Spatially Homogeneous Cosmology: Background and Dynamics

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EXPLANATION OF HANDWRITTEN MANUSCRIPT (I)

lower case: a b c d e f g h i j k l m n o p q r s t u v w x y z

upper case: A B C D E F G H I J K L M N O P Q R S T U V W X Y Z

numerals: 0 1 2 3 4 5 6 7 8 9

Roman numerals: I II III IV V VI VII VIII IX

lower case Greek used: $\alpha\ \beta\ \gamma\ \delta\ \epsilon\ \rho\ \mu\ \omega\ \sigma\ \kappa\ \theta\ \psi\ \varphi\ \xi\ \eta\ \zeta\ \pi\ \lambda\ \tau$

upper case Greek used: $\Omega\ \Sigma\ \Phi\ \Delta$

upper case Gothic (German) used: $\mathfrak{F}\ \ \mathfrak{X}$

lower case Gothic (German) used, indicated by longhand written (not printed) letters:

$a\ c\ d\ f\ g\ h\ i\ k\ l\ o\ s\ u\ t$

upper case "handwritten type" used:

$\mathcal{C}\ \mathcal{D}\ \mathcal{G}\ \mathcal{H}\ \mathcal{I}\ \mathcal{K}\ \mathcal{L}\ \mathcal{M}\ \mathcal{R}\ \mathcal{S}\ \mathcal{T}\ \mathcal{X}$

sanserif lower case g (not g) indicated by the symbol g .

all matrices are underlined to indicate BOLDFACE TYPE .

SYMBOLS

( )	parentheses
[ ]	square brackets
{ }	set brackets
$\vert\ \vert$	absolute value brackets
/	slash
+	plus sign
−	minus sign
=	equal sign
$\neq$	not equal sign
$\equiv$	"is defined to be"
$\subset$	set inclusion symbol
$\in$	"is an element of"
$\oplus$	direct sum symbol
$\otimes$	tensor product symbol
$\times$	set product symbol
$\int$	integration symbol
< >	"bra", "ket" brackets

$\longrightarrow$	mapping symbol between spaces
$\longmapsto$	mapping symbol between elements
$\pounds$	Lie derivative symbol
δ	functional derivative symbol
$\not\delta$	slashed functional derivative symbol
∂	partial derivative symbol
$\not\partial$	slashed partial derivative symbol
∞	infinity
$\perp$	perpendicular symbol
Tr	trace of matrix
ln	natural logarithm
$\mathbb{R}$	real line
$f\circ g$	composition of maps symbol
$\cong$	isomorphism symbol
$\wedge$	wedge symbol
$\binom{p}{q}$	long paretheses ?
$\pm$	plus-or-minus sign

EXPLANATION OF HANDWRITTEN MANUSCRIPT (2)

SYMBOLS (ABOVE AND BELOW)

A^* ↖ asterisk	S^T ↖ capital T for matrix transpose
$A^\#$ ↖ sharp symbol	S^{-1} ↖ minus one = inverse symbol
$\bar{A}$ ↖ bar	A' ↖ prime
$\tilde{e}$ ↖ tilde	$\mathbb{R}^+$ ↖ plus sign
$\bar{\tilde{e}}$ ↖ barred tilde	$N_{a\|b}$ ↖ vertical bar
$\vec{N}$ ↖ vector arrow symbol	$N_{;a}$ ↖ semicolon
$\dot{q}$ ↖ dot	$\mathcal{M}_D$, $\mathcal{M}_{S(a)}$, $\mathcal{M}_{T(a)}$, $\mathcal{M}_I$ ↖ uppercase letters
$\hat{e}$ ↖ "caret" or "hat"	$N_{,a}$ ↖ comma
$\dot{\tilde{\omega}}$ ↖ dotted tilde	

The entire manuscript is written with the intention of a 3 line system, namely a base line, a subscript line and a superscript line, REGARDLESS OF HOW THE HANDWRITTEN SYMBOLS MIGHT APPEAR.

FOR EXAMPLE: $C^a{}_{bc} \Rightarrow C^a{}_{bc}$

With certain exceptions noted below, a superscript and a subscript are never intended to appear in the same vertical line

$C^a{}_b{}_c \quad R^{ab}{}_{cd} \quad \Gamma^a{}_{bc} \quad I^a{}_b{}^c{}_d \quad S^{-1}{}^a{}_b \qquad \boxed{\delta^a_b \rightarrow \delta^a{}_b}$ important!!

EXCEPTIONS δ^{ab}_{cd}, $\delta^{c_1\cdots c_d}_{d_1\cdots d_n}$, $F^{-1}_{x_0}$, ξ^i_a, ω^i_a, γ^{-1}_{abcd}, $\rho^{p_1q}_{w}$ R^{-1}_a

OTHER IRREGULARITIES $C^{a_b}{}_{c_d}{}^{e_f}$ ← pairs ^a_b in an index position

All momentum symbols p_a p_0 p_+ p_- p_i are lowercase $\bar{p}_a$ $\bar{p}_0$ etc

except for P_a, P_1, P_2, P_3, which are upper case.

EXPLANATION OF HANDWRITTEN SYMBOLS (3)

The lower case "a" used to denote points in a Lie group appears in the subscript position in the following places:

$$AD_a \quad L_a \quad R_a \quad f_a$$

where it should not be confused with a smaller index letter, i.e. it should be normal size.

SPATIALLY HOMOGENEOUS COSMOLOGY: BACKGROUND AND DYNAMICS

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§0. Introduction

Spatially homogeneous cosmology is a topic in which the disciplines of differential geometry, Lie group theory and classical mechanics unite to provide not only an interesting example for the application of ideas of theoretical relativity but a physically useful model for ideas about the nature of our universe. Unfortunately spatially homogeneous dynamics has not been dealt with systematically, its symmetries have been ignored and its classical mechanical structure misunderstood, resulting in the generation of a host of special cases (and occasional errors) lacking any coherence and conveying little general understanding. The present work attempts to remedy this while providing a complete and essentially self-contained introduction to the topic and the relevant mathematics.

Because there do not appear to be available expositions of Lie group theory which make use of modern differential geometry without being too sophistocated for the physicist who only has a grasp of the basics, we first develop this subject in that spirit. It is not done rigorously and global questions are ignored but hopefully a pedestrian understanding is conveyed. The material is then applied to spatially homogeneous cosmology, which when treated carefully, represents an extremely rich and beautiful dynamical system.

To be more specific this system is interpreted as an ordinary classical mechanical system in which a nonconservative force is present, allowing modified Lagrangian and Hamiltonian techniques to be used. By exploring the action of the automorphism and adjoint groups of the Lie algebra of each Bianchi type on the associated configuration and phase spaces, one is led to construct a class of coordinate systems on those spaces adapted to the symmetry properties of the dynamics which considerably simplifies the entire discussion of the problem. A great deal of insight is also gained in interpreting the momentum constraints on this system in terms of the natural automorphism action, the shift freedom on the associated spacetimes and the flow of energy-momentum of the source of the gravitational field parallel to the spatially homogeneous slicing of those spacetimes. In short the techniques introduced allow a somewhat unified approach to the dynamics of spatially homogeneous spacetimes adapted to the specific features of each Bianchi type.

The book GRAVITATION by Misner, Thorne and Wheeler[13] (hereafter referred to as MTW) is used frequently as a source of background material and unless otherwise specified, its notation and conventions are respected. Appendix A summarizes the mathematics and notation used in this chapter. The author thanks Remo Ruffini for encouraging this work and Michael P. Ryan and Abraham Taub for helpful discussion and for laying the foundation upon which the present discussion of spatially homogeneous dynamics is based.

§1. A First Look at Lie Groups and Lie Algebras

A Lie group G is an r-dimensional real analytic manifold (points symbolized by a) with a closed multiplication of its points: $a_1a_2 = \varphi(a_1,a_2) \in G$; this multiplication must be associative, an identity element a_0 must exist and every $a \in G$ must have an inverse $\psi(a) = a^{-1}$. Both φ for fixed a_1 or a_2 and ψ are analytic diffeomorphisms of G into itself. Since the inverse has the property $(a_1a_2)^{-1} = a_2^{-1}a_1^{-1}$, the inverse map satisfies $\psi(a_1a_2) = \psi(a_2)\psi(a_1)$. Any diffeomorphism of G with this property will be called a transposition. (Any transposition ψ satisfies $\psi(a_0) = a_0$, $\psi(a^{-1}) = \psi(a)^{-1}$; an involutive transposition like the inverse satisfies $\psi^2 = \psi\circ\psi = Id$.)

Let M be an n-dimensional real analytic manifold (points symbolized by x). We assume that G acts on M on the left (unless otherwise stated) as a group of analytic diffeomorphisms:

(1.1) $\quad x \mapsto a\cdot x = f(x,a)$

$$a_1\cdot(a_2\cdot x) = (a_1a_2)\cdot x\ , \quad a_0\cdot x = x.$$

The latter requirements insure that the action has the group property and that a_0 acts as the identity transformation Id on M; f is also analytic in both its arguments. We always assume (unless otherwise stated) that G acts effectively on M, meaning that only a_0 leaves every point of M fixed. The set $G\cdot x = \{a\cdot x \mid a \in G\}$ is called the orbit of x. Points lying on the same orbit can be mapped into each other under the action of the group. The action is called transitive if M consists of a single orbit and intransitive otherwise.

For each $x \in M$ we may define a map $F_x : G \to M$ by $F_x(a) = f(x,a)$. Its differential $dF_x(a)$ maps TG_a into $TM_{a\cdot x}$; $\xi_x = dF_x(a_0)$ therefore provides us with a map from the tangent space at the identity of the group to the tangent space at each point $x \in M$. A corresponding map ξ from TG_{a_0} into $\mathfrak{X}(M)$, the vector space of analytic vector fields on M, is obtained by setting $\xi(X)(x) = \xi_x(X)$ where $X \in TG_{a_0}$.

It is convenient to introduce the notation f_a for the diffeomorphism $x \mapsto f_a(x) = f(x,a)$ of M into itself. To say that G acts on the left or on the right simply means that $f_{a_1} \circ f_{a_2}$ equals $f_{a_1 a_2}$ or $f_{a_2 a_1}$ respectively. As the second line of (1.1) shows, the notation $a \cdot x$ is natural for a left action while $x \cdot a$ would be appropriate for a right action. We will use a tilde to signal right actions. For example, a left action is changed into a right action by composing f with a transposition like the inverse map:

(1.2) $\tilde{f}_a = f_{a^{-1}}$, $\tilde{f}_{a_1} \circ \tilde{f}_{a_2} = \tilde{f}_{a_2 a_1}$.

G naturally acts on itself effectively as a group of analytic diffeomorphisms in two ways, on the left by left translation $a \mapsto L_{a_1}(a) = a_1 a$ and on the right by right translation $a \mapsto R_{a_1}(a) = a a_1$. The associativity of the multiplication implies that right and left translation commute:

(1.3) $L_{a_2} \circ R_{a_1}(a) = a_2 a a_1 = R_{a_1} \circ L_{a_2}(a)$.

G also acts on itself by inner automorphism:

(1.4) $AD_{a_1}(a) = a_1 a a_1^{-1} = L_{a_1} \circ R_{a_1^{-1}}(a) = R_{a_1^{-1}} \circ L_{a_1}(a)$

$AD_{a_1} \circ AD_{a_2} = AD_{a_1 a_2}$.

By choosing $a_1 a a_1^{-1}$ rather than $a_1^{-1} a a_1$ we obtain a left action (adjoint action). Let $AD(G) = \{AD_a \mid a \in G\}$ be called the adjoint group or group of inner automorphisms of G. The adjoint action is not necessarily effective. The set $C(G) = \{a \in G \mid AD_a = Id\}$ is a subgroup of G called its center and consists of exactly those elements which commute with every other element of G, since $a_1 a a_1^{-1} = a$ implies $a_1 a = a a_1$. The adjoint action is effective only when $C(G)$ is trivial, i.e. $C(G) = \{a_0\}$. Note that the adjoint group interchanges left and right translations:

(1.5) $AD_a \circ R_a = L_a$, $AD_{a^{-1}} \circ L_a = R_a$.

A homomorphism is a map h from G to another group $\bar{G}$ which preserves the group multiplication:

(1.6) $h(a_1 a_2) = h(a_1) h(a_2)$.

(Note that $h(a_0) = \bar{a}_0$ and $h(a^{-1}) = h(a)^{-1}$.) The kernel of a homomorphism is defined to be the inverse image of $\bar{a}_0$: $\ker h = \{a \in G \mid h(a) = \bar{a}_0\}$. When h is a diffeomorphism it is called an isomorphism and G and $\bar{G}$ are called isomorphic, written $G \cong \bar{G}$. If in addition $G = \bar{G}$, h is called

The set $Aut(G)$ of all automorphisms of G is clearly a subgroup of $\mathcal{D}(G)$, namely, the group of automorphisms of G. It is in fact a finite-dimensional Lie group.[1]

an automorphism. (The composition of two transpositions is an automorphism.) For example, the adjoint diffeomorphisms are automorphisms of G since:

(1.7) $AD_a(a_1 a_2) = AD_a(a_1)\, AD_a(a_2)$,

and hence $AD(G) \subset Aut(G)$.

They are called inner automorphisms since they involve the group multiplication. Similarly (1.4) shows that AD considered as a map from G onto $AD(G)$ is a homomorphism. By definition its kernel is $C(G)$.

A homomorphism ρ from G into the group $GL(V)$ of invertible linear transformations of an n-dimensional vector space V into itself is called a representation of G. (The multiplication making this "general linear group" $GL(V)$ into a group is composition of its elements.) The identity a_0 is a fixed point of the diffeomorphism AD_a for all $a \in G$ so its differential $Ad(a) = (d\,AD_a)(a_0)$ is an element of $GL(TG_{a_0})$. By the chain rule the differential of the composition of two maps is the composition of the differentials so (1.4) implies the relation $Ad(a_1 a_2) = Ad(a_1) \circ Ad(a_2)$. Ad is therefore a homomorphism from G into $GL(TG_{a_0})$ called the adjoint representation of G. The image of the map, denoted by $Ad(G)$, is a subgroup of $GL(TG_{a_0})$ which should be called the linear adjoint group but will also be referred to as the adjoint group. This sloppiness is justified by the fact that $AD(G) \cong Ad(G)$, as will be shown in section 5.

Let us adopt the notation $HK = \{a_1 a_2 \mid a_1 \in H, a_2 \in K\}$ for the product of arbitrary subsets H, K of G and review some general facts about groups. A subgroup H is simply a subset which is closed under multiplication ($HH = H$) and contains all inverses ($H^{-1} = H$). The sets aH and Ha are called left and right cosets of H in G and represent the orbits of the point a under right and left translation by the subgroup H. Denote the set of left cosets by G/H. If $h: G \to \bar{G}$ is a homomorphism then $h(H)$ is a subgroup of $\bar{G}$ for any subgroup H of G while $\ker h$ is a subgroup of G. For example, AD_a is an automorphism so $AD_a(H)$ for each $a \in G$ is a subgroup of G called conjugate to H. If $AD_a(H) = H$ for all $a \in G$, H is called a normal or invariant subgroup. (The kernel of a homomorphism is a normal subgroup.) The left and right cosets of such a subgroup coincide since $aHa^{-1} = H$ implies $aH = Ha$. In this case G/H inherits a group structure of its own from G:

$$a_1 H a_2 H = a_1 a_2 HH = a_1 a_2 H.$$

Thus G/H is itself a group with identity element $a_0 H = H$. The map π_H projecting a point of G to the coset to which it belongs is therefore a homomorphism (with kernel H). A very useful fact is that if $H = \ker h$ for some homomorphism h, then $G/H \cong h(G)$. For example, the center of G is the kernel of the homomorphism AD so $G/C(G) \cong AD(G)$.

As another example, consider the infinite-dimensional group $\mathcal{D}(M)$ of diffeomorphisms of M into itself, with composition as its multiplication. The property $f_{a_1 a_2} = f_{a_1} \circ f_{a_2}$ reveals f to be a homomorphism from G into $\mathcal{D}(M)$, i.e. a left action of a group G on a manifold M is just a homomorphism from G into $\mathcal{D}(M)$. Its kernel is just: $\ker f = \{a \in G \mid f_a = Id\}$. An effective action has trivial kernel and is an isomorphism onto its image $f_G \subset \mathcal{D}(M)$ while $G/\ker f$ acts effectively on M in a natural way if the action is ineffective (by defining $(a \ker f) \cdot x = a \cdot x$). Similarly left translation and inner automorphism yield an isomorphism L and homomorphism AD from G into $\mathcal{D}(G)$, while right actions (like right translation) are "homomorphisms up to a transposition", namely maps between groups which become homomorphisms when composed with a transposition such as the inverse (as in (1.2)).

A Lie algebra $\mathfrak{g}$ is an r-dimensional vector space over the real numbers $\mathbb{R}$ with an alternating bilinear bracket operation $[\,,\,]$ satisfying the Jacobi identity (elements of $\mathfrak{g}$ will be denoted by X, Y, Z etc.):

(1.8) $[X,Y] = -[Y,X]$, $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$.

The set $gl(V)$ of linear transformations of an n-dimensional vector space V into itself is naturally a Lie algebra where the bracket of two transformations X, Y is just their commutator $[X,Y] = XY - YX$. (The Jacobi identity is identically satisfied for brackets defined by commutators, as may be seen by expanding it out into twelve terms which cancel in pairs.) A choice of basis $\{e_a\}$ of V with dual basis $\{\omega^a\}$ (i.e. $\omega^a(e_b)=\delta^a{}_b$) maps $\mathfrak{gl}(V)$ isomorphically onto the vector space of n-dimensional real square matrices $\mathfrak{gl}(n,\mathbb{R})$ and $GL(V)$ onto the group $GL(n,\mathbb{R})$ of nonsingular elements of $\mathfrak{gl}(n,\mathbb{R})$:

(1.9) $A \mapsto \underline{A} = A^a{}_b \,\hat{e}^b{}_a \,, \qquad A^a{}_b = \omega^a(A(e_b)).$

For notation see §6 where we study these spaces.

A transposition of a Lie algebra is defined exactly as for a group, namely a map $\beta \in GL(\mathfrak{g})$ which satisfies $\beta([X,Y]) = [\beta(Y),\beta(X)] = -[\beta(X),\beta(Y)]$. The natural transposition ψ' corresponding to the inverse transposition of G is reflection about the origin: $\psi'(X) = -X$.

With $X \in \mathfrak{g}$ we may associate an element $ad(X)$ of $\mathfrak{gl}(\mathfrak{g})$ by defining $ad(X)Y = [X,Y]$. Rewriting the Jacobi identity and noting that it holds for all $Z \in \mathfrak{g}$ we obtain: $ad([X,Y]) = [ad(X), ad(Y)]$. This says that the map $ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is a Lie algebra homomorphism, namely a linear map σ from $\mathfrak{g}$ into another Lie algebra $\bar{\mathfrak{g}}$ satisfying:

(1.10) $\sigma([X,Y]) = [\sigma(X),\sigma(Y)].$

The kernel of σ is defined by $\ker\sigma = \{X \in \mathfrak{g} \mid \sigma(X) = 0\}$. When σ is also a vector space isomorphism ($\ker\sigma = \{0\}$), it is called a Lie algebra isomorphism and we write $\mathfrak{g} \cong \bar{\mathfrak{g}}$, while if in addition $\mathfrak{g} = \bar{\mathfrak{g}}$ it is called a Lie algebra automorphism. The set of all $A \in GL(\mathfrak{g})$ which are automorphisms is a subgroup denoted by $Aut(\mathfrak{g})$. A Lie subalgebra of $\mathfrak{g}$ is a subvector space $\mathfrak{h}$ closed under the bracket or in an obvious notation $[\mathfrak{h},\mathfrak{h}] \subset \mathfrak{h}$, while an invariant Lie subalgebra or ideal is one for which $ad(\mathfrak{g})\mathfrak{h} = [\mathfrak{g},\mathfrak{h}] \subset \mathfrak{h}$. The kernel of a homomorphism $\sigma : \mathfrak{g} \to \bar{\mathfrak{g}}$ is an ideal in $\mathfrak{g}$ while its image $\sigma(\mathfrak{g})$ is a Lie subalgebra of $\bar{\mathfrak{g}}$. The kernel of the map ad is the set of all $X \in \mathfrak{g}$ which commute with all other elements of $\mathfrak{g}$, an ideal $c(\mathfrak{g})$ called the center of $\mathfrak{g}$, and its image $ad(\mathfrak{g}) \subset \mathfrak{gl}(\mathfrak{g})$ is called the adjoint Lie algebra. When the center of $\mathfrak{g}$ is trivial, $\mathfrak{g} \cong ad(\mathfrak{g})$. A Lie algebra representation σ is a homomorphism from $\mathfrak{g}$ into $\mathfrak{gl}(V)$ for some V. ad is called the adjoint representation of $\mathfrak{g}$. As a last example, the set of all brackets $[\mathfrak{g},\mathfrak{g}]$ is an ideal called the derived Lie algebra.

Let $\{e_a\}$ be a basis of a Lie algebra $\mathfrak{g}$ with dual basis $\{\omega^a\}$. The components of the bracket in this basis are defined by:

(1.11) $[e_a, e_b] = C^c{}_{ab}\, e_c \qquad C^c{}_{ab} = \omega^c([e_a, e_b])$.

They are the components of a $\binom{1}{2}$-tensor over $\mathfrak{g}$ called the structure constant tensor (sometimes abbreviated by SCT): $C = \frac{1}{2} C^a{}_{bc}\, e_a \otimes (\omega^b \wedge \omega^c)$.
It is antisymmetric in its covariant arguments and satisfies the following relations imposed by the Jacobi identity:

(1.12) $0 = C^d{}_{ab} C^e{}_{cd} + C^d{}_{bc} C^e{}_{ad} + C^d{}_{ca} C^e{}_{bd} = 3\, C^d{}_{[ab} C^e{}_{c]d}$.

The matrix of $ad(e_a)$ in this basis, denoted by $\underline{k}_a$, has components $k_a{}^b{}_c = \omega^b(ad(e_a)e_c) = C^b{}_{ac}$, while the matrix form of the homomorphism relation $[ad(e_a), ad(e_b)] = ad([e_a, e_b])$ is:

(1.13) $[\underline{k}_a, \underline{k}_b] = \underline{k}_c C^c{}_{ab}$.

The matrices $\{\underline{k}_a\}$ therefore generate a matrix Lie algebra which is the matrix representation of the adjoint Lie algebra in the basis $\{e_a\}$.

> It is useful to introduce a subscript notation to indicate the matrix representations with respect to the basis $\{e_a\}$ of $\mathfrak{g}$ of the adjoint homomorphism ad and its image and the automorphism group, namely, ad_e, $ad_e(\mathfrak{g})$ and $Aut_e(\mathfrak{g})$. Thus $\underline{k}_a = ad_e(e_a)$.

Define the following quantities:

(1.14) $2a_b = C^a{}_{ba} = Tr\, \underline{k}_b \,, \quad \gamma_{ab} = C^d{}_{ac} C^c{}_{bd} = Tr\, \underline{k}_a \underline{k}_b = \gamma_{ba}$.

These are the components of a covector $Tr\, ad$ over $\mathfrak{g}$ and a symmetric second rank covariant tensor over $\mathfrak{g}$ called the Killing form:

$$Kill(X, Y) = Tr\, ad(X) ad(Y) = \gamma_{ab} X^a Y^b.$$

These are important in the classification of Lie algebras. By contracting the indices (e, c) in (1.12) and using the symmetry of γ_{ab} (or by taking the trace of (1.13)) we obtain the "contracted Jacobi identities":

(1.15) $a_d C^d{}_{ab} = 0$.

Let $A: \mathfrak{g} \to \bar{\mathfrak{g}}$ be a Lie algebra isomorphism and $A^b{}_a = \bar{\omega}^b(A(e_a))$ the components of its matrix with respect to bases $\{e_a\}$, $\{\bar{e}_a\}$ and $C^a{}_{bc}$, $\bar{C}^a{}_{bc}$ the respective SCT components in these bases. The component form of (1.10) yields the result:

(1.16) $\bar{C}^a{}_{bc} = A^a{}_d C^d{}_{fg} A^{-1f}{}_b A^{-1g}{}_c$.

The SCT components of isomorphic Lie algebras are related exactly as are the components of the same SCT in two different bases for $\mathfrak{g}$ which would be the interpretation of (1.16) if $\mathfrak{g} = \bar{\mathfrak{g}}$ and A were the identity transformation. If $\bar{\mathfrak{g}} = \mathfrak{g}$ and we take $\bar{e}_a = e_a$ so that A is an automorphism of $\mathfrak{g}$, then:

(1.17) $\quad C^a{}_{bc} = A^a{}_d C^d{}_{fg} A^{-1f}{}_b A^{-1g}{}_c, \qquad \underline{A}\, \underline{k}_a \underline{A}^{-1} = \underline{k}_b A^b{}_a.$

The second equation is an equivalent matrix expression.

Let $\mathfrak{X}(M)$ be the infinite-dimensional real vector space of vector fields on a manifold M. Since the Lie bracket $[X,Y]$ of two vector fields X,Y is again a vector field (see (A.5)) and is defined by a commutator expression, $\mathfrak{X}(M)$ is an infinite-dimensional Lie algebra of vector fields on M. Suppose $h: M \to N$ is a diffeomorphism and $hX \in \mathfrak{X}(N)$ for $X \in \mathfrak{X}(M)$ is defined by $(hX)(h(x)) = dh(x)X(x)$; then $h: \mathfrak{X}(M) \to \mathfrak{X}(N)$ is a Lie algebra isomorphism:

(1.18) $\quad h[X,Y] = [hX, hY].$

This can be generalized to the case when h is not a diffeomorphism. Although one usually cannot push X forward to a vector field on N, if $\bar{X} \in \mathfrak{X}(N)$ is such that $dh(x)X(x) = \bar{X}(h(x))$ for all $x \in M$, $\bar{X}$ is called h-related to X and the statement analogous to (1.18) is that if $\bar{X}, \bar{Y}$ are h-related to X, Y their brackets are also h-related.

Finite-dimensional Lie subalgebras of $\mathfrak{X}(M)$ for some M are particularly important to Lie group actions. Three such r-dimensional Lie algebras play a fundamental role in the action of a group G on itself by left and right translation and on another manifold M. Let $X \in TG_{a_0}$ be a tangent vector at the identity of G and $c(t)$ any parametrized curve such that $c(0) = a_0$ and $c'(0) = X$. The prime indicates the tangent vector to the curve (see (A.4)). Now let this curve act on G by left and right translation and on M by a left action, generating three families of orbits whose tangents at $t=0$ may be evaluated by the chain rule (A.3):

(1.19) $\quad \tilde{c}(a,t) = L_{c(t)}(a) = R_a(c(t)) \qquad \tilde{X}(a) = \tilde{c}'(a,0) = dR_a(a_0)X \in TG_a$

$\quad c(a,t) = R_{c(t)}(a) = L_a(c(t)) \qquad X(a) = c'(a,0) = dL_a(a_0)X \in TG_a$

$\quad \bar{c}(x,t) = c(t)\cdot x = F_x(c(t)) \qquad \xi(X)(x) = \bar{c}'(a,0) = dF_x(a_0)X \in TM_x.$

Thus for each tangent vector X at the identity we obtain two vector fields X and $\tilde{X}$ on G satisfying $X(a_0) = X = \tilde{X}(a_0)$ and a vector field $\xi(X)$ on M. These have the interpretation that if a curve with tangent X at the identity acts in its various ways, it begins by pushing the points along the corresponding "generating vector fields" or "generators of the action" at these points. This observation provides us with two linear maps from TG_{a_0} into $\mathfrak{X}(G)$ and a linear map ξ into $\mathfrak{X}(M)$ called the generating maps of their respective actions:

$$X \in TG_{a_0} \mapsto \mathbf{X}, \tilde{\mathbf{X}} \in \mathfrak{X}(G);\ \xi(\mathbf{X}) \in \mathfrak{X}(M).$$

Denote the images of these maps by $\mathfrak{g}, \tilde{\mathfrak{g}}$ and $\xi(TG_{a_0})$. The first two are r-dimensional real vector spaces having been constructed isomorphic to TG_{a_0}; we delay proving that $\xi(TG_{a_0})$ is also r-dimensional over $\mathbb{R}$, a claim which depends on the assumption that G acts effectively on M.

We next establish that these vector spaces are isomorphic Lie algebras. Some preliminary results are needed first. From the definitions and the chain rule:

$$(1.20)\quad \mathbf{X} \circ L_{a_1}(a) = dL_{a_1 a}(a_0)X = dL_{a_1}(a)\, dL_a(a_0)X = dL_{a_1}(a)\mathbf{X}(a) = (L_{a_1}\mathbf{X})(a_1 a)$$

$$\tilde{\mathbf{X}} \circ R_{a_1}(a) = \cdots = dR_{a_1}(a)\tilde{\mathbf{X}}(a) = (R_{a_1}\tilde{\mathbf{X}})(a a_1).$$

The first states that $\mathbf{X} \in \mathfrak{g}$ is L_a-related to itself for all $a \in G$ and that the vector field $L_a\mathbf{X} = (dL_a\mathbf{X}) \circ L_{a^{-1}}$ coincides with $\mathbf{X}$. (From $(L_{a_1}\mathbf{X}) \circ L_{a_1}(a) = dL_{a_1}(a)\mathbf{X}(a) = (dL_{a_1}\mathbf{X})(a)$, composition with $L_{a_1^{-1}}$ and changing notation we obtain the quoted form of $L_a\mathbf{X}$.) $\mathbf{X}$ is therefore invariant under the action of left translation or simply "left invariant." Conversely if $\mathbf{X}$ is any left invariant vector field it satisfies $dL_a(a_0)\mathbf{X}(a_0) = \mathbf{X}(a)$, which is how we defined the elements of $\mathfrak{g}$ in the first place. Similarly $\tilde{\mathbf{X}} \in \tilde{\mathfrak{g}}$ is a right invariant vector field. In short, $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ are the vector spaces of left and right invariant vector fields on G (generating the right and left translations respectively).

Since the elements of $\mathfrak{g}$ are L_a-related to themselves, the Lie bracket of two elements of $\mathfrak{g}$ is also L_a-related to itself for all $a \in G$ and hence is also an element of $\mathfrak{g}$. Thus $\mathfrak{g}$ and similarly $\tilde{\mathfrak{g}}$ are closed under the Lie bracket and therefore are r-dimensional Lie algebras.

Let $c_X(t), c_Y(t)$ be curves with tangents X, Y at the identity of G where $c_X(0) = a_0 = c_Y(0)$. $\mathbf{X}(a), \tilde{\mathbf{Y}}(a)$ were defined to be the tangents to the curves $R_{c_X(t)}(a)$ and $L_{c_Y(t)}(a)$ at a; for example, if f is a function on G then by (A.4):

(1.21) $\quad \tilde{X}(a)f = (d/dt)|_0 \, f \circ R_{C_X(t)}(a).$

Using the commutivity of the left and right translations we calculate the Lie bracket of X and $\tilde{Y}$ acting on a function f:

$$X(a)(\tilde{Y}f) = (d^2/dt_1 dt_2)|_0 \, f \circ L_{C_Y(t_2)} \circ R_{C_X(t_1)}(a)$$
$$= (d^2/dt_1 dt_2)|_0 \, f \circ R_{C_X(t_1)} \circ L_{C_Y(t_2)}(a) = \tilde{Y}(a)(Xf).$$

Since f is arbitrary $[X, \tilde{Y}] = 0$ proving that $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ are mutually commuting Lie subalgebras of $\mathcal{X}(G)$.

Now manipulate the expression $[X,Y](a_0)f$ using this fact and the agreement of corresponding left and right invariant vector fields at the identity (and the definition (A.5) of the Lie bracket):

$$\widetilde{[X,Y]}(a_0)f = [X,Y](a_0)f = X(a_0)(Yf) - Y(a_0)(Xf) = \tilde{X}(a_0)(Yf) - \tilde{Y}(a_0)(Xf)$$
$$= Y(a_0)(\tilde{X}f) - X(a_0)(\tilde{Y}f) = \tilde{Y}(a_0)(\tilde{X}f) - \tilde{X}(a_0)(\tilde{Y}f) = [\tilde{Y}, \tilde{X}](a_0)f.$$

Since $\widetilde{[X,Y]}$ and $[\tilde{Y}, \tilde{X}]$ are both right invariant vector fields which agree at the identity, they are equal and therefore:

(1.22) $\quad \widetilde{[X,Y]} = [\tilde{Y}, \tilde{X}] \quad , \quad [-\tilde{X}, -\tilde{Y}] = -\widetilde{[X,Y]}.$

This states that the map $X \mapsto -\tilde{X}$ is a Lie algebra isomorphism from $\mathfrak{g}$ onto $\tilde{\mathfrak{g}}$.

Next we make the following observations:

(1.23) $\quad F_{a \cdot x}(a_1) = a_1 \cdot (a \cdot x) = (a_1 a) \cdot x = F_x \circ R_a(a_1)$

$$dF_{a \cdot x}(a_0)X = dF_x(a)\, dR_a(a_0)X = dF_x(a)\tilde{X}(a)$$
$$\xi(X) \circ F_x(a) = \xi(X)(a \cdot x) = dF_{a \cdot x}(a_0)X = dF_x(a)\tilde{X}(a).$$

This says $\xi(X)$ is F_x-related to $\tilde{X}$ and therefore $[\xi(X), \xi(Y)]$ is F_x-related to $[\tilde{X}, \tilde{Y}]$:

(1.24) $\quad [\xi(X), \xi(Y)] \circ F_x(a) = dF_x(a)\,[\tilde{X}, \tilde{Y}](a) = \xi([\tilde{X}, \tilde{Y}]) \circ F_x(a).$

We adopt the convention that when the argument of ξ is an element of $\mathfrak{g}$ or $\tilde{\mathfrak{g}}$ it is to be evaluated at the identity so $\xi(X) = \xi(\tilde{X})$. This allows us to interpret ξ as a map from either TG_{a_0}, $\mathfrak{g}$ or $\tilde{\mathfrak{g}}$. Evaluation of (1.24) at the identity then leads to the following relation which together with (1.22) yields another:

(1.25) $\quad \xi([\tilde{X}, \tilde{Y}]) = [\xi(\tilde{X}), \xi(\tilde{Y})] \quad , \quad \xi([X,Y]) = -[\xi(X), \xi(Y)].$

ξ is therefore a homomorphism from $\tilde{\mathfrak{g}}$ onto its image $\xi(\tilde{\mathfrak{g}})$. For an effective action we will later show this to be an isomorphism. If

G had acted on M on the right, we would have obtained a homomorphism $\tilde{\xi}$ from $\mathfrak{g}$ onto its image satisfying (1.25) with tildes interchanged.

Finally we may convert TG_{a_0} into a Lie algebra by its identification with $\mathfrak{g}$. If $X, Y \in TG_{a_0}$ let $[X,Y] = [X,Y](a_0)$ where X, Y in the right bracket are the corresponding left invariant vector fields. The identification we have made by using the same symbol for $X \in TG_{a_0}$ and its image in $\mathfrak{g}$ is therefore complete. $TG_{a_0} \cong \mathfrak{g}$ is referred to as the Lie algebra of the Lie group G. However, occasionally it is convenient to distinguish these two spaces which we do when necessary by an explicit map $\wedge$ from $\mathfrak{g}$ onto $TG_{a_0} = \hat{\mathfrak{g}}$, namely evaluation at a_0. A tangent vector $\hat{X} \in \hat{\mathfrak{g}}$ then generates left and right invariant vector fields X and $\tilde{X}$.

Let $\{\hat{e}_a\}$ be a basis of TG_{a_0} with dual basis $\{\hat{\omega}^a\}$. It naturally induces bases $\{e_a\}$, $\{\tilde{e}_a\}$ and $\{\xi_a\}$ of $\mathfrak{g}$, $\tilde{\mathfrak{g}}$ and $\xi(\mathfrak{g})$ where $\xi_a = \xi(e_a)$. If $C^a{}_{bc}$ are the components of the SCT of $\mathfrak{g}$ in the basis $\{e_a\}$ then by (1.22) and (1.25):

(1.26) $[e_a, e_b] = C^c{}_{ab} e_c \;, \quad [\tilde{e}_a, \tilde{e}_b] = -C^c{}_{ab} \tilde{e}_c \;, \quad [\xi_a, \xi_b] = -C^c{}_{ab} \xi_c \,.$

Since the translations are diffeomorphisms, their differentials at the identity are vector space isomorphisms, so $\{e_a\}$ and $\{\tilde{e}_a\}$ are each global frames on G. However, $\{\xi_a\}$ is required to be linearly independent as a set only over $\mathbb{R}$ and not over $\mathcal{F}(M)$ and hence spans a $p \leq r$ dimensional subspace of TM_x at each $x \in M$; in some cases p may even vary over M.

We may introduce the 1-form frames dual to $\{e_a\}$ and $\{\tilde{e}_a\}$ by $\omega^a(e_b) = \delta^a{}_b = \tilde{\omega}^a(\tilde{e}_b)$. By (A.11) these satisfy:

(1.27) $d\omega^a = -\tfrac{1}{2} C^a{}_{bc} \omega^b \wedge \omega^c \;, \quad d\tilde{\omega}^a = \tfrac{1}{2} C^a{}_{bc} \tilde{\omega}^b \wedge \tilde{\omega}^c.$

By defining $\hat{\mathfrak{g}}$-valued 1-forms $\omega = \omega^a \hat{e}_a$ and $\tilde{\omega} = \tilde{\omega}^a \hat{e}_a$ and using the notation $[\alpha \wedge \beta]$ for the combined wedge and Lie bracket of $\hat{\mathfrak{g}}$-valued forms, we may write these relations in a basis independent way:

(1.28) $d\omega + \tfrac{1}{2}[\omega \wedge \omega] = 0 = d\tilde{\omega} - \tfrac{1}{2}[\tilde{\omega} \wedge \tilde{\omega}].$

Note that evaluating ω or $\tilde{\omega}$ on an element $X = X^a e_a \in \mathfrak{g}$ or $\tilde{X} = X^a \tilde{e}_a \in \tilde{\mathfrak{g}}$ respectively is equivalent to evaluating those elements at the identity:

(1.29) $\omega(X) = X^a \hat{e}_a = \hat{X} = \tilde{\omega}(\tilde{X})$.

The transpose of the differential (defined in (A.2)) of a left translation L_{a_1} maps the cotangent space at $L_{a_1}(a)$ onto the cotangent space at a. By the left invariance of $\{e_a\}$:

$$[dL_{a_1}(a)^* \omega^a(a_1a)](e_b(a)) = \omega^a(a_1a)(dL_{a_1}(a)e_b(a))$$
$$= \omega^a(a_1a)(e_b(a_1a)) = \delta^a{}_b = \omega^a(a)(e_b(a)).$$

This establishes the result:

(1.30) $dL'_{a_1}(a)^* \omega^a(a_1a) = \omega^a(a)$.

We naturally call the 1-forms ω^a left invariant. Any left invariant 1-form σ is determined by its value at the identity:

$$\sigma(a) = dL_{a^{-1}}(a)^* \sigma(a_0).$$

The r-dimensional vector space of left invariant 1-forms for which $\{\omega^a\}$ provides a basis is the dual vector space to $\mathfrak{g}$ and will be denoted by $\mathfrak{g}^*$. Similarly $\{\tilde{\omega}^a\}$ is a basis for the dual vector space $\tilde{\mathfrak{g}}^*$ of right invariant 1-forms. ω and $\tilde{\omega}$ are the uniquely defined $\hat{\mathfrak{g}}$-valued invariant 1-forms which map $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ onto $\hat{\mathfrak{g}}$ according to our correspondence.

Modern differential geometry provides a very simple framework to develop Lie group theory. However in any application and in most classical texts on the subject one usually deals with local coordinate systems. It is therefore important to make a connection.

Let $\{a^a\}$ and $\{X^\mu\}$ be local coordinates on G and M' and define the functions $\varphi^a = a^a \circ \varphi$ and $f^\mu = X^\mu \circ f$. With the shorthand notation $\partial_a = \partial/\partial a^a$, $\partial_a^{(i)} = \partial/\partial a^{(i)}$, $\partial_\mu = \partial/\partial X^\mu$, the components of the left and right differentials of φ and the differential of F_X are given by the expressions: $[\partial_a^{(1)}\varphi^b](a_1,a_2)$, $[\partial_a^{(2)}\varphi^b](a_1,a_2)$, $[\partial_a f^\mu](x,a)$. The coordinate basis $\hat{e}_a = \partial_a$ of TG_{a_0} induces bases $\{e_a\}\{\tilde{e}_a\}$ and $\{\xi_a\}$ defined by (1.19):

(1.31) $e_a(a) = dL_a(a_0)\partial_a = [\partial_a^{(2)}\varphi^b](a,a_0)\partial_b$

$\tilde{e}_a(a) = dR_a(a_0)\partial_a = [\partial_a^{(1)}\varphi^b](a_0,a)\partial_b$

$\xi_a(x) = dF_x(a_0)\partial_a = [\partial_a f^\mu](x,a_0)\partial_\mu$.

One may construct the matrices of components of the dual frames $\omega^a{}_b(a)$ and $\tilde{\omega}^a{}_b(a)$ as the inverse matrices to the component matrices of e_a and $\tilde{e}_a$:

(1.32) $\omega^a{}_c e^c{}_b = \delta^a{}_b = e^a{}_c \omega^c{}_b$, $\tilde{\omega}^a{}_c \tilde{e}^c{}_b = \delta^a{}_b = \tilde{e}^a{}_c \tilde{\omega}^c{}_b$.

The left invariance of the frame $\{e_a\}$, namely $dL_{a_1}(a)\, e_a(a) = e_a \circ L_{a_1}(a)$, has the coordinate expression:

$$[\partial^{(2)}_d \varphi^c](a_1,a)\, e^d{}_a(a) = e^c{}_a \circ \varphi(a_1,a)$$

(1.33) $$[\partial^{(2)}_d \varphi^c](a_1,a) = e^c{}_a \circ \varphi(a_1,a)\, \omega^a{}_d(a)$$

$$[\partial^{(1)}_d \varphi^c](a,a_1) = \tilde{e}^c{}_a \circ \varphi(a,a_1) \tilde{\omega}^a{}_d(a).$$

The second line uses (1.32) while the third follows from a parallel discussion for the right invariant frame. Similarly from (1.23), namely $dF_x(a)\tilde{e}_a(a) = \xi_a \circ F_x(a)$, we obtain:

(1.34) $$[\partial_b f^\mu](x,a) = \xi^\mu_d \circ f(x,a)\, \tilde{\omega}^d{}_b(a).$$

The second line of (1.33) is a system of partial differential equations for the multiplication functions $\varphi^c(a_1,a)$ with integration constants $a_1^c = \varphi^c(a_1,a_0)$, while (1.34) is a system for the functions $f^\mu(x,a)$ with integration constants $x^\mu = f^\mu(x,a_0)$. In classical presentations the commutators of the coordinate induced bases of $\mathfrak{g}$ and $\xi(\mathfrak{g})$ are obtained from the integrability conditions for these systems.

§2. Integral Curves and Diffeomorphism Groups

Let $\mathfrak{X}$ be an analytic vector field on an analytic manifold M. An integral curve of $\mathfrak{X}$ is a curve whose tangent coincides with the value of $\mathfrak{X}$ at each point along it. Let $C_{\mathfrak{X}}(x,t)$ be the integral curve which passes through x at t=0:

(2.1) $C'_{\mathfrak{X}}(x,t) = \mathfrak{X} \circ C_{\mathfrak{X}}(x,t) \qquad C_{\mathfrak{X}}(x,0) = x.$

In a coordinate system $\{x^\mu\}$ the components of $\mathfrak{X}$ at x are $\mathfrak{X}^\mu(x) = \mathfrak{X}(x)\, x^\mu$. Using the notation $\bar{x}^\mu(x,t) = x^\mu \circ C_{\mathfrak{X}}(x,t)$, (2.1) becomes:

(2.2) $(d/dt)\, \bar{x}^\mu(x,t) = \mathfrak{X}^\mu(\bar{x}) \qquad \bar{x}^\mu(x,0) = x^\mu.$

This has the power series solution:

(2.3) $\bar{x}^\mu(x,t) = (e^{t\mathfrak{X}} x^\mu)(x),$

where the exponential operator is defined by the usual series expansion. (2.3) is just a Taylor series expansion of $\bar{x}^\mu(x,t)$ about t=0 valid for $|t|$ sufficiently small and obtained using iterations of (2.2) evaluated at t=0 (making use of the old fashioned chain rule):

$$(d/dt)^n|_0\, \bar{x}^\mu(x,t) = (\mathfrak{X}^n x^\mu)(x).$$

We may interpret $x \mapsto C_{\mathfrak{X}}(x,t)$ as a diffeomorphism $\mathfrak{X}_t$ of M into itself for each t and for sufficiently nice $\mathfrak{X}$. By varying t from zero to a finite value the points of M flow along the integral curves or streamlines of $\mathfrak{X}$. $\mathfrak{X}_t$ may be interpreted as a parametrized curve in $\mathfrak{D}(M)$ passing through the identity diffeomorphism since $\mathfrak{X}_0 = Id$. This curve is called the flow of $\mathfrak{X}$. The coordinate power series representation (2.3) may be written:

(2.4) $x^\mu \circ \mathfrak{X}_t(x) = (e^{t\mathfrak{X}} x^\mu)(x).$

From this (ignoring global problems) it is clear that these diffeomorphisms form a 1-parameter abelian group and hence a 1-dimensional subgroup of $\mathfrak{D}(M)$:

(2.5) $\mathfrak{X}_{t_1} \circ \mathfrak{X}_{t_2} = \mathfrak{X}_{t_1+t_2}, \qquad \mathfrak{X}_t \circ \mathfrak{X}_{-t} = \mathfrak{X}_0 = Id.$

This representation also makes evident the useful property $\mathfrak{X}_t = (t\mathfrak{X})_1$. (When global statements hold $\mathfrak{X}$ is said to be complete.)

Let F be a function on M. Then we may powerseries expand $F \circ \mathfrak{X}_t$ about t=0 exactly as we did $x^\mu \circ \mathfrak{X}_t$ to obtain locally:

(2.6) $F \circ \mathfrak{X}_t(x) = (e^{t\mathfrak{X}} F)(x).$

Suppose $\mathfrak{Y}$ is another vector field and consider $\mathfrak{Y}_s \circ \mathfrak{X}_t$, introducing the notation $C^\mu_{\mathfrak{Y}}(x,t) = x^\mu \circ C_{\mathfrak{Y}}(x,t) = x^\mu \circ \mathfrak{Y}_t(x)$. By an application of (2.6) and (2.3):

(2.7) $x^\mu \circ Y_s \circ X_t(x) = C^\mu_Y(X_t(x), s) = e^{tX} C^\mu_Y(x,s) = (e^{tX} e^{sY} x^\mu)(x)$.

Note the reversal in order of the exponential operators relative to the corresponding maps. Suppose we naively multiply the exponential expansions of e^X and e^Y and insert the product into the power series for the logarithm. The result valid locally is:

(2.8) $Z[X,Y] = \log e^X e^Y = X + Y + 1/2\,[X,Y] + 1/12\,([X,[X,Y]] - [Y,[Y,X]]) + \cdots$.

The important feature of this result is that each term in the series is built from brackets of X and Y and is therefore itself a vector field on M (belonging to the Lie subalgebra of $\mathfrak{X}(M)$ generated by X and Y). When the series converges, it therefore converges to a vector field $Z[X,Y]$. We call $Z[\,,\,]$ the Campbell-Baker-Haussdorff functional or C-B-H functional for short.

Suppose X and Y belong to an r-dimensional Lie subalgebra $\mathfrak{g} \subset \mathfrak{X}(M)$. Since $\mathfrak{g}$ is closed under the Lie bracket each term in the C-B-H series belongs to $\mathfrak{g}$ and when the series converges, $Z[X,Y]$ will also be in the Lie algebra. By (2.7) and (2.8), $Y_1 \circ X_1 = Z[X,Y]_1$ holds for all $X, Y \in \mathfrak{g}$ suitably near the origin. In other words the set of diffeomorphisms $\{X_1 | X \in \mathfrak{g}\}$ is locally closed under composition and inverses as well since $X_1^{-1} = (-X)_1$. The products (i.e. compositions) of all such diffeomorphisms with any number of factors form an r-parameter diffeomorphism group (i.e. r-dimensional subgroup of $\mathfrak{D}(M)$) which acts on M by pushing its points along the integral curves of the elements of its Lie algebra $\mathfrak{g}$. Let $\{e_a\}$ be a basis of $\mathfrak{g}$, $\{\omega^a\}$ its dual basis and $C^a{}_{bc}$ its SCT components. We may define canonical coordinates $\{u^a\}$ on the group by:

(2.9) $u^a(X_1) = \omega^a(X) = X^a$.

The C-B-H functional then provides a power series representation for the multiplication function in these coordinates:

(2.10) $\varphi^a(u_1, u_2) = \omega^a(Z[u_1^a e_a, u_2^b e_b]_1) = u_1^a + u_2^a + \frac{1}{2} C^a{}_{bc} u_1^b u_2^c + \cdots$.

Note that if two vector fields commute, $Z[X,Y] = X + Y$ and their 1-parameter groups commute.

If we choose an arbitrary curve $X(t)$ in the Lie algebra $\mathfrak{g}$ rather than a straight line through its origin, its action on M may also be described in coordinates by a formal solution analogous to (2.4). $X(t)$ is called a time-dependent vector field. Let $C(x,t)$ be the curve whose tangent is $\underline{X}(t)\circ c(x,t)$ and such that $c(x,0)=x$, i.e. the integral curve of $\underline{X}(t)$ through x:

(2.11) $$c'(x,t) = \underline{X}(t)\circ c(x,t)$$

$$(d/dt)\, c^{\mu}(x,t) = \underline{X}(t)\circ c(x,t)\; x^{\mu} \quad , \quad c^{\mu}(x,0)=x^{\mu}.$$

These equations have the usual iteration solution familiar from perturbation theory:

$$c^{\mu}(x,t) = \left(T \exp\left(\int_0^t d\tau\, \underline{X}(\tau)\right) x^{\mu}\right)(x).$$

exp rather than e is used for typographical reasons and T indicates t-ordering along the path $\underline{X}(t)$. When $[\underline{X}(t),\underline{X}(t')]=0$ it may be dropped. This is exactly the case for the curves $\underline{X}(t)=t\underline{X}$, yielding the previous formula for the orbits of 1-parameter subgroups. The 1-parameter family of diffeomorphisms h_t which map x onto $c(x,t)$ is called the flow of the time-dependent vector field $\underline{X}(t)$ and represents a curve in $\mathcal{D}(M)$.