A^{d_n}{}_{c_n} = (1/n)\,\Delta^c{}_d(\underline{A})\,A^d{}_c\,.$$

Closely associated with the latter is a map $\underline{A}\mapsto\underline{A}^{-1}$ defined on the support of the determinant function:

(6.3) $\quad A^{-1\,c}{}_d = (\det\underline{A})^{-1}\Delta^c{}_d(\underline{A})$.

We use the obvious shorthand $A^{-1\,c}{}_d = a^c{}_d(\underline{A}^{-1}) = a^{-1\,c}{}_d(\underline{A})$ and $\underline{a}^{-1} = a^{-1a}{}_b\underline{\hat{e}}^b{}_a$.

The transpose map of $g\ell(n,\mathbb{R})$ into itself is defined by:

(6.4) $\quad \underline{A}\mapsto\underline{A}^T \qquad A^{T\,a}{}_b = A^b{}_a$.

This enables us to decompose $g\ell(n,\mathbb{R})$ into a direct sum of two subvector spaces which are respectively even and odd under this map, namely the $n(n+1)/2$-dimensional space $S(n,\mathbb{R})$ of symmetric matrices and the $n(n-1)/2$-dimensional space of antisymmetric matrices $o(n,\mathbb{R})$. Similarly the trace function leads to the direct sum decomposition into the space $sl(n,\mathbb{R})$ of traceless matrices and the 1-dimensional space of multiples of the identity matrix $\underline{1} = \hat{\underline{e}}^a{}_a$ (having components $\delta^a{}_b$), which may be abbreviated by span $\{\underline{1}\}$. We may also break $gl(n,\mathbb{R})$ into the diagonal matrices $d(n,\mathbb{R})$ and the off-diagonal matrices $offd(n,\mathbb{R})$. (The notation diag $(A^{(1)},\dots,A^{(n)})$ indicates a diagonal matrix $\underline{A}$ with components $A^{(b)}$ along the diagonal.) In fact the following direct sum decomposition will soon find application:

(6.5) $$gl(n,\mathbb{R}) = \mathrm{span}\{\underline{1}\} \oplus sd(n,\mathbb{R}) \oplus (S(n,\mathbb{R}) \cap offd(n,\mathbb{R})) \oplus o(n,\mathbb{R})$$

Here $sd(n,\mathbb{R}) = d(n,\mathbb{R}) \cap sl(n,\mathbb{R})$ consists of the traceless diagonal matrices. (s always signals the traceless subset.) In this direct sum the first two summands represent $d(n,\mathbb{R})$, the first three $S(n,\mathbb{R})$, the last two $offd(n,\mathbb{R})$ and the last three $sl(n,\mathbb{R})$.

Matrix multiplication is defined on the natural basis by:

(6.6) $$\hat{\underline{e}}^a{}_b \hat{\underline{e}}^c{}_d = \delta^a{}_d \hat{\underline{e}}^c{}_b ,$$

and extended by bilinearity to all matrices yielding $(\underline{A}\underline{B})^a{}_b = A^a{}_c B^c{}_b$. $gl(n,\mathbb{R})$ is naturally a Lie algebra with the bracket defined to be the matrix commutator: $[\underline{A},\underline{B}] = \underline{A}\underline{B} - \underline{B}\underline{A}$. (6.6) helps evaluate the components of the structure constant tensor in the natural basis:

(6.7) $$[\hat{\underline{e}}^a{}_b, \hat{\underline{e}}^c{}_d] = (\delta^a{}_d \delta^c{}_g \delta^f{}_b - \delta^c{}_b \delta^a{}_g \delta^f{}_d)\, \hat{\underline{e}}^g{}_f = C^{f}{}_{g}{}^{a}{}_{b}{}^{c}{}_{d}\, \hat{\underline{e}}^g{}_f .$$

The familiar properties:

(6.8) $$\mathrm{Tr}\, \underline{A}\underline{B} = \mathrm{Tr}\, \underline{B}\underline{A} , \qquad (\underline{A}\underline{B})^T = \underline{B}^T \underline{A}^T ,$$

make it easy to see that $o(n,\mathbb{R})$ and $sl(n,\mathbb{R})$ are each Lie subalgebras, as are span$\{\underline{1}\}$, $sd(n,\mathbb{R})$ and $d(n,\mathbb{R})$. Span$\{\underline{1}\}$ is the center of $gl(n,\mathbb{R})$. The first property shows that $\mathrm{Tr}\,[\underline{A},\underline{B}] = 0 = [\mathrm{Tr}\,\underline{A}, \mathrm{Tr}\,\underline{B}]$ and hence $\mathrm{Tr} : gl(n,\mathbb{R}) \to \mathbb{R} \cong gl(1,\mathbb{R})$ is a representation of $gl(n,\mathbb{R})$; its kernel $sl(n,\mathbb{R})$ is therefore an ideal. The second property shows that $[\underline{A},\underline{B}]^T = [\underline{B}^T, \underline{A}^T]$ so the transpose map is a transposition of $gl(n,\mathbb{R})$.

$gl(n,\mathbb{R})$ has a natural inner product $\langle\, ,\rangle$ defined by the trace of the matrix product: $\langle \underline{A},\underline{B}\rangle = \mathrm{Tr}\, \underline{A}\underline{B}$. The direct sum $gl(n,\mathbb{R}) = S(n,\mathbb{R}) \oplus o(n,\mathbb{R})$ is orthogonal with respect to this trace inner product which is positive-definite (space like) on $S(n,\mathbb{R})$ and negative-definite (time like) on $o(n,\mathbb{R})$. In fact,

the detailed direct sum decomposition (6.5) is orthogonal with respect to $\langle , \rangle$.
A pseudo-orthonormal basis adapted to the decomposition $gl(n,\mathbb{R}) = d(n,\mathbb{R}) \oplus (S(n,\mathbb{R}) \cap \mathrm{offd}(n,\mathbb{R})) \oplus o(n,\mathbb{R})$ is:

(6.9) $\{\hat{e}^1{}_1, \ldots, \hat{e}^n{}_n ; (\hat{e}^1{}_2 + \hat{e}^2{}_1)/\sqrt{2}, \ldots ; (\hat{e}^1{}_2 - \hat{e}^2{}_1)/\sqrt{2}, \ldots \}$.

A related inner product might appropriately be named after DeWitt:

(6.10) $\langle \underline{A}, \underline{B} \rangle_{DW} = \mathrm{Tr}\, \underline{A}\underline{B} - \mathrm{Tr}\underline{A}\, \mathrm{Tr}\underline{B}$.

This reduces to $\langle , \rangle$ on $sl(n,\mathbb{R})$ and changes $\underline{1}$ from a spacelike to a timelike direction while leaving unchanged its orthogonality to $sl(n,\mathbb{R})$. ($\langle \underline{1}, \underline{1} \rangle = n$ and $\langle \underline{1}, \underline{1} \rangle_{DW} = n - n^2$, so $n > 1$ for the first statement to hold.)

The exponential map exp is defined by:

(6.11) $\exp \underline{A} = \sum_{n=0}^{\infty} (\underline{A})^n / n! = e^{\underline{A}}$, $\underline{A}^0 = \underline{1}$.

Using the obvious property (6.22) $\underline{B} e^{\underline{A}} \underline{B}^{-1} = e^{\underline{B}\underline{A}\underline{B}^{-1}}$ of the exponential series and the fact that for every $\underline{A} \in gl(n,\mathbb{R})$, there exists a $\underline{B} \in gl(n,\mathbb{R})$ such that $\underline{B}\underline{A}\underline{B}^{-1}$ has zeros below the diagonal, one easily proves the formula [1]:

(6.12) $\det \circ \exp = \exp \circ \mathrm{Tr}$.

The exp on the right is the ordinary exponential on $\mathbb{R} = gl(1,\mathbb{R})$. As a result, the exponential of a traceless matrix has unit determinant. Unit determinant matrices are called unimodular.

The general linear group $GL(n,\mathbb{R})$, inheriting its manifold structure from $gl(n,\mathbb{R})$ as the open set where det is nonzero, is naturally a Lie group under matrix multiplication. $\underline{1}$ is the identity and $\underline{A}^{-1}$ the inverse of $\underline{A}$. The closure of the multiplication is a consequence of the familiar formula:

$$\det \underline{A}\underline{B} = \det \underline{A} \det \underline{B} .$$

(Its consequence $1 = \det \underline{1} = \det \underline{A} \det \underline{A}^{-1}$ shows $GL(n,\mathbb{R})$ to be closed under the inverse map.) In fact this shows that $\det : GL(n,\mathbb{R}) \to \mathbb{R} - \{0\} = GL(1,\mathbb{R})$ is a representation of $GL(n,\mathbb{R})$; its kernel is the normal subgroup of unimodular matrices $SL(n,\mathbb{R})$. Also since $\det \underline{A}^T = \det \underline{A}$, the transpose map restricts to a transposition of $GL(n,\mathbb{R})$. $GL(n,\mathbb{R})$ consists of two disconnected pieces corresponding to the two possible signs of the determinant function. The component $GL(n,\mathbb{R})^+$ for which the sign is positive contains the identity and is a subgroup of $GL(n,\mathbb{R})$.

The multiplication function of the component coordinates is $\varphi^a{}_b(a_1, a_2) = a_1{}^a{}_c\, a_2{}^c{}_b$ and therefore by (1.31) the left and right invariant frames generated by the coordinate frame at the identity are:

(6.13) $e^a{}_b = (\partial^{(2)a}{}_b \varphi^c{}_d)(\underline{1}, a)\, \partial^d{}_c = a^c{}_b\, \partial^a{}_c$

$$\tilde{e}^a{}_b = (\partial^{(1)}{}^a{}_b \varphi^c{}_d)(\underline{a}, \underline{1})\,\partial^d{}_c = a^a{}_c\,\partial^c{}_b .$$

The dual frames may be constructed from the duality relations $\omega^a{}_b(e^c{}_d) = \delta^a_d \delta^c_b = \tilde{\omega}^a{}_b(\tilde{e}^c{}_d)$:

(6.14) $\quad \omega^a{}_b = a^{-1a}{}_c\, da^c{}_b \qquad \tilde{\omega}^a{}_b = da^a{}_f\, a^{-1f}{}_b .$

A compact notation may be achieved by defining matrix-valued vector fields and -forms:

(6.15) $\quad \underline{\partial} = \hat{e}^a{}_b \partial^b{}_a , \quad \underline{e} = \hat{e}^b{}_a e^a{}_b , \quad \underline{\omega} = \hat{e}^b{}_a \omega^a{}_b , \ldots$

With matrix multiplication understood (6.13) and (6.14) may be written:

(6.16) $\quad \underline{e} = \underline{\partial}\,\underline{a} \qquad \tilde{\underline{e}} = \underline{a}\,\underline{\partial} \qquad \underline{\omega} = \underline{a}^{-1} d\underline{a} \qquad \tilde{\underline{\omega}} = d\underline{a}\,\underline{a}^{-1}.$

From $d^2 = 0$ it follows that:

$$0 = d^2\underline{a} = d(\underline{a}\,\underline{\omega}) = d\underline{a} \wedge \underline{\omega} + \underline{a}\, d\underline{\omega}$$

$$0 = d^2\underline{a} = d(\tilde{\underline{\omega}}\,\underline{a}) = d\tilde{\underline{\omega}}\,\underline{a} - \tilde{\underline{\omega}} \wedge d\underline{a} .$$

Multiplying through by $\underline{a}^{-1}$ and using the definitions leads to the result:

(6.17) $\quad d\underline{\omega} = -\underline{\omega} \wedge \underline{\omega} \qquad d\tilde{\underline{\omega}} = \tilde{\underline{\omega}} \wedge \tilde{\underline{\omega}} .$

Another short computation using (6.1) and the definitions shows that:

(6.18) $\quad [e^a{}_b, e^c{}_d] = C^{fg}{}_{a}{}^{b}{}_{c}{}^{d}\, e^g{}_f .$

Comparison with (6.7) shows that the linear map "boldface ∘ $\wedge$" sending $A = A^a{}_b e^b{}_a$ into $\underline{\hat{A}} = A^a{}_b \hat{\underline{e}}^b{}_a$ is a Lie algebra isomorphism from the Lie algebra $\mathfrak{g}$ of left invariant vector fields on $GL(n,\mathbb{R})$ to $\mathfrak{gl}(n,\mathbb{R})$:

$$[A,B] = [\underline{\hat{A}}, \underline{\hat{B}}]^a{}_b\, e^b{}_a \qquad A, B \in \mathfrak{g} .$$

We may therefore identify $\mathfrak{gl}(n,\mathbb{R})$ (with its natural Lie algebra structure) with the Lie algebra of the group $GL(n,\mathbb{R})$. The left and right invariant vector fields corresponding to $\underline{\hat{A}} \in \mathfrak{gl}(n,\mathbb{R})$ are $A = \mathrm{Tr}\, \underline{\hat{A}}\,\underline{e}$ and $\tilde{A} = \mathrm{Tr}\, \underline{\hat{A}}\,\tilde{\underline{e}}$ in the notation (6.16). The Lie brackets in this notation are:

(6.19) $\quad [\mathrm{Tr}\,\underline{A}\,\underline{e}, \mathrm{Tr}\,\underline{B}\,\underline{e}] = \mathrm{Tr}\,[\underline{A},\underline{B}]\,\underline{e} \qquad [\mathrm{Tr}\,\underline{A}\,\underline{e}, \mathrm{Tr}\,\underline{B}\,\tilde{\underline{e}}] = 0 .$

$\quad\quad\quad [\mathrm{Tr}\,\underline{A}\,\tilde{\underline{e}}, \mathrm{Tr}\,\underline{B}\,\tilde{\underline{e}}] = -\mathrm{Tr}\,[\underline{A},\underline{B}]\,\tilde{\underline{e}}$

If $\hat{A} \in TG_{a_0}$, then $\underline{\omega}(\hat{A})(a_0) = \underline{\hat{A}}$ is its matrix of components in the component frame, making explicit the map "boldface" from TG_{a_0} onto $\mathfrak{gl}(n,\mathbb{R})$ whose composition with the map $\wedge : \mathfrak{g} \to TG_{a_0}$ yields the isomorphism $A \mapsto \underline{\hat{A}} = \underline{\omega}(A)$ between $\mathfrak{g}$ and $\mathfrak{gl}(n,\mathbb{R})$. $\underline{\omega}$ and $\tilde{\underline{\omega}}$ are the left and right invariant $\hat{\mathfrak{g}}$-valued 1-forms composed with "boldface".

For $A = A^a{}_b e^b{}_a \in \mathfrak{g}$, the relation $e^a{}_b\, a^c{}_d = \delta^a_d\, a^c{}_b$ (see (6.13), (6.1)) enables us to evaluate (2.4):

(6.20) $A a^c{}_d = A^b{}_a \delta^a{}_d a^c{}_b = [\underline{a}\hat{A}]^c{}_d$

$[A_t \underline{B}]^c{}_d = (e^{tA} a^c{}_d)(\underline{B}) = (\underline{a} e^{t\hat{A}})(\underline{B}) = \underline{B} e^{t\hat{A}}$

$A_t \underline{B} = \underline{B} e^{t\hat{A}}$.

Thus the integral curves of the left invariant vector field A are simply right multiplication by $e^{t\hat{A}}$, with a similar statement holding for right invariant vector fields. In particular the exponential map from the Lie algebra of the group coincides with the matrix exponential if we identify $g\ell(n,\mathbb{R})$ with that algebra:

(6.21) $\exp A = A_1 \underline{1} = e^{\hat{A}} = \exp \hat{A}$.

(4.8) and (6.12) show that the trace representation Tr of $g\ell(n,\mathbb{R})$ is the representation $\det{}'$ associated with the determinant representation of $GL(n,\mathbb{R})$.

$s\ell(n,\mathbb{R})$ is the Lie algebra of the special linear group $SL(n,\mathbb{R})$ and since $e^{-\underline{A}} = e^{\underline{A}^T} = (e^{\underline{A}})^T$ for $A \in o(n,\mathbb{R})$, $o(n,\mathbb{R})$ is the Lie algebra of the orthogonal group $O(n,\mathbb{R}) = \{\underline{O} \in GL(n,\mathbb{R}) \mid \underline{O}^T\underline{O} = \underline{1}\}$. The special orthogonal group $SO(n,\mathbb{R}) = O(n,\mathbb{R}) \cap SL(n,\mathbb{R})$ is the component of $O(n,\mathbb{R})$ with unit determinant. (The unimodular subgroup of any matrix group is indicated by the the modifier "special".) Similarly $d(n,\mathbb{R})$ is the Lie algebra of the abelian subgroup of nonsingular diagonal matrices $D(n,\mathbb{R})$. The component $D(n,\mathbb{R})^+$ connected to the identity is the subgroup of positive definite diagonal matrices or "scale group", since it acts on $\mathbb{R}^n$ by scaling the natural basis. This group is the direct product of the pure dilation subgroup $\{c\underline{1} \mid c \in \mathbb{R}^+\}$ (the center of $GL(n,\mathbb{R})$) and the unimodular "shear" subgroup $SD(n,\mathbb{R})$ generated by $sd(n,\mathbb{R})$.

The adjoint formulas (5.1) and (5.5) are familiar matrix relations:

(6.22) $\underline{B} e^{\underline{A}} \underline{B}^{-1} = e^{\underline{B}\underline{A}\underline{B}^{-1}} \qquad e^{\underline{A}} \underline{B} e^{-\underline{A}} = e^{ad(\underline{A})} \underline{B}$.

Another useful formula is:

(6.23) $d \ln \det \underline{a} = \mathrm{Tr}\, \underline{a}^{-1} d\underline{a} = \mathrm{Tr}\, \underline{\omega} = \mathrm{Tr}\, \tilde{\omega}$.

Its proof is simple; using an abbreviated notation:

$$d\Delta = (1/n!)\, \delta^{c_1 \cdots c_n}_{d_1 \cdots d_n} (da^{d_1}{}_{c_1} \cdots a^{d_n}{}_{c_n} + \cdots + a^{d_1}{}_{c_1} \cdots da^{d_n}{}_{c_n})$$

$$= \Delta^c{}_d\, da^d{}_c = \Delta\, a^{-1c}{}_d\, da^d{}_c.$$

Similar formulas resulting from the exterior derivative of $\underline{a}\underline{a}^{-1} = \underline{1}$ are:

(6.24) $\underline{a}\, d\underline{a}^{-1} = -d\underline{a}\, \underline{a}^{-1}, \qquad d\underline{a}^{-1} = -\underline{a}^{-1} d\underline{a}\, \underline{a}^{-1}$.

Let $\{\hat{e}_A\}$ be a basis of an r-dimensional Lie subalgebra $\mathfrak{g}$ of $\mathfrak{gl}(n,\mathbb{R})$ generating an r-dimensional subgroup G of $GL(n,\mathbb{R})$ and let $C^A{}_{BC}$ be the components of the structure constant tensor in this basis. Let $\underset{\sim}{S}$ denote the restriction of the matrix-valued function $\underset{\sim}{a}$ to the submanifold G. In the same way that $\underset{\sim}{a} = a^a{}_b \hat{e}^b{}_a$ is parametrized in terms of coordinates on $GL(n,\mathbb{R})$, $\underset{\sim}{S}$ may be parametrized locally in terms of local coordinates $\{X^A\}$ on G, since the restrictions of the functions $a^a{}_b$ to G (namely $S^a{}_b$) then become functions of the coordinates on G. Let $\underset{\sim}{S}(X)$ denote an explicit parametrization of $\underset{\sim}{S}$ in terms of local coordinates $\{X^A\}$ on G. For example, canonical coordinates of the first and second kinds lead to the parametrizations:

(6.25) $\underset{\sim}{S}(x) = \exp X^A \hat{e}_A$, $\underset{\sim}{S}(x) = \exp X^1 \hat{e}_1 \cdots \exp X^r \hat{e}_r$

$$\det \underset{\sim}{S}(x) = \exp X^A \,\mathrm{TR}\, \hat{e}_A .$$

Let $\underset{\sim}{\Omega} = \underset{\sim}{S}^{-1} d\underset{\sim}{S}$, $\tilde{\underset{\sim}{\Omega}} = d\underset{\sim}{S}\,\underset{\sim}{S}^{-1}$ denote the restrictions of the left, right invariant matrix-valued forms $\omega, \tilde{\omega}$ to G, let $e_A, \tilde{e}_A$ be the invariant vector fields on $GL(n,\mathbb{R})$ (and hence on G by inclusion) corresponding to $\hat{e}_A$ and let $\omega^A, \tilde{\omega}^A$ be the corresponding invariant forms on G. Since d and $\wedge$ commute with restriction to submanifolds (6.17) remains true:

(6.26) $d\underset{\sim}{\Omega} = -\underset{\sim}{\Omega} \wedge \underset{\sim}{\Omega}$ $\quad d\tilde{\underset{\sim}{\Omega}} = \tilde{\underset{\sim}{\Omega}} \wedge \tilde{\underset{\sim}{\Omega}}$,

and of course $\underset{\sim}{\Omega}$ and $\tilde{\underset{\sim}{\Omega}}$ are also invariant as matrix-valued forms on G. In fact restricting the relations $\underset{\sim}{\omega}(e_A) = \hat{e}_A = \tilde{\underset{\sim}{\omega}}(\tilde{e}_A)$ to the subgroup shows that $\underset{\sim}{\Omega}(e_A) = \hat{e}_A = \hat{e}_B \omega^B(e_A)$, etc., and hence:

(6.27) $\underset{\sim}{\Omega} = \hat{e}_B \omega^B = \underset{\sim}{S}^{-1} d\underset{\sim}{S}$ $\quad \tilde{\underset{\sim}{\Omega}} = \hat{e}_B \tilde{\omega}^B = d\underset{\sim}{S}\,\underset{\sim}{S}^{-1}$.

The usual exterior derivative relations for $\omega^A, \tilde{\omega}^A$ follow from (6.26), (6.27) and $[\hat{e}_A, \hat{e}_B] = C^C{}_{AB} \hat{e}_C$. (6.27) is a very useful result. It enables one to read off a basis of invariant 1-forms in any coordinate system on G after a simple though possibly tedious computation; the dual invariant vector fields may then be constructed easily from the duality relations.

Let $\gamma(t)$ be a curve in G with tangent $\gamma'(t)$ and let $X^A(t) = X^A \circ \gamma(t)$ and $\dot{\underset{\sim}{\gamma}}(t) = d/dt\, \underset{\sim}{\gamma}(t)$:

(6.28) $\underset{\sim}{\gamma}(t) = \underset{\sim}{S}(X(t))$

$\gamma'(t) = \dot{S}^a{}_b(X(t))\, \partial^b{}_a = \dot{X}^A(t)\, \partial/\partial X^A \in TG_{\gamma(t)}$,

$\omega^A(\gamma'(t)) = \omega^A{}_B(\underset{\sim}{\gamma}(t))\, \dot{X}^B(t)$, ...

Evaluating (6.27) and the restriction of (6.23) to G on $\gamma'(t)$:

(6.29) $\qquad \underline{\gamma}^{-1}\dot{\underline{\gamma}} = \hat{\underline{e}}_A\, \omega^A(\underline{\gamma}') \qquad\qquad \dot{\underline{\gamma}}\underline{\gamma}^{-1} = \hat{\underline{e}}_A\, \tilde{\omega}^A(\underline{\gamma}')$

$$(\ln\det\gamma)^{\cdot} = \mathrm{Tr}\,\hat{\underline{e}}_A\, \omega^A(\gamma') = \mathrm{Tr}\,\hat{\underline{e}}_A\, \tilde{\omega}^A(\gamma').$$

Conversely, given any functions $N^A(t)$, then $\hat{e}_A N^A(t)$ is a curve in the Lie algebra $\mathfrak{g}$ of G and $\underline{\gamma}^{-1}\dot{\underline{\gamma}} = \hat{\underline{e}}_A N^A$ and $\dot{\underline{\gamma}}\underline{\gamma}^{-1} = \hat{\underline{e}}_A N^A$ are each equations for curves in G.

The adjoint matrix Lie algebra with respect to the basis $\{\hat{\underline{e}}_A\}$ is generated by the (not necessarily linearly independent) matrices $\underline{k}_A = C^B{}_{AC}\,\hat{\underline{e}}^C{}_B \in gl(r,\mathbb{R})$ which satisfy (1.13). By exponentiation we obtain the adjoint matrix group for this basis. The homomorphism from G to its adjoint matrix group may be expressed in local canonical coordinates of the first kind by:

$$\underline{S}(x) = \exp X^A\hat{\underline{e}}_A \mapsto \underline{R}(x) = \exp X^A \underline{k}_A.$$

Expressing (6.22) in this notation yields the result:

(6.30) $\qquad \underline{S}\,\hat{\underline{e}}_A\,\underline{S}^{-1} = \hat{\underline{e}}_B\, R^B{}_A.$

Equation (5.9) holds since the adjoint matrix group of an adjoint matrix group is itself.

Let $\underline{B}$ be an element of the Lie algebra of the matrix automorphism group of the Lie algebra $\mathfrak{g}$ with respect to the basis $\{\hat{\underline{e}}_A\}$. Then $\exp t\underline{B}$ satisfies (1.17); the derivative of this equation at $t=0$ and its trace are:

(6.31) $\qquad [\underline{B}, \underline{k}_A] = \underline{k}_B B^B{}_A \qquad\qquad 0 = C^C{}_{BC}\, B^B{}_A.$

Coordinates on G which involve a product of single exponentials $\exp X^A\hat{\underline{e}}_B$ are often more convenient for calculations since the exponential series may usually be summed easily and the exterior derivative is trivial:

(6.32) $\quad \exp(-X^A\hat{\underline{e}}_B)\, d(\exp X^A\hat{\underline{e}}_B) = \hat{\underline{e}}_B\, dX^A = d(\exp X^A\hat{\underline{e}}_B)\, \exp(-X^A\hat{\underline{e}}_B).$

We may then use the adjoint identity (6.30) to compute the invariant forms. The following illustrates this technique for a 3-dimensional subgroup with canonical coordinates of the second kind:

(6.33) $\qquad \underline{S}(x) = e^{x^1\hat{\underline{e}}_1}\, e^{x^2\hat{\underline{e}}_2}\, e^{x^3\hat{\underline{e}}_3} \qquad \underline{S}^{-1}(x) = e^{-x^3\hat{\underline{e}}_3}\, e^{-x^2\hat{\underline{e}}_2}\, e^{-x^1\hat{\underline{e}}_1}$

$$d\underline{S}\,\underline{S}^{-1} = \left((d e^{x^1\hat{\underline{e}}_1})\, e^{x^2\hat{\underline{e}}_2}\, e^{x^3\hat{\underline{e}}_3} + \dots \right) \underline{S}^{-1}(x)$$

$$= \hat{\underline{e}}_1 dx^1 + (e^{x^1\hat{\underline{e}}_1}\,\hat{\underline{e}}_2\, e^{-x^1\hat{\underline{e}}_1})\, dx^2 + (e^{x^1\hat{\underline{e}}_1} e^{x^2\hat{\underline{e}}_1}\, \hat{\underline{e}}_3\, e^{-x^2\hat{\underline{e}}_2} e^{-x^3\hat{\underline{e}}_3})\, dx^3$$

$$= \hat{\underline{e}}_1 dx^1 + \hat{\underline{e}}_A\left((e^{x^1\underline{k}_1})^A{}_2\, dx^2 + (e^{x^1\underline{k}_1} e^{x^2\underline{k}_2})^A{}_3\, dx^3 \right).$$

Suppose G acts on $\mathbb{R}^n$ (with cartesian coordinates $\{X^a\}$) as a linear transformation group:

(6.34) $X^a(\underline{A}\cdot x) = A^a{}_b X^b(x)$, $\underline{A} \in G$.

The generators of this action are easily computed using (1.31):

(6.35) $\xi_A = \xi(\hat{e}_A) = \hat{e}_A{}^a{}_b X^b \partial/\partial X^a$.

$GL(n,\mathbb{R})$ itself acts as a linear transformation group on $gl(n,\mathbb{R})$ by extending left and right translation to actions on $gl(n,\mathbb{R})$. Application of this formula in the component coordinates yields (6.13). Extension of the adjoint action to $gl(n,\mathbb{R})$ leads to a generating basis given by:

(6.36) $\xi^a{}_b = C^{a}{}_{b}{}^{c}{}_{d}{}^{f}{}_{g}\, a^c{}_d\, \partial^f{}_g = \tilde{e}^a{}_b - e^a{}_b$.

This agrees with (5.11). In fact consider the adjoint action on the Lie algebra $\mathfrak{g}$ of any Lie group G as a linear transformation group. In cartesian coordinates u^a on $\mathfrak{g}$ determined by the basis $\{e_a\}$, (6.35) yields for the generating basis:

(6.37) $\xi_a = C^b{}_{ac} u^c \partial/\partial u^b$.

Because the transpose map is a transposition, an alternative left action of $GL(n,\mathbb{R})$ on itself (extending to a linear action on $gl(n,\mathbb{R})$) may be obtained from the adjoint action by using the transpose rather than inverse map in its definition. This left "transpose action" is given by:

(6.38) $\underline{B} \mapsto f_{\underline{A}}(\underline{B}) = \underline{A}\underline{B}\underline{A}^T$.

This can be made into a right action by composing f with either the transpose or inverse map or into another left action by composition with both in succession (in either order since they commute). It is convenient to redefine the transpose action into a right action by the first method:

(6.39) $\underline{B} \mapsto \tilde{f}_{\underline{A}}(\underline{B}) = \underline{A}^T\underline{B}\underline{A} = \tilde{F}_{\underline{B}}(\underline{A})$.

The corresponding map $\tilde{F}_{\underline{B}}$ defined in §1 satisfies:

(6.40) $\tilde{F}_{\underline{1}} \circ R_{\underline{B}} = \tilde{f}_{\underline{B}} \circ \tilde{F}_{\underline{1}}$.

Define the metric submanifold $\mathcal{M}^n \subset S(n,\mathbb{R}) \cap GL(n,\mathbb{R})$ of $GL(n,\mathbb{R})$ to be the set of all positive definite symmetric matrices and the unimodular metric submanifold $\bar{\mathcal{M}}^n \subset S(n,\mathbb{R}) \cap SL(n,\mathbb{R})$ to be the unimodular subset of $\mathcal{M}^n$. $\mathcal{M}^n$ and $\bar{\mathcal{M}}^n$ are the orbits of $\underline{1}$ under the transpose action of $GL(n,\mathbb{R})$ and $SL(n,\mathbb{R})$ respectively, and the isotropy groups at $\underline{1}$ are by definition the orthogonal and special orthogonal groups respectively:

$$\mathcal{M}^n = \{\, \underline{A}^T\underline{A} = \tilde{f}_{\underline{A}}(\underline{1}) = \tilde{F}_{\underline{1}}(\underline{A}) \mid \underline{A} \in GL(n,\mathbb{R}) \,\}$$
$$\bar{\mathcal{M}}^n = \mathcal{M}^n \cap SL(n,\mathbb{R})$$
$$O(n,\mathbb{R}) = \{\, \underline{A} \in GL(n,\mathbb{R}) \mid \tilde{f}_{\underline{A}}(\underline{1}) = \underline{1} \,\} = I_{\underline{1}}$$
$$SO(n,\mathbb{R}) = O(n,\mathbb{R}) \cap SL(n,\mathbb{R}) = \bar{I}_{\underline{1}}.$$

Since the restrictions of these actions to these orbit submanifolds are effective and by definition transitive, $\mathcal{M}^n$ and $\bar{\mathcal{M}}^n$ are respectfully diffeomorphic to $GL(n,\mathbb{R})/O(n,\mathbb{R})$ and $SL(n,\mathbb{R})/SO(n,\mathbb{R})$. Restricting $\tilde{F}_{\underline{1}}$ to the coset manifolds gives the respective diffeomorphisms. Similarly the diagonal submanifolds $\mathcal{M}^n_D, \bar{\mathcal{M}}^n_D$ of $\mathcal{M}^n, \bar{\mathcal{M}}^n$ are the orbits of the scale groups $D(n,\mathbb{R})^+$ and $SD(n,\mathbb{R})^+$ and actually coincide with these groups.

According to (6.40), composing $\tilde{F}_{\underline{1}}$ with right translation yields the transpose action of $GL(n,\mathbb{R})$ on $\mathcal{M}^n$ or of $SL(n,\mathbb{R})$ on $\bar{\mathcal{M}}^n$. Composing $\tilde{F}_{\underline{1}}$ with left translation yields an interesting new phenomena which is not an action but nevertheless has well-defined orbits. Let m be the number:

$$\dim \bar{\mathcal{M}}^n = n(n+1)/2 - 1 = (n-1) + n(n-1)/2 = \dim SD(n,\mathbb{R}) + \dim SO(n,\mathbb{R}).$$

For fixed $\underline{A} \in SL(n,\mathbb{R})$ an m-parameter family of curves in $\bar{\mathcal{M}}^n$ emanating from the point $\tilde{F}_{\underline{1}}(\underline{A}) = \underline{A}^T\underline{A} \in \bar{\mathcal{M}}^n$ is defined by:

$$(6.41)\quad c(\underline{A}, \underline{N}_D, \underline{O}, t) = \tilde{F}_{\underline{1}} \circ L_{\exp t\underline{N}_D/2}(\underline{O}\underline{A})$$
$$= \underline{A}^T\underline{O}^T(\exp t\underline{N}_D)\underline{O}\underline{A} = \underline{A}^T(\exp t\underline{N})\underline{A}$$
$$\underline{N} = \underline{O}^T\underline{N}_D\underline{O}, \quad \underline{N}_D \in sd(n,\mathbb{R}), \quad \underline{O} \in SO(n,\mathbb{R}).$$

(The scale of $\underline{N}_D$ is a redundant parameter when $\underline{N}_D \neq 0$ and may be eliminated by requiring that $\mathrm{Tr}\, \underline{N}_D^2 = \mathrm{Tr}\, \underline{N}^2 = 1$.) Choosing a new $\underline{A}$ in the same right coset of $SO(n,\mathbb{R})$ as the original leads to a corresponding new dependence on $SO(n,\mathbb{R})$ but the same orbits result:

$$c(\underline{O}_1\underline{A}, \underline{N}_D, \underline{O}\underline{O}_1, t) = c(\underline{A}, \underline{N}_D, \underline{O}, t).$$

DeWitt[37] shows that these are the geodesics of his metric on $\mathcal{M}^n$ restricted to $\bar{\mathcal{M}}^n$. The geodesics of $\mathcal{M}^n$ are closely related. The geometry of $\mathcal{M}^3$ will be explored in §11.

If $\tilde{\xi}$ is the generating map for the transpose action, a basis of generators is defined by $\tilde{\xi}^a{}_b = \tilde{\xi}(e^a{}_b) = \tilde{\xi}(\partial^a{}_b)$ and a matrix-valued vector field by $\underset{\sim}{\tilde{\xi}} = \tilde{\xi}(\underset{\sim}{e}) = \tilde{\xi}^a{}_b \underset{\sim}{\hat{e}}^b{}_a$. By an application of (1.31) we may evaluate them in the component coordinates. The result is analogous to (6.36):

(6.42) $\tilde{\xi}^a{}_b = e^a{}_b + \tilde{e}^b{}_a = 2E^a{}_b$, $\underset{\sim}{\tilde{\xi}} = \underset{\sim}{e} + \underset{\sim}{\tilde{e}}^T = 2\underset{\sim}{E}$.

$\underset{\sim}{E} = \underset{\sim}{\tilde{\xi}}/2$ is introduced for later convenience. Since the transpose action is a right action, $\{\tilde{\xi}^a{}_b\}$ will have the same brackets as $\{e^a{}_b\}$; these may be written as in (6.19):

(6.43) $[\mathrm{Tr}\,\underset{\sim}{A}\underset{\sim}{\tilde{\xi}}, \mathrm{Tr}\,\underset{\sim}{B}\underset{\sim}{\tilde{\xi}}] = \mathrm{Tr}\,[\underset{\sim}{A},\underset{\sim}{B}]\underset{\sim}{\tilde{\xi}}$.

Since $\mathcal{M}^n$ is an orbit of the transpose action its generators are tangent to $\mathcal{M}^n$ and hence $\underset{\sim}{E}$ is a matrix-valued vector field on $\mathcal{M}^n$. It is also useful to define a matrix-valued 1-form $\underset{\sim}{W}$ on $\mathcal{M}^n$ by restricting either $\underset{\sim}{\omega}$, $\underset{\sim}{\tilde{\omega}}^T$ or $\frac{1}{2}(\underset{\sim}{\omega} + \underset{\sim}{\tilde{\omega}}^T)$ to $\mathcal{M}^n$. (They all give the same result since $\underset{\sim}{g} = \underset{\sim}{g}^T$ on $\mathcal{M}^n$.) $\underset{\sim}{E}$ and $\underset{\sim}{W}$ enable us to associate a matrix-valued function $\underset{\sim}{X}$ or $\underset{\sim}{\sigma}$ on $\mathcal{M}^n$ with each vector field X or 1-form σ on $\mathcal{M}^n$ called the matrix of the field by:

(6.44) $\underset{\sim}{X} = \underset{\sim}{W}(X)$ $\qquad \underset{\sim}{\sigma} = \sigma(\underset{\sim}{E})$.

These provide us with natural "boldface" maps from $T\mathcal{M}^n_{\underset{\sim}{y}}$ and $T\mathcal{M}^{n*}_{\underset{\sim}{y}}$ onto isomorphic subspaces of $gl(n,\mathbb{R})$. Inner products on $gl(n,\mathbb{R})$ may therefore be used to induce metrics on $\mathcal{M}^n$ by composition with "boldface". For example, the DeWitt inner product induces a metric $\bar{\mathscr{G}}$:

(6.45) $\bar{\mathscr{G}}(X,Y) = \langle \underset{\sim}{X}, \underset{\sim}{Y} \rangle_{DW} = \mathrm{Tr}\,\underset{\sim}{X}\underset{\sim}{Y} - \mathrm{Tr}\,\underset{\sim}{X}\,\mathrm{Tr}\,\underset{\sim}{Y}$

$$= (\mathrm{Tr}\,\underset{\sim}{W} \otimes \underset{\sim}{W} - \mathrm{Tr}\,\underset{\sim}{W} \otimes \mathrm{Tr}\,\underset{\sim}{W})(X,Y).$$

The contravariant metric $\bar{\mathscr{G}}^{-1}$ is:

(6.46) $\bar{\mathscr{G}}^{-1}(\sigma,\delta) = \langle \underset{\sim}{\sigma}, \underset{\sim}{\delta} \rangle^*_{DW} = \mathrm{Tr}\,\underset{\sim}{\sigma}\underset{\sim}{\delta} - \frac{1}{n-1}\mathrm{Tr}\,\underset{\sim}{\sigma}\,\mathrm{Tr}\,\underset{\sim}{\delta}$

$$= (\mathrm{Tr}\,\underset{\sim}{E} \otimes \underset{\sim}{E} - \tfrac{1}{n-1}\mathrm{Tr}\,\underset{\sim}{E} \otimes \mathrm{Tr}\,\underset{\sim}{E})(\sigma,\delta).$$

$\langle\,,\rangle^*_{DW}$ is the dual inner product on $gl(n,\mathbb{R})^*$ which is identified with $gl(n,\mathbb{R})$ as in the first paragraph of this section.

The tangent space at a point $\underset{\sim}{y} \in T\bar{\mathcal{M}}^n$ consists of those tangent vectors X with traceless matrices $\underset{\sim}{X}$, so $\bar{\mathscr{G}}(X,Y) = \langle \underset{\sim}{X}, \underset{\sim}{Y} \rangle$ for $X, Y \in T\bar{\mathcal{M}}^n_{\underset{\sim}{y}}$. Restricting $\bar{\mathscr{G}}$ to $\bar{\mathcal{M}}^n$ yields the metric whose geodesics were discussed above.

Let $g_{ab} = g_{ba}$ be the restriction of the component function $g^a{}_b$ to $\mathcal{M}^n$, $\underline{g} = g_{ab}\hat{\underline{e}}^b{}_a$ the restriction of $\underline{g}$ and $g = \det \underline{g}$ the restriction of the determinant function. Let $g^{ab} = g^{ba}$ be the composition of g_{ab} with the inverse map and $\underline{g}^{-1} = g^{ab}\hat{\underline{e}}^b{}_a$ the restriction of $\underline{g}^{-1}$ to $\mathcal{M}^n$. Thus if $\underline{y} = y_{ab}\hat{\underline{e}}^b{}_a \in \mathcal{M}^n$, then $g_{ab}(\underline{y}) = y_{ab}$, $g(\underline{y}) = \det \underline{y}$ and $g^{ab}(\underline{y}) = g_{ab}(\underline{y}^{-1}) = y^{ab}$. (We will also occasionally use the symbol $\underline{g}$ for a generic point of $\mathcal{M}^n$ but its meaning should always be clear from the context.) Restriction of $\underline{\omega} = \underline{g}^{-1}d\underline{g}$ to $\mathcal{M}^n$ yields:

(6.47) $\quad \underline{W} = \underline{g}^{-1}d\underline{g}$,

while restriction of the relations (6.23), (6.24) yields:

(6.48) $$d\ln g = \mathrm{Tr}\, \underline{g}^{-1}d\underline{g} = \mathrm{Tr}\, \underline{W} = g^{ab}dg_{ab}$$
$$d\underline{g}^{-1} = -\underline{g}^{-1}d\underline{g}\,\underline{g}^{-1}.$$

The transpose action $\underline{y} \mapsto \tilde{f}_{\underline{A}}(\underline{y}) = \underline{A}^T\underline{y}\underline{A}$ has the component form $y_{ab} \mapsto A^c{}_a y_{cd} A^d{}_b$ which explains the choice of $\tilde{f}_{\underline{A}}$ rather than $f_{\underline{A}}(\underline{y}) = \underline{A}\underline{y}\underline{A}^T$ which would break our index conventions. Under dragging along by $\tilde{f}_{\underline{A}}$ the functions $\underline{g}$ and g satisfy:

(6.49) $$\tilde{f}_{\underline{A}}\,\underline{g} = \underline{g}\circ\tilde{f}_{\underline{A}^{-1}} = \underline{A}^{-1T}\underline{g}\,\underline{A}^{-1}$$
$$\tilde{f}_{\underline{A}}\, g = (\det \underline{A}^{-1})^2\, g.$$

Let $\mathbb{R}^{r,s}$ be the space of components in the natural basis of $\binom{r}{s}$-tensor densities over $\mathbb{R}^n$. A point $T \in \mathbb{R}^{r,s}$ is a set of components $\{T^{a\cdots}{}_{b\cdots}\}$. The weight W tensor density representation of $GL(n,\mathbb{R})$ is the map $\rho^{r,s}_W : GL(n,\mathbb{R}) \to GL(\mathbb{R}^{r,s})$ defined by:

(6.50) $$(\rho^{r,s}_W(\underline{A})T)^{a\cdots}{}_{b\cdots} = (\det \underline{A}^{-1})^W A^a{}_c \cdots A^{-1\,d}{}_b \cdots T^{c\cdots}{}_{d\cdots}$$

When W is not integral $|\det \underline{A}^{-1}|$ is understood. The associated representation of $gl(n,\mathbb{R})$ is:

(6.51) $$(\rho'^{\,r,s}_W(\underline{A})T)^{a\cdots}{}_{b\cdots} = A^a{}_c T^{c\cdots}{}_{b\cdots} + \cdots - A^d{}_b T^{a\cdots}{}_{d\cdots} - \cdots - (\mathrm{Tr}\,\underline{A})T^{a\cdots}{}_{b\cdots}.$$

A special class of $\mathbb{R}^{r,s}$-valued functions on $\mathcal{M}^n$ designated by $\rho^{r,s}_W(\mathcal{F}(\mathcal{M}))$ are those which satisfy:

(6.52) $$\tilde{f}_{\underline{A}}T = \rho^{r,s}_W(\underline{A})T.$$

Each $\rho^{r,s}_W(\mathcal{F}(\mathcal{M}^n))$ is a finite-dimensional vector space. For example $\rho^{0,2}_0, \rho^{2,0}_0, \rho^{0,0}_1, \rho^{1,1}_0$ are each 1-dimensional with bases g_{ab}, g^{ab}, $g^{1/2}$, $\delta^a{}_b$ respectively (see (6.49)). In general these functions are constructed from g_{ab}, g^{ab} and powers of g by the manipulations of tensor analysis. The derivative of such a field by the vector field $\tilde{\xi}(\underline{A}) = 2\,\mathrm{Tr}\,\underline{A}\,\underline{E}$ follows from (6.52) and the definitions:

(6.53) $\xi(\underline{A})T = (d/dt)|_0\, T \circ \tilde{f}_{\underline{A}} = -\rho_W'^{r,s}(A)\, T.$

As an example we have:

(6.54) $\xi^a{}_b\, g_{cd} = 2\,\delta^a_{(c} g_{d)b} = 2E^a{}_b\, g_{cd}.$

$g^{1/2}$ is really an example of an oriented representation:

$$\rho_4^{0,0\,(+)}(\underline{A})\, g^{1/2} = (\det(\underline{A}^{-1T}\underline{g}\,\underline{A}^{-1}))^{1/2} = |\det A^{-1}|\, g^{1/2},$$

but we will ignore this distinction in notation.

We are only interested in $\mathcal{M}^n$ for $n=3$, so that the factor $(n-1)^{-1}$ appearing in the inverse of the DeWitt metric is just $\frac{1}{2}$. In the remainder of the section we assume $n=3$ and drop the superscript on $\mathcal{M}^n$.

The component functions $\{g_{ab}\}$ may be used as a "generalized" coordinate system on $\mathcal{M}$ (the component coordinate system). Rather than choosing six independent components (say $\{g_{ab}\}$ with $b \geq a$), all nine are used with the off-diagonal ones identified in pairs. The role of the Kronecker delta is then played by the symmetrizer:

(6.55) $I_{ab}{}^{cd} = \delta^c_{(a}\delta^d_{b)} = dg_{ab}(\partial^{cd}) = \partial^{cd} g_{ab}$

$$\partial^{cd} = \partial/\partial g_{cd} = \partial^{dc}.$$

The usual number of indices are doubled and covariant and contravariant positioning are interchanged. For example, if $X \in \mathcal{X}(\mathcal{M})$ then $X = X_{ab}\,\partial^{ab}$ with $X_{ab} = X g_{ab} = X_{ba}$. (6.54) shows that $E^a{}_b = \delta^a_{\{c} g_{d)b}\,\partial^{cd} = g_{bc}\partial^{ca}$, while (6.47) shows $W^a{}_b = g^{ac}\, dg_{cb}$. As is customary, indices are raised or lowered from their natural positions using g^{ab}, g_{ab}. This is how the "generalized" frame $\{E^a{}_b\}$ with dual frame $\{W^a{}_b\}$ is obtained from the component coordinate frame. The "generalized" duality relations are $W^a{}_b(E^c{}_d) = I^a{}_b{}^c{}_d$. Components in this "frame" are related to component coordinate components by index raising and lowering:

(6.56) $X_{ab}\,\partial^{ab} = X^a{}_b E^b{}_a \qquad X^a{}_b = W^a{}_b(X) = I^a{}_b{}^c{}_d X^d{}_c$

$$\sigma^{ab}\, dg_{ab} = \sigma^a{}_b W^b{}_a \qquad \sigma^a{}_b = \sigma(E^a{}_b) = I^a{}_b{}^c{}_d\,\sigma^d{}_c.$$

$X^a{}_b$ and $\sigma^a{}_b$ are the components of the matrices $\underline{X} = \underline{W}(X)$ and $\underline{\sigma} = \sigma(\underline{E})$

of the fields X and σ.

Taking components of the equations (6.48) enables one to evaluate the following useful partial derivatives:

(6.57) $\partial^{cd} g^{ab} = -g^{a(c} g^{d)b} = -I^{ab\,cd}$

$\partial^{cd} g^{W} = W g^{W} g^{cd}$.

The partial derivatives of any element of $\rho_W^{r,s}(\mathcal{F}(\mathcal{M}))$ may then be easily evaluated using these and (6.55).

If $\underline{A} \in gl(3,\mathbb{R})$, $A^a{}_b$ may be interpreted as constant functions on $\mathcal{M}$. Lowering the upper index ($A_{ab} = g_{ac}A^c{}_b$) and symmetrizing ($A^\#_{ab} = A_{(ab)}$) yields the components of a vector field $A^\#$ on $\mathcal{M}$ with matrix $\underline{A}^\# = A^{\#a}{}_b \hat{e}^b{}_a$:

(6.58) $\underline{A}^\# = \frac{1}{2}(\underline{A} + \underline{g}^{-1}\underline{A}^T\underline{g})$

$A^{\#a}{}_b = I^a{}_b{}^c{}_d A^d{}_c$ $\qquad$ $\mathrm{Tr}\,\underline{A}^\# = \mathrm{Tr}\,\underline{A}$

$A^\# = \mathrm{Tr}\,\underline{A}\underline{E} = \mathrm{Tr}\,\underline{A}^\#\underline{E} = \frac{1}{2}\,\xi(A)$.

$2A^\#$ is the transpose generator corresponding to $\underline{A}$. Note that $\underline{A}^\#$ is also the matrix of the 1-form $\mathrm{Tr}\,\underline{A}\underline{W} = \mathrm{Tr}\,\underline{A}^\#\underline{W}$.

Let $\rho_W^{r,s}(T^{p,q}(\mathcal{M}))$ be the vectorspace of $\mathbb{R}^{r,s}$-valued $\binom{p}{q}$-tensor fields T on $\mathcal{M}$ which satisfy:

(6.59) $\tilde{f}_{\underline{A}} T = \rho_W^{r,s}(\underline{A}) T$,

and therefore:

(6.60) $\mathcal{L}_{\xi(A)} T = -(d/dt)|_0 \tilde{f}_{\exp t\underline{A}} T = -\rho'^{r,s}_W(\underline{A}) T$.

Since the transpose action is essentially linear, as long as we use the component coordinate system it is easy to see how tensor fields behave under dragging along A. dg_{ab} behaves exactly like g_{ab}, and ∂^{ab} which is dual to dg_{ab} therefore behaves like g^{ab} so $dg_{ab} \in \rho_0^{0,2}(\mathcal{X}^*(\mathcal{M}))$ and $\partial^{ab} \in \rho_0^{2,0}(\mathcal{X}(\mathcal{M}))$. In general, the elements of $\rho_W^{r,s}(T^{p,q}(\mathcal{M}))$ are exactly those elements of $T^{p,q}(\mathcal{M})$ whose components in the component coordinate frame are elements of $\rho_W^{r+2q,\,s+2p}(\mathcal{F}(\mathcal{M}))$. $E^a{}_b$ and $W^a{}_b$ are elements of $\rho_0^{1,1}(\mathcal{X}(\mathcal{M}))$ and $\rho_0^{1,1}(\mathcal{X}^*(\mathcal{M}))$ respectively, i.e. $\underline{E}$ and $\underline{W}$ transform according to the adjoint representation under dragging along.

Elements of $\rho_0^{0,0}$ are exactly those fields which are invariant under the transpose action, while $\rho_W^{0,0}$ are those fields which simply scale under that action.

The DeWitt metric $\mathcal{G} = g^{1/2}\bar{\mathcal{G}}$ provides $\mathcal{M}$ with a pseudo-Riemannian structure closely related to the dynamics of general relativity. (See (37) for an excellent treatment of the geometry of $(\mathcal{M},\mathcal{G})$.) Its contravariant form or inverse is $\mathcal{G}^{-1} = g^{-1/2}\bar{\mathcal{G}}^{-1}$. From (6.45), (6.46) the components of these metrics in the component coordinate system are:

(6.61) $$\mathcal{G}^{abcd} = g^{1/2}\bar{\mathcal{G}}^{abcd} = g^{1/2}(g^{a(c}g^{d)b} - g^{ab}g^{cd})$$
$$\mathcal{G}^{-1}{}_{abcd} = g^{-1/2}\bar{\mathcal{G}}^{-1}{}_{abcd} = g^{-1/2}(g_{a(c}g_{d)b} - \tfrac{1}{2} g_{ab}g_{cd})$$
$$\mathcal{G}^{-1}{}_{abfg}\mathcal{G}^{fgcd} = \bar{\mathcal{G}}^{-1}{}_{abfg}\bar{\mathcal{G}}^{fgcd} = I_{ab}{}^{cd}.$$

$\bar{\mathcal{G}}$ is favored by Misner (18) and Ryan (14) who also prefer an inverse component coordinate system. DeWitt (37) essentially uses a generalized coordinate system $\{\mathcal{J}, \bar{g}_{ab}\}$ defined by:

(6.62) $$\kappa\mathcal{J} = g^{1/4} \qquad \kappa = (3/32)^{1/2}$$
$$\underline{\bar{g}} = g^{-1/3}\underline{g} \qquad \underline{g} = (\kappa\mathcal{J})^{4/3}\underline{\bar{g}}$$

$\underline{\bar{g}}$ is unimodular and $\underline{\bar{W}} = \underline{\bar{g}}^{-1}d\underline{\bar{g}}$ is traceless since $\mathrm{Tr}\,\underline{\bar{W}} = d\ln\det\underline{\bar{g}} = 0$.

A straightforward matrix calculation shows:

(6.63) $$\underline{W} = (4/3\, d\ln\mathcal{J})\,\underline{1} + \underline{\bar{W}}.$$

Insertion of this into (6.45) yields:

(6.64) $$\mathcal{G} = -d\mathcal{J}\otimes d\mathcal{J} + (\kappa\mathcal{J})^2\,\mathrm{Tr}(\underline{\bar{W}}\otimes\underline{\bar{W}})$$
$$\mathrm{Tr}(\underline{\bar{W}}\otimes\underline{\bar{W}}) = \bar{g}^{ac}\bar{g}^{bd}\,d\bar{g}_{ab}\otimes d\bar{g}_{cd},$$

explaining the introduction of κ. If $\{\bar{g}_{ab}\}$ are interpreted as generalized coordinates on $\bar{\mathcal{M}}$, the second line represents the induced metric on $\bar{\mathcal{M}}$. The correspondence $\underline{g} \leftrightarrow (\mathcal{J},\underline{\bar{g}})$ decomposes $\mathcal{M}$ into the cross product $\mathbb{R}^+\times\bar{\mathcal{M}}$ where $\mathbb{R}^+$ is the positive real axis. $\mathcal{J}=0$, called the frontier F by DeWitt, is the boundary of $\mathcal{M}$ on which the determinant vanishes. These coordinates are very convenient for finding the geodesics of $(\mathcal{M},\mathcal{G})$ in terms of those of $(\bar{\mathcal{M}}, \mathrm{Tr}\,\underline{\bar{W}}\otimes\mathrm{Tr}\,\underline{\bar{W}})$. (See Appendix B.)

$\mathcal{G}^{abcd}$ is clearly an element of $\rho_1^{4,0}(F(\mathcal{M}))$ while $\bar{\mathcal{G}}^{abcd} \in \rho_0^{4,0}(F(\mathcal{M}))$ so $\bar{\mathcal{G}} \in \rho_0^{0,0}(T^{0,2}(\mathcal{M}))$ and $\mathcal{G} \in \rho_1^{0,0}(T^{0,2}(\mathcal{M}))$. Thus $\bar{\mathcal{G}}$ is invariant under the action of $GL(3,\mathbb{R})$ which is therefore an isometry group of $(\mathcal{M},\bar{\mathcal{G}})$ and $\tilde{\xi}(gl(3,\mathbb{R}))$ a vector space of Killing vector fields of $\bar{\mathcal{G}}$ with basis $\{E^a{}_b\}$. (See §8 for the definitions.) Since $\mathcal{G}$ scales conformally by the factor $|\det \underline{A}|$ only $SL(3,\mathbb{R})$ is an isometry group of the DeWitt metric on $\mathcal{M}$ and $\tilde{\xi}(sl(3,\mathbb{R}))$ is the corresponding Killing vector space. The generator $\tilde{\xi}(\underline{1}) = 2\,\mathrm{Tr}\,\underline{E}$ satisfies:

$$\tilde{\xi}(\underline{1})_t\, \mathcal{G} = e^{3t}\mathcal{G} \qquad \mathcal{L}_{\xi(\underline{1})}\mathcal{G} = -3$$

and is therefore a conformal Killing vector field of $\mathcal{G}$.

The orbits of the action of $SL(3,\mathbb{R})$ on $\mathcal{M}$ are just the hypersurfaces for which g has constant values. A tangent space to one of these hypersurfaces is spanned by the tangent vectors with traceless matrices. Let $_sT\mathcal{M}_{\underline{g}} = \{X \in T\mathcal{M}_{\underline{g}} \mid \mathrm{Tr}\,\underline{X} = 0\}$. In fact, if $\{\hat{\underline{e}}_M\}$ is an orthonormal basis of traceless symmetric matrices (on which $\langle , \rangle_{DW}$ is positive-definite and coincides with $\langle , \rangle$), then $\{\,|\det \underline{A}|^{-1/2}\,\underline{A}^{-1}\hat{\underline{e}}_M\underline{A}\,\}$ are the matrices of an orthonormal basis of $_sT\mathcal{M}_{\underline{A}^T\underline{A}}$. This subspace is orthogonal to the timelike vector $(\mathrm{Tr}\,\underline{E})(\underline{A}^T\underline{A})$ and hence $\mathrm{Tr}\,\underline{E}$ is proportional to the field of timelike normals to the spacelike hypersurfaces of constant g values. The signature of $\mathcal{G}$ is therefore $(-+++++)$ and hence $\mathrm{Tr}\,\overline{W}\otimes\overline{W}$ is a Riemannian metric on $\overline{\mathcal{M}}$.

The DeWitt metric may be used to "raise and lower" indices. If $X = X_{ab}\,\partial^{ab}$ and $\sigma = \sigma^{ab}\,dg_{ab}$, we may associate with them a 1-form $X_* = \mathcal{G}^{abcd}X_{cd}\,dg_{ab}$ and a vector field $\sigma_* = \mathcal{G}^{-1}{}_{abcd}\,\sigma^{cd}\,\partial^{ab}$ whose matrices are:

$$(6.65)\qquad \underline{X}_* = g^{1/2}(\underline{X} - \underline{1}_3\,\mathrm{Tr}\,\underline{X}) \qquad \underline{\sigma}_* = g^{-1/2}(\underline{\sigma} - \tfrac{1}{2}\underline{1}_3\,\mathrm{Tr}\,\underline{\sigma})$$

$$\underline{X} = g^{-1/2}(\underline{X}_* - \tfrac{1}{2}\underline{1}_3\,\mathrm{Tr}\,\underline{X}_*) \qquad \underline{\sigma} = g^{1/2}(\underline{\sigma}_* - \underline{1}_3\,\mathrm{Tr}\,\underline{\sigma}_*).$$

Inner products may then be written:

$$(6.66)\qquad \mathcal{G}(X,Y) = \mathrm{Tr}\,\underline{X}\,\underline{Y}_* \qquad \mathcal{G}^{-1}(\sigma,\delta) = \mathrm{Tr}\,\underline{\sigma}\,\underline{\delta}_*$$

§7. Actions Lifted To TM and T*M

A left action of G on M may be lifted to an action on any of the tensor bundles $TM^{r,s}$ by dragging along the spaces $TM_x^{r,s}$, $x \in M$. Our interest lies only in TM and T*M which are the arenas of Lagrangian and Hamiltonian mechanics. If M is the configuration space of a classical mechanical system, then $TM = \{(x, X) \mid X \in TM_x\}$ is the velocity phase space while $T^*M = \{(x,\sigma) \mid \sigma \in T^*M_x\}$ is the momentum phase space.

Local coordinates $\{x^i\}$ on M lift naturally to "induced coordinates" $\{q^i, \dot{q}^i\}$ and $\{q^i, p_i\}$ on TM and T*M respectively, corresponding to taking components in the coordinate frame:

(7.1) $$q^i((x,X)) = x^i(x) \qquad \dot{q}^i((x,X)) = dx^i(X) = X^i$$
$$q^i((x,\sigma)) = x^i(x) \qquad p_i((x,\sigma)) = \sigma(\partial/\partial x^i) = \sigma_i \quad .$$

In fact any functions on M lift up to functions on TM or T*M by composition with the projection maps $\pi_v((x,X)) = x = \pi_p((x,\sigma))$ in terms of which $q^i = x^i \circ \pi$; these have the same functional dependence on $\{q^i\}$ as the original functions had on $\{x^i\}$. It is convenient to sloppily use the same symbol for both functions letting the context distinguish them (except in the case of q^i and x^i).

Similarly symmetric covariant and contravariant tensor fields on M lift naturally to functions on TM and T*M respectively:

(7.2) $$S = S_{ij\ldots}\, dx^i \otimes dx^j \otimes \cdots \mapsto f(S) = S_{ij\ldots}\, \dot{q}^i \dot{q}^j \cdots \in \mathcal{F}(TM)$$
$$T = T^{ij\ldots}\, \partial_i \otimes \partial_j \otimes \cdots \mapsto f(T) = T^{ij\ldots}\, p_i p_j \in \mathcal{F}(T^*M).$$

For example $\dot{q}^i = f(dx^i)$ and $p_i = f(\partial_i)$ so that for a vector field $\xi = \xi^i \partial_i$ and a 1-form $\sigma = \sigma_i\, dx^i$, $f(\xi) = \xi^i p_i$ and $f(\sigma) = \sigma_i \dot{q}^i$. Suppose $e_a = e^i{}_a \partial_i$ are such that $\{e_a\}$ is a frame over the region where the coordinates $\{x^i\}$ are valid, with dual frame $\{\omega^a\}$ where $\omega^a = \omega^a{}_i\, dx^i$. New local coordinates $\{q^i, \dot{\omega}^a, P_a\}$ on TM and T*M corresponding to taking components in this frame are defined by:

(7.3) $$\dot{\omega}^a = f(\omega^a) = \omega^a{}_i \dot{q}^i \quad \cdot \qquad P_a = f(e_a) = e^i{}_a p_i$$
$$\omega^a{}_i e^i{}_b = \delta^a{}_b \qquad e^i{}_b \omega^b{}_j = \delta^i{}_j \,.$$

As another example, a metric $g = g_{ij}\, dx^i \otimes dx^j$ and its inverse $g^{-1} = g^{ij} \partial_i \otimes \partial_j$ lift up to $T = \frac{1}{2} f(g) = \frac{1}{2} g_{ij} \dot{q}^i \dot{q}^j$ and ${}^*T = \frac{1}{2} f(g^{-1}) = \frac{1}{2} g^{ij} p_i p_j$ which are a Lagrangian and its corresponding Hamiltonian for the geodesics of the metric. Any nondegenerate

Lagrangian provides a correspondence between TM and T^*M through the relation:
$$p_i(q,\dot q) = (\partial L/\partial \dot q^i)(q,\dot q)$$
which may be inverted to yield $\dot q^i(q,p)$. This also provides a correspondence between the functions on these manifolds by "reexpressing" functions on TM through replacing $\dot q^i$ by $\dot q^i(q,p)$ and vice versa. In our metric example, the relations between velocities and momenta are just the images of "index lowering and raising" on vector fields and 1-forms:
$$p_i = g_{ij}\dot q^j \qquad \dot q^i = g^{ij}p_j .$$
T and $^*T$ are corresponding functions, each called the kinetic energy.

If $h \in \mathcal{D}(M)$, it may be lifted to $\bar h \in \mathcal{D}(TM)$ and $\bar h^* \in \mathcal{D}(T^*M)$ by the dragging along of tangent and cotangent spaces:

(7.4) $$\bar h((x,X)) = (h(x),\ dh(x)X)$$
$$\bar h^*((x,\sigma)) = (h(x),\ dh^{-1}(h(x))^*\sigma).$$

Let $\xi = \xi^i\partial_i$ be a vector field on M and consider its flow ξ_t which lifts in this way to flows $\bar\xi_t$ and $\bar\xi_t^*$ of vector fields $\bar\xi$ and $\bar\xi^*$ on TM and T^*M. By (3.9) we have the relations:

(7.5) $$(d/dt)|_0\ x^i\circ\xi_t(x) = \xi^i(x)$$
$$(d/dt)|_0\ dx^i(d\xi_t(x)X) = \xi^i{}_{,j}(x)\,X$$
$$(d/dt)|_0\ (d\xi_{-t}(\xi_t(x))^*\sigma)(\partial_j) = -\sigma_i\,\xi^i{}_{,j}(x)$$

Using these to compute the tangents to these flows in the induced coordinates we find:

(7.6) $$\bar\xi = \xi^i\,\partial/\partial q^i + \xi^i{}_{,j}\,\dot q^j\,\partial/\partial \dot q^i$$
$$\bar\xi^* = \xi^i\,\partial/\partial q^i - p_i\,\xi^i{}_{,j}\,\partial/\partial p_j$$

These enable us to lift any vector field ξ on M to vector fields $\bar\xi$ and $\bar\xi^*$ on TM and T^*M whose integral curves project down to integral curves of ξ on M. Suppose G acts on M on the left and $\xi : \mathfrak{g} \to \mathcal{X}(M)$ is the corresponding generating map. The generating maps of the lifted actions $\bar\xi : \mathfrak{g} \to \mathcal{X}(TM)$ and $\bar\xi^* : \mathfrak{g} \to \mathcal{X}(T^*M)$ are just the lifts of ξ as defined by (7.6). We also have a third linear map $\eta = f\circ\xi : \mathfrak{g} \to \mathcal{F}(T^*M)$ called the moment whose significance requires preliminary discussion.

Poisson brackets of any functions $f,g \in \mathcal{F}(T^*M)$ are defined by the following expression in an induced coordinate system:

(7.7) $$\{f,g\} = \partial f/\partial q^i\ \partial g/\partial p_i - \partial g/\partial q^i\ \partial f/\partial p_i = -\mathrm{ad}(g)f = X_g f,$$

where we have introduced the vector field $X_g = -ad(g) \in \mathfrak{X}(T^*M)$ defined by:

(7.8) $X_g = \partial g/\partial p_i \, \partial/\partial q^i - \partial g/\partial q^i \, \partial/\partial p_i$.

The Poisson bracket is "antisymmetric" and satisfies the Jacobi identity and therefore converts $\mathfrak{F}(T^*M)$ into an infinite-dimensional Lie algebra. Inner derivations $ad(g)$ for $g \in \mathfrak{F}(T^*M)$ are elements of $\mathfrak{X}(T^*M)$ which is also a Lie algebra. The map $g \mapsto -X_g = ad(g)$ is a Lie algebra homomorphism from $\mathfrak{F}(T^*M)$ into $\mathfrak{X}(T^*M)$. Its kernel is just the set of constant functions which is also the center of $\mathfrak{F}(T^*M)$. The adjoint homomorphism relation (1.11) may be rewritten in the X_g notation:

(7.9) $X_{\{f,g\}} = -[X_f, X_g]$.

Introduce the canonical 2-form Ω and its dual or inverse Ω^{-1} by the following expressions in an induced coordinate system:

(7.10) $\Omega = dq^i \wedge dp_i \qquad d\Omega = 0$

$\Omega^{-1} = \partial/\partial q^i \otimes \partial/\partial p_i - \partial/\partial p_i \otimes \partial/\partial q^i$

$-\Omega^{-1} \lrcorner \, \Omega = \mathrm{id} = \partial/\partial q^i \otimes dq^i + \partial/\partial p_i \otimes dp_i$,

where the contraction refers to adjoining arguments of the tensor product. By the rules for contraction one may deduce the relations:

(7.11) $X_g \lrcorner \, \Omega = dg \qquad \Omega^{-1} \lrcorner \, dg = X_g$

$X_f \lrcorner \, (X_g \lrcorner \, \Omega) = \Omega(X_g, X_f) = \{g, f\}$.

The first line shows that X_g is the contravariant form of dg in the symplectic geometry determined by Ω.

Since Ω and $X_g \lrcorner \, \Omega$ are closed (vanishing exterior derivative), formula (3.20) shows that the adjoint vector fields leave Ω invariant under dragging:

(7.12) $\mathcal{L}_{X_g} \Omega = X_g \lrcorner \, d\Omega + d(X_g \lrcorner \, \Omega) = 0$.

Any vector field X satisfying $\mathcal{L}_X \Omega = 0$ is a derivation of $\mathfrak{F}(T^*M)$ since one may show that this property implies:

(7.13) $X\{f,g\} = \{Xf, g\} + \{f, Xg\}$.

For adjoint vector fields $X_g = -ad(g)$ this is true since by definition these are inner derivations of the Lie algebra $\mathfrak{F}(T^*M)$. In §5 we showed that derivations of a Lie algebra exponentiate to automorphisms. Since the action of a derivation X of $\mathfrak{F}(T^*M)$ on that space by dragging along is given by $\mathfrak{X}_t g = g \circ \mathfrak{X}_{-t} = e^{-t\mathfrak{X}} g$, these are automorphisms, i.e. canonical transformations:

(7.14) $\qquad \mathfrak{X}_t \{f, g\} = \{\mathfrak{X}_t f, \mathfrak{X}_t g\}.$

Induced coordinates are canonical coordinates on T^*M, namely coordinates characterized by the brackets:

$$\{q^i, p_j\} = \delta^i{}_j \qquad \{q^i, q^j\} = 0 = \{p_i, p_j\}.$$

By (7.14), these relations are invariant under dragging along by a derivation $\mathfrak{X}$, so the dragged along coordinates are also canonical (and induced by the dragged along coordinates $\{\mathfrak{X}_t x^i\}$ on M).

Dragging $\mathfrak{F}(T^*M)$ by lifted vector fields provides a subgroup of the inner automorphisms (corresponding to lifts of tranformations on M) since the lifting map is just $-\mathrm{ad} \circ f$:

(7.15) $\qquad \bar{\xi}^* = -\mathrm{ad}(f(\xi)) = \mathfrak{X}_{f(\xi)}, \qquad \xi \in \mathfrak{X}(M).$

In particular, if $\xi : \mathfrak{g} \to \mathfrak{X}(M)$ is the generating map for a left action of G on M, we obtain a linear map $\eta = f \circ \xi : \mathfrak{g} \to \mathfrak{F}(T^*M)$ which is called a moment for the lifted action. This map generates the dragging action of G on $\mathfrak{F}(T^*M)$ through inner automorphism:

(7.16) $\qquad \bar{\xi}^*(\mathfrak{X})_t g = e^{-t\bar{\xi}^*(\mathfrak{X})} g = e^{t\,\mathrm{ad}(\eta(\mathfrak{X}))} g.$

η is in fact a homomorphism:

(7.17) $\qquad \eta([\mathfrak{X}, \mathfrak{Y}]) = \{\eta(\mathfrak{X}), \eta(\mathfrak{Y})\}$

If G had acted on M on the right, we would have obtained a map $\tilde{\eta} = f \circ \tilde{\xi}$ satisfying (7.17) with an additional minus sign. If $\{e_a\}$ is a basis of $\mathfrak{g}$, it follows that $\eta_a = \eta(e_a) = f(\xi_a) = \xi^i{}_a p_i$ satisfy:

(7.18) $\qquad \{\eta_a, \eta_b\} = C^c{}_{ab} \eta_c.$

For example, suppose G is a matrix group with generating basis $\{\hat{e}_A\}$ and parametrization $\underline{S}(\underline{\theta}) = e^{\theta^A \hat{e}_A}$ acting like a group of linear transformations on M in the global coordinates $\{x^i\}$ as in (6.34) and (6.35) (different index symbols). The moment functions and lifted generators are:

(7.19) $\qquad \eta_A = \hat{e}_A{}^i{}_j q^j p_i \qquad \bar{\xi}^*_A = \hat{e}_A{}^i{}_j (q^j \partial/\partial q_i - p_i \partial/\partial p_j).$

For the rotation group $SO(3, \mathbb{R})$ acting on $\mathbb{R}^3$, $\hat{e}_a{}^i{}_j = C^i{}_{aj} = \epsilon_{iaj}$ and $\eta_a = \epsilon_{aij} q^i p_j$ are the components of angular momentum for a classical mechanical system whose configuration space is $\mathbb{R}^3$. (The components of linear momentum p_i arise from translations.)