§3. Dragging; Lie and Lagrange Derivatives

Let $h: M \to N$ be a diffeomorphism with inverse h^{-1}. The natural action of h on vectorfields and contravariant tensor fields via its differential dh is from M to N (pushing forward) while its natural action on covariant fields via the transpose of the differential dh^* is from N to M (pulling back). By using dh^{-1*} on the latter fields, we push everything forward and are therefore able to map tensor fields of mixed valence. (This is exactly what one does in transforming fields by coordinate transformations, which are diffeomorphisms between open sets in $\mathbb{R}^n$.) Given a tensor field T on M we then obtain a field hT on N which we call the field dragged along by h. If f, X and σ are a function, vectorfield and 1-form on M, then the following rules enable one to determine hT for any arbitrary tensorfield T (together with the fact that h is linear over the real numbers):

$$(3.1) \qquad hf = f \circ h^{-1}, \quad (hX)(y) = dh(h^{-1}(y))\, X(h^{-1}(y))$$
$$(h\sigma)(y) = dh^{-1}(y)^*\, \sigma(h^{-1}(y))$$
$$h(f\, X \otimes \cdots \otimes \sigma \cdots) = hf\; hX \otimes \cdots \otimes h\sigma \cdots .$$

We call the h appearing in hT the dragging operator associated with the diffeomorphism h. It satisfies the property:

$$(3.2) \qquad T(X, \ldots, \sigma, \ldots)(x) = hT(hX, \ldots, h\sigma, \ldots)(h(x)),$$

since contraction cancels dh against dh^{-1}:

$$(h\sigma)(h(x))\,(hX(h(x))) = [dh^{-1}(h(x))^*\, \sigma(x)]\,(dh(x) X(x))$$
$$= \sigma(x)\,(dh^{-1}(h(x))\, dh(x) X(x)) = \sigma(x)(X(x)).$$

If $h_1: M \to N$ and $h_2: N \to P$ are diffeomorphisms, then by the chain rule it follows that the dragging operator of the composed map $h_2 \circ h_1 : M \to P$ is the product of the dragging operators in the same order:

$$(3.3) \qquad h_2 h_1 T = (h_2 \circ h_1) T.$$

If h is a diffeomorphism of a manifold M into itself, we obtain a new tensor field hT on M whose value at x is dragged along from $h^{-1}(x)$. A field which satisfies $hT = T$ is called invariant under dragging along by h or simply h-invariant. On a Lie group the left (right) invariant vector fields and 1-forms are invariant under dragging along by all left (right) translations.

and hence so are all tensor products of such fields. A Lie group G is special in that a unique left (right) translation exists which takes the identity to any given point of G and we may therefore use dragging by left (right) translations to left (right) translate an element of $TG^{r\,s}_{a_0}$ all over G to yield a left (right) invariant $\binom{r}{s}$-tensor field.

If $\{x^\mu\}$ are local coordinates on M, an explicit formula is easily written down for the dragging operation, completely analogous to a coordinate transformation, enabling us to extend the notion of dragging to tensor densities (or if we wish to any geometric object). Making use of the notation $h^\mu = x^\mu \circ h$ and $f_{,\mu} = \partial f / \partial x^\mu$, the components of a tensor density of weight W behave under dragging in the following way:

(3.4) $$(hT)^{\mu\cdots}{}_{\nu\cdots}(x) = J(x)^W \, h^\mu{}_{,\alpha}(h^{-1}(x)) \cdots h^{-1\beta}{}_{,\nu}(x) \cdots T^{\alpha\cdots}{}_{\beta\cdots}(h^{-1}(x))$$
$$J = \det(h^{-1\mu}{}_\nu).$$

Let ξ_t be the 1-parameter group of diffeomorphisms associated with an analytic vector field ξ on M. The above discussion applies with $h = \xi_t$ and $h^{-1} = \xi_{-t}$. Define the Lie derivative of a tensor density T with respect to ξ by:

(3.5) $$\pounds_\xi T = -d/dt|_0 \, \xi_t T = d/dt|_0 \, \xi_{-t} T.$$

It is also a tensor density of the same type as T. Using the observations:

(3.6) $$d/dt \; \xi_t T = d/ds|_0 \, \xi_{t+s} T = d/ds|_0 \, \xi_s \xi_t T = -\pounds_\xi \, \xi_t T,$$
$$(d/dt)^n \xi_t T = (-\pounds_\xi)^n (\xi_t T) \quad , \quad \xi_0 T = T,$$

a power series expansion of $\xi_t T$ about $t=0$ yields locally:

(3.7) $$\xi_t T = e^{-t\pounds_\xi} T.$$

From the properties of the dragging operator and its definition, the Lie derivative with respect to a fixed vector field satisfies the Liebnitz rule (or derivation property) for tensor products (and hence exterior products of forms) and contractions of tensor products (see (A.6)). In addition one may show that if $h: M \to N$ is a diffeomorphism:

(3.8) $$h \, \pounds_\xi T = \pounds_{h\xi} \, hT.$$

In local coordinates $\{x^\mu\}$ we may derive from (3.4) a formula for the components of $\mathcal{L}_\xi T$ by substituting $h = \xi_t$ and using consequences of (2.4) and (2.6). With $h^{\pm 1}$ standing simultaneously for h and h^{-1}:

(3.9) $$h^{\pm 1\,\mu} = e^{\pm t\xi} x^\mu \quad , \qquad f \circ h^{-1} = e^{-t\xi} f \, ,$$
$$d/dt|_0 \, h^{\pm 1\,\mu} = \pm \xi x^\mu = \pm \xi^\mu \, , \qquad d/dt|_0 \, f \circ h^{-1} = -\xi f \, ,$$
$$d/dt|_0 \, h^{\pm 1\,\mu}{}_{,\alpha} = (d/dt|_0 \, h^{\pm 1\,\mu})_{,\alpha} = \pm \xi^\mu{}_{,\alpha}$$
$$d/dt|_0 \, J^W = -W \xi^\mu{}_{,\mu} \, .$$

The last relation is a consequence of (6.23) evaluated at the identity matrix. Taking the derivative of (3.4) then yields:

(3.10) $$(\mathcal{L}_\xi T)^{\mu\cdots}{}_{\nu\cdots} = T^{\mu\cdots}{}_{\nu\cdots,\alpha}\xi^\alpha - \xi^\mu{}_{,\alpha} T^{\alpha\cdots}{}_{\nu\cdots} - \cdots + \xi^\beta{}_{,\nu} T^{\mu\cdots}{}_{\beta\cdots} + \cdots$$
$$+ W\xi^\alpha{}_{,\alpha} T^{\mu\cdots}{}_{\nu\cdots} \, .$$

When $W = 1$ the first and last terms may be written $(T^{\mu\cdots}{}_{\nu\cdots}\xi^\alpha)_{,\alpha}$. For functions, vector fields and 1-forms this leads to the following coordinate free expressions:

(3.11) $$\mathcal{L}_\xi f = \xi f \, , \quad \mathcal{L}_\xi X = [\xi, X] \, , \quad (\mathcal{L}_\xi \sigma)(X) = \xi\,\sigma(X) - \sigma([\xi, X]) \, .$$

The last expression (and corresponding ones for higher rank covariant tensor fields) may also be obtained directly from the first two and the derivation property (A.6); for example see (A.14). One often writes $(\mathcal{L}_\xi T)^{\mu\cdots}{}_{\nu\cdots}$ as $\mathcal{L}_\xi T^{\mu\cdots}{}_{\nu\cdots}$ as long as it is understood that it represents the components of the Lie derivative and not the derivative of the components.

In the first section we introduced the infinite-dimensional group $\mathcal{D}(M)$ of diffeomorphisms of M into itself whose multiplication is composition. Since $h_2(h_1(x)) = h_2 \circ h_1(x)$, $\mathcal{D}(M)$ acts on M on the left. The 1-dimensional subgroups of $\mathcal{D}(M)$ are the 1-parameter diffeomorphism groups generated by elements of $\mathcal{X}(M)$. This leads to a natural identification of the Lie algebra of $\mathcal{D}(M)$ with $\mathcal{X}(M)$.

Consider the infinite-dimensional vector space $T^{r,s}(M)$ of $\binom{r}{s}$-tensor fields on M; being a vector space we may identify its tangent spaces with itself. A parametrized curve in $T^{r,s}(M)$ is a time-dependent tensor field T_t. Its tangent may then be identified with the ordinary derivative $T_t' = d/dt\, T_t$. By dragging along, the action of $\mathcal{D}(M)$ on M lifts to an action on this space, an action which by (3.3) is also a left action. According to the definition (3.5),

$-\pounds_X T$ is just the tangent to the orbit through T of the action of the 1-dimensional subgroup generated by X. Therefore the map ξ from the Lie algebra of $\mathcal{D}(M)$ to vector fields on $T^{r,s}(M)$ may be identified with the map $X \mapsto -\pounds_X$. The negative of this map is as it should be a homomorphism according to the identity (A.6).

Suppose we have a symmetric connection arising from a metric g on our manifold M. In a coordinate system the components of the connection $\Gamma^\gamma{}_{\alpha\beta} = \Gamma^\gamma{}_{\beta\alpha}$ are just the Christoffel symbols and the covariant derivative of a weight W tensor density T by the vector field ξ has components:

$$(3.12)\quad (\nabla_\xi T)^{\alpha\cdots}{}_{\beta\cdots} = T^{\alpha\cdots}{}_{\beta\cdots;\gamma}\xi^\gamma$$
$$= T^{\alpha\cdots}{}_{\beta\cdots,\gamma}\xi^\gamma + \xi^\gamma \Gamma^\alpha{}_{\gamma\delta} T^{\delta\cdots}{}_{\beta\cdots} + \cdots - \xi^\gamma \Gamma^\delta{}_{\gamma\beta} T^{\alpha\cdots}{}_{\delta\cdots} - \cdots - W\xi^\gamma \Gamma^\delta{}_{\gamma\delta} T^{\alpha\cdots}{}_{\beta\cdots}$$
$$= (\pounds_\xi T)^{\alpha\cdots}{}_{\beta\cdots} + \xi^\alpha{}_{;\delta} T^{\delta\cdots}{}_{\beta\cdots} + \cdots - \xi^\delta{}_{;\beta} T^{\alpha\cdots}{}_{\delta\cdots} - \cdots - W\xi^\gamma{}_{;\gamma} T^{\alpha\cdots}{}_{\beta\cdots}.$$

We have used (3.10) to replace the first term and then grouped the remaining terms together to obtain covariant derivatives of ξ. This gives a formula for the Lie derivative in terms of covariant rather than ordinary derivatives:

$$(3.13)\quad (\pounds_\xi T)^{\alpha\cdots}{}_{\beta\cdots} = T^{\alpha\cdots}{}_{\beta\cdots;\gamma}\xi^\gamma - \xi^\alpha{}_{;\delta} T^{\delta\cdots}{}_{\beta\cdots} - \cdots + \xi^\delta{}_{;\beta} T^{\alpha\cdots}{}_{\delta\cdots} + \cdots + W\xi^\delta{}_{;\delta} T^{\alpha\cdots}{}_{\beta\cdots}.$$

This is exactly (3.10) with commas replaced by semicolons, and it is a covariant formula so it holds even for components taken in a noncoordinate frame. It is very useful.

The Lie derivative of the metric (which is covariant constant) is:

$$(3.14)\quad (\pounds_\xi g)_{\alpha\beta} = g_{\alpha\beta;\gamma}\xi^\gamma + 2g_{\gamma(\alpha}\xi^\gamma{}_{;\beta)} = 2\xi_{(\alpha;\beta)},$$

while the Lie derivative of a weight one scalar density Ψ is a covariant divergence:

$$(3.15)\quad \pounds_\xi \Psi = \Psi_{;\alpha}\xi^\alpha + \xi^\alpha{}_{;\alpha}\Psi = (\Psi\xi^\alpha)_{;\alpha}.$$

If $\phi^{\alpha\cdots}{}_{\beta\cdots}$ are the components of an $\binom{r}{s}$-tensor Φ and $\Pi_{\alpha\cdots}{}^{\beta\cdots}$ those of an $\binom{s}{r}$-tensor density Π of weight one whose indices have the corresponding symmetries of those of ϕ, we call Π conjugate to ϕ. $\Psi = \phi^{\alpha\cdots}{}_{\beta\cdots}\Pi_{\alpha\cdots}{}^{\beta\cdots}$ is a weight one scalar density obtained by contracting corresponding indices of Φ and Π. By (3.15) and the Liebnitz rule:

(3.16) $(\psi\xi^\alpha)_{;\alpha} = \mathcal{L}_\xi\psi = \Pi_{\alpha\ldots}{}^{\beta\ldots}\mathcal{L}_\xi\phi^{\alpha\ldots}{}_{\beta\ldots} + (\mathcal{L}_\xi\Pi_{\alpha\ldots}{}^{\beta\ldots})\phi^{\alpha\ldots}{}_{\beta\ldots}$.

An important application of this formula occurs for the pair g, Π where Π is conjugate to g :

$$(g_{\alpha\beta}\Pi^{\alpha\beta}\xi^\gamma)_{;\gamma} = \Pi^{\alpha\beta}\mathcal{L}_\xi g_{\alpha\beta} + g_{\alpha\beta}\mathcal{L}_\xi\Pi^{\alpha\beta}.$$

The first term is:

$$\Pi^{\alpha\beta}\mathcal{L}_\xi g_{\alpha\beta} = 2\Pi^{\alpha\beta}\xi_{\alpha;\beta} = (2\Pi^{\alpha\beta}\xi_\alpha)_{;\beta} - 2\Pi^{\alpha\beta}{}_{;\beta}\xi_\alpha$$

From these two relations and the definition:

(3.17) $\mathcal{H}_\alpha = -2\Pi_\alpha{}^\beta{}_{;\beta}$,

we obtain:

(3.18) $\xi^\alpha\mathcal{H}_\alpha = \Pi^{\alpha\beta}\mathcal{L}_\xi g_{\alpha\beta} - (2\Pi^{\alpha\beta}\xi_\alpha)_{;\beta}$

$$= -g_{\alpha\beta}\mathcal{L}_\xi\Pi^{\alpha\beta} + (g_{\alpha\beta}\Pi^{\alpha\beta}\xi^\gamma)_{;\gamma} - (2\Pi^{\alpha\beta}\xi_\alpha)_{;\beta}.$$

These are important in the ADM formulation of the dynamics of general relativity. Notice that the terms not involving Lie derivatives are covariant divergences.

Let α be a differential form and X a vector field. Then $X \lrcorner \alpha$ is a differential form of one less degree arising from α by evaluating its first argument on X (contraction by X). For example, if σ is a 1-form and f a function:

(3.19) $X \lrcorner \sigma = \sigma(X) \qquad X \lrcorner df = df(X) = Xf.$

If α, β are p- and q-forms then the following identity holds:

(3.20) $X \lrcorner (\alpha\wedge\beta) = (X\lrcorner\alpha)\wedge\beta + (-1)^p\alpha\wedge(X\lrcorner\beta).$

One may show that when acting on differential forms the Lie derivative is related to the exterior derivative and contraction in the following way:

(3.21) $\mathcal{L}_X\alpha = d(X\lrcorner\alpha) + X\lrcorner d\alpha.$

When $d\alpha = 0$, as occurs when the degree of α equals the dimension of the manifold, the last term drops out leaving an exact form.

Let $\{e_a\}$ be a frame on an n-dimensional manifold M and let $\{\omega^a\}$ be its dual frame:

(3.22) $\omega^a(e_b) = e_b\lrcorner\omega^a = \delta^a{}_b$,

$$[e_a, e_b] = C^c{}_{ab}e_c \quad , \quad d\omega^a = -\tfrac{1}{2}C^a{}_{bc}\,\omega^b\wedge\omega^c.$$

$C^a{}_{bc}$ are the structure functions for the frame. By (3.11):

(3.23) $\mathcal{L}_{e_a}e_b = C^c{}_{ab}e_c \, , \quad \mathcal{L}_{e_a}\omega^b = (\mathcal{L}_{e_a}\omega^b)(e_c)\,\omega^c = -C^b{}_{ac}\,\omega^c.$

The n-form $\omega^{1\cdots n} = \omega^1 \wedge \cdots \wedge \omega^n$ satisfies $d\omega^{1\cdots n} = 0$ so by (3.21):

(3.24) $$d(e_a \lrcorner\, \omega^{1\cdots n}) = \mathcal{L}_{e_a}\omega^{1\cdots n} = \mathcal{L}_{e_a}\omega^1 \wedge \cdots \wedge \omega^n + \cdots + \omega^1 \wedge \cdots \wedge \mathcal{L}_{e_a}\omega^n = -C^b{}_{ab}\,\omega^{1\cdots n}.$$

Let $\chi = \chi^a e_a$ be a weight one vector density and introduce the notation $\partial_a f = e_a f$ for the frame derivatives of functions and let $\gamma_a = \partial_a - C^c{}_{ac}$. Using properties of the exterior derivative and the contraction identity (3.19) in the form:

$$0 = e_a \lrcorner\, (\omega^b \wedge \omega^{1\cdots n}) = \delta^a{}_b\,\omega^{1\cdots n} - \omega^b \wedge (e_a \lrcorner\, \omega^{1\cdots n}),$$

we find:

(3.25) $$d(\chi \lrcorner\, \omega^{1\cdots n}) = d\chi^a \wedge (e_a \lrcorner\, \omega^{1\cdots n}) + \chi^a d(e_a \lrcorner\, \omega^{1\cdots n}) = \partial_b \chi^a\, \omega^b \wedge (e_a \lrcorner\, \omega^{1\cdots n}) - \chi^a C^b{}_{ab}\,\omega^{1\cdots n} = (\gamma_a \chi^a)\,\omega^{1\cdots n}.$$

This is an important relation; $\gamma_a \chi^a$ is called an ordinary divergence. The operator γ_a does not satisfy the Liebnitz rule. The rule for integrating by parts is:

(3.26) $$\gamma_a(f\chi^a) = \chi^a \partial_a f + f\gamma_a \chi^a.$$

If a metric and its connection are available it is often more convenient to use covariant divergences. In (A.15) we show that for a weight one vector density χ with frame components χ^a the ordinary and covariant divergences agree:

(3.27) $$\gamma_a \chi^a = \chi^a{}_{;a}.$$

Suppose we have a variational principle involving the integral of an n-form over a region C of M:

(3.28) $$S[\phi, C] = \int_C F[\phi]\,\omega^{1\cdots n}.$$

Here F is the component in the frame $\{e_a\}$ of a weight one scalar density which is some explicit function of the frame components of a collection of fields ϕ^A (A standing for the frame indices and an index labeling the individual fields) and the frame components of their derivatives $\partial_a \phi^A, \ldots,$ and of the structure functions of the frame and possibly their derivatives. All of this we symbolize by the square brackets in $F[\phi]$. If in our manipulations of the coefficient $F[\phi]$ we obtain a term $\gamma_a \chi^a$, then (3.25) makes possible an application of Stoke's Theorem to the corresponding term in the integral:

(3.29) $\int_c \partial_a X^a\, \omega^{1\cdots n} = \int_c d(X \lrcorner\, \omega^{1\cdots n}) = \int_{\partial c} X \lrcorner\, \omega^{1\cdots n}$.

We are now ready to define the components of a Lagrange derivative in a frame. Restrict $F[\phi]$ to depend only on frame derivatives up to the second order:

(3.30) $F[\phi] = F(\phi^A, \partial_a\phi^A, \partial_a\partial_b\phi^A, C^a{}_{bc}, \partial_d C^a{}_{bc})$.

Consider a 1-parameter family or curve of fields $\phi(\lambda)$ where $\phi = \phi(0)$ is some fixed set of fields and ϕ' is the tangent to the curve at $\lambda = 0$:

$$\phi' = d/d\lambda|_0\, \phi(\lambda).$$

A second order differential operator $DF[\phi]$ (or first order if F is independent of second derivatives) is defined by computing the derivative of $F[\phi(\lambda)]$ at $\lambda=0$ using the old fashioned chain rule:

(3.31) $DF[\phi]\cdot\phi' = \partial F/\partial\phi^A\, \phi'^A + \partial F/\partial\partial_a\phi^A\, \partial_a\phi'^A + \partial F/\partial\partial_a\partial_b\phi\, \partial_a\partial_b\phi'^A$.

We have used $[d/d\lambda, \partial_a] = 0$ since the frame is assumed to be held fixed. Integrating by parts using (3.26) we may remove the derivatives from ϕ'^A to obtain the components of the Lagrange derivative of $F[\phi]$ with respect to ϕ in our frame and a divergence:

(3.32) $DF[\phi]\cdot\phi' = \delta F[\phi]/\delta\phi^A\, \phi'^A + \eth_a(G^a[\phi]\cdot\phi')$

$$\delta F[\phi]/\delta\phi^A = \partial F/\partial\phi^A - \eth_a\, \partial F/\partial\partial_a\phi^A + \eth_b\eth_a\, \partial F/\partial\partial_a\partial_b\phi^A$$

$$G^a[\phi]\cdot\phi' = (\partial F/\partial\partial_a\phi^A - \eth_b\,\partial F/\partial\partial_b\partial_a\phi^A)\,\phi'^A + \partial F/\partial\partial_a\partial_b\phi^A\, \partial_b\phi'^A.$$

In a coordinate frame these are familiar formulas. If ϕ is a collection of tensor fields then $\delta F[\phi]/\delta\phi$ is a collection of conjugate tensor densities and $\delta F[\phi]/\delta\phi^A$ are simply their components in a frame rather than in a coordinate system. Since the Lagrange derivative annihilates divergences, we are free to drop them from $F[\phi]$ if we are only interested in its Lagrange derivative.

As an instructive example let $N = N^a e_a$ be a vector field and g_{ab}, π^{ab} the frame components of a metric and a symmetric weight one tensor density. We defined in (3.17) the quantity:

(3.33) $\mathcal{H}_a[g,\pi] = \mathcal{H}_a(g_{cd}, \partial_b g_{cd}, \pi^{cd}, \partial_b \pi^{cd}, C^b{}_{cd}) = -2\pi_a{}^b{}_{;b}$.

If we let $\approx$ mean equality to within a divergence then (3.18) says:

(3.34) $N^c \mathcal{H}_c \approx \pi^{ab} \mathcal{L}_N g_{ab} \approx - g_{ab} \mathcal{L}_N \pi^{ab}$.

Since $\mathcal{L}_N \pi^{ab}$ is independent of g_{ab} we have the following Lagrange derivatives:

(3.35) $\delta/\delta\pi^{ab}(N^c\mathcal{H}_c) = \mathcal{L}_N g_{ab}$ $\qquad \delta/\delta g_{ab}(N^c\mathcal{H}_c) = -\mathcal{L}_N \pi^{ab}$

$$\delta/\delta N^a (N^c\mathcal{H}_c) = \mathcal{H}_a.$$

We may define the components in the frame $\{e_a\}$ of the functional derivative of the functional (3.28) with respect to $\phi(x)$ for $x \in C$ by:

(3.36) $\delta S[\phi, C]/\delta\phi^A(x) = (\delta F[\phi]/\delta\phi^A)(x)$.

Suppose ϕ is a set of fields which extremizes the functional $S[\phi, C]$. This means that if $\phi(\lambda)$ is any curve of fields for which $\phi = \phi(0)$ and $\phi' = 0$ on ∂C, then $S[\phi(\lambda), C]$ has vanishing λ-derivative at $\lambda = 0$:

(3.37) $0 = d/d\lambda|_0 \, S[\phi, C] = \int_C DF[\phi]\cdot\phi' \, \omega^{1\cdots n}$

$$= \int_C \delta F[\phi]/\delta\phi^A \, \phi'^A \, \omega^{1\cdots n}$$

The vanishing of ϕ' on ∂C is necessary to complete the last step in (3.37) using (3.32) and (3.29). Now we may reason that since ϕ' is arbitrary in C, its coefficient must vanish there. It follows that in C, ϕ must satisfy the field equations:

(3.38) $\delta F[\phi]/\delta\phi^A = 0$

$$\delta F[\phi]/\delta\phi(x) = 0, \quad x \in C.$$

Suppose $G[\phi]$ is a tensor-valued function of ϕ (in the above sense of the bracket). We may compute its Lie derivative using the original definition (3.5) and the old fashioned chain rule:

(3.39) $\mathcal{L}_X G[\phi] = -d/dt|_0 \, G[X_t\phi] = -DG[\phi]\cdot d/dt|_0 \, X_t\phi$

$$= DG[\phi]\cdot\mathcal{L}_X\phi.$$

For example, consider the Riemann tensor $R[g]$ of a metric g on M:

$$\mathcal{L}_X R[\phi] = DR[g]\cdot\mathcal{L}_X g.$$

Therefore the curvature tensor has vanishing Lie derivative when the metric tensor does. The same applies to the Ricci and Einstein tensors and the scalar curvature.

§4. Lie Groups Revisited

We now return to our discussion of §1 armed with the mathematics developed in the intervening sections. Suppose G is a 1-dimensional group with coordinate a^1 and multiplication function $\varphi^1(a_1,a_2) = a^1 \circ \varphi(a_1,a_2)$. The left invariant vector field e_1 and dual 1-form ω^1 induced by the coordinate derivative at the identity a_0 will have the form:

$$e_1 = E(a^1)\, d/da^1 \qquad \omega^1 = da^1/E(a^1).$$

Define a new "canonical" coordinate t by:

$$t = \int_{a_0^1}^{a^1} d\tau/E(\tau),$$

so that $\omega^1 = dt$ and $e_1 = \partial/\partial t$, and let $\theta(t_1,t_2)$ be the multiplication function expressed in terms of the new coordinate. Note that the identity a_0 with old coordinate a_0^1 has the new coordinate $t_0 = 0$. Because the components of e_1, ω^1 are unity in this coordinate system, equation (1.33) and its initial condition and unique solution are:

$$(\partial\theta/\partial t^1)(t_1,t_2) = 1 \quad , \quad \theta(t_1,0) = t_1$$

$$\theta(t_1,t_2) = t_1 + t_2.$$

Thus every 1-dimensional group is locally like the real line $\mathbb{R}$ under addition. The canonical coordinate $t: G \to \mathbb{R}$ is a local homomorphism defined up to a multiplicative constant. Globally, however, G may have the topology of either the real line or the circle.

Next let the curve $c(t)$ with $c(0) = a_0$ and $c'(0) = X$ be a 1-dimensional subgroup of an r-dimensional group G and assume that t is a canonical parameter for this subgroup. In other words $c(t_1+t_2) = \varphi(c(t_1), c(t_2))$ is an identity and

$$c(t+t_1) = \varphi(c(t_1), c(t)) = L_{c(t_1)}(c(t))$$

is a reparametrization of the curve $c(t)$. Computing its tangent by the chain rule one finds

$$c'(t+t_1) = dL_{c(t_1)}(c(t))\, c'(t)$$

$$c'(t_1) = dL_{c(t_1)}(a_0)X = X \circ c(t_1),$$

where the final X refers to the image of X in $\mathfrak{g}$. This states that $c(t)$ is the integral curve of $X \in \mathfrak{g}$ passing through the identity. We may repeat the entire argument to find that $c(t)$ is also an integral curve of $\tilde{X}$:

$c(t) = X_t(a_0) = \tilde{X}_t(a_0)$.

The 1-dimensional subgroups of G are the integral curves of either the left or right invariant vector fields which pass through the identity. This allows us to define an important map exp from $TG_{a_0} \cong \mathfrak{g}$ to the group G called the exponential map:

(4.1) $\exp X = X_1(a_0) = \tilde{X}_1(a_0)$.

The 1-dimensional subgroups of G are by definition the curves $C_X(t) = \exp tX = X_t(a_0)$, from which it follows that:

(4.2) $\exp t_1 X \ \exp t_2 X = \exp(t_1+t_2)X$.

The tangents to these curves are:

(4.3) $C_X'(t) = X \circ C_X(t) = \tilde{X} \circ C_X(t)$.

By choosing a basis $\{e_a\}$ of $TG_{a_0} \cong \mathfrak{g}$ with dual basis $\{\omega^a\}$, we may use the exponential map to define "canonical coordinates of the first kind" on G by:

$$u^a(\exp X) = \omega^a(X) = X^a.$$

These coordinates are defined only on that neighborhood of the identity covered by the 1-parameter subgroups and on which no intersections occur. (2.10) is a power series expansion for the multiplication function $\varphi^a(u_1, u_2)$ in these coordinates. "Canonical coordinates of the second kind" may also be defined:

$$v^a(\exp t^1 e_1 \cdots \exp t^r e_r) = t^a.$$

Suppose we let a 1-dimensional subgroup $C_X(t)$ act on G by left and right translation and on M on the left, generating families of curves $\tilde{C}_X(a,t)$, $C_X(a,t)$ and $\bar{C}_X(x,t)$ as in (1.19). Using (4.3), the chain rule and (1.23) we may compute their tangents at any t:

$$\tilde{C}_X'(a,t) = dR_a(C_X(t))\,\tilde{X} \circ C_X(t) = \tilde{X} \circ R_a(C_X(t)) = \tilde{X} \circ \tilde{C}_X(a,t)$$

$$C_X'(a,t) = dL_a(C_X(t))\,X \circ C_X(t) = X \circ L_a(C_X(t)) = X \circ C_X(a,t).$$

$$\bar{C}_X'(x,t) = dF_x(C_X(t))\,\tilde{X} \circ C_X(t) = \xi(X) \circ \bar{C}_X(x,t).$$

Thus the orbits are the integral curves of $\tilde{X}$, X and $\xi(X)$ respectively:

(4.4) $L_{\exp tX} = \tilde{X}_t \qquad R_{\exp tX} = X_t \qquad f_{\exp tX} = \xi(X)_t$.

In this sense the right (left) invariant vector fields on G generate the left (right) translations and the generators of the left action of G on M generate that action. In each case one has an r-dimensional real Lie algebra of vector fields and a diffeomorphism action which consists of pushing the points of the manifold along the integral curves of those vector fields, as discussed in §2.

Let $X, Y \in TG_{a_0} \cong \mathfrak{g}$. From above and §2 it follows that:

(4.5) $\exp X \exp Y = R_{\exp Y}(\exp X) = Y_1 \circ X_1(a_0) = Z[X,Y]_1(a_0) = \exp Z[X,Y]$.

In this context the expression for $Z[X,Y] \in TG_{a_0} \cong \mathfrak{g}$ is called the Campbell-Hausdorff formula. It is this formula which leads to the formal expansion (2.10) for the multiplication function in canonical coordinates of the first kind. Similarly:

(4.6) $\exp X \exp Y = L_{\exp X}(\exp Y) = \tilde{X}_1 \circ \tilde{Y}_1(a_0)$.

The interpretation of this equation and the last is that the point $\exp X \exp Y$ may be reached from the identity by traveling along unit segments of integral curves of X and then Y: or of $\tilde{Y}$ and then $\tilde{X}$. (Since every integral curve has a natural parameter defined up to an additive constant, the notion of a unit segment is well defined.) Although every point of a connected Lie group G may not lie on a 1-dimensional subgroup, it is true that every point may always be reached by traveling along a very small number of segments of invariant vector fields, i.e. we may represent every point as a product of a few exponentials.

A p-dimensional distribution D on a manifold M is a smooth choice of a p-dimensional subspace $D_x \subset TM_x$ at each point x of M. If D is a Lie subalgebra of $\mathfrak{X}(M)$, i.e. if the bracket of any two vector fields lying in D is again in D, then the distribution is called involutive or completely integrable. An integral manifold of D is a p-dimensional submanifold of M whose tangent space coincides with the distribution subspace at each point of that submanifold. The Frobenius theorem states that an involutive distribution determines a family of integral manifolds filling M. (For $p=1$ this is a trivial statement about integral curves of vector fields.) This theorem enables us to extend our discussion of 1-dimensional subgroups of G to p-dimensional subgroups.

Consider a p-dimensional Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ of the Lie algebra of left invariant vector fields on a group G. $\mathfrak{h}$ is a p-dimensional involutive distribution and therefore determines a family of p-dimensional submanifolds covering the group. The same is true of $\tilde{\mathfrak{h}}$. The unique integral manifold H of $\mathfrak{h}$ containing the identity is locally swept out by all integral curves of elements of $\mathfrak{h}$ emanating from that point, i.e. the 1-dimensional subgroups corresponding to those elements. Because of the relations (4.5) and (4.6) between integral curves of $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ emanating from the identity, it should be no surprise that H is also the integral manifold of $\tilde{\mathfrak{h}}$ containing the identity. In other words H is locally the image of $\mathfrak{h}$ under exponentiation ($\exp \mathfrak{h} \subset H$) while globally $H = \{\exp X \exp Y \cdots \mid X, Y, \ldots \in \mathfrak{h}\}$. It should be clear that this submanifold is a p-dimensional subgroup of G with Lie algebra $\mathfrak{h}$ which is said to generate H. The remaining integral manifolds of $\mathfrak{h}$ are the orbits of the action of the subgroup H on G by right translation, namely the left cosets aH of H in G, while the integral manifolds of $\tilde{\mathfrak{h}}$ are the right cosets; these coincide only if H is normal.

The importance of the exponential map is that it establishes a one-to-one correspondence between Lie subgroups of G and Lie subalgebras of $\mathfrak{g}$. For example, one may show that the center of $\mathfrak{g}$ generates the center of G (as should be clear from the Campbell-Haussdorff formula) and that an ideal $\mathfrak{h}$ in $\mathfrak{g}$ generates a normal subgroup H of G. Transpositions of a Lie algebra exponentiate to transpositions of the group ($\exp \circ \psi' = \psi \circ \exp$) while homomorphisms of Lie algebras exponentiate to Lie group homomorphisms. Even the multiplicative structure of G is completely determined locally by the algebraic structure of $\mathfrak{g}$ through the Campbell-Haussdorff formula.

Let x_0 be a fixed point of M and consider the map $\delta_{x_0} : \mathfrak{g} \to TM_{x_0}$ which was defined by $\delta_{x_0}(X) = \delta(X)(x_0)$. The kernel of δ_{x_0} consists of those $X \in \mathfrak{g}$ for which $\delta(X)$ vanishes at x_0. Since $-\delta$ is a homomorphism:

$$\delta([X,Y])(x_0) = -[\delta(X), \delta(Y)](x_0).$$

If $X, Y \in \ker \delta_{x_0}$, then the right hand side vanishes and $[X,Y] \in \ker \delta_{x_0}$ which means that $\ker \delta_{x_0}$ is a Lie subalgebra of $\mathfrak{g}$. It therefore generates a subgroup $I_{x_0} \subset G$ called the isotropy group at x_0, all of whose elements leave the point x_0 fixed (since a point is invariant under the flow of a vector field which vanishes there):

$$I_{x_0} = \{ a \in G \mid f_a(x_0) = x_0 \}.$$

The differential $df_a(x_0)$ therefore maps TM_{x_0} into itself for $a \in I_{x_0}$, and by the chain rule applied to the relation $f_{a_1} \circ f_{a_2} = f_{a_1 a_2}$ we obtain:

$$df_{a_1}(x_0) \circ df_{a_2}(x_0) = df_{a_1 a_2}(x_0),$$

which means "$df(x_0)$": $I_{x_0} \to GL(TM_{x_0})$ is a homomorphism and hence its image $I'_{x_0} = df_{I_{x_0}}(x_0)$ is a subgroup of $GL(TM_{x_0})$ called the linear isotropy group at x_0. (For example, $AD(G)$ and $Ad(G)$ are the isotropy and linear isotropy groups at a_0 of the action of $AD(G)$ on G.) Since $(a a_1 a^{-1}) \cdot (a \cdot x_0) = (a \cdot x_0)$ for $a_1 \in I_{x_0}$, $AD_a(I_{x_0}) \subset I_{a \cdot x_0}$ but similarly $AD_{a^{-1}}(I_{a \cdot x_0}) \subset I_{x_0}$ so in fact $AD_a(I_{x_0}) = I_{a \cdot x_0}$, hence the isotropy groups at different points of the same orbit are conjugate subgroups of G.

Suppose G acts transitively on M (but not necessarily effectively) and consider the map $F_{x_0} : G \to M$. Every element of the left coset aI_{x_0} maps onto the same point $F_{x_0}(a) \in M$ since if $a_1 \in I_{x_0}$:

$$F_{x_0}(a a_1) = a \cdot (a_1 \cdot x_0) = a \cdot x_0 = F_{x_0}(a).$$

Therefore it makes sense to restrict F_{x_0} to a map $\hat{F}_{x_0} = F_{x_0} \circ \pi_{I_{x_0}}^{-1}$: $G/I_{x_0} \to M$ where $\pi_{I_{x_0}} : G \to G/I_{x_0}$ is the natural projection map $\pi_{I_{x_0}}(a) = aI_{x_0}$. Furthermore, distinct left cosets map onto distinct points of M since $\hat{F}_{x_0}(a_1 I_{x_0}) = \hat{F}_{x_0}(a_2 I_{x_0})$ implies the following sequence: $a_1 \cdot x_0 = a_2 \cdot x_0$, $(a_2^{-1} a_1) \cdot x_0 = x_0$, $a_2^{-1} a_1 \in I_{x_0}$, $a_1 I_{x_0} = a_2 I_{x_0}$. Since the action is transitive, $\hat{F}_{x_0}$ is a one-to-one map between G/I_{x_0} and M. It should be plausible that this is a diffeomorphism, i.e. if G acts transitively on M, then M is diffeomorphic to G/I_{x_0} for any $x_0 \in M$. The diffeomorphism pulls back the left action of G on M to its natural left action on the set of left cosets since:

$$f_{a_1} \circ F_{x_0} = F_{x_0} \circ L_{a_1}.$$

Even when G does not act transitively on M we may apply this result to each of the orbits, on which G does act transitively by definition.

Consider the kernel of the homomorphism $\xi : \mathfrak{g} \to \mathfrak{X}(M)$, namely $\ker \xi = \{ X \in \mathfrak{g} \mid \xi(X) = 0 \}$. This is an ideal in $\mathfrak{g}$ which generates

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the normal subgroup $I_M \subset G$ of elements which act as the identity on M since the flow of the zero vector field is just Id :

$$I_M = \{a \in G \mid f_a = Id\}.$$

For an effective action $I_M = \{a_0\}$ and $\ker \xi = \{0\}$ so ξ is an isomorphism onto $\xi(g)$ which is therefore an p-dimensional Lie algebra.

When G acts transitively on M with trivial isotropy group ($I_x = \{a_0\}$ for all $x \in M$), this action is called simply transitive and G is diffeomorphic to M with the left action on M pulling back to left translation on G. Otherwise the action is called multiply transitive. Manifolds of the form G/H for a subgroup H of G or which are diffeomorphic to such manifolds are called homogeneous spaces.

$\xi(g)$ is a subalgebra of $\mathfrak{X}(M)$. Let q be an integer-valued function on M whose value at x is the rank of the linear transformation $\xi_x : g \to TM_x$ or equivalently the dimension of the image subspace $\xi_x(g)$. Clearly $q \leq r$ and $q \leq n$. On some open submanifold M_s of M consisting of "regular points", q has maximum value s and represents an s-dimensional involutive distribution on M_s whose integral manifolds are orbits. When $s = r$ the action of G on each orbit is called simply transitive (and each such orbit is diffeomorphic to G); otherwise G acts multiply transitively on these orbits. On submanifolds of lesser dimension than M_s (whose points are called singular) for which q has a fixed value p, $\xi(g)$ represents an involutive distribution of dimension p whose integral manifolds are orbits. For example, the generators of rotations about an axis or point of $\mathbb{R}^n$ determine one- or (n-1)-dimensional distributions everywhere except on the axis or point where the dimension is zero.

Suppose T is a tensor field on either G or M which is invariant under an action of G. Suppose for instance that T lives on M. Then for all $X \in g$ and sufficiently small t for each such X :

$$T = \xi(X)_t T = e^{-t \pounds_{\xi(X)}} T, \quad 0 = -(d/dt)|_0 T = \pounds_{\xi(X)} T.$$

In other words a tensor field is invariant under the action of G if and only if its Lie derivative with respect to all generators of that action vanishes. On G itself a basis $\{e_a\}$ of $TM_{a_0} \cong \mathfrak{g}$ naturally induces both a left invariant and right invariant basis for $\binom{p}{q}$- tensor fields over G, namely $\{e_{a_1} \otimes \cdots \otimes e_{a_p} \otimes \omega^{b_1} \otimes \cdots \otimes \omega^{b_q}\}$ and its counterpart. A tensor field which is left (right) invariant has constant components in a left (right) invariant frame, namely its components at a_0. In this way we obtain an isomorphism between the tensor algebra over TG_{a_0} and the tensor algebra over $\mathfrak{g}$ or $\tilde{\mathfrak{g}}$ of left or right invariant tensor fields on G. Another situation arises when a tensor field is invariant under the left (right) action of a subgroup H. In this case its components in a left (right) invariant frame are constant on each right (left) coset of H in G.

Let $\rho : \underline{G} \to GL(V)$ be a representation of G and define $\rho' : \mathfrak{g} \to \mathfrak{gl}(V)$ by:

(4.7) $\qquad \rho'(X) = (d/dt)|_0 \, \rho_{\exp tX} \quad , \quad X \in \mathfrak{g} \cong TG_{a_0}$.

One may show that ρ' is a representation of $\mathfrak{g}$. By a calculation in the same spirit as (3.6) a power series expansion yields (setting $t=1$):

(4.8) $\qquad \rho_{\exp X} = \exp \rho'(X) \quad , \quad \rho \circ \exp = \exp \circ \rho'$.

Now consider ρ as a $GL(V)$-valued function on G; we compute its derivative by $X \in \mathfrak{g}$ with image $\hat{X} = \omega(X) \in TG_{a_0}$:

$$d\rho(X)(a) = X(a)\rho = (d/dt)|_0 \, \rho_{a \exp tX} = (d/dt)|_0 \, \rho_a \rho_{\exp tX}$$
$$= \rho_a \rho'(\hat{X}) = \rho_a \rho'(\omega(X)).$$

From this and a similar computation we obtain:

(4.9) $\qquad \rho^{-1} d\rho = \rho'(\omega) \qquad d\rho\, \rho^{-1} = \rho'(\tilde{\omega})$,

where ω and $\tilde{\omega}$ are the special $\hat{\mathfrak{g}}$-valued invariant 1-forms on the group. If $\{\hat{e}_a\}$ is a basis of $\hat{\mathfrak{g}}$, then since ρ' is linear, $\rho'(\omega) = \rho'(\hat{e}_a)\,\omega^a$, etc.

§5. The Adjoint Action and Automorphism Groups

We now explore the adjoint action in more detail. Suppose $C_X(t) = \exp tX$ is a 1-dimensional subgroup of G. Since AD_a is an automorphism, $AD_a C_X(t)$ is also a 1-dimensional subgroup with canonical parameter t and may therefore be represented as $\exp tY$ where Y is its tangent at a_0; by the chain rule and definition of Ad:

$$Y = dAD_a(a_0)\, C'_X(0) = Ad(a)X.$$

This proves the first adjoint formula:

(5.1) $$AD_a(\exp X) = \exp Ad(a)X.$$

Using this one may convince oneself that the homomorphism $AD \to Ad$ is an isomorphism. As another application:

(5.2) $$(\exp X)a = a(a^{-1}\exp X\, a) = a\exp(Ad(a)^{-1}X)$$

$$\tilde{X}_t a = (Ad(a)^{-1}X)_t a.$$

This says that the integral curve of $\tilde{X} \in \tilde{g}$ through a coincides with the integral curve of $Ad(a)^{-1}X \in g$ through a. In particular they have the same tangent at a:

(5.3) $$\tilde{X}(a) = (Ad(a)^{-1}X)(a).$$

By our identification of TG_{a_0} and g, the adjoint group $Ad(G)$ has become a subgroup of $GL(g)$. We now show that it also represents the action of $AD(G)$ on g by dragging along. To do this we distinguish $TG_{a_0} = \hat{g}$ and g using the $\wedge$ notation. Since a_0 is a fixed point of AD, $Ad(a)\hat{X} = (AD_a X)(a_0)$. Once we show that $AD_a X \in g$, it will have to equal $Ad(a)X \in g$ since they agree at the identity. But for $X, Y \in g$, recalling that $ad(X)Y = [X,Y] = \mathcal{L}_X Y$:

(5.4) $$AD_a X = R_{a^{-1}} L_a X = R_{a^{-1}} X$$

$$AD_{\exp X} Y = R_{\exp -X} Y = X_{-1} Y = e^{\mathcal{L}_X} Y = e^{ad(X)} Y \in g.$$

Thus in the unhatted notation $AD_a X = Ad(a)X$ and the last formula becomes:

(5.5) $$Ad(\exp X) = e^{ad(X)}.$$

Notice that conjugation and right translation have the same effect on g, so the (linear) adjoint group represents the action of right translation on the left invariant Lie algebra g.

Replacing X by tX in (5.5) and differentiating:

(5.6) $$ad(X) = (d/dt)|_0\, Ad(\exp tX) = d(Ad)(a_0)X.$$

Thus $ad = d(Ad)(a_0)$ may also be defined as the differential of Ad at the identity. It is also Ad' according to the previous section making (5.5) a special case of (4.8).

The Jacobi identity on $\mathfrak{g}$ may be rewritten in the form:

(5.7) $\quad ad(X)[Y,Z] = [ad(X)Y, Z] + [Y, ad(X)Z]$.

Any element of $g\ell(\mathfrak{g})$ with this derivative property is called a derivation; $ad(X)$ is called an inner derivation since it involves the bracket multiplication. Using (5.7) and (5.5) from which it follows that $d/dt \; Ad(\exp tX) = ad(X)\, Ad(\exp tX)$, one may evaluate the following derivative:

$$d/dt\,[Ad(\exp tX)Y, Ad(\exp tX)Z] = ad(X)\,[Ad(\exp tX)Y, Ad(\exp tX)Z].$$

From this and its iterations, a power series expansion evaluated at $t=1$ yields via (5.5) the formula:

(5.8) $\quad Ad(\exp X)[Y,Z] = [Ad(\exp X)Y, Ad(\exp X)Z]$.

This shows $Ad(\exp X) = e^{ad(X)}$ is a Lie algebra automorphism of $\mathfrak{g}$ and hence $Ad(G) \subset Aut(\mathfrak{g}) \subset GL(\mathfrak{g})$ is a subgroup of the group of automorphisms of $\mathfrak{g}$. Since Ad is related to inner derivations by exponentiation, the elements

REPLACE WITH CORRECTION

[illegible]

$\rho = Ad$ is a $GL(\mathfrak{g})$-valued function on G with $\rho' = ad$ its associated $g\ell(\mathfrak{g})$-valued function. Let $\underline{R}$ be the matrix-valued function on G which represents ρ in the basis $\{e_a\}$ with SCT components $C^a{}_{bc}$. By the discussion following (1.12), $\rho'(e_a) = ad(e_a)$ has the matrix representation $\underline{k}_a$ where $k_a{}^b{}_c = C^b{}_{ac}$. According to (4.8), (4.9) and (1.17) the following relations then hold where $X = X^a e_a \in \mathfrak{g}$:

(5.9) $\quad \underline{R}(\exp X) = e^{X^a \underline{k}_a} \qquad \underline{R}\,\underline{k}_a \underline{R}^{-1} = \underline{k}_b R^b{}_a$

$\qquad \underline{R}^{-1} d\underline{R} = \underline{k}_a \omega^a \qquad d\underline{R}\,\underline{R}^{-1} = \underline{k}_a \tilde{\omega}^a$.

Equation (5.3) gives the transformation between the two frames $\{e_a\}$ and $\{\tilde{e}_a\}$ on G:

(5.10) $\quad \tilde{e}_a = e_b R^{-1\,b}{}_a \qquad \tilde{\omega}^a = R^a{}_b \omega^b$.

Correction for page 5.2, 4^{Th} line after Eq. (5.8):

of $Ad(G)$ are called inner automorphisms of $\mathfrak{g}$, so that we may write suggestively $Ad(G) = IAut(\mathfrak{g})$, where $IAut(\mathfrak{g}) \subset Aut(\mathfrak{g})$ is defined to be the subgroup one obtains by exponentiating $ad(\mathfrak{g}) = \mathfrak{iaut}(\mathfrak{g})$. Let $der(\mathfrak{g}) = \mathfrak{aut}(\mathfrak{g}) \subset \mathfrak{gl}(\mathfrak{g})$ be the subspace of derivations of the Lie algebra $\mathfrak{g}$, easily seen to be a Lie subalgebra containing $ad(\mathfrak{g})$ as an ideal. The above manipulations hold for any derivation D and show that e^D is an automorphism of $\mathfrak{g}$. In fact the component of $Aut(\mathfrak{g})$ connected to the identity is obtained by exponentiating $der(\mathfrak{g})$. Derivations and automorphisms which are not inner are called outer automorphisms.

Let $\omega^{1\cdots r} = \omega^1\wedge\cdots\wedge\omega^r$ and $\tilde{\omega}^{1\cdots r} = \tilde{\omega}^1\wedge\cdots\wedge\tilde{\omega}^r$ be the basis r-forms induced by these frames. By the definition of the determinant:

$$\tilde{\omega}^{1\cdots r} = (\det \underline{R})\, \omega^{1\cdots r}.$$

$\omega^{1\cdots r}$ and $\tilde{\omega}^{1\cdots r}$ are bases of the 1-dimensional vector spaces $\Lambda^r(\mathfrak{g})$ and $\Lambda^r(\tilde{\mathfrak{g}})$ of left and right invariant r-forms respectively. These vector spaces provide G with invariant measures defined up to constant scale factors. These measures agree yielding a bi-invariant measure only when $\det \underline{R} = 1$, i.e., when the adjoint group is unimodular.

In fact the Ad-invariant tensors over $\mathfrak{g}$ correspond exactly to the bi-invariant tensor fields on G. Suppose $T^{a\cdots}{}_{b\cdots}$ are the (constant) components of an Ad-invariant tensor over $\mathfrak{g}$:

$$R^a{}_c\cdots R^{-1d}{}_b\cdots T^{c\cdots}{}_{d\cdots} = T^{a\cdots}{}_{b\cdots}.$$

It then follows that:

$$T = T^{a\cdots}{}_{b\cdots}\, e_a\otimes\cdots\otimes\omega^b = T^{a\cdots}{}_{b\cdots}\, \tilde{e}_a\otimes\cdots\otimes\tilde{\omega}^b,$$

showing that the tensor field T is bi-invariant; the reverse statement is also easily verified. The Killing form induces a bi-invariant symmetric tensor field $\gamma_{ab}\omega^a\otimes\omega^b$ which when non-degenerate (semi-simple Lie groups) provides G with a natural bi-invariant pseudo-Riemannian metric unique up to a constant scale factor. On an abelian group any inner product on $\mathfrak{g}$ is Ad-invariant (since Ad(G) consists only of the identity) and induces a bi-invariant metric on the group. A little thought shows that the only kinds of groups which admit bi-invariant metrics are direct products of abelian and semi-simple groups; this class of groups may be used for gauge field theories in which an Ad-invariant inner product is essential.

Since $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ mutually commute, the C-B-H functional satisfies $Z[X,\tilde{Y}] = X+\tilde{Y}$ for $X\in\mathfrak{g}$ and $\tilde{Y}\in\tilde{\mathfrak{g}}$ and hence:

(5.11) $$AD_{\exp tX}(a) = L_{\exp tX}\circ R_{\exp -tX}(a) = (\tilde{X}_t\circ X_{-t})(a) = (\tilde{X}-X)_t(a).$$

(To continue the paragraph at the end of page 5.3)

The vector space

$$(5.12) \qquad iaut(G) = \{\tilde{X} - X \mid X \in \mathfrak{g}\}$$

is therefore the generating Lie algebra for the (left) adjoint action, namely, the left action of G on itself by inner automorphism. The corresponding generating map is given by $X \in \mathfrak{g} \mapsto \tilde{X} - X$. $AD(G) \subset \mathfrak{D}(G)$ is the diffeomorphism group corresponding to this action.

Let $Aut(G) \subset \mathfrak{D}(G)$ be the group of automorphisms of G, containing $AD(G)$ as a subgroup. One may easily show that $Aut(G)$ is a finite-dimensional Lie group.[1] Let $aut(G)$ be the generating Lie algebra for the action of $Aut(G)$ on G, containing $iaut(G)$ as a Lie subalgebra. Extending the ad notation by the definition:

$$(5.13) \qquad ad(\xi)X = [\xi, X] = \mathcal{L}_\xi X \qquad \xi \in aut(G),\ X \in \mathfrak{g},$$

one may show that $ad : aut(G) \rightarrow aut(\mathfrak{g})$ is a 1-1 Lie algebra homomorphism. This is in fact an isomorphism if, for example, G is simply connected.[1] In that case $Aut(G) \cong Aut(\mathfrak{g})$ also holds and the Lie group isomorphism is just the action of $Aut(G)$ on $\mathfrak{g}$ by dragging along. It is convenient to extend also the notation of §1 used to indicate matrix representations. Let $ad_e(\xi)$ be the matrix with components

$$(5.14) \qquad ad_e(\xi)^a{}_b = \omega^a(ad(\xi)e_b),$$

i.e. the matrix of the linear transformation $ad(\xi)$ with respect to the basis $\{e_a\}$ of $\mathfrak{g}$. Similarly let $aut_e(\mathfrak{g})$ be the matrix representation of $aut(\mathfrak{g})$.

Let

$$(5.15) \qquad \mathfrak{X}(\mathfrak{g}) = aut(G) \oplus \tilde{\mathfrak{g}} = aut(G) \oplus \mathfrak{g} \subset \mathfrak{X}(\mathfrak{g})$$

and extend the ad notation to this space as in (5.13). The equality in (5.15) holds due to the definition (5.12) of $iaut(G) \subset aut(G)$. This "semi-direct sum" Lie algebra $\mathfrak{X}(\mathfrak{g})$ is the generating Lie algebra of the "semi-direct product" diffeomorphism group:

$$(5.16) \qquad \mathfrak{D}(\mathfrak{g}) = Aut(G) \times_s L(G) = Aut(G) \times_s R(G) \subset \mathfrak{D}(G).$$

Here the equality is due to the relation (1.5) and the fact that $AD(G) \subset Aut(G)$. $\mathcal{D}(g)$ is the largest subgroup of $\mathcal{D}(G)$ which preserves the left or right invariance of tensor fields under dragging along, namely, the group of translations and automorphisms of G.