LECTURE NOTES
IN PHYSICS

Yaghjian

Relativistic Dynamics of a Charged Sphere

Updating the Lorentz-Abraham Model
Second Edition

"This is a remarkable book. [. . .] A fresh and novel approach to old problems and to their solution." –Fritz Rohrlich, Professor Emeritus of Physics, Syracuse University

This book takes a fresh, systematic approach to determining the equation of motion for the classical model of the electron introduced by Lorentz more than 100 years ago. The original derivations of Lorentz, Abraham, Poincaré and Schott are modified and generalized for the charged insulator model of the electron to obtain an equation of motion consistent with causal solutions to the Maxwell-Lorentz equations and the equations of special relativity. The solutions to the resulting equation of motion are free of pre-acceleration and runaway behavior. Binding forces and a total stress–momentum–energy tensor are derived for the charged insulator model. Appendices provide simplified derivations of the self-force and power at arbitrary velocity.

In this Second Edition, the method used for eliminating the noncausal pre-acceleration from the equation of motion has been generalized to eliminate pre-deceleration as well. The generalized method is applied to obtain the causal solution to the equation of motion of a charge accelerating in a uniform electric field for a finite time interval. Alternative derivations of the Landau-Lifshitz approximation are given as well as necessary and sufficient conditions for the Landau-Lifshitz approximation to be an accurate solution to the exact Lorentz-Abraham-Dirac equation of motion.

The book is a valuable resource for students and researchers in physics, engineering, and the history of science.

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Relativistic Dynamics of a Charged Sphere

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2nd edition

Springer

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To

LUCRETIA

Foreword

This is a remarkable book. Arthur Yaghjian is by training and profession an electrical engineer; but he has a deep interest in fundamental questions usually reserved for physicists. He has studied the relevant papers of an enormous literature that accumulated for longer than a century. The result is a fresh and novel approach to old problems providing better solutions and contributing to their understanding.

Physicists since Lorentz in the late nineteenth century have looked at the equations of motion of a charged object primarily as a description of a fundamental particle, typically the electron. Since the limitations of classical physics due to quantum mechanics have long been known, Yaghjian considers a macroscopic object, a spherical insulator with a surface charge. He thus avoids the pitfalls that have misguided research in the field since Dirac's famous paper of 1938.

The first edition of this book, published in 1992, was an apt tribute to the centennial of Lorentz's seminal paper of 1892 in which he first proposed the extended model of the electron. In the present second edition, attention is also paid to very recent work on the equation of motion of a classical charged particle. Mathematical approximations for specific applications are clearly distinguished from the physical validity of their solutions. It is remarkable how these results call for empirical tests yet to be performed at the necessarily extreme conditions and with sufficiently high accuracy. In these important ways, the present book thus revives interest in the classical dynamics of charged objects.

Syracuse University
2005

Fritz Rohrlich

Preface to the Second Edition

Chapters 1 through 6 and the Appendices in the Second Edition of the book remain the same as in the First Edition except for the correction of a few typographical errors, for the addition and rewording of some sentences, and for the reformatting of some of the equations to make the text and equations read more clearly. A convenient three-vector form of the equation of motion has been added to Chapter 7 that is used in expanded sections of Chapter 7 on hyperbolic and runaway motions, as well as in Chapter 8. Several references and an index have also been added to the Second Edition of the book.

The method used in Chapter 8 of the First Edition for eliminating the noncausal pre-acceleration from the equation of motion has been generalized in the Second Edition to eliminate pre-deceleration as well. The generalized method is applied to obtain the causal solution to the equation of motion of a charge accelerating in a uniform electric field for a finite time interval. Alternative derivations of the Landau-Lifshitz approximation to the Lorentz-Abraham-Dirac equation of motion are also given in Chapter 8 along with Spohn's elegant solution of this approximate equation for a charge moving in a uniform magnetic field. A necessary and sufficient condition is found for this Landau-Lifshitz approximation to be an accurate solution to the exact Lorentz-Abraham-Dirac equation of motion.

Many of the additions that have been made to the Second Edition of the book have resulted from illuminating discussions with Professor W.E. Baylis of the University of Windsor, Professor Dr. H. Spohn of the Technical University of Munich, and Professor Emeritus F. Rohrlich of Syracuse University. Dr. A. Nachman of the United States Air Force Office of Scientific Research supported and encouraged much of the research that led to the Second Edition of the book.

Concord, Massachusetts
2005

Arthur D. Yaghjian

Preface to the First Edition

This re-examination of the classical model of the electron, introduced by H. A. Lorentz 100 years ago, serves as both a review of the subject and as a context for presenting new material. The new material includes the determination and elimination of the basic cause of the pre-acceleration, and the derivation of the binding forces and total stress-momentum-energy tensor for a charged insulator moving with arbitrary velocity. Most of the work presented here was done while on sabbatical leave as a guest professor at the Electromagnetics Institute of the Technical University of Denmark.

I am indebted to Professor Jesper E. Hansen and the Danish Research Academy for encouraging the research. I am grateful to Dr. Thorkild B. Hansen for checking a number of the derivations, to Marc G. Cote for helping to prepare the final camera-ready copy of the manuscript, and to Jo-Ann M. Ducharme for typing the initial version of the manuscript.

The final version of the monograph has benefited greatly from the helpful suggestions and thoughtful review of Professor F. Rohrlich of Syracuse University, and the perceptive comments of Professor T. T. Wu of Harvard University.

Concord, Massachusetts
April, 1992

Arthur D. Yaghjian

Contents

1

Introduction and Summary of Results

The primary purpose of this work is to determine an equation of motion for the classical Lorentz model of the electron that is consistent with causal solutions to the Maxwell-Lorentz equations, the relativistic generalization of Newton's second law of motion, and Einstein's mass-energy relation. (The latter two laws of physics were not discovered until after the original works of Lorentz, Abraham, and Poincaré. The hope of Lorentz and Abraham for deriving the equation of motion of an electron from the self force determined by the Maxwell-Lorentz equations alone was not fully realized.) The work begins by reviewing the contributions of Lorentz, Abraham, Poincaré, and Schott to this century-old problem of finding the equation of motion of an extended electron. Their original derivations, which were based on the Maxwell-Lorentz equations and assumed a zero bare mass, are modified and generalized to obtain a nonzero bare mass and consistent force and power equations of motion. By looking at the Lorentz model of the electron as a charged insulator, general expressions are *derived* for the binding forces that Poincaré *postulated* to hold the charge distribution together. A careful examination of the classic Lorentz-Abraham derivation reveals that the self electromagnetic force must be modified during a short time interval after the external force is first applied and after all other nonanalytic points in time of the external force. The resulting modification to the equation of motion, although slight, eliminates the noncausal pre-acceleration (and pre-deceleration) that has plagued the solution to the Lorentz-Abraham equation of motion. As part of the analysis, general momentum and energy relations are derived and interpreted physically for the solutions to the equation of motion, including "hyperbolic" and "runaway" solutions. Also, a stress-momentum-energy tensor that includes the binding, bare-mass, and electromagnetic momentum-energy densities is derived for the charged insulator model of the electron, and an assessment is made of the redefinitions of electromagnetic momentum-energy that have been proposed in the past to obtain a consistent equation of motion.

Many fine articles have been written on the classical theories of the electron, such as [7], [32], [41], [42], [52], [71], and [72], to complement the original works by Lorentz [4], Abraham [3], Poincaré [19], and Schott [16]. However, in returning to the original derivations of Lorentz, Abraham, Poincaré, and Schott, re-examining them in detail, modifying them when necessary, and supplementing them with the results of special relativity not contained explicitly in the Maxwell-Lorentz equations, it is possible to clarify and resolve a number of the subtle problems that have remained with the classical theory of the Lorentz model of the extended electron.

An underlying motivation to the present analysis is the idea that one can separate the problem of deriving the equation of motion of the extended model of the electron from the question of whether the model approximates an actual electron. Hypothetically, could not one enter the classical laboratory, distribute a charge e uniformly on the surface of an insulating sphere of radius a, apply an external electromagnetic field to the charged insulator and observe a causal motion predictable from the relativistically invariant equations of classical physics? Moreover, the short-range polarization forces binding the excess charge to the surface of the insulator need not be postulated, but should be derivable from the relativistic generalization of Newton's second law of motion applied to both the charge and insulator, and from the requirement that the charge remain uniformly distributed on the spherical insulator in its proper inertial frame of reference. A summary of the results in each of the succeeding chapters follows.

Chapter 2 introduces the original Lorentz-Abraham force and power equations of motion for Lorentz's relativistically rigid model of the electron moving without rotation[1] with arbitrary velocity. Lorentz and Abraham derived their force equation of motion by determining the self electromagnetic force induced by the moving charge distribution upon itself, and setting the sum of the externally applied and self electromagnetic force equal to zero, that is, they assumed a zero "bare mass." Similarly, they derived their power equation of motion by setting the sum of the externally applied and self electromagnetic power (work done per unit time by the forces on the charge distribution) equal to zero.

To the consternation of Abraham and Lorentz, these two equations of motion were not consistent. In particular, the scalar product of the velocity of the charge center with the self electromagnetic force (force equation of motion) did not equal the self electromagnetic power (power equation of motion). Merely introducing a nonzero bare mass into the equations of motion does not remove this inconsistency between the force and power equations of motion. Moreover, it is shown that the apparent inconsistency between self electromagnetic force and power is not a result of the electromagnetic mass in

[1] The work of Nodvik [8, eq. (7.28)] shows that the effect of a finite angular velocity of rotation on the self force and power of the Lorentz model approaches zero to the order of the radius of the charge as it approaches zero and thus classical rotational effects are of the same order as the higher order terms neglected in the Lorentz-Abraham equations of motion.

the equations of motion equaling 4/3 the electrostatic mass, nor a necessary consequence of the electromagnetic momentum-energy not transforming like a four-vector. The 4/3 factor occurs in both the force and power equations of motion, (2.1) and (2.4), and it was of no concern to Abraham, Lorentz, or Poincaré in their original works which, as mentioned above, appeared before Einstein proposed the mass-energy relationship.

Neither the self electromagnetic force-power nor the momentum-energy transforms as a four-vector. (For this reason, they are referred to herein as force-power and momentum-energy rather than four-force and four-momentum.) However, there are any number of force and power functions that could be added to the electromagnetic momentum and energy that would make the total momentum-energy (call it G^i) transform like a four-vector, and yet not satisfy $\mathrm{d}G^i/\mathrm{d}s\ u_i = 0$, so that the inconsistency between the force and power equations of motion would remain. Conversely, it is possible for the proper time derivatives of momentum and energy (force-power) to transform as a four-vector and satisfy $\mathrm{d}G^i/\mathrm{d}s\ u_i = 0$ without the momentum-energy G^i itself transforming like a four-vector. In fact, Poincaré introduced binding forces that removed the inconsistency between the force and power equations of motion, and restored the force-power to a four-vector, without affecting the 4/3 factor in these equations or requiring the momentum and energy of the charged sphere to transform as a four-vector.

The apparent inconsistency between the self electromagnetic force and power is investigated in detail in Chapter 3 by reviewing the Abraham-Lorentz derivation and rigorously rederiving the electromagnetic force and power for a charge moving with arbitrary velocity. For the Lorentz model of the electron moving with *arbitrary* velocity, one finds that the Abraham-Lorentz derivation depends in part on differentiating with respect to time the velocity in the electromagnetic momentum and energy determined for a charge distribution moving with *constant* velocity. Although Lorentz and Abraham give a plausible argument for the validity of this procedure, the first rigorous derivation of the self electromagnetic force and power for the Lorentz electron moving with arbitrary velocity was given by Schott in 1912, several years after the original derivations of Lorentz and Abraham. Because Schott's rigorous derivation of the electromagnetic force and power, obtained directly from the Liénard-Wiechert potentials for an arbitrarily moving charge, is extremely involved and difficult to repeat, a much simpler, yet rigorous derivation is provided in Appendix B.

It is emphasized in Section 3.1 that the self electromagnetic force and power are equal to the internal Lorentz force and power densities integrated over the charge-current distribution of the extended electron, and thus one has no a priori guarantee that they will obey the same relativistic transformations as an external force and power applied to a point mass. An important consequence of the rigorous derivations of the electromagnetic force and power of the extended electron, with arbitrary velocity, is that the integrated self electromagnetic force, and thus the Lorentz-Abraham force equation of motion of the extended electron, is shown to transform as an external force applied to a point mass. However, the rigorous derivations also reveal that the integrated self electromagnetic power, and thus the Lorentz-Abraham power equation of motion, for the relativistically rigid model of the extended electron do not transform as the power delivered to a moving point mass. This turns out to be true even when the radius of the charged sphere approaches zero, because the internal fields become singular as the radius approaches zero and the velocity of the charge distribution is not the same at each point on a moving, relativistically rigid shell. Thus, it is not permissible to use the simple point-mass relativistic transformation of power to find the integrated self electromagnetic power of the extended electron in an arbitrarily moving inertial reference frame from its small-velocity value. (This is unfortunate because the proper-frame and small-velocity values of self electromagnetic force and power, respectively, are much easier to derive than their arbitrary-frame values from a series expansion of the Liénard-Wiechert electric fields; see Appendix A.)

The rigorous derivations of self electromagnetic force and power in Chapter 3 critically confirm the discrepancy between the Lorentz-Abraham force and power equations of motion. Chapter 4 introduces a more detailed picture of the Lorentz model of the electron as a charge uniformly distributed on the surface of a nonrotating insulator that remains spherical with radius a in its proper inertial reference frame. (The values of the permittivity and permeability inside the insulating sphere are assumed to equal those of free space.) Applying the relativistic version of Newton's second law of motion to the surface charge and insulator separately, we prove the remarkable conclusion of Poincaré that the discrepancy between the Lorentz-Abraham force and power equations of motion is caused by the neglect of the short-range polarization forces binding the charge to the surface of the insulator. Even though these short-range polarization forces need not contribute to the total self force or rest energy of formation, they add to the total self power an amount that exactly cancels the discrepancy between the Lorentz-Abraham force and power equations of motion. Moreover, the power equation of motion modified by the addition of the power delivered by the binding forces now transforms relativistically like power delivered to a point mass. With the addition of Poincaré binding forces, the power equation of motion of the Lorentz model of the electron derives from the Lorentz-Abraham force equation of motion, and no longer needs separate consideration.

Of course, Poincaré did not know what we do today about the nature of these surface forces when he published his results in 1906, so he simply assumed the necessity of "other forces or bonds" that transformed like the electromagnetic forces. Also, Poincaré drew his conclusions from the analysis of the fields and forces of a charged sphere moving with constant velocity; see Section 4.1. The derivation in Section 4.2 from the relativistic version of Newton's second law of motion reveals, in addition to the original Poincaré stress, both "inhomogeneous" and "homogeneous" surface stresses that are required to keep the surface charge bound to the insulator moving with arbitrary center velocity. The extra inhomogeneous stress integrates to zero when calculating the total binding force and power. The extra homogeneous binding force and power just equal the negative of the time rate of change of momentum and energy needed to accelerate the mass of the uncharged insulator. It also vanishes when the mass of the uncharged insulator is zero.

The mass of the uncharged insulator should not be confused with the "bare mass" of the surface charge. Today the bare mass should be viewed as simply a mathematically defined mass required to make the Lorentz-Abraham force equation of motion compatible with the relativistic version of Newton's second law of motion and the Einstein mass-energy relation. Also, the analysis in Section 4.2 confirms the original results of Poincaré that the forces binding the charge to the insulator remove the inconsistency between the Lorentz-Abraham force and power equations of motion (that is, between self force and power), but do not remove the 4/3 factor multiplying the electrostatic mass in the equations of motion or require the momentum-energy to transform as a four-vector. With the addition of the binding forces, the force-power, but not the momentum-energy, transforms as a four-vector.

Chapter 5 determines the relationships between the various masses (electromagnetic, electrostatic, bare, measured, and insulator masses) involved with the analysis of the classical model of the electron as a charged insulator. Specifically, the Einstein mass-energy relation demands that the measured mass of the charged insulator equals the sum of the electrostatic mass and the mass of the uncharged insulator (which can include any mass, positive or negative, due to gravitational fields and the short-range polarization forces binding the charge to the insulator, if their contribution to the rest energy of formation is not negligible). The relativistic version of Newton's second law of motion then demands that the momentum of the so-called bare mass equals the difference between the momentum of the electromagnetic mass and the electrostatic mass, regardless of the value of the mass of the insulator. Thus, the final analysis shows what one might expect initially, namely, that the self force derived from the Maxwell-Lorentz equations determines the radiation reaction term in the Lorentz-Abraham (or renormalized Lorentz-Abraham-Dirac) equation of motion but not the correct mass in the relativistic Newtonian acceleration term (whether or not the Poincaré binding forces are included).

It is the negative bare mass that removes the 4/3 factor from the electrostatic mass in the Lorentz-Abraham(-Poincaré) equation of motion and makes the momentum of the charged insulator compatible with the electrostatic rest energy of formation. With the inclusion of both the bare mass and binding stresses, the momentum-energy as well as force-power transform as four-vectors. Why Lorentz, Abraham, and the general physics community assumed as late as 1915 that the bare mass was zero is explained in Sec. 5.1.2.

The final result of the analysis of Chapter 5 is an equation of motion (5.12) for a charged insulator compatible with the Maxwell-Lorentz equations, the relativistic version of Newton's second law of motion, and the Einstein mass-energy relation. (The possibility, considered by Dirac, of extra momentum-energy terms in the relativistic version of Newton's second law of motion for charged particles, and the conditions these terms should satisfy, are discussed in Section 5.1.1.)

Chapter 6 begins by summarizing the transformation properties of the different force-powers and momentum-energies, and deriving a total stress-momentum-energy tensor that accounts for the binding forces and bare mass, as well as the electromagnetic self force for the charged insulator model of the electron. We then consider the redefinitions of electromagnetic momentum-energy that have been proposed to obtain consistent momentum and energy equations of motion without introducing specific binding forces and bare masses. With the exception of the momentum-energy of Schwinger's tensors [23], the redefined momentum-energy densities can be found for the Lorentz model of the electron by multiplying the four-velocity of the center of the extended charge by an invariant function of the electromagnetic field. The total momentum-energy of the charge distribution moving with constant velocity then transforms as a four-vector, and for arbitrary velocity predicts consistent $1/a$ terms for the self force and self power, that is, consistent $1/a$ terms in the force and power equations of motion. However, these invariant redefinitions of electromagnetic momentum-energy do not predict the correct radiation reaction terms in the equations of motion.

Schwinger's method [23] consists of writing the force-power density as the divergence of a tensor that depends on the charge-current distribution for charge moving with constant velocity. This charge-current tensor is subtracted from the original electromagnetic stress-momentum-energy tensor, to obtain a divergenceless stress-momentum-energy tensor (when the velocity is constant) and a total momentum-energy that transforms as a four-vector. This method produces the correct radiation reaction terms as well as consistent $1/a$ terms in the force and power equations of motion for arbitrary velocity. The tensor resulting from this method is ambiguous to within an arbitrary divergenceless tensor. Schwinger concentrates on two tensors which, for a thin shell of charge, are equivalent to the stress-momentum-energy tensor derived for the charged insulator when the value of the mass of the insulator is chosen equal to zero and $m_{es}/3$, where m_{es} is the electrostatic mass.

None of these methods of redefining the electromagnetic momentum-energy require the removal of the 4/3 factor multiplying the electrostatic mass in the original equations of motion. They have the drawback for the Lorentz model of the electron of requiring unknown self force and power (electromagnetic or otherwise) that do not equal the Lorentz force and power. Also, none of the redefined stress-momentum-energy tensors recover the secondary binding forces necessary to hold the accelerating charge to the surface of the insulator. Thus, redefining the electromagnetic momentum-energy seems an unattractive alternative to the deterministic binding forces, bare mass, and total stress-momentum-energy tensor derived for the charged-insulator model of the extended electron.

In Chapter 7, general expressions for the momentum and energy of the moving charge are derived from the equation of motion. The reversible kinetic momentum-energy, the reversible Schott acceleration momentum-energy, and the irreversible radiation momentum-energy are separated in both three and four-vector notation. After the application of an external force to the charged particle, all the momentum-energy that has been supplied by the external force has been converted entirely to kinetic and radiated momentum-energy. However, while the external force is being applied, the momentum-energy is converted to Schott acceleration momentum-energy, as well as kinetic and radiated momentum-energy.

An understanding of the "Schott acceleration momentum-energy" as reactive momentum-energy may be gained by looking at time harmonic motion and comparing the energy of the oscillating charge with the reactive energy of an antenna. It is also confirmed that the conservation of momentum-energy is not violated by a charge in hyperbolic motion (relativistically uniform acceleration), or by the homogeneous runaway solutions to the equation of motion.

By writing the three-vector equation of motion in an especially compact form, it is proven that the only possible solution to the equation of motion for relativistically uniform acceleration is rectilinear "hyperbolic motion" of the charge under a constant externally applied force in some inertial reference frame. This is the only externally applied force for which the radiation reaction force is zero and the Lorentz-Abraham-Dirac equation of motion reduces to the relativistic version of Newton's second law of motion.

Chapter 8 begins by solving the equation of motion for the extended charge in rectilinear motion. When one neglects the higher order terms (in radius a) of the equation of motion, one obtains the well-known pre-acceleration solution under the two asymptotic conditions that the acceleration approaches zero in the distant future (when the external force approaches zero in the distant future) and the velocity approaches zero in the remote past. It is shown that this pre-acceleration solution, which violates causality, is not a strictly valid solution to the equation of motion of the extended charge because the pre-acceleration does not satisfy the requirement that the neglected higher order terms in a are negligible. Unfortunately, when higher order terms in the Lorentz-Abraham(-Poincaré) equation of motion are retained, the noncausal pre-acceleration remains; its time dependence merely changes.

In Section 8.2.1 the root cause of the noncausal pre-acceleration solution is traced to the assumption in the classical derivation of the self electromagnetic force that the position, velocity, and acceleration of each element of charge at the retarded time can be expanded in a Taylor series about the present time. With a finite external force that is zero for all time less than zero and yet an analytic function of time about the real t axis for all time greater than zero, these Taylor series expansions are valid for all time except during the initial short time interval light takes to traverse the charge distribution ($0 \leq t \leq \Delta t_a$). It is shown in Section 8.2.2 that when the derivation of the self force is done properly near $t = 0$, a correction force $\mathbf{f}_a(t)$ that is nonzero only in the "transition interval" $[0, \Delta t_a]$ must be included in the equation of motion.[2] *This small correction force in the equation of motion removes the noncausal pre-acceleration from the solution to the equation of motion without destroying the covariance of the equation of motion.* In Section 8.2.3 the correction force $\mathbf{f}_a(t)$ is determined for rectilinear motion in terms of the change in the velocity of the charge across the transition interval.

In Section 8.2.4, the corrected equation of motion is applied to the problem of determining the motion of a charge that is accelerated by a uniform electric field for a finite time interval, for example, between the parallel plates of a charged capacitor. With the addition of correction forces $f_{a1}(t)$ and $f_{a2}(t)$ at the two nonanalytic points of time in the external force (one when the charge enters the first plate and one when it exits the second plate) both pre-acceleration and pre-deceleration are eliminated.

Section 8.2.5 reveals that the removal of the noncausal pre-acceleration and pre-deceleration in the equation of motion comes at a cost. Unless the magnitude of the externally applied force is bounded by a finite (though extremely large) value, no change in velocity across the transition intervals can be chosen to avoid a negative energy radiated during the transition intervals while maintaining causality. For the charged spherical insulator (extended model of the electron), this restriction on the magnitude of the external force is of little concern because it is identical to a proper-frame condition required for neglecting the terms of higher order in the equation of motion than the radiation reaction term. However, for the Lorentz-Abraham-Dirac equation of motion of a point charge (considered in Section 8.5), obtained by letting the radius of the charge approach zero while renormalizing the mass to a fixed finite value, the higher order terms vanish. Thus, the Lorentz-Abraham-Dirac equation of motion of a mass-renormalized point charge corrected by the transition forces can be made to satisfy causality but at the expense of producing an unphysical negative radiated energy during the transition intervals if the externally applied force becomes extraordinarily large.

If one is not concerned with the correct behavior of the solution to the equation of motion during the time immediately after the external force is first applied, one can obtain a convenient power series solution to the equation of motion. Specifically, power series solutions and conditions for their convergence are derived in Section 8.3 by the method of successive substitutions for the proper-frame, the rectilinear, and the general equation of motion. The first two terms of the power series solution to the general equation of motion are converted to the approximation derived by Landau and Lifshitz [51, sec. 76] to the Lorentz-Abraham-Dirac equation of motion. For the special case of a charge moving in a uniform magnetic field, the solution first derived by Spohn [38] is given for the Landau-Lifshitz approximation to the Lorentz-Abraham-Dirac equation of motion. This solution emerges in a simple form convenient for determining the long-term motion of the charge as well as its change in energy and radius of curvature per unit time. The necessary and sufficient condition on the magnitude of the applied uniform magnetic field is found for this Landau-Lifshitz solution to be an accurate approximation to the exact solution of the Lorentz-Abraham-Dirac equation of motion.

Section 8.4 considers the finite difference equation of motion of the extended charge that has been proposed as an alternative to the differential equation of motion. We find that there is little justification to accept the finite difference equation as a valid equation of motion because it neglects all nonlinear terms (in the proper frame of the charge) involving products of the time derivatives of the velocity, and retains a homogeneous runaway solution that leads to pre-acceleration.

Section 8.5 concentrates on the mass-renormalized or Lorentz-Abraham-Dirac (LAD) equation of motion for a point charge corrected by the transition forces to remove noncausal behavior at nonanalytic points in time of the externally applied force. For a finite external force that is an analytic function of time about the real t axis for all t except for a finite number of nonanalytic points in time, the radiation reaction term in the LAD equation of motion is exact for all t except during the infinitesimal transition intervals following the nonanalytic points in time. Thus, the corrected LAD equation of motion merely equates the sum of the radiation reaction and external forces on a point charge outside the transition intervals to the relativistic Newtonian acceleration force, and adds delta-function transition forces during the transition intervals to eliminate the noncausal pre-acceleration and pre-deceleration. Unfortunately, as mentioned above, these causal solutions to the mass-renormalized corrected equation of motion of a point charge can predict an unphysical negative radiated energy during the transition intervals if the magnitude of the external force is large enough. Consequently, and quite startlingly, a classical causal equation of motion of a mass-renormalized point charge that maintains a non-negative radiated energy during the transition intervals for arbitrarily large values of the external force must involve a more complicated joining of the external, radiation reaction, and Newtonian acceleration forces than just their summation. A fully satisfactory classical equation of motion of a point charge does not result from the corrected equation of motion for an extended classical model of a charged particle by simply renormalizing the diverging mass to a finite value as the radius of the charge is allowed to approach zero.

[2] It is assumed throughout the book that the fundamental equation of motion for a charged particle is ultimately obtained by equating the sum of the external force and the radiation reaction part of the self force to the rest mass of the particle times the relativistic acceleration. During the transition interval this fundamental equation of motion differs from the equation of motion one would obtain by equating the sum of the external force and the total self force to a constant "bare mass" times the relativistic acceleration. This difference is important because it allows for a causal (no pre-acceleration or pre-deceleration) initial-value solution to the equation of motion.

2

Lorentz-Abraham Force and Power Equations

2.1 Force Equation of Motion

Toward the end of the nineteenth century Lorentz modeled the electron ("vibrating charged particle," as he called it) by a spherical shell of uniform surface charge density and set about the difficult task of deriving the equation of motion of this electron model by determining, from Maxwell's equations and the Lorentz force law, the retarded self electromagnetic force that the fields of the accelerating charge distribution exert upon the charge itself [1]. (This initial work of Lorentz in 1892 on a moving charged sphere appeared five years before J.J. Thomson's "discovery" of the electron. It is summarized in English by J.Z. Buchwald [2, app. 7].) With the help of Abraham,[1] a highly successful theory of the moving electron model was completed by the early 1900's [3], [4]. Before Einstein's papers [5], [6] on special relativity appeared in 1905, they had derived the following force equation of motion

$$\mathbf{F}_{\text{ext}} = \frac{e^2}{6\pi\epsilon_0 ac^2}\frac{\mathrm{d}}{\mathrm{d}t}(\gamma\mathbf{u}) - \frac{e^2\gamma^2}{6\pi\epsilon_0 c^3}\left\{\ddot{\mathbf{u}} + \frac{3\gamma^2}{c^2}(\mathbf{u}\cdot\dot{\mathbf{u}})\dot{\mathbf{u}} + \frac{\gamma^2}{c^2}\left[\mathbf{u}\cdot\ddot{\mathbf{u}} + \frac{3\gamma^2}{c^2}(\mathbf{u}\cdot\dot{\mathbf{u}})^2\right]\mathbf{u}\right\} + \mathbf{O}(a) \tag{2.1}$$

with

$$\gamma = \left(1 - u^2/c^2\right)^{-\frac{1}{2}}, \quad \mathbf{O}(a^m) \equiv \sum_{n=m}^{\infty} \boldsymbol{\alpha}_n(\mathbf{u})a^n$$

for a "relativistically rigid" spherical shell of total charge e and radius a, moving with arbitrary *center* velocity, $\mathbf{u} = \mathbf{u}(t)$, and externally applied force

[1] Abraham was the first author to obtain the equations of motion in (2.1) and (2.4) for the charge moving with arbitrary velocity. Nonetheless, I refer to them as the Lorentz-Abraham (rather than the Abraham-Lorentz) equations of motion because Lorentz first obtained the proper-frame equation of motion corresponding to (2.1) many years earlier in 1892 and was the first to propose the relativistically rigid (contracting) model of the electron.

$\mathbf{F}_{\text{ext}}(t)$. The speed of light and permittivity in free space are denoted by c and ϵ_0, respectively. The rationalized mksA international system of units is used throughout, and dots over the velocity denote differentiation with respect to time.

"Relativistically rigid" refers to the particular model of the electron, proposed originally by Lorentz, that remains spherical in its proper (instantaneous rest) frame, and in an arbitrary inertial frame is contracted in the direction of velocity to an oblate spheroid with minor axis equal to $2a/\gamma$. Lorentz, however, used the word "deformable" to refer to this model of the electron. (Even a relativistically rigid finite body cannot strictly exist because it would transmit motion instantaneously throughout its finite volume. Nonetheless, one makes the assumption of relativistically "rigid motion" to avoid the possibility of exciting vibrational modes within the extended model of the electron [7, pp. 131-132].)

The derivation of the differential equation of motion (2.1) requires that the externally applied force be an analytic function of time (in a neighborhood of the real time axis) for all time. This is discussed in Chapter 8 when dealing with the problem of pre-acceleration.

The infinite summation of order a in the equation of motion (2.1) goes to zero as the radius a approaches zero. For a charged sphere of finite radius a moving with arbitrary velocity, it is difficult to determine sufficient conditions on the velocity and its derivatives or on the externally applied force for these $\mathbf{O}(a)$ terms to be negligible. However, the inequalities (8.24) in Section 8.2 give the conditions on the time derivatives of velocity in the proper inertial frame of the charged sphere sufficient for neglecting the $\mathbf{O}(a)$ terms. Specifically, in the proper inertial frame it is sufficient that 1) the fractional changes in the first and higher time derivatives of velocity be small during the time it takes light to travel across the charge distribution, and 2) the velocity changes by a small fraction of the speed of light in this time. Alternatively, the inequalities in (8.90) of Section 8.3 show that the $\mathbf{O}(a)$ terms are negligible in the proper inertial frame if 1) the fractional changes in the externally applied force and its first and higher time derivatives are small during the time light traverses the charge, and 2) the external force is not large enough to change the velocity by a significant fraction of c in this time.

In the original analyses of the Lorentz model of the electron, as well as throughout this book, it is assumed that the charged sphere moves "without rotation," that is, the angular velocity of each point on the sphere is zero in its proper frame of reference. Nodvik [8] has generalized the derivation of the classical equation of motion to include rotation. This generalized equation of motion [8, eq. (7.28)] shows that a finite angular velocity produces a self electromagnetic force of order a and thus (2.1) remains a valid classical equation of motion if the charged sphere has a finite angular velocity. (Schott [9] also investigates the motion of a "spinning" sphere, but he left the general expressions for the self force and couple in terms of integrals that discourage a direct comparison with Nodvik's results.)

The right-hand side of (2.1) is the negative of the self electromagnetic force $\mathbf{F}_{\text{em}}$ determined by Lorentz and Abraham for the moving charge distribution. Thus (2.1) expresses Newton's second law of motion for the shell of charge when the unknown "bare" mass, or "material" mass as Lorentz called it, in Newton's second law of motion is set equal to zero. (With the acceptance of special relativity [5] and, in particular, the Einstein mass-energy equivalence relation [6], it is no longer valid to assume, as did Lorentz and Abraham, that the bare mass is independent of the electrostatic energy of formation, that is, independent of the total charge e and radius a. We shall return in Chapter 5 to the subject of the bare mass and the question of why Lorentz et al. believed the bare mass of the electron was negligible.)

Remarkably, the special relativistic factor γ in the time rate of change of momentum (first term on the right-hand side of (2.1)) and the *radiation reaction* part of the self force with coefficient $e^2/(6\pi\epsilon_0 c^3)$ that doesn't depend on the size or shape of the charge (second term on the right-hand side of (2.1)) were both correctly revealed, so that (2.1) is invariant to a relativistic transformation from one inertial reference frame to another. That is, both sides of the force equation of motion (2.1) transform covariantly. Moreover, one could choose the radius a such that the inertial electromagnetic rest mass

$$m_{\text{em}} = \frac{e^2}{6\pi\epsilon_0 a c^2} \tag{2.2}$$

equaled the measured rest mass of the electron. (This value of a equals 4/3 times the "classical radius of the electron.")

2.2 Power Equation of Motion

As long as Lorentz and Abraham limited themselves to the derivation of the force equation of motion (2.1), they saw no inconsistencies in the Lorentz model of the electron. Lorentz was unconcerned with the terms of order a that are neglected in the self force because he assumed the predicted radius of the electron was both realistic and small enough that only the "next term of the series [the radiation reaction term in (2.1)] makes itself felt" [4, sec. 37].

Lorentz and Abraham were also unconcerned with the electromagnetic mass m_{em} in (2.1) equaling 4/3 the electrostatic mass m_{es}, defined as the energy of formation of the spherical charge divided by c^2

$$m_{\text{es}} = \frac{e^2}{8\pi\epsilon_0 a c^2} \tag{2.3}$$

because they derived the equation of motion (2.1) before Einstein's 1905 papers on relativistic electrodynamics [5] and the mass-energy relation [6]. In neither of the original editions of their books [3], [4] do they mention the 4/3 factor in the inertial electromagnetic mass of (2.1) being incompatible with the electrostatic energy of formation, or, conversely, the energy of formation of the electron having to equal c^2 times the inertial electromagnetic mass.[2]

In 1904, however, Abraham [10], [3, secs. 15 and 22], [4, sec. 180] derived the following power equation of motion for the Lorentz relativistically rigid model of the electron by determining from Maxwell's equations the time rate of change of work done by the internal electromagnetic forces (self electromagnetic power)

$$\mathbf{F}_{\text{ext}} \cdot \mathbf{u} = \frac{e^2}{6\pi\epsilon_0 a}\frac{\mathrm{d}}{\mathrm{d}t}\left(\gamma - \frac{1}{4\gamma}\right) - \frac{e^2\gamma^4}{6\pi\epsilon_0 c^3}\left[\mathbf{u}\cdot\ddot{\mathbf{u}} + \frac{3\gamma^2}{c^2}(\mathbf{u}\cdot\dot{\mathbf{u}})^2\right] + O(a) \quad (2.4)$$

with

$$O(a^m) \equiv \sum_{n=m}^{\infty} \alpha_n(\mathbf{u})a^n.$$

As Abraham and Lorentz pointed out, the power equation of motion (2.4) is not consistent with the force equation of motion (2.1). Specifically, taking the scalar product of the center velocity $\mathbf{u}$ with equation (2.1) gives

$$\mathbf{F}_{\text{ext}} \cdot \mathbf{u} = \frac{e^2}{6\pi\epsilon_0 a}\frac{\mathrm{d}\gamma}{\mathrm{d}t} - \frac{e^2\gamma^4}{6\pi\epsilon_0 c^3}\left[\mathbf{u}\cdot\ddot{\mathbf{u}} + \frac{3\gamma^2}{c^2}(\mathbf{u}\cdot\dot{\mathbf{u}})^2\right] + O(a) \quad (2.5)$$

which differs from (2.4) by the term

$$-\frac{e^2}{24\pi\epsilon_0 a}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\gamma}\right). \quad (2.6)$$

This is the discrepancy between the force equation of motion and the power equation of motion for the Lorentz model that concerned Abraham and Lorentz, namely, that the scalar product of $\mathbf{u}$ with the time rate of change of the electromagnetic momentum did not equal the time rate of change of the work done by the internal electromagnetic forces.

Unlike the force equation of motion (2.1), the left- and right-hand sides of the power equation of motion (2.4) do not transform covariantly; see Appendix A. Moreover, neither the force-power on the right-hand sides of (2.1) and (2.4) nor the momentum-energy transforms as a four-vector; see Section 6.1. (Lorentz and Abraham did not mention and were probably not aware of this noncovariance because these equations were discussed outside the general framework and without the correct velocity transformations of special relativity; compare [11] with [5].)

After the derivation of (2.4), they still saw no problem with the 4/3 factor in the inertial electromagnetic mass, nor with the conventional electromagnetic momentum-energy per se (before taking the time derivative) failing to

[2] The second edition (1908) of Abraham's book added to the first edition [3] a discussion of the theory of relativity and a section 49 in which he mentions the 4/3 factor.

transform as a relativistic four-vector. Moreover, if one rewrites the Lorentz-Abraham equation of motion (2.1) in four-vector notation

$$F^i_{\text{ext}} = \frac{e^2}{6\pi\epsilon_0 a}\frac{\mathrm{d}u^i}{\mathrm{d}s} - \frac{e^2}{6\pi\epsilon_0}\left(\frac{\mathrm{d}^2u^i}{\mathrm{d}s^2} + u^i\frac{\mathrm{d}u_j}{\mathrm{d}s}\frac{\mathrm{d}u^j}{\mathrm{d}s}\right) + O(a) \quad (2.7)$$

one recovers equation (2.1) and (2.5) from (2.7) and misses entirely the discrepancy introduced by the power equation of motion (2.4) derived from Maxwell's equations by Abraham. If the mass in (2.7) is "renormalized" to a finite value as the radius of the charge approaches zero and the $O(a)$ terms vanish, then (2.7) becomes identical to the Lorentz-Abraham-Dirac equation of motion [12], [13]; see Section 8.5. Apparently, the radiation reaction, $[e^2/(6\pi\epsilon_0)][\mathrm{d}^2u^i/\mathrm{d}s^2 + u^i(\mathrm{d}u_j/\mathrm{d}s)(\mathrm{d}u^j/\mathrm{d}s)]$, in the equation of motion was first written in four-vector form by von Laue [14]. Early use of the four-vector notation for the radiation reaction in the equation of motion (2.7) can be found in Pauli's article on relativity theory [7, sec. 32]. Herein we use the four-vector notation of Panofsky and Phillips [13], who normalized the four-velocity to be dimensionless.

Throughout this book the entire von Laue term is referred to as the "radiation reaction" rather than just the $u^i(\mathrm{d}u_j/\mathrm{d}s)(\mathrm{d}u^j/\mathrm{d}s)$ part of this term. That is, the "Schott term" ($\mathrm{d}^2u^i/\mathrm{d}s^2$ term) is considered part of the radiation reaction. Only then does the "radiation reaction" in an arbitrary inertial frame reduce to the original $\ddot{\mathbf{u}}$ radiation reaction term (see (3.3)) derived by Lorentz in the proper ($\mathbf{u} = 0$) inertial frame of the charged sphere.

3

Derivation of Force and Power Equations

The inconsistency between the power and force equations of motion, (2.4) and (2.1) or (2.5), is so surprising that one is tempted to question the Lorentz-Abraham derivation ([10], [3, secs. 15 and 22], [4, sec. 180]) of (2.1) and (2.4). Thus, let us take a careful look at their method of derivation.

The right-hand side of (2.1) is the negative of the self electromagnetic force, $\mathbf{F}_{\rm em}$, and the right-hand side of (2.4) is the negative of the work done per unit time, $P_{\rm em}$, by the internal electromagnetic forces (self electromagnetic power) on the moving shell of charge; specifically

$$\mathbf{F}_{\rm em}(t) = \int\limits_{\rm charge} \rho(\mathbf{r},t)[\mathbf{E}(\mathbf{r},t) + \mathbf{u}(\mathbf{r},t) \times \mathbf{B}(\mathbf{r},t)]\mathrm{d}V = -\frac{\mathrm{d}}{\mathrm{d}t}\epsilon_0 \int\limits_{\rm all\ space} \mathbf{E} \times \mathbf{B}\,\mathrm{d}V \quad (3.1)$$

$$P_{\rm em}(t) = \int\limits_{\rm charge} \rho(\mathbf{r},t)\mathbf{u}(\mathbf{r},t) \cdot \mathbf{E}(\mathbf{r},t)\mathrm{d}V = -\frac{\mathrm{d}}{\mathrm{d}t}\frac{\epsilon_0}{2} \int\limits_{\rm all\ space} (E^2 + c^2 B^2)\mathrm{d}V \quad (3.2)$$

where $\rho(\mathbf{r},t)$ and $\mathbf{u}(\mathbf{r},t)$ are the density and velocity of the charge distribution in the shell, and $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$ are the electric and magnetic fields produced by this moving charge distribution. The magnetic field does not appear in the first integral of (3.2) because the magnetic force is perpendicular to the velocity. (Some authors refer to the radiation reaction term alone in (2.1) or (2.4) as the self force or self power, respectively. However, this seems undesirable because (3.1) and (3.2) clearly define the Lorentz self electromagnetic force and power, and they are equal to the negative of the right-hand side of (2.1) and (2.4), respectively.)

The second equations in (3.1) and (3.2) are, of course, identities derived from Maxwell's equations, assuming there are no radiation fields beyond a finite distance from the charge distribution [15, sec. 2.5, eq.(25) and sec. 2.19, eq.(6)].

For the Lorentz relativistically rigid model of the electron, the charge density and velocity of each part of the shell cannot be the same for an arbitrarily moving shell if the shell is to maintain its spherical shape and uniform charge density in its proper frame of reference (inertial frame at rest instantaneously with respect to the center of the electron). In particular, the relativistic contraction of the moving Lorentz model of the electron, from a spherical to an oblate spheroidal shell, demands that the velocity of its charge distribution cannot be uniformly equal (except in the proper frame) to the velocity of the center of the shell denoted simply by $\mathbf{u} = \mathbf{u}(t)$ in our previous equations (see Appendix A). If $\mathbf{u}(\mathbf{r}, t)$ did not depend on the position $\mathbf{r}$ within the shell, as in Abraham's noncontracting (nonrelativistically rigid) model of the electron [3], $\mathbf{u}(\mathbf{r}, t)$ could be brought outside the charge integrals in (3.1) and (3.2), P_{em} would equal $\mathbf{F}_{\text{em}} \cdot \mathbf{u}$, and the discrepancy (2.6) between (2.4) and (2.5) or (2.1) would vanish. Such a model is unrealistic because it would have a single preferred inertial frame of reference in which it were spherical (with fixed radius a) and its major axis would stretch to an infinite length in its proper frame when its velocity with respect to the preferred frame approached the speed of light.

Still we can ask if the variable velocity in the charge integrals of (3.1) and (3.2) for the Lorentz model of the electron actually produces the discrepancy (2.6) between equations (2.4) and (2.5) or (2.1). For a charge with velocity other than zero, both Abraham and Lorentz derived the first terms on the right-hand sides of (2.1) and (2.4), the terms in question, not from the charge integrals in (3.1) and (3.2) but by evaluating the momentum and energy integrals (second integrals) in (3.1) and (3.2) for a charge moving with constant velocity with respect to time, then differentiating the resulting functions of velocity with respect to time [3, sec. 22], [4, sec. 180]. We know that falsely setting the charge velocity $\mathbf{u}(\mathbf{r}, t)$ independent of $\mathbf{r}$ in the first integrals of (3.1) and (3.2) eliminates the discrepancy (2.6). Therefore, is it really justifiable, as Lorentz [4, sec. 183] and Abraham [3, sec. 23] argue, to assume a charge velocity constant in time in the second integrals of (3.1) and (3.2) to derive the first terms of (2.1) and (2.4), the terms that produce the discrepancy (2.6)?

Apparently, this question was not decided with certainty until the work of Schott [16] who derived both the force and power equations of motion, (2.1) and (2.4), by evaluating directly the integrals in (3.1) and (3.2) over the charge distribution for the Lorentz model of the electron moving (without rotation) with arbitrary center velocity $\mathbf{u}$. In particular, his evaluation of the charge integral in (3.2) indeed yielded the power equation of motion (2.4) to prove that the discrepancy (2.6) with the force equation of motion (2.1) actually existed. In fact, Schott's book appears to be the first reference in which either the force or power equation of motion can be found in the general form of (2.1) and (2.4). To obtain these equations from the work of Lorentz and Abraham, one has to piece together the results of a number of their papers or various sections of their books (e.g., secs. 28, 32, 37, 179, and 180 of [4] plus secs. 15 and 22 of [3]).

Schott's derivations of the force and power equations of motion, (2.1) and (2.4), from the charge integrals of (3.1) and (3.2) involve extremely tedious manipulations of the double integrations of the Liénard-Wiechert potentials for an arbitrarily moving charge distribution. They are so involved that Schott's rigorous approach to the analysis of the Lorentz model of the electron has not appeared or been repeated, as far as I am aware, in any subsequent review or textbook. Page [17] also derives the force equation of motion (2.1) by evaluating and integrating directly the self electromagnetic fields over the charge distribution. However, Page's derivation does not show explicitly the variation in velocity of the charge distribution throughout the shell, and thus it cannot be used to derive the power equation of motion (2.4).

3.1 General Equations of Motion from Proper-Frame Equations

Lorentz also derived the force equation of motion from the charge integral for electromagnetic force in (3.1) by means of a double integral of the Liénard-Wiechert potentials, but only in the proper frame of the electron where the velocity of the charge is zero and the derivation simplifies greatly to yield the well-known result [4], [13] (derived in Appendix A)

$$\mathbf{F}_{\text{ext}} = \frac{e^2}{6\pi\epsilon_0 ac^2}\dot{\mathbf{u}} - \frac{e^2}{6\pi\epsilon_0 c^3}\ddot{\mathbf{u}} + \mathbf{O}(a), \quad u = 0 \tag{3.3}$$

to which the general force equation of motion reduces when the velocity $\mathbf{u}$ in (2.1) is set equal to zero or when $(u/c)^2 \ll 1$. (Equation (3.3) was first derived in 1892 [1] for a charged sphere even though the electron had not been officially "discovered" by J.J. Thomson until 1897.)

For a velocity much less than the speed of light, a derivation performed in Appendix A, similar to Lorentz's derivation of (3.3), but applied to the charge integral for electromagnetic power in (3.2), yields the small-velocity power equation of motion

$$\mathbf{F}_{\text{ext}} \cdot \mathbf{u} = \frac{5e^2}{24\pi\epsilon_0 ac^2}\mathbf{u} \cdot \dot{\mathbf{u}} - \frac{e^2}{6\pi\epsilon_0 c^3}\mathbf{u} \cdot \ddot{\mathbf{u}} + O(a), \quad (u/c)^2 \ll 1 \tag{3.4}$$

to which the general power equation of motion (2.4) reduces when only the first order terms in u/c are retained. Note once again that the scalar product of $\mathbf{u}$ with the force equation (3.3) does not yield the power equation (3.4). Section A.1.2 of Appendix A shows explicitly that the variation of the velocity over the charge distribution, even for $(u/c)^2 \ll 1$, must be taken into account to derive the correct expression (3.4) for the small-velocity electromagnetic power.

Now equations (3.3) and (3.4) raise an important question. Since the force and power equations of motion, (3.3) and (3.4), are derived rigorously from (3.1) and (3.2) for $\mathbf{u}$ approaching zero, why not simply apply the relativistic transformation to the velocity, its time derivatives, and the external force in (3.3) and (3.4) to obtain the general equations of motion (2.1) and (2.4) in an arbitrary frame. Thereby, one would avoid the difficult evaluation of the self force and power directly from (3.1) and (3.2) for a relativistically rigid shell of charge moving with arbitrary center velocity $\mathbf{u}$.

Indeed a relativistic transformation of $\dot{\mathbf{u}}$, $\ddot{\mathbf{u}}$, and $\mathbf{F}_{\text{ext}}$ in the proper-frame force equation of motion (3.3) produces the general force equation of motion (2.1) [14], [18], [7]. However, the same relativistic transformations applied to (3.4) produce the equation (see Appendix A)

$$\mathbf{F}_{\text{ext}} \cdot \mathbf{u} = \frac{5e^2}{24\pi\epsilon_0 a}\frac{d\gamma}{dt} - \frac{e^2\gamma^4}{6\pi\epsilon_0 c^3}\left[\mathbf{u}\cdot\ddot{\mathbf{u}} + \frac{3\gamma^2}{c^2}(\mathbf{u}\cdot\dot{\mathbf{u}})^2\right] + O(a) \tag{3.5}$$

which does not agree with either the general power equation of motion (2.4) or the equation (2.5) obtained from the scalar product of $\mathbf{u}$ with the force equation of motion (2.1).

This apparent paradox is explained by returning to (3.1) and (3.2). Since the self force $\mathbf{F}_{\text{em}}$ and self power P_{em} in (3.1) and (3.2) are quantities obtained by integrating over a finite distribution of charge and are not the force and power applied to a point mass, it is not valid to apply the point relativistic transformations of force and center velocity (and derivatives of velocity) to determine the general values of the integrals in (3.1) and (3.2) from their proper-frame or small-velocity values. For the force equation of motion, the integrated self force (3.1) maintains the transformation properties of a point force, and thus the point relativistic transformations can still be applied to obtain the general integrated self force (3.1) in an arbitrary inertial frame from its proper-frame value on the right-hand side of (3.3). (Unfortunately, one proves this fact by performing the difficult evaluation of (3.1) in the arbitrary inertial frame.) For the power equation of motion, however, the integrated power (3.2) does not transform as the time rate of change of energy of a moving point mass (see Appendix A), even as the radius of the charged shell approaches zero, and thus the point relativistic transformations applied to the small-velocity power (right-hand side of (3.4) as $u \to 0$) do not give the correct value of the power in an arbitrary inertial frame (right-hand side of (2.4)).

From the viewpoint of the electromagnetic stress-momentum-energy tensor (discussed in Chapter 6), it is not surprising that the power equation of motion does not transform covariantly, because the electromagnetic stress-momentum-energy tensor of a charged shell is not divergenceless and the electromagnetic momentum-energy does not transform as a four-vector.

In summary, then, since the point relativistic transformations do not necessarily apply to an integrated force or power (and the electromagnetic stress-momentum-energy tensor is not divergenceless), it is not mathematically rigorous to use these transformations to find the integrated self force and power, (3.1) and (3.2), in an arbitrarily moving inertial reference frame from their proper-frame or small-velocity expressions (3.3) and (3.4). Moreover, as explained above, the classic Lorentz-Abraham derivation of (2.1) and (2.4) for arbitrary $\mathbf{u}$ also lacks rigor because it depends upon the evaluation of the

momentum and energy of a shell of charge moving with constant, rather than arbitrary time varying velocity. Thus, it appears that Schott's book [16] contains the only rigorous derivation to date of both the force equation of motion (2.1) and the power equation of motion (2.4).

Since this highly commendable derivation by Schott is also extremely tedious and difficult to repeat or check, a much shorter, simpler, yet rigorous derivation of the self electromagnetic force and power is given in Appendix B by applying the relativistic transformations of the electromagnetic fields at each point within the arbitrarily moving shell of charge before performing the integrations in (3.1) and (3.2). (All these derivations depend upon expanding the position, velocity, and acceleration of each element of the charge at the retarded time in a power series about the present time. When the external force is applied at $t = 0$, having been zero for $t < 0$, these series expansions are invalid between $t = 0$ and $t = \Delta t_a$, the short time interval it takes light to traverse the charge distribution. In Section 8.2 it is shown that the addition of a transition correction force during this short time interval $[0, \Delta t_a]$ eliminates the noncausal pre-acceleration from the solution to the uncorrected equation of motion.)

4

Internal Binding Forces

In Appendix B, we have critically confirmed the evaluation of the self electromagnetic force and power, (3.1) and (3.2), leading to the force and power equations of motion (2.1) and (2.4). Yet (2.1) and (2.4) are inconsistent, since taking the scalar product of **u** with (2.1) gives (2.5), which differs from (2.4) by the term (2.6). Not only the self electromagnetic momentum-energy but also the self electromagnetic force-power fails to transform as a four-vector. What has gone wrong?

To see clearly the problem and its resolution, it helps to divorce the analysis of the moving spherical shell of charge from the question of whether it models the electron. The analysis is based entirely upon classical fields, forces, and charges, and the extent to which it describes the internal structure of the electron is irrelevant to the question of the inconsistency between the force equation of motion (2.1) and the power equation of motion (2.4). We could enter our classical laboratory, distribute a charge uniformly on the surface of an arbitrarily small, massless (or nearly massless), "relativistically rigid," insulating sphere, accelerate this charged sphere, and, assumably, get consistent results between the force that is required to accelerate the sphere and the power delivered to the sphere.

4.1 Poincaré Binding Forces

Poincaré visualized such a model in his 1906 paper on the dynamics of the electron [19]. (Actually, Poincaré [19, sec. 6] mentions the charge distributed on a conductor rather than an insulator. We choose the insulator model to avoid the possibility of the charge redistributing itself when the sphere moves. Also, we assume that the values of the relative permittivity and permeability within the spherical insulator are equal to unity so that there is free-space propagation of electromagnetic waves inside as well as outside the sphere.) He argued that the only way the charge could remain on the sphere was for there to exist binding forces exerted on the charge by the insulator that would exactly cancel the repulsive portion of the electromagnetic forces. These internal binding forces are not optional, they are necessary in a stable classical Lorentz model. They are the short-range polarization[1] forces that must exist at the surface of the insulator to hold the excess charge to the surface. Although Poincaré did not have today's knowledge of the nature of the internal binding forces, he assumed they existed. To quote the English translation of Poincaré, "Therefore it is indeed necessary to assume [in the Lorentz model] that in addition to electromagnetic forces [of the excess charge alone], there are other forces or bonds" [19, sec. 1].

Thus the total force exerted on the charge in both the force and power equations of motion, (2.1) and (2.4), must include these internal binding forces (which, for the insulator model, are also electromagnetic in origin) as well as the internal electromagnetic forces of the excess charge.

For a stationary charged sphere, as Poincaré explained, the binding forces exerted by the relativistically rigid insulator on the excess charge must be equal and opposite the repulsive electromagnetic forces produced by the excess charge distribution. However, in order to include the binding forces in the force and power equations of motion, one has to know the value of the binding forces for an arbitrarily moving shell of charge. Poincaré determined the internal binding forces on a moving shell by assuming a "postulate of relativity", namely that the "impossibility of experimentally demonstrating the absolute movement of the earth would be a general law of nature"; and, in particular, hypothesized with Lorentz [11, sec. 8] that the internal forces in the Lorentz model would obey the same transformations that Maxwell's equations implied for the electromagnetic forces [19, Introduction]. (Poincaré did not have the benefit of Einstein's relativity papers [5], [6] when he submitted his paper [19] in July 1905, or the knowledge that the binding forces could be short-range polarization forces of electromagnetic origin.)

As a consequence of this latter hypothesis, Poincaré drew a startling conclusion. The internal binding forces that canceled the internal self electrostatic forces of the excess charge on the sphere at rest, when transformed to a moving shell, *would not* contribute to the total self force on the moving charge but *would* contribute to the total time rate of change of energy (power) delivered to the charge in the Lorentz model of the moving charge. Specifically, when Poincaré assumed with Lorentz that the spherical shell compressed to the shape of an oblate spheroid in the direction of its velocity by a factor of $\sqrt{1-u^2/c^2}$, the time rate of change of the binding self energy just canceled the discrepancy (2.6) in the power equation of motion (2.4).

To see how Poincaré arrived at this remarkable result, begin with the electrostatic force per unit surface charge

[1] The surface forces that bind the excess charge to the surface of the spherical insulator can be regarded classically as resulting from electric polarization induced at the surface of the insulator material; see Section 4.2.1.

$$\mathbf{f}^0_{\rm em} = \frac{e}{8\pi\epsilon_0 a^2}\hat{\mathbf{r}} \tag{4.1}$$

for a stationary sphere of radius a and total charge e. The binding force per unit charge required to hold the charge on the stationary sphere is then given by the negative of $\mathbf{f}^0_{\rm em}$ or

$$\mathbf{f}^0_{\rm b} = -\frac{e}{8\pi\epsilon_0 a^2}\hat{\mathbf{r}}\,. \tag{4.2}$$

Now let the charged sphere move with a constant velocity $\mathbf{u}$ and contract in the direction of $\mathbf{u}$ to an oblate spheroid with minor axis equal to $a\sqrt{1-u^2/c^2} = a/\gamma$. The Lorentz force law and Maxwell's equations applied to this moving oblate spheroid predict that the electrostatic force per unit charge in (4.1) and thus the binding force per unit charge in (4.2) transforms to

$$\mathbf{f}_{\rm b} = \mathbf{f}^0_{\rm b\parallel} + \mathbf{f}^0_{\rm b\perp}/\gamma \tag{4.3}$$

where the subscripts, $\parallel$ and $\perp$, refer to the three-vector components parallel and perpendicular to the velocity $\mathbf{u}$. The transformed binding force in (4.3) is directed along the normal into the surface of the oblate spheroid.

The binding force per unit charge (4.3) integrated over the surface charge of the oblate spheroid, because of its symmetry, gives a total binding force $\mathbf{F}_{\rm b}$ equal to zero as in the case of the stationary sphere, that is

$$\mathbf{F}_{\rm b} = \int\limits_{\rm charge} \mathbf{f}_{\rm b} de = \int\limits_{\rm charge} \left(\mathbf{f}^0_{\rm b\parallel} + \mathbf{f}^0_{\rm b\perp}/\gamma\right) de = 0\,. \tag{4.4}$$

However, the work taken by the binding forces from the charge distribution as the charge accelerates from zero to velocity $\mathbf{u}$, if we can assume (4.3) is valid for the accelerating charge as well as the charge moving with constant velocity, would be

$$W_{\rm b} = -\int\limits_{\rm charge}\left[\int\limits_{a\cos\theta}^{a(\cos\theta)/\gamma} \mathbf{f}_{\rm b}\cdot d\mathbf{r}_\parallel\right] de = -\int\limits_{\rm charge}\left[\int\limits_{a\cos\theta}^{a(\cos\theta)/\gamma} \mathbf{f}^0_{\rm b}\cdot d\mathbf{r}_\parallel\right] de \tag{4.5}$$

where θ is the angle between the position vector $\mathbf{r}$ to the element of charge de and the velocity $\mathbf{u}$. The charge element de can be expressed as the product of the surface charge density on the sphere $(e/(4\pi a^2))$ and the projection of the surface area element of the sphere onto the plane perpendicular to $\mathbf{u}$

$$de = \frac{e}{4\pi a^2}\frac{dA_\perp}{\cos\theta}\,. \tag{4.6}$$

From (4.2), the integrand of (4.5) can be rewritten as

$$\mathbf{f}^0_{\rm b}\cdot d\mathbf{r}_\parallel = -\frac{e}{8\pi\epsilon_0 a^2}\cos\theta\, dr_\parallel\,. \tag{4.7}$$

Substitution of (4.6) and (4.7) into (4.5) gives

$$W_b = \frac{e^2}{32\pi^2\epsilon_0 a^4} \int_{\text{sphere}} \left[\int_{a\cos\theta}^{a(\cos\theta)/\gamma} dr_{\|} \right] dA_{\perp} = -\frac{e^2}{32\pi^2\epsilon_0 a^4} \int_{\text{spheroid}}^{\text{sphere}} dV \tag{4.8}$$

or

$$W_b = \frac{e^2}{24\pi\epsilon_0 a}\left(\frac{1}{\gamma} - 1\right) . \tag{4.9}$$

Equations (4.8) and (4.9) reveal that the work taken by the internal binding forces as the spherical charge distribution accelerates and contracts to the shape of an oblate spheroid is the same as the work taken by a constant pressure, $e^2/(32\pi^2\epsilon_0 a^4)$, on a sphere that is compressed to an oblate spheroid. In the words of the English translation of Poincaré, "I have attempted to determine this force, and I found that it can be compared to a constant external pressure acting on the deformable and compressible electron, the work of which is proportional to the variations of the volume of this electron" [19, Introduction].

The negative of the time derivative of (4.9) determines the work done per unit time, P_b, by the internal binding forces on the moving charge

$$P_b = -\frac{e^2}{24\pi\epsilon_0 a}\frac{d}{dt}\left(\frac{1}{\gamma}\right) \tag{4.10}$$

that must be subtracted from the right-hand side of the power equation of motion (2.4). Comparing (4.10) with (2.6), we see, as Poincaré did, that the time rate of change of the work done on the charge by the binding force required to keep the charge on the insulator just cancels the discrepancy (2.6) in power between the power equation of motion (2.4) and the force equation of motion (2.1). As (4.4) shows, the Poincaré binding forces do not alter, however, the total force on the charge distribution, *and thus the force equation of motion (2.1), including the 4/3 factor multiplying the electrostatic mass (2.3), remains unaffected by the Poincaré binding forces. Neither does the power (4.10) delivered by the Poincaré binding forces remove the 4/3 factor from the power equation of motion (2.4), nor do these binding forces change the rest energy of the charged sphere because* W_b *in (4.9) vanishes when* $\mathbf{u}$ *is zero.* Of course, when the charge is first placed on the insulator, the short-range attractive polarization forces holding the charge to the insulator may contribute a negative work of formation to the charge that can subtract from the total rest mass of the charge. The negative mass contributed by these short-range polarization forces binding the charge to the insulator and by other possible attractive forces such as gravity or the short-range forces holding the insulator material together will be included as part of the uncharged insulator mass m_{ins} introduced in the next section.

4.2 Binding Forces at Arbitrary Velocity

The formulation and integrations of the Poincaré binding forces in the previous section are based on the fields and forces of charges in uniform motion. It is uncertain that these results obtained assuming a constant velocity are valid for a shell of charge moving with arbitrary velocity, especially when taking the time derivative of (4.9) to determine the contribution (4.10) of the internal binding forces to the power equation of motion. Thus, we shall derive the polarization binding forces needed to keep the charge on an insulator moving with arbitrary velocity, assuming that the charge remains uniformly distributed on the spherical insulator in its proper inertial frame of reference. (Incidentally, the question raised by Abraham and Lorentz [4, sec. 182] of what keeps the electron in *stable* equilibrium can be answered for the charged insulator model as the nonclassical energy configurations keeping the insulating material and excess charge "rigid" in the proper reference frame; see Section 4.2.1.)

Consider the shell of total charge e in its proper frame as a uniform distribution of volume charge density located between the radius a and $a+\delta$, where δ is the infinitesimally thin thickness of the spherical shell (see Fig. 4.1). At

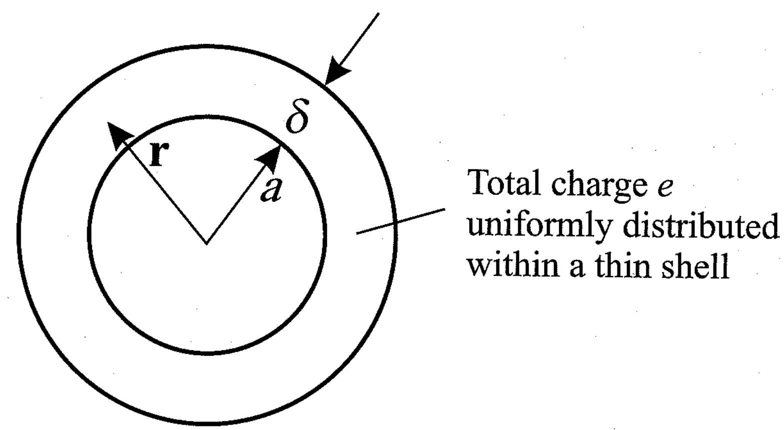

Fig. 4.1. Lorentz model of the electron viewed in its proper frame $[\mathbf{u}(\mathbf{r},t) = 0,\ \dot{\mathbf{u}}(\mathbf{r},t), \ddot{\mathbf{u}}(\mathbf{r},t) \ldots \neq 0]$.

the one instant of time t in its proper frame, the velocity $\mathbf{u}(\mathbf{r},t)$ of the charge at every position $\mathbf{r}$ within the shell is zero, but the acceleration $\dot{\mathbf{u}}(\mathbf{r},t)$ and higher time derivatives of velocity are not necessarily zero nor independent of position $\mathbf{r}$ within the shell.

In Appendix C we determine the internal electric and magnetic fields in the proper frame of the accelerating shell of charge, and, in particular, find the self electromagnetic force per unit charge within the shell to equal

$$\mathbf{f}_{\text{em}}(\mathbf{r},t) = \frac{e}{4\pi\epsilon_0}\left[\frac{(r-a)}{\delta a^2}\hat{\mathbf{r}} - \frac{2\dot{\mathbf{u}}}{3ac^2} + \frac{2\ddot{\mathbf{u}}}{3c^3} + \frac{4}{5c^4}\hat{\mathbf{r}}\cdot\left(\dot{\mathbf{u}}\dot{\mathbf{u}} - \frac{\bar{\mathbf{I}}}{3}|\dot{\mathbf{u}}|^2\right)\right] + \mathbf{O}(a), \quad u = 0\,. \tag{4.11}$$

(In (4.11), as throughout, when **u** and its time derivatives are written without the explicit functional dependence $(\mathbf{r}, t)$, they refer to the velocity and its time derivatives of the center of the shell.)

The force on any volume element of charge in the shell is the sum of the externally applied force, the internal electromagnetic force, and the internal binding force on that element. From Newton's second law of motion, we assume the sum of these three forces equals an unknown "bare" mass of that charge element multiplied by the acceleration (see Section 5.1). Specifically

$$\mathbf{f}_{\text{ext}}(\mathbf{r},t) + \mathbf{f}_{\text{em}}(\mathbf{r},t) + \mathbf{f}_{\text{b}}(\mathbf{r},t) = \frac{M_0}{e}\dot{\mathbf{u}}(\mathbf{r},t), \quad u = 0 \tag{4.12}$$

where $\mathbf{f}_{\text{ext}}(\mathbf{r},t)$, $\mathbf{f}_{\text{em}}(\mathbf{r},t)$, and $\mathbf{f}_{\text{b}}(\mathbf{r},t)$ are the external, internal self electromagnetic, and internal binding forces per unit charge, respectively, at the position **r** in the shell at the instant of time t in the proper frame $(\mathbf{u}(\mathbf{r},t) = \mathbf{u}(t) = 0)$.

The so-called bare mass M_0, which Lorentz set equal to zero, should not be associated with the uncharged mass of the insulator on which the charge is placed. In principle, the mass of the insulator can be made negligible, but M_0 on the right-hand side of (4.12) is dependent upon the charge despite its traditional label as "bare" mass. The following derivation shows that the binding force is independent of the value of the bare mass M_0. (The determination of the mass M_0 and the reason Lorentz thought it was negligible are discussed in Section 5.1.)

In (4.12) we assume the bare mass M_0 of the charge is uniformly distributed with the charge in its proper frame so that the bare mass per unit charge at each point in the spherical shell is M_0/e. Similarly, we shall assume that the variation of the external force is negligible over the charge distribution so that it is applied uniformly (to order a) throughout the proper-frame shell, that is

$$\mathbf{f}_{\text{ext}}(\mathbf{r},t) = \frac{\mathbf{F}_{\text{ext}}(t)}{e} + \mathbf{O}(a)\,. \tag{4.13}$$

As a consequence of the shell remaining spherical in its proper inertial frame of reference, we have from equation (A.8) of Appendix A that the acceleration $\dot{\mathbf{u}}(\mathbf{r},t)$ of the charge element at **r** is related to the acceleration, $\dot{\mathbf{u}} = \dot{\mathbf{u}}(t)$, of the center of the nonrotating shell by the formula

$$\dot{\mathbf{u}}(\mathbf{r},t) = \dot{\mathbf{u}} - \frac{a}{c^2}(\hat{\mathbf{r}}\cdot\dot{\mathbf{u}})\dot{\mathbf{u}} + \mathbf{O}(a^2)\,. \tag{4.14}$$

Inserting the external force (4.13), the internal self electromagnetic force (4.11), and the acceleration from (4.14) into the equation (4.12), we obtain

$$\frac{\mathbf{F}_{\text{ext}}}{e} - \left(\frac{e}{6\pi\epsilon_0 ac^2} + \frac{M_0}{e}\right)\dot{\mathbf{u}} + \frac{e}{6\pi\epsilon_0 c^3}\ddot{\mathbf{u}} + \mathbf{f}_{\text{b}}(\mathbf{r},t) = -\frac{e(r-a)}{4\pi\epsilon_0\delta a^2}\hat{\mathbf{r}} - \frac{e}{5\pi\epsilon_0 c^4}\hat{\mathbf{r}}\cdot\left(\dot{\mathbf{u}}\dot{\mathbf{u}} - \frac{\bar{\mathbf{I}}}{3}|\dot{\mathbf{u}}|^2\right) + \mathbf{O}(a), \quad u = 0. \tag{4.15}$$

Next, integrate (4.15) over the entire charge on the shell to get

$$\mathbf{F}_{\text{ext}} - \left(\frac{e^2}{6\pi\epsilon_0 ac^2} + M_0\right)\dot{\mathbf{u}} + \frac{e^2}{6\pi\epsilon_0 c^3}\ddot{\mathbf{u}} + \int_{\text{charge}} \mathbf{f}_{\text{b}}(\mathbf{r},t)de + \mathbf{O}(a) = 0, \quad u = 0 \tag{4.16}$$

since the integral of $\hat{\mathbf{r}}$ over the uniform spherical charge distribution is zero. Divide (4.16) by the total charge e and subtract the result from (4.15) to show that the binding force has to satisfy the equation

$$\mathbf{f}_{\text{b}}(\mathbf{r},t) - \frac{1}{e}\int_{\text{charge}} \mathbf{f}_{\text{b}}(\mathbf{r},t)de = -\frac{e(r-a)}{4\pi\epsilon_0\delta a^2}\hat{\mathbf{r}} - \frac{e}{5\pi\epsilon_0 c^4}\hat{\mathbf{r}}\cdot\left(\dot{\mathbf{u}}\dot{\mathbf{u}} - \frac{\bar{\mathbf{I}}}{3}|\dot{\mathbf{u}}|^2\right) + \mathbf{O}(a), \quad u = 0. \tag{4.17}$$

The most general solution to (4.17) can be found by letting the binding force equal the right-hand side of (4.17) plus an homogeneous solution $\mathbf{f}_{\text{bh}}(\mathbf{r},t)$

$$\mathbf{f}_{\text{b}}(\mathbf{r},t) = -\frac{e(r-a)}{4\pi\epsilon_0\delta a^2}\hat{\mathbf{r}} - \frac{e}{5\pi\epsilon_0 c^4}\hat{\mathbf{r}}\cdot\left(\dot{\mathbf{u}}\dot{\mathbf{u}} - \frac{\bar{\mathbf{I}}}{3}|\dot{\mathbf{u}}|^2\right) + \mathbf{f}_{\text{bh}}(\mathbf{r},t) + \mathbf{O}(a), \quad u = 0\,. \tag{4.18}$$

Substituting $\mathbf{f}_{\text{b}}(\mathbf{r},t)$ from (4.18) into (4.17) and again noting that the integral of $\hat{\mathbf{r}}$ over the charge distribution is zero, one sees that the homogeneous solution must satisfy the condition

$$\mathbf{f}_{\text{bh}}(\mathbf{r},t) = \frac{1}{e}\int_{\text{charge}} \mathbf{f}_{\text{bh}}(\mathbf{r},t)de\,. \tag{4.19}$$

The right-hand side of (4.19) is not a function of position **r**, so the left-hand side, $\mathbf{f}_{\text{bh}}(\mathbf{r},t)$, cannot be a function of **r**, that is

$$\mathbf{f}_{\text{bh}}(\mathbf{r},t) = \mathbf{f}_{\text{bh}}(t) \tag{4.20}$$

and (4.19) reduces to an identity.

Since we have proven that the homogeneous solution $\mathbf{f}_{\text{bh}}$ for the binding force is independent of the position of the charge element within the shell, it does not average to zero when integrated over the charge unless it is identically zero. This homogeneous binding force is exerted on the insulator in the opposite direction. Specifically, if the rest mass of the uncharged insulator is m_{ins} (assumed uniformly distributed over the sphere), $\mathbf{f}_{\text{bh}}$ is given simply as

$$\mathbf{f}_{\rm bh}(t) = -\frac{m_{\rm ins}}{e}\dot{\mathbf{u}} \tag{4.21}$$

from Newton's second law of motion applied to the insulator in its proper frame. (The inhomogeneous binding force in (4.18) is also exerted in the opposite direction on the insulator but because its total integrated value is zero it does not contribute to the acceleration of the rigid insulator.) With the addition of the homogeneous binding force (4.21), the binding force (4.18) per unit charge needed to keep the charge on the moving insulator is given by

$$\mathbf{f}_{\rm b}(\mathbf{r},t) = -\frac{e(r-a)}{4\pi\epsilon_0\delta a^2}\hat{\mathbf{r}} - \frac{e}{5\pi\epsilon_0 c^4}\hat{\mathbf{r}}\cdot\left(\dot{\mathbf{u}}\dot{\mathbf{u}} - \frac{\bar{\mathbf{I}}}{3}|\dot{\mathbf{u}}|^2\right) - \frac{m_{\rm ins}}{e}\dot{\mathbf{u}} + \mathbf{O}(a), \quad u=0\,. \tag{4.22}$$

Equation (4.22) shows that the binding force is independent of the bare mass M_0.

The first term on the right-hand side of (4.22), when integrated over the thickness of the shell of charge, produces the static binding force (4.2) per unit charge given by Poincaré [19].

The second term on the right-hand side of (4.22) is a binding force that does not appear in Poincaré's analysis using a charged shell moving with constant velocity. It is required to cancel the self electromagnetic acceleration forces in (4.11) that vary with position $\mathbf{r}$ about the shell.

The third term on the right-hand side of (4.22) accounts for the force exerted on the charge to accelerate the mass of the uncharged insulator. If the short-range polarization forces binding the charge to the insulator, or gravitational fields [20], [21], or other attractive forces such as those that hold the insulator material together contribute to the rest energy of formation, this mass can be included in the mass $m_{\rm ins}$ of the uncharged insulator.

When we integrate the force per unit charge in (4.22) over the shell, the first two terms on the right-hand side of (4.22) vanish to give a total binding force equal to the homogeneous binding force

$$\mathbf{F}_{\rm b} = \int_{\rm charge} \mathbf{f}_{\rm b}(\mathbf{r},t)de = m_{\rm ins}\dot{\mathbf{u}} + \mathbf{O}(a), \quad u=0 \tag{4.23}$$

needed to accelerate the insulator in the proper frame. Furthermore, because the first two terms of the internal binding force (4.22) at every point within the charge shell equal the negative of the internal electromagnetic force (4.11), except for the terms in (4.11) that are independent of $\hat{\mathbf{r}}$, the analyses in Appendices A and B can also be applied to these internal binding forces to obtain the total binding force and the total power delivered to the charge by the binding forces in an arbitrary frame of reference. In particular, the generalization of the second term on the right-hand side of (4.22) to an arbitrary inertial frame integrates to zero when finding the total binding force, and leads to a term of order a when finding the total power delivered to the charge by the binding forces. The first term on the right-hand side of (4.22) also integrates to zero in an arbitrary inertial frame but contributes to the power delivered to the charge by the amount given in (2.6) or (4.10) when multiplied by the velocity $\mathbf{u}(\mathbf{r},t)$ and integrated over the charge. And, of course, the third term in (4.22) generalizes immediately to $-m_{\rm ins}d(\gamma\mathbf{u})/dt$, which contributes $-m_{\rm ins}c^2 d\gamma/dt$ to the power delivered to the charge.

Thus, the total binding force and power, contributed by the rigorously derived internal binding force per unit charge needed to keep the charge on an insulator moving with arbitrary velocity, are identical to those given in (4.4) and (4.10) by Poincaré (for a massless insulator, $m_{\rm ins}=0$) using binding forces inferred from the fields and forces of a charge distribution moving with constant velocity, that is

$$\mathbf{F}_{\rm b}(t) = \int_{\rm charge} \mathbf{f}_{\rm b}(\mathbf{r},t)de = -m_{\rm ins}\frac{d(\gamma\mathbf{u})}{dt} + \mathbf{O}(a), \quad u=0 \tag{4.24a}$$

$$P_{\rm b}(t) = \int_{\rm charge} \mathbf{f}_{\rm b}(\mathbf{r},t)\cdot\mathbf{u}(\mathbf{r},t)de = -\frac{e^2}{24\pi\epsilon_0 a}\frac{d}{dt}\left(\frac{1}{\gamma}\right) - m_{\rm ins}c^2\frac{d\gamma}{dt} + O(a). \tag{4.24b}$$

Recall that the velocity $\mathbf{u}(\mathbf{r},t)$ for each portion of the charge distribution cannot equal the velocity $\mathbf{u}(t)$ of the center of the shell (except when $\mathbf{u}(t)=0$) if the shell is to remain spherical in its proper frame of reference (see Appendix A). Thus $\mathbf{u}(\mathbf{r},t)$ in the charge integral of (4.24b) cannot be taken outside the integral sign. Also, we rely on the indirect procedures of Appendices A and B to determine the charge integrals in (4.24) for an arbitrarily moving shell, rather than transform the proper-frame binding force per unit charge (4.22) to obtain the general binding force per unit charge $\mathbf{f}_{\rm b}(\mathbf{r},t)$ in an arbitrary inertial frame. The reason for this indirect procedure is that (4.22) holds for different spatial points within the shell at one instant of time only in the proper frame, but the relativistic transformation of (4.22) to an arbitrary inertial frame for different spatial points within the shell requires the force over an interval of time in the original (proper) frame of reference, even as the radius a approaches zero, because of the $1/a^2$ term in (4.22).

Equations (4.24a) and (4.24b) critically confirm that the rigorously derived binding forces for charge on a relativistically rigid insulator moving with arbitrary center velocity, like the original Poincaré binding forces (4.2), remove the discrepancy (2.6) between the power equation of motion (2.4) and the force equation of motion (2.1), while having no effect (except for the addition of the mass of the uncharged insulator) on the force equation of motion (2.1), or the 4/3 factor in the electromagnetic mass. With the addition of the binding stresses to the self electromagnetic stresses, the force-power, but not the momentum-energy, transforms as a four-vector; see Section 6.1.

4.2.1 Electric Polarization Producing the Binding Forces

One can find a particular polarization at the surface of the insulator that will produce the static binding forces derived in the previous section. When the charge is at rest, the electric field for the binding forces is given by the first term of (4.18) within the shell of charge ($a < r < a + \delta$) and zero everywhere else. An electric polarization that would produce this internal binding electric field is given simply by a radial polarization density, $\mathbf{P}(\mathbf{r})$, within the thin shell of charge proportional to the binding electric field

$$\mathbf{P}(\mathbf{r}) = \begin{cases} \dfrac{e(r-a)}{4\pi\delta a^2}\hat{\mathbf{r}}\,, & a < r < a + \delta \\ 0 \,, & a + \delta < r < a\,. \end{cases} \tag{4.25}$$

The total electric field, $\mathbf{E}_{\mathrm{T}}(\mathbf{r})$, is the sum of the electric field of the free charge and the electric field produced by the radial polarization density. For the charge at rest, it is given by

$$\mathbf{E}_{\mathrm{T}}(\mathbf{r}) = \begin{cases} 0 \,, & r < a + \delta \\ \dfrac{e}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}\,, & r > a + \delta\,. \end{cases} \tag{4.26}$$

In other words, the polarization adds a bound volume charge density ($-\nabla\cdot\mathbf{P}$) that cancels the free-charge density within the surface layer ($a \le r < a+\delta$), and a compensating bound surface charge density ($\mathbf{P}\cdot\hat{\mathbf{r}}$) at the outer surface ($r = a + \delta$). The total charge (free charge plus bound polarization charge) reduces to that of Lorentz's original free-charge shell model of the electron as the thickness δ of the shell approaches zero.

As the thickness δ of the shell approaches zero, the electrostatic energy of formation of the free charge and polarization distribution is the same as the free charge alone; thereby confirming that it is not mandatory for the rest energy and mass contributed by the short-range dipolar binding forces to have values other than zero. For the shell at rest, there is no net force exerted on the free charge by the polarization. (When the charge is moving, the results of Sections 4.1 and 4.2 show that the polarization binding forces supply a net force and power to the free charge given in (4.24).)

One can also determine an effective molecular polarizability required to produce the polarization that holds the free charge on the stationary insulator. For a linear, homogeneous, isotropic dielectric insulator, the polarization density is proportional to the local field

$$\mathbf{P} = \alpha_v\left(\epsilon_0\mathbf{E}_{\mathrm{T}} + \frac{\mathbf{P}}{3}\right) = \alpha_v\frac{\mathbf{P}}{3}, \quad a < r < a + \delta \tag{4.27}$$

where the proportionality constant α_v is the molecular polarizability per unit volume [13, ch. 2]. This last equation shows that the effective molecular polarizability at the surface of the insulator must be equal to 3.0 in order for the free-charge distribution to excite the required polarization density. This value of effective molecular polarizability corresponds to an infinite electric susceptibility at the surface of the insulator.

In brief, the free charge uniformly distributed in a thin layer at the surface of the insulator induces an effective dielectric polarizability of 3.0 within this layer and a polarized field that cancels the self repulsive forces that the free charge exerts on itself. Opposite forces are, of course, exerted on the polarized material of the dielectric insulator. The insulator material does not fly apart because it is held together in the stable energy configurations described by nonclassical physics (quantum mechanics rather than the classical electrodynamics employed here). Inside the sphere ($r < a$) it is assumed throughout the book that the values of the permittivity and permeability are equal to those of free space, so that the speed of electromagnetic wave propagation inside as well as outside the sphere is equal to c.

Before leaving this chapter on the internal binding forces, let us summarize with hindsight the origin and elimination of the discrepancy in power (2.6) between the Lorentz-Abraham force and power equations of motion. When the charged sphere is stationary, each element of the charge experiences a repulsive force due to its own electric field. This electrostatic force integrated over the charge contributes nothing to the total force on the charge. When the charged sphere moves, this static self force transforms relativistically, but still integrates to give a zero total force. However, the moving charged sphere contracts relativistically in the direction of the velocity by an amount proportional to $1/\gamma$, while the component of the static self force per unit charge parallel to the velocity remains unchanged. Thus, the component of the static self force parallel to the velocity does work at a rate proportional to the time rate of change of $1/\gamma$, as exhibited by the negative of the power in (2.6).

In addition to the self electrostatic force on the stationary charge distribution, each element of charge is held to the insulator by a binding force that just cancels the electrostatic force. When the charged sphere moves, this binding force exerted on the charge contributes exactly the negative of the power delivered to the charge by the electrostatic force, thereby canceling the discrepancy in power (2.6) between the force and power equations of motion.

Lastly, consider a question concerning the mass of the insulator. Even if the rest mass of the insulator is negligible, the insulator exerts a binding force on the charge distribution that does work on the moving charge at the rate given by (2.6) or (4.10). The negative of this binding force is exerted on the insulator by the charge. Consequently, work is done on the moving insulator at the rate given by the negative of (2.6). Thus, one might ask if this work done on the moving insulator changes the total mass of the moving charged sphere. The answer to this question is certainly no because of the Einstein mass-energy relation. That is, the relativistic mass of the body can be changed by externally applied forces but not by internal forces that do not involve energy that leaves or enters the body. From an alternative viewpoint, the work done by the binding forces exerted on the insulator surface by the charge distribution is canceled by the work done by the stresses inside the insulator material that must oppose the charge-induced binding forces to keep the insulator relativistically rigid.

5

Electromagnetic, Electrostatic, Bare, Measured, and Insulator Masses

As a means of discussing the various masses, let us summarize the basic results that have been derived so far in our re-examination of the Lorentz model of the electron. We begin with a specific model that we can, in principle, realize in our classical laboratory, namely, a charge e uniformly distributed on the surface of an insulator (insulating material continuum) which remains spherical with constant radius a in its proper inertial frame of reference. Whether or not the model actually approximates the internal structure of the electron is irrelevant to its analysis which is based on Maxwell's equations with retarded (causal) solutions only, the Lorentz force law, the relativistic generalization of Newton's second law of motion, the Einstein mass-energy relation, and the short-range polarization forces binding the charge to the insulator surface.

When an external force is applied to the shell of charge, for example, by means of an external electric field, the charge distribution moves with velocity $\mathbf{u}(\mathbf{r}, t)$. Except when the insulator has zero velocity, the velocity of the charge at different positions $\mathbf{r}$ on the surface of the insulator cannot have the same velocity $\mathbf{u} = \mathbf{u}(t)$ as the center of the insulator if the insulator remains spherical in its proper frame. (The relationship between $\mathbf{u}(\mathbf{r}, t)$ and the center velocity $\mathbf{u}(t)$ is given in equation (A.13) for $(u/c)^2 \ll 1$, and equation (B.32) for arbitrary u/c.)

The force on each differential element de of the moving charge is the sum of the externally applied force per unit charge $\mathbf{f}_{\rm ext}(\mathbf{r}, t)$, the internal electromagnetic force per unit charge $\mathbf{f}_{\rm em}(\mathbf{r}, t)$ generated by the charge itself, and the polarization binding forces per unit charge $\mathbf{f}_{\rm b}(\mathbf{r}, t)$ holding the charge to the insulator, that is

$$[\mathbf{f}_{\rm ext}(\mathbf{r}, t) + \mathbf{f}_{\rm em}(\mathbf{r}, t) + \mathbf{f}_{\rm b}(\mathbf{r}, t)]\, de\ . \tag{5.1a}$$

Similarly, the work done per unit time by these forces on the element of charge de moving with velocity $\mathbf{u}(\mathbf{r}, t)$ is

$$[\mathbf{f}_{\rm ext}(\mathbf{r}, t) + \mathbf{f}_{\rm em}(\mathbf{r}, t) + \mathbf{f}_{\rm b}(\mathbf{r}, t)] \cdot \mathbf{u}(\mathbf{r}, t) de\ . \tag{5.1b}$$

The internal self electromagnetic force is determined by the Lorentz force law in terms of the self electric and magnetic fields excited by the moving charge. The self electromagnetic fields in the charge distribution derive from Maxwell's equations with retarded (causal) potentials to give the self electromagnetic force per unit charge in (4.11) in the proper frame. The binding force per unit charge was derived in Section 4.2 by applying Newton's second law of motion to each differential element of charge under the requirements that the charge remains uniformly distributed on the relativistically rigid spherical insulator in its proper frame of reference (instantaneous rest frame) and that the mass of the charge distribution is uniformly distributed with the charge in its proper frame. The binding force per unit charge exerted on the charge in the proper frame by the short-range polarization forces holding the charge to the insulator is given in (4.22). It is emphasized that this binding force is not speculated but deduced from the specific model of the charge residing on the surface of a nonrotating insulator that maintains its spherical shape and uniform charge distribution in its proper frame.

The total force $\mathbf{F}(t)$ exerted on the charge and the total power $P(t)$ delivered to the charge are found by integrating (5.1a) and (5.1b), respectively, over the charge distribution

$$\mathbf{F}(t) = \int_{\text{charge}} \mathbf{f}_{\text{ext}}(\mathbf{r},t)de + \int_{\text{charge}} \mathbf{f}_{\text{em}}(\mathbf{r},t)de + \int_{\text{charge}} \mathbf{f}_{\text{b}}(\mathbf{r},t)de \tag{5.2a}$$

$$P(t) = \int_{\text{charge}} \mathbf{f}_{\text{ext}}(\mathbf{r},t)\cdot\mathbf{u}(\mathbf{r},t)de + \int_{\text{charge}} \mathbf{f}_{\text{em}}(\mathbf{r},t)\cdot\mathbf{u}(\mathbf{r},t)de + \int_{\text{charge}} \mathbf{f}_{\text{b}}(\mathbf{r},t)\cdot\mathbf{u}(\mathbf{r},t)de. \tag{5.2b}$$

By definition

$$\int_{\text{charge}} \mathbf{f}_{\text{ext}}(\mathbf{r},t)de = \mathbf{F}_{\text{ext}}(t)\,. \tag{5.3a}$$

Also, because the radius a is assumed small enough that the externally applied force varies negligibly with the position over the charge distribution (see (4.13)), the integral involving the external force in (5.2b) becomes

$$\begin{aligned}\int_{\text{charge}} \mathbf{f}_{\text{ext}}(\mathbf{r},t)\cdot\mathbf{u}(\mathbf{r},t)de &= \mathbf{u}(t)\cdot\int_{\text{charge}} \mathbf{f}_{\text{ext}}(\mathbf{r},t)de + O(a^2)\\ &= \mathbf{F}_{\text{ext}}(t)\cdot\mathbf{u}(t) + O(a^2).\end{aligned} \tag{5.3b}$$

The expression (B.32) in Appendix B for $\mathbf{u}(\mathbf{r},t)$ in terms of the velocity $\mathbf{u}(t)$ of the center of the shell has been used to perform the integration in (5.3b).

The integrals of the self electromagnetic Lorentz force and power in (5.2a) and (5.2b), shown explicitly in (3.1) and (3.2) and evaluated rigorously in Appendix B for the arbitrarily moving shell of charge, are just the negative of the right-hand sides of the Lorentz-Abraham force equation of motion (2.1) and the Lorentz-Abraham power equation of motion (2.4), respectively. That is

$$\begin{aligned}\int_{\text{charge}} \mathbf{f}_{\text{em}}(\mathbf{r},t)de =& -\frac{e^2}{6\pi\epsilon_0 ac^2}\frac{d}{dt}(\gamma\mathbf{u}) + \frac{e^2\gamma^2}{6\pi\epsilon_0 c^3}\left\{\ddot{\mathbf{u}} + \frac{3\gamma^2}{c^2}(\mathbf{u}\cdot\dot{\mathbf{u}})\dot{\mathbf{u}}\right.\\ &\left.+\frac{\gamma^2}{c^2}\left[(\mathbf{u}\cdot\ddot{\mathbf{u}}) + \frac{3\gamma^2}{c^2}(\mathbf{u}\cdot\dot{\mathbf{u}})^2\right]\mathbf{u}+\right\} + \mathbf{O}(a)\end{aligned} \tag{5.4a}$$

and

$$\begin{aligned}\int_{\text{charge}} \mathbf{f}_{\text{em}}(\mathbf{r},t)\cdot\mathbf{u}(\mathbf{r},t)de =& -\frac{e^2}{6\pi\epsilon_0 a}\frac{d}{dt}\left(\gamma - \frac{1}{4\gamma}\right)\\ &+\frac{e^2\gamma^4}{6\pi\epsilon_0 c^3}\left[(\mathbf{u}\cdot\ddot{\mathbf{u}}) + \frac{3\gamma^2}{c^2}(\mathbf{u}\cdot\dot{\mathbf{u}})^2\right] + O(a)\end{aligned} \tag{5.4b}$$

where, as throughout, $\mathbf{u}$ and its derivatives on the right-hand sides of (5.4) refer to the velocity $\mathbf{u}(t)$ of the center of the shell.

The integrals of the binding force and power in (5.2a) and (5.2b) were evaluated in Section 4.2 and are given explicitly in (4.24a) and (4.24b), respectively. The total binding force (4.24a) involves the mass m_{ins} of the insulator, and the power delivered by the binding force (4.24b) to the charge just cancels the electromagnetic power term in the right-hand side of (5.4b) that creates the discrepancy (2.6) between the right-hand side of (5.4b) and $\mathbf{u}$ dotted into the right-hand side of (5.4a). Thus, as a result of adding (5.3), (5.4) and (4.24), the total force (5.2a) and power (5.2b) become

$$\begin{aligned}\mathbf{F}(t) =& \mathbf{F}_{\text{ext}}(t) - (m_{\text{em}} + m_{\text{ins}})\frac{d}{dt}(\gamma\mathbf{u}) + \frac{e^2\gamma^2}{6\pi\epsilon_0 c^3}\left\{\ddot{\mathbf{u}} + \frac{3\gamma^2}{c^2}(\mathbf{u}\cdot\dot{\mathbf{u}})\dot{\mathbf{u}}\right.\\ &\left.+\frac{\gamma^2}{c^2}\left[(\mathbf{u}\cdot\ddot{\mathbf{u}}) + \frac{3\gamma^2}{c^2}(\mathbf{u}\cdot\dot{\mathbf{u}})^2\right]\mathbf{u}\right\} + \mathbf{O}(a)\end{aligned} \tag{5.5a}$$

and

$$\begin{aligned}P(t) = \mathbf{F}(t)\cdot\mathbf{u}(t) =& \mathbf{F}_{\text{ext}}(t)\cdot\mathbf{u}(t) - (m_{\text{em}} + m_{\text{ins}})c^2\frac{d\gamma}{dt}\\ &+\frac{e^2\gamma^4}{6\pi\epsilon_0 c^3}\left[(\mathbf{u}\cdot\ddot{\mathbf{u}}) + \frac{3\gamma^2}{c^2}(\mathbf{u}\cdot\dot{\mathbf{u}})^2\right] + O(a)\end{aligned} \tag{5.5b}$$

where the electromagnetic mass $m_{\text{em}} = e^2/(6\pi\epsilon_0 ac^2)$. Because the binding force has removed the discrepancy between (5.5a) and (5.5b), these two equations can also be written concisely in the four-vector notation given in (2.7). Also, all the information in both (5.5a) and (5.5b) is contained in (5.5a) alone.

5.1 Bare Mass in Terms of Electromagnetic and Electrostatic Masses

In (5.5a) we have derived the total force $\mathbf{F}(t)$, internal plus external, experienced by the charge moving with arbitrary center velocity $\mathbf{u}(t)$. What should this total force equal? One's first thought might be that the total force on the charge should equal the time rate of change of momentum $(m_{\text{es}} + m_{\text{ins}})\mathrm{d}(\gamma\mathbf{u})/dt$ since the electrostatic mass m_{es} is the rest mass of the charge and m_{ins} is the rest mass of the insulator; see discussion between (5.8) and (5.9) below. Alternatively, one might think, as Lorentz and Abraham concluded from the measurements of Kaufmann (see Section 5.1.2), that the total force should be zero so that the externally applied force equals the time rate of change of the electromagnetic momentum when m_{ins} is zero. But neither of these alternatives is correct if one accepts the relativistic generalization of Newton's second law of motion [22], [7, sec. 29] that says the total *external* force applied to a particle should equal, apart from the radiation reaction and forces of order a of a charged particle, the time derivative of momentum of the particle, that is

$$\mathbf{F}_{\text{ext}}(t) = m\frac{\mathrm{d}}{\mathrm{d}t}[\gamma\mathbf{u}(t)] - \begin{pmatrix}\text{radiation}\\ \text{reaction}\end{pmatrix} + \mathbf{O}(a) \tag{5.6a}$$

or in four-vector form

$$F^i_{\text{ext}} = mc^2\frac{\mathrm{d}u^i}{\mathrm{d}s} - \begin{pmatrix}\text{radiation}\\ \text{reaction}\end{pmatrix}^i + O(a) \tag{5.6b}$$

where m is the measured rest mass of the particle (charge plus insulator) and

$$\begin{pmatrix}\text{radiation}\\ \text{reaction}\end{pmatrix} = \frac{e^2\gamma^2}{6\pi\epsilon_0 c^3}\left\{\ddot{\mathbf{u}} + \frac{3\gamma^2}{c^2}(\mathbf{u}\cdot\dot{\mathbf{u}})\dot{\mathbf{u}} + \frac{\gamma^2}{c^2}\left[(\mathbf{u}\cdot\ddot{\mathbf{u}}) + \frac{3\gamma^2}{c^2}(\mathbf{u}\cdot\dot{\mathbf{u}})^2\right]\mathbf{u}\right\} \tag{5.6c}$$

$$\begin{pmatrix}\text{radiation}\\ \text{reaction}\end{pmatrix}^i = \frac{e^2}{6\pi\epsilon_0}\left(\frac{\mathrm{d}^2u^i}{\mathrm{d}s^2} + u^i\frac{\mathrm{d}u_j}{\mathrm{d}s}\frac{\mathrm{d}u^j}{\mathrm{d}s}\right). \tag{5.6d}$$

Accepting the rest mass term in (5.6a)–(5.6b) from the relativistic generalization of Newton's second law of motion (possible extra terms are discussed in the next subsection 5.1.1), one sees that (5.5) is compatible with (5.6) if the total force in (5.5) equals the time rate of change of momentum

$$\mathbf{F}(t) = M_0\frac{\mathrm{d}}{\mathrm{d}t}(\gamma\mathbf{u}) + \mathbf{O}(a) \tag{5.7a}$$

or in four-vector form

$$F^i = M_0c^2\frac{\mathrm{d}u^i}{\mathrm{d}s} + O(a) \tag{5.7b}$$

with the "bare" mass M_0 related to the electromagnetic rest mass (2.2) and the measured rest mass by

$$M_0 = m - m_{\text{em}} - m_{\text{ins}}\,. \tag{5.8}$$

Furthermore, the measured rest mass m of the charge shell can be determined theoretically. Assume the charge is initially distributed uniformly on a spherical surface of infinite radius where it has zero mass. The work required to assemble this charge quasi-statically from infinity to the surface of the insulator of radius a is determined from a simple electrostatic calculation [15, sec. 2.7] as $e^2/(8\pi\epsilon_0 a)$. By the Einstein mass-energy relation, the rest mass of the charged insulator will then be this electrostatic energy of formation divided by c^2, or what is called the electrostatic mass in (2.3), plus the mass m_{ins} of the uncharged insulator. (If gravitational fields [20], [21], or short-range polarization forces binding the charge to the insulator, or short-range forces holding the insulator material together contribute non-negligibly to the rest energy of formation, this mass can be included in m_{ins}.) Thus, the measured rest mass of the charged insulator equals the sum of the electrostatic mass and the mass of the insulator

$$m = m_{\text{es}} + m_{\text{ins}} \tag{5.9}$$

and the bare mass in (5.8) can be written simply as

$$M_0 = m_{\text{es}} - m_{\text{em}} \tag{5.10}$$

or from (2.2) and (2.3)

$$M_0 = -\frac{e^2}{24\pi\epsilon_0 ac^2}\,. \tag{5.11}$$

Emphatically, the value of the bare mass does not depend on the mass m_{ins} of the insulator.

The final form of the equation of motion in (5.6) or, equivalently, in (5.5) and (5.7), can now be written for the charged insulator as

$$\mathbf{F}_{\text{ext}}(t) = m\frac{\mathrm{d}}{\mathrm{d}t}(\gamma\mathbf{u}) - \frac{e^2\gamma^2}{6\pi\epsilon_0 c^3}\left\{\ddot{\mathbf{u}} + \frac{3\gamma^2}{c^2}(\mathbf{u}\cdot\dot{\mathbf{u}})\dot{\mathbf{u}} + \frac{\gamma^2}{c^2}\left[(\mathbf{u}\cdot\ddot{\mathbf{u}}) + \frac{3\gamma^2}{c^2}(\mathbf{u}\cdot\dot{\mathbf{u}})^2\right]\mathbf{u}\right\} + \mathbf{O}(a) \tag{5.12a}$$

$$\mathbf{F}_{\text{ext}}\cdot\mathbf{u} = mc^2\frac{\mathrm{d}\gamma}{\mathrm{d}t} - \frac{e^2\gamma^4}{6\pi\epsilon_0 c^3}\left[(\mathbf{u}\cdot\ddot{\mathbf{u}}) + \frac{3\gamma^2}{c^2}(\mathbf{u}\cdot\dot{\mathbf{u}})^2\right] + O(a) \tag{5.12b}$$

where $m = (m_{\text{es}} + m_{\text{ins}})$ and $m_{\text{es}} = e^2/(8\pi\epsilon_0 ac^2)$. Of course, (5.12b) is redundant because it is consistent with the equation obtained by taking the dot product of $\mathbf{u}$ with (5.12a). The negative bare mass M_0 in (5.11) eliminates the 4/3 factor in the inertial rest mass of the original Lorentz-Abraham equation of motion (2.1) in which the bare mass was assumed zero. With the addition of the bare-mass force-power to the binding and electromagnetic force-power, the total force-power and the total momentum-energy transform as four vectors; see Section 6.1.

The mass of the insulator ($m_{\rm ins}$) allows the equations of motion (5.12) for the charged insulator to be written with an arbitrary value of the inertial rest mass. Thus, (5.12) conforms with the results of Schwinger [23], who shows that stress-momentum-energy tensors with covariant momentum-energy for stable charge-current distributions can be constructed with or without the 4/3 factor. Setting $m_{\rm ins}$ equal to $m_{\rm es}/3$ and zero, respectively, corresponds to Schwinger's tensors with and without the 4/3 factor; see Section 6.2. (The mass $m_{\rm ins}$ can even be negative since, as mentioned above, it includes gravitational and other attractive-force formation energies which, in general, are negative.)

The rest mass of the charge has been taken as $m_{\rm es}$ and that of the insulator as $m_{\rm ins}$. In fact, all we can determine experimentally is the total mass m of the charge and insulator together as one body. Ultimately, we do not know where the mass resides—how much in the field, in the charge, in the insulator. All we know is the rest mass of the insulator before it is charged and after it is charged. After the insulator is charged, we can ascribe any part of the rest mass to the insulator and the remaining part to the charge, and the above analysis goes through to give the same final equations (5.12). Of course, there is a certain appeal and convenience in keeping, as we have above, the rest mass of the charge equal to its electrostatic energy of formation mass $m_{\rm es}$ after the charge is placed on the insulator so that $m_{\rm ins}$ refers to the difference $(m - m_{\rm es})$.

5.1.1 Extra Momentum-Energy in Newton's Second Law of Motion for Charged Particles

The relativistic generalization of Newton's second law of motion in (5.6) for a charged particle is not determined uniquely from the nonrelativistic version of Newton's second law for uncharged particles. From purely theoretical considerations, any four-vector function of velocity and its time derivatives that vanishes when the charge is zero could be added to the right-hand side of (5.6). If, however, we assume that the only irreversible loss of momentum-energy of the charged particle is the radiated momentum-energy (so that when the initial and final velocities and their time derivatives are the same, the only change in momentum-energy will be that which is radiated) then this extra four-vector function must be expressible as the time derivative of a momentum-energy function. In addition, since all the functions in (5.6b) satisfy the condition that their scalar product with u_i equals zero (that is, the time rate of change of momentum and energy components of (5.6b) are compatible) the extra function must also satisfy this condition. Thus, on theoretical grounds (5.6b) can be further generalized to [12]

$$F^i_{\rm ext} = mc^2\frac{du^i}{ds} - \begin{pmatrix}\text{radiation}\\ \text{reaction}\end{pmatrix}^i + \frac{dG^i_{\rm extra}}{ds} + O(a) \tag{5.13}$$

where $dG^i_{\rm extra}/ds$ is a four-vector function of velocity and its derivatives that exists only for charged particles and satisfies

$$\frac{dG^i_{\rm extra}}{ds}u_i = 0\,. \tag{5.14}$$

Of course, $dG^i_{\rm extra}/ds$ is a function of the charge e, since it vanishes when the charge vanishes and may be a function of the radius a of the charge distribution.[1]

If we also assume that the only irreversible loss in angular momentum-energy of the charged particle is the radiated angular momentum-energy, since the shell of charge is assumed to translate without rotation, then $\mathbf{u}\times\mathbf{G}_{\rm extra}$ and its four-vector version, $u^iG^j_{\rm extra} - u^jG^i_{\rm extra}$, must be expressible as the time derivative of an angular momentum-energy function [24]. This follows from taking the cross product of the position vector $\mathbf{r}$ of the center of the particle with the three-vector equation of motion in (5.13) to get

$$\begin{aligned}\mathbf{r}\times\mathbf{F}_{\rm ext} &= m\frac{d}{dt}(\mathbf{r}\times\gamma\mathbf{u}) - \mathbf{r}\times\begin{pmatrix}\text{radiation}\\ \text{reaction}\end{pmatrix}\\ &\quad + \frac{d}{dt}(\mathbf{r}\times\mathbf{G}_{\rm extra}) - \mathbf{u}\times\mathbf{G}_{\rm extra} + \mathbf{O}(a)\end{aligned} \tag{5.15a}$$

or in four-vector form

$$\begin{aligned}x^iF^j_{\rm ext} - x^jF^i_{\rm ext} &= mc^2\frac{d}{ds}(x^iu^j - x^ju^i) - \begin{pmatrix}\text{angular radiation}\\ \text{reaction}\end{pmatrix}^{ij}\\ &+\frac{d}{ds}\left(x^iG^j_{\rm extra} - x^jG^i_{\rm extra}\right) - \left(u^iG^j_{\rm extra} - u^jG^i_{\rm extra}\right) + O(a).\end{aligned} \tag{5.15b}$$

When the initial and final position, velocity, and higher derivatives of the position of the center of the particle are the same, the only change in angular momentum will be in the radiated angular momentum if $\mathbf{u}\times\mathbf{G}_{\rm extra}$ is a perfect time differential of an angular momentum function ($\mathbf{L}_{\rm extra}$)

$$\mathbf{u}\times\mathbf{G}_{\rm extra} = \frac{d}{dt}\mathbf{L}_{\rm extra} \tag{5.16a}$$

[1] The reversible and irreversible parts of the radiation reaction term in (5.13) are given in (5.6d) as $[e^2/(6\pi\epsilon_0)]d^2u^i/ds^2$ and $[e^2/(6\pi\epsilon_0)]u^i(du_j/ds)(du^j/ds)$, respectively. In Chapter 8 we show that the series expansions of the self force that lead to this radiation reaction term do not hold during the short proper time interval Δt_a after the external force is first applied. During this transition interval $[0,\Delta t_a]$, the self force and thus the right-hand side of (5.13) may contain a "transition force" $(-f^i_a)$ with both reversible and irreversible parts that removes the pre-acceleration from the solution to the equation of motion. The $O(a)$ terms in (5.13) also contain both reversible and irreversible parts.

or in four-vector form

$$u^i G^j_{\text{extra}} - u^j G^i_{\text{extra}} = \frac{\mathrm{d}}{\mathrm{d}s} L^{ij}_{\text{extra}} \,. \tag{5.16b}$$

There is apparently no experimental evidence for the existence of an extra momentum-energy function in the equation of motion of a charged particle at least to order a, and, as Dirac said, "they are all much more complicated than [$mc^2 \mathrm{d}u^i/\mathrm{d}s$], so that one would hardly expect them to apply to a simple thing like an electron" [12, p. 154]. Thus, except for the transition force f^i_a explained in Chapter 8 (see Footnote 1 of the present chapter), we will assume G^i_{extra} is zero and accept (5.6) as the correct generalization of Newton's second law of motion for the charged shell.

5.1.2 Reason for Lorentz Setting the Bare Mass Zero

All the tools of special relativity [5], [6] were not available to Lorentz and Abraham when they originally derived the total force on the moving Lorentz model of the electron. In particular, the Einstein mass-energy relationship [6] and the relativistic version of Newton's second law of motion [22] had not appeared. However, Lorentz did assume the pre-relativistic form of Newton's second law of motion and thus set the total force equal to a constant bare mass M, which Lorentz called the "material" mass, times the acceleration $\dot{\mathbf{u}}$ [4, secs. 28, 32, and 179] to get

$$\mathbf{F}_{\text{ext}}(t) = \frac{e^2}{6\pi\epsilon_0 a c^2}\frac{\mathrm{d}}{\mathrm{d}t}(\gamma\mathbf{u}) + M\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} - \begin{pmatrix}\text{radiation}\\ \text{reaction}\end{pmatrix} + \mathbf{O}(a) \,. \tag{5.17}$$

(For the charged insulator model, Lorentz's bare mass M in (5.17) would include the mass of the uncharged insulator, that is, $M = M_0 + m_{\text{ins}}$.)

The key feature of (5.17) is that Lorentz assumed the constant bare mass M in (5.17) was multiplied by $\mathrm{d}\mathbf{u}/\mathrm{d}t$ rather than $\mathrm{d}(\gamma\mathbf{u})/\mathrm{d}t$ (even though he and Abraham had discovered the γ factor in the time rate of change of the electromagnetic momentum in (5.17) before 1905).

Between 1901 and 1905, Kaufmann [25] performed experiments to determine the charge to mass ratio for "fast moving" electrons. Lorentz and Abraham hoped that these experiments would decide between Lorentz's contracting (relativistically rigid) model of the electron and Abraham's noncontracting (nonrelativistically rigid) model. Although his experiments were not accurate enough to settle this question [26], Kaufmann's experiments showed clearly that the preponderance of momentum in the electron varied as $\mathrm{d}(\gamma\mathbf{u})/\mathrm{d}t$ rather than $\mathrm{d}\mathbf{u}/\mathrm{d}t$. Thus Lorentz accepted Kaufmann's results as experimental evidence that the bare mass in (5.17) was negligible. To quote Lorentz [4, sec. 32], "Of course we are free to believe, if we like, that there is some small material [bare] mass attached to the electron, say equal to one hundredth part of the electromagnetic one, but with a view to simplicity, it will be best to admit Kaufmann's conclusion, or hypothesis, if we prefer so to call it, that the negative electrons have no material [bare] mass at all. This is certainly one of the most important results of modern physics, ..." (Abraham also concluded from Kaufmann's experiments that the bare mass of the electron was zero [3, sec. 16].)

As late as 1912, Schott continued to "suppose M zero, in accordance with the most recent measurements" [16, p. 178]. Even after experiments by Bucherer [27] in 1909, Neumann [28] in 1914, and Bohr [29] in 1915 decided in favor of Lorentz's contracting model over Abraham's noncontracting model of the electron, and thus also confirmed the prediction of special relativity, at least for "electrical systems," the bare mass was generally assumed outside the jurisdiction of special relativity and these experiments were regarded as confirming that the bare mass was zero. Richardson [30, ch. 11] summarizes the general consensus in 1915:

"These experiments [Bucherer's] appear to dispose effectually of the rigid [Abraham's nonrelativistically rigid or noncontracting] electron and they may be regarded as making it reasonably certain that Thomson's corpuscles are devoid of mass except such as is due to the charge that they carry. For this reason we shall always refer to them in the sequel as negative electrons. We shall find later that the relation between [the moving mass] and [the rest mass] characteristic of the Lorentz contractible electron is true of all electrical systems according to the principle of relativity. Bucherer's experiment may therefore be regarded as evidence in favor of that principle. A remarkable confirmation of the relativity expression for the mass of a moving particle has recently been obtained by N. Bohr from consideration of the decrease of velocity of α and β rays in passing through matter."

Cunningham [31] also gives a very readable account of the conclusions drawn in 1914 from the experiments of Kaufmann et al.

By 1920, however, it was generally accepted that the principle of relativity applied to all mass, and Pauli would write, "The old idea that one could distinguish between the constant 'true' [bare] mass and the 'apparent' electromagnetic mass, by means of deflection experiments on cathode rays, can therefore not be maintained" [7, sec. 29].

Thus, one cannot accept (5.17) or continue to assume a bare mass M_0 equal to zero, for our specific model of the electron as a charged insulator, without violating the equivalence of mass and energy and the relativistic version of Newton's second law of motion, which imply the negative bare mass in (5.11) for this model. Also the bare mass, as pointed out in Section 4.2, should not be confused with the uncharged mass of the insulator. However, because Lorentz's bare mass corresponds to $(M_0 + m_{\text{ins}})$ in our analysis of the charged insulator, Lorentz's bare mass M can still be zero in the special case when the mass m_{ins} of the insulator equals $-M_0$, that is, $m_{\text{es}}/3$. In that special case the total mass of the charged insulator would be $(4/3)m_{\text{es}} = m_{\text{em}}$.

6

Transformation and Redefinition of Force-Power and Momentum-Energy

In chapter 4 it was shown that the specific model of an electron as a charged insulator demands polarization forces, binding the charge to the insulator, that just cancel the discrepancy (2.6) between the Lorentz-Abraham force and power equations of motion, (2.1) and (2.4). In Chapter 5 we saw that the relativistic generalization of Newton's second law of motion, together with the Einstein mass-energy equivalence relation, require the negative bare mass in (5.11) that eliminates the factor of 4/3 multiplying the electrostatic mass in the original equation of motion (2.1). In this chapter we summarize the transformation properties of the electromagnetic, binding, and bare-mass force-powers and momentum-energies, derive a total stress-momentum-energy tensor for the charged insulator model of the electron, and review the redefinitions of electromagnetic momentum-energy that have been proposed in the past for the extended electron.

6.1 Transformation of Electromagnetic, Binding, and Bare-Mass Force-Power and Momentum-Energy

In order to summarize the transformation properties of the electromagnetic, binding, and bare-mass momentum and energy as well as their time derivatives, force and power, for the charged insulator model of the electron, it will be helpful first to make a concise list of these quantities. The self electromagnetic, binding, and bare-mass forces exerted on the charge, and the associated powers delivered to the charge can be written from the preceding chapters as

$$\mathbf{F}_{\rm em} = -\frac{d\mathbf{G}_{\rm em}}{dt} = -\frac{4}{3}m_{\rm es}\frac{d(\gamma\mathbf{u})}{dt} + \mathbf{O}(1) \tag{6.1a}$$

$$P_{\rm em} = -\frac{dW_{\rm em}}{dt} = -\frac{4}{3}m_{\rm es}c^2\frac{d}{dt}\left(\gamma - \frac{1}{4\gamma}\right) + O(1) \tag{6.1b}$$

$$\mathbf{F}_b = -\frac{d\mathbf{G}_b}{dt} = -m_{ins}\frac{d(\gamma\mathbf{u})}{dt} + \mathbf{O}(a) \tag{6.2a}$$

$$P_b = -\frac{dW_b}{dt} = -m_{ins}c^2\frac{d\gamma}{dt} - \frac{4}{3}m_{es}c^2\frac{d}{dt}\left(\frac{1}{4\gamma}\right) + O(a) \tag{6.2b}$$

$$\mathbf{F}_0 = -\frac{d\mathbf{G}_0}{dt} = \frac{1}{3}m_{es}\frac{d(\gamma\mathbf{u})}{dt} \tag{6.3a}$$

$$P_0 = -\frac{dW_0}{dt} = \frac{1}{3}m_{es}c^2\frac{d\gamma}{dt} \tag{6.3b}$$

where the electrostatic mass is given in (2.3), and the dominant parts of the $\mathbf{O}(1)$ term in (6.1a) and the $O(1)$ term in (6.1b) are the radiation reaction force and power, respectively. Adding the externally applied force and power to the sum of the electromagnetic, binding, and bare-mass forces and powers in (6.1), (6.2), and (6.3), and setting the total force and power equal to zero give the equations of motion (5.12) for the charged insulator.

The momentum and energy of the charged insulator system as a whole can be found by integrating the expressions (6.1), (6.2), and (6.3) of force and power with respect to time for zero initial velocity. For zero initial velocity, the initial electromagnetic momentum, $\epsilon_0\int \mathbf{E}\times\mathbf{B}dV$ over all space, is zero and the binding and bare-mass momenta are zero. The initial electromagnetic energy, $(\epsilon_0/2)\int(E^2+c^2B^2)dV$ over all space, equals the rest energy of formation of the charge $(m_{es}c^2)$ and the initial binding energy is chosen equal to the rest energy of the mass of the insulator $(m_{ins}c^2)$. Then, the initial energy of the negative bare mass is zero because the total rest energy of formation of the charged insulator is assumed equal to the sum of the electrostatic and insulator rest energies. (If it is more appealing to have the initial energy of the bare mass equal to $-m_{es}c^2/3$, one can choose the initial binding energy equal to $m_{ins}c^2 + m_{es}c^2/3$. Such a change would add and subtract $m_{es}c^2/3$ in the following expressions for W_b and W_0, respectively.) The momenta and energies corresponding to (6.1)–(6.3) are

$$\mathbf{G}_{em} = \frac{4}{3}m_{es}\gamma\mathbf{u} + \mathbf{O}(1) \tag{6.4a}$$

$$W_{em} = \frac{4}{3}m_{es}c^2\left(\gamma - \frac{1}{4\gamma}\right) + O(1) = m_{es}c^2\gamma\left(1 + \frac{u^2}{3c^2}\right) + O(1) \tag{6.4b}$$

$$\mathbf{G}_b = m_{ins}\gamma\mathbf{u} + \mathbf{O}(a) \tag{6.5a}$$

$$W_b = m_{ins}c^2\gamma + \frac{1}{3}m_{es}c^2\left(\frac{1}{\gamma} - 1\right) + O(a) \tag{6.5b}$$

$$\mathbf{G}_0 = -\frac{1}{3}m_{es}\gamma\mathbf{u} \tag{6.6a}$$

$$W_0 = -\frac{1}{3}m_{es}c^2(\gamma - 1)\,. \tag{6.6b}$$

From these expressions of force-power and momentum-energy, one draws the following conclusions about their transformation properties. Neither the electromagnetic momentum-energy $(c\mathbf{G}_{em}, W_{em})$ nor its time derivative, the electromagnetic force-power $\gamma(c\mathbf{F}_{em}, P_{em})$, transforms as a four-vector. Similarly, neither the binding momentum-energy $(c\mathbf{G}_b, W_b)$ nor the binding force-power $\gamma(c\mathbf{F}_b, P_b)$ transforms as a four-vector. Also $(\mathbf{F}_{em}\cdot\mathbf{u} - P_{em})$ and $(\mathbf{F}_b\cdot\mathbf{u} - P_b)$ are not equal to zero. Even the sum of the electromagnetic and binding momentum-energy does not transform as a four-vector. However, the sum of the electromagnetic and binding force-power transforms as a four-vector and satisfies $(\mathbf{F}_{em}+\mathbf{F}_b)\cdot\mathbf{u}-(P_{em}+P_b) = 0$. The bare-mass force-power $\gamma(c\mathbf{F}_0, P_0)$ also transforms as a four-vector satisfying $\mathbf{F}_0\cdot\mathbf{u} - P_0 = 0$; whereas, the bare-mass momentum-energy $(c\mathbf{G}_0, W_0)$ does not transform as a four-vector, but contributes to the electromagnetic and binding momentum-energy to yield a total momentum-energy that is free of the 4/3 factor and transforms as a four vector. (If, as mentioned above, the initial binding energy were chosen equal to $m_{ins}c^2 + m_{es}c^2/3$, so that the initial energy of the bare mass equaled $-m_{es}c^2/3$, then both the bare-mass momentum-energy and the sum of the electromagnetic and binding momentum-energy would transform as four-vectors.)

It may still be disconcerting that the total momentum and energy of a charged massless insulator is *not* given by the conventional electromagnetic momentum and energy

$$\mathbf{G}_{em} = \epsilon_0\int_{\text{all space}} \mathbf{E}\times\mathbf{B}\,dV \tag{6.7a}$$

$$W_{em} = \frac{\epsilon_0}{2}\int_{\text{all space}} (E^2 + c^2B^2)dV \tag{6.7b}$$

or that the total momentum of a charged massless insulator is not given by the conventional electromagnetic momentum alone even when the velocity of the charge is much less than the speed of light, but contains also the momentum of a negative bare mass. However, one can take consolation in realizing that no law of physics is violated by the conventional electromagnetic momentum not equaling the total momentum of the charge. What we know from Einstein's mass-energy relation and the relativistic version of Newton's second law of motion is that the total momentum equals (in addition to the radiation momentum) the electrostatic mass (m_{es}, rest energy of formation divided by c^2) times the velocity ($\gamma\mathbf{u}$). However, what we know from Maxwell's equations and the Lorentz force law is merely that the sum of the external and self electromagnetic forces on the charge is $\mathbf{F}_{ext} - (d/dt)\epsilon_0\int\mathbf{E}\times\mathbf{B}dV$. Only if this force on the charge equals zero, can the total momentum of the particle be given entirely by the conventional electromagnetic momentum. Since $(d/dt)\epsilon_0\int\mathbf{E}\times\mathbf{B}dV$ equals $(4/3)m_{es}d(\gamma\mathbf{u})/dt$ (plus radiation terms) rather than $m_{es}d(\gamma\mathbf{u})/dt$, the Einstein mass-energy relation and Newton's second law for relativistic motion demand that this force not be zero but equal $(-1/3)m_{\rm es}{\rm d}(\gamma\mathbf{u})/{\rm d}t$, and consequently, that the total momentum of the moving charge not be equal to its conventional electromagnetic momentum alone.

From the standpoint of the electromagnetic stress-momentum-energy tensor, it is not surprising that the conventional electromagnetic momentum-energy does not represent the total momentum-energy of the moving charge distribution. Because the electromagnetic stress-momentum-energy tensor is not divergenceless when charge-current is present, the associated momentum-energy will not, in general, be a four-vector. Thus the electromagnetic momentum-energy could not, in general, be expected to represent the total momentum-energy of the system.

6.1.1 Total Stress-Momentum-Energy Tensor for the Charged Insulator

The four-divergence of the electromagnetic stress-momentum-energy tensor $T^{ij}_{\rm em}(\mathbf{r},t)$ equals the force-power density [13], that is

$$\frac{\partial T^{ij}_{\rm em}}{\partial x^j} = -f^i_{\rm em} \tag{6.8}$$

where

$$f^i_{\rm em} \equiv \rho(\mathbf{r},t)\left[\mathbf{f}_{\rm em}(\mathbf{r},t), \mathbf{f}_{\rm em}(\mathbf{r},t)\cdot\mathbf{u}(\mathbf{r},t)/c\right] \tag{6.9}$$

and $T^{ij}_{\rm em}$ can be written out as

$$T^{ij}_{\rm em} = \left[\begin{array}{c|c} -\bar{\mathbf{T}}_{\rm em} & c\mathbf{g}_{\rm em} \\ \hline c\mathbf{g}_{\rm em} & w_{\rm em}\end{array}\right] \tag{6.10}$$

$$\bar{\mathbf{T}}_{\rm em} \equiv \epsilon_0\left[(\mathbf{EE}-\bar{\mathbf{I}}E^2/2) + c^2(\mathbf{BB}-\bar{\mathbf{I}}B^2/2)\right] \tag{6.11a}$$

$$\mathbf{g}_{\rm em} = \epsilon_0\mathbf{E}\times\mathbf{B} \tag{6.11b}$$

$$w_{\rm em} = \frac{\epsilon_0}{2}(E^2 + c^2B^2)\,. \tag{6.11c}$$

One can also construct stress-momentum-energy tensors with divergences equal to the binding and bare-mass force-power densities, that is

$$\frac{\partial T^{ij}_{\rm b}}{\partial x^j} = -f^i_{\rm b}(\mathbf{r},t),\quad f^i_{\rm b} \equiv \rho(\mathbf{r},t)\left[\mathbf{f}_{\rm b}(\mathbf{r},t), \mathbf{f}_{\rm b}(\mathbf{r},t)\cdot\mathbf{u}(\mathbf{r},t)/c\right] \tag{6.12}$$

$$\frac{\partial T^{ij}_0}{\partial x^j} = -f^i_0(\mathbf{r},t),\quad f^i_0 \equiv \frac{e}{24\pi\epsilon_0 ac^2}\rho(\mathbf{r},t)\left[\frac{{\rm d}(\gamma\mathbf{u})}{{\rm d}t}, c\frac{{\rm d}\gamma}{{\rm d}t}\right]\,. \tag{6.13}$$

(As usual, when $\mathbf{u}=\mathbf{u}(t)$ appears without the functional dependence $(\mathbf{r},t)$, it refers to the velocity of the center of the charged shell.) Adding the binding and bare-mass tensors to the electromagnetic tensor would then produce a total stress-momentum-energy tensor whose momentum-energy density would form a four-vector when integrated over all space. Taking the time rate of change of this four-vector momentum-energy produces a four-vector force-power that, when set equal to the externally applied force, results in the force and power equations of motion (5.12). If no external force is applied to the charged insulator, so that its velocity is constant, the total stress-momentum-energy tensor is divergenceless and the associated four-vector momentum-energy is conserved.

First, let us construct the bare-mass tensor T^{ij}_0, from its following three-vector equations corresponding to (6.13)

$$-\nabla\cdot\bar{\mathbf{T}}_0 + \frac{\partial\mathbf{g}_0}{\partial t} = -\frac{e}{24\pi\epsilon_0 ac^2}\rho\frac{{\rm d}(\gamma\mathbf{u})}{{\rm d}t} \tag{6.14a}$$

$$c\nabla\cdot\mathbf{g}_0 + \frac{1}{c}\frac{\partial w_0}{\partial t} = -\frac{e}{24\pi\epsilon_0 ac}\rho\frac{{\rm d}\gamma}{{\rm d}t}\,. \tag{6.14b}$$

A fairly obvious solution to (6.14) is

$$\mathbf{g}_0 = -\frac{e}{24\pi\epsilon_0 ac^2}\gamma\rho\mathbf{u} \tag{6.15a}$$

$$w_0 = -\frac{e}{24\pi\epsilon_0 a}\gamma\rho \tag{6.15b}$$

$$\bar{\mathbf{T}}_0 = \frac{e}{24\pi\epsilon_0 ac^2}\gamma\rho\mathbf{uu} \tag{6.15c}$$

or in four-vector notation

$$T^{ij}_0 = -\frac{e}{24\pi\epsilon_0 a\gamma}\rho u^i u^j\,. \tag{6.16}$$

Rohrlich [32, sec. 6-1] includes the bare-mass tensor (6.16) as part of the "cohesion" or binding stress-momentum-energy tensor. However, for the charged insulator model, it seems preferable to separate the bare-mass tensor from the binding tensor, because we found in Chapters 4 and 5 that the binding forces do not make the inertial mass compatible with the rest energy of formation.

It is easily shown that the solution (6.15) satisfies (6.14), or that (6.16) satisfies (6.13); specifically we have

$$-\nabla\cdot\bar{\mathbf{T}}_0 = -\frac{e}{24\pi\epsilon_0 ac^2}\gamma[\nabla\cdot(\rho\mathbf{u})]\mathbf{u} \tag{6.17a}$$

$$\begin{aligned}\frac{\partial\mathbf{g}_0}{\partial t} &= -\frac{e}{24\pi\epsilon_0 ac^2}\left[\rho\frac{\partial(\gamma\mathbf{u})}{\partial t} + \gamma\mathbf{u}\frac{\partial\rho}{\partial t}\right]\\ &= -\frac{e}{24\pi\epsilon_0 ac^2}\left[\rho\frac{{\rm d}(\gamma\mathbf{u})}{{\rm d}t} - \gamma[\nabla\cdot(\rho\mathbf{u})]\mathbf{u}\right]\end{aligned} \tag{6.17b}$$

$$c\nabla\cdot\mathbf{g}_0 = -\frac{e}{24\pi\epsilon_0 ac}\gamma\nabla\cdot(\rho\mathbf{u}) \tag{6.17c}$$

$$\begin{aligned}\frac{1}{c}\frac{\partial w_0}{\partial t} &= -\frac{e}{24\pi\epsilon_0 ac}\left[\rho\frac{\partial\gamma}{\partial t}+\gamma\frac{\partial\rho}{\partial t}\right]\\ &= -\frac{e}{24\pi\epsilon_0 ac}\left[\rho\frac{\partial\gamma}{\partial t}-\gamma\nabla\cdot(\rho\mathbf{u})\right]\end{aligned} \tag{6.17d}$$

which produce identities when inserted into the left-hand sides of (6.14a) and (6.14b).

The binding stress-momentum-energy tensor must satisfy the following three-vector equations corresponding to (6.12)

$$-\nabla\cdot\bar{\mathbf{T}}_\mathrm{b}+\frac{\partial\mathbf{g}_\mathrm{b}}{\partial t} = \frac{e^2}{32\pi^2\epsilon_0 a^4}\gamma\delta(r_0-a)\hat{\mathbf{r}}_0+\frac{m_\mathrm{ins}}{e}\rho\frac{\mathrm{d}}{\mathrm{d}t}(\gamma\mathbf{u}) \tag{6.18a}$$

$$c\nabla\cdot\mathbf{g}_\mathrm{b}+\frac{1}{c}\frac{\partial w_\mathrm{b}}{\partial t} = \frac{e^2}{32\pi^2\epsilon_0 a^4 c}\gamma\delta(r_0-a)\hat{\mathbf{r}}_0\cdot\mathbf{u}(\mathbf{r},t)+\frac{m_\mathrm{ins}c}{e}\rho\frac{\mathrm{d}\gamma}{\mathrm{d}t}. \tag{6.18b}$$

The charge density in the first terms on the right-hand sides of (6.18) has been expressed as a function of the static charge density, that is

$$\rho(\mathbf{r},t)=\gamma\rho_0(\mathbf{r}_0)=\gamma\delta(r_0-a)\frac{e}{4\pi a^2} \tag{6.19}$$

where $\mathbf{r}_0$ is given in terms of $\mathbf{r}$ at the time t by the Lorentz transformation

$$\mathbf{r}_0=(\mathbf{r}-\mathbf{r}_\mathrm{c})_\perp+\gamma(\mathbf{r}-\mathbf{r}_\mathrm{c})_\parallel \tag{6.20a}$$

and the position $\mathbf{r}_\mathrm{c}$ of the center of the charged shell can be written in terms of the velocity of the center as

$$\mathbf{r}_\mathrm{c}=\int^t\mathbf{u}(t')\mathrm{d}t'\,. \tag{6.20b}$$

(The subscripts $\perp$ and $\parallel$ mean perpendicular and parallel, respectively, to the center velocity $\mathbf{u}(t)$ at the time t; and $\delta(x)$ is the Dirac delta function.) The binding force per unit charge in (6.18) is equal to the exact binding force per unit charge in (4.22) with the first term on the right-hand side of (4.22) averaged over the thickness of the shell and generalized to an arbitrary inertial reference frame. The second term on the right-hand side of (4.22), which is present when the velocity of the charge is not constant, is not included in (6.18). Also, the expressions (6.19) and (6.20a) neglect terms of second order and higher in $(\mathbf{r}-\mathbf{r}_\mathrm{c})$ when the velocity of the charge is not constant; see (B.30). These secondary binding forces are necessary to hold the accelerating charge to the insulator, but they are inconsequential to the integrated force and power because the results of Chapter 4 (specifically, equations (4.24)) show that their integrals over the charge distribution are of $O(a)$. (In principle, T_b^{ij} could be modified to include the secondary binding stresses, but in practice it may be rather tedious to construct the necessary, relativistically invariant modification.)

A particularly simple solution to (6.18) is

$$\mathbf{g}_\mathrm{b}=\frac{m_\mathrm{ins}}{e}\gamma\rho\mathbf{u} \tag{6.21a}$$

$$w_\mathrm{b}=\frac{e^2}{32\pi^2\epsilon_0 a^4}h(a-r_0)+\frac{m_\mathrm{ins}c^2}{e}\gamma\rho \tag{6.21b}$$

$$\bar{\mathbf{T}}_\mathrm{b}=\frac{e^2}{32\pi^2\epsilon_0 a^4}h(a-r_0)\bar{\mathbf{I}}-\frac{m_\mathrm{ins}}{e}\gamma\rho\mathbf{u}\mathbf{u} \tag{6.21c}$$

or in four-vector notation

$$T_\mathrm{b}^{ij}=\frac{e^2}{32\pi^2\epsilon_0 a^4}h(a-r_0)g^{ij}+\frac{m_\mathrm{ins}c^2}{e\gamma}\rho u^i u^j \tag{6.22}$$

where g^{ij} is the metric tensor

$$g^{ij}\equiv\begin{pmatrix}-1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix} \tag{6.23}$$

and h is the unit step function.

The preceding solution for T_b^{ij} can be used to prove immediately that the m_ins part of the solution in (6.21) satisfies (6.18). To see that the remaining part of the solution in (6.21) satisfies (6.18), evaluate $\nabla\cdot\bar{\mathbf{T}}_\mathrm{b}$ and $\partial w_\mathrm{b}/\partial t$ for that part to get

$$-\nabla\cdot\bar{\mathbf{T}}_\mathrm{b}=-\frac{e^2}{32\pi^2\epsilon_0 a^4}\nabla h(a-r_0)=-\frac{e^2}{32\pi^2\epsilon_0 a^4}\left[\frac{\partial h}{\partial r_\parallel}\hat{\mathbf{r}}_\parallel+\frac{\partial h}{\partial r_\perp}\hat{\mathbf{r}}_\perp\right]$$

or

$$\begin{aligned}-\nabla\cdot\bar{\mathbf{T}}_\mathrm{b}&=-\frac{e^2}{32\pi^2\epsilon_0 a^4}\left[\gamma\frac{\partial h}{\partial r_{0\parallel}}\hat{\mathbf{r}}_{0\parallel}+\frac{\partial h}{\partial r_{0\perp}}\hat{\mathbf{r}}_{0\perp}\right]\\&=-\frac{e^2}{32\pi^2\epsilon_0 a^4}\nabla_0 h\cdot\left[\gamma\hat{\mathbf{r}}_{0\parallel}\hat{\mathbf{r}}_{0\parallel}+\hat{\mathbf{r}}_{0\perp}\hat{\mathbf{r}}_{0\perp}\right]\end{aligned}$$

or

$$-\nabla\cdot\bar{\mathbf{T}}_\mathrm{b}=\frac{e^2\delta(r_0-a)\gamma}{32\pi\epsilon_0 a^5}\left[\mathbf{r}_{0\parallel}+\frac{\mathbf{r}_{0\perp}}{\gamma}\right]=-\rho\mathbf{f}_\mathrm{b} \tag{6.24a}$$

and

$$\frac{\partial w_\mathrm{b}}{\partial t}=\frac{e^2}{32\pi\epsilon_0 a^4}\frac{\partial h(a-r_0)}{\partial t}=\frac{e^2}{32\pi\epsilon_0 a^4}\nabla_0 h\cdot\frac{\partial\mathbf{r}_0}{\partial t}=-\frac{e^2\delta(r_0-a)}{32\pi\epsilon_0 a^4}\hat{\mathbf{r}}_0\cdot\frac{\partial\mathbf{r}_0}{\partial t}$$

or since from (6.20)

$$\frac{\partial \mathbf{r}_0}{\partial t} = -\left(\frac{\partial \mathbf{r}_c}{\partial t}\right)_\perp + \frac{\partial}{\partial t}[\gamma(\mathbf{r}-\mathbf{r}_c)]_\| = -\gamma\mathbf{u} + (\mathbf{r}-\mathbf{r}_c)_\| \frac{\partial \gamma}{\partial t} = -\gamma\mathbf{u} + \frac{\mathbf{r}_{0\|}}{\gamma}\frac{\partial\gamma}{\partial t}$$

having made use of $(\partial \mathbf{r}_c/\partial t)_\perp = \mathbf{u}_\perp = 0$, so that

$$\frac{\partial \mathbf{r}_0}{\partial t} = -\gamma\mathbf{u}\left[1 + \frac{(\mathbf{u}\cdot\mathbf{r}_0)}{u^2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\gamma}\right)\right]$$

then

$$\frac{1}{c}\frac{\partial w_\mathrm{b}}{\partial t} = \frac{e^2\delta(r_0-a)\gamma}{32\pi\epsilon_0 a^4 c}\hat{\mathbf{r}}_0\cdot\mathbf{u}\left[1+\frac{\mathbf{u}\cdot\mathbf{r}_0}{u^2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\gamma}\right)\right] = -\rho\mathbf{f}_\mathrm{b}\cdot\mathbf{u}_\|(\mathbf{r},t)/c. \quad (6.24\mathrm{b})$$

Inserting (6.24a) and (6.24b) into (6.18) shows that the binding stress-momentum-energy tensor in (6.21) indeed satisfies its defining equations (6.18), or equivalently, that (6.22) satisfies (6.12).

Equations (B.32) and (A.21) have been used to prove in (6.24b) that (to order r_0^2)

$$\mathbf{u}\left[1+\frac{\mathbf{u}\cdot\mathbf{r}_0}{u^2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\gamma}\right)\right] = \mathbf{u}_\|(\mathbf{r},t)\,. \quad (6.25)$$

Thus, the time derivative of w_b in (6.21b) equals $-\rho\mathbf{f}_\mathrm{b}\cdot\mathbf{u}_\|(\mathbf{r},t)$ rather than $-\rho\mathbf{f}_\mathrm{b}\cdot\mathbf{u}(\mathbf{r},t)$. However, the difference is inconsequential with respect to the integral over all space of the power density, because

$$\int_{\text{all space}} \rho\mathbf{f}_\mathrm{b}\cdot\mathbf{u}_\perp(\mathbf{r},t)\mathrm{d}V = 0 \quad (6.26)$$

so that

$$\int_{\text{all space}} \rho\mathbf{f}_\mathrm{b}\cdot\mathbf{u}_\|(\mathbf{r},t)\mathrm{d}V = \int_{\text{all space}} \rho\mathbf{f}_\mathrm{b}\cdot\mathbf{u}(\mathbf{r},t)\mathrm{d}V = -\frac{e^2}{24\pi\epsilon_0 a}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\gamma}\right) \quad (6.27)$$

exactly the right value to cancel the discrepancy (2.6) between the electromagnetic force and power. Note that if we had assumed $\mathbf{u}$ were constant in our derivation of the binding force tensor, $\mathbf{u}_\|(\mathbf{r},t)$ would equal $\mathbf{u}$, and the total power obtained by integrating the power density would erroneously equal zero, that is

$$\int_{\text{all space}} \rho\mathbf{f}_\mathrm{b}\cdot\mathbf{u}_\|\mathrm{d}V = \mathbf{u}\cdot\int_{\text{all space}}\rho\mathbf{f}_\mathrm{b}\mathrm{d}V = 0 \quad (6.28)$$

as explained previously in Chapters 3 and 4. (In (6.24) through (6.30) below, the terms involving the mass of the insulator are ignored since they are irrelevant to this discussion.)

One can also obtain the result (6.27) by integrating the energy density w_b of the binding tensor over all space to get

$$W_\mathrm{b} = \int_{\text{all space}} w_\mathrm{b}\mathrm{d}V = \frac{e^2}{24\pi\epsilon_0 a}\frac{1}{\gamma} \quad (6.29)$$

and taking the negative of the time derivative. Note that W_b in (6.29) differs by a constant $(e^2/(24\pi\epsilon_0 a))$ from its value in (4.9) or (6.5b) (with $m_\mathrm{ins}=0$). This is because W_0 calculated from

$$W_0 = \int_{\text{all space}} w_0\mathrm{d}V = -\frac{e^2}{24\pi\epsilon_0 a}\gamma \quad (6.30)$$

differs from W_0 in equation (6.6b) by the negative of the same constant $(-e^2/(24\pi\epsilon_0 a))$, so that the sum, $W_\mathrm{b}+W_0$, remains the same whether it is calculated by adding (6.29) and (6.30) or (6.5b) and (6.6b). As mentioned in Section 6.1, an arbitrary constant energy can be added and subtracted from the binding and bare-mass energies, W_b and W_0, respectively, without changing the total energy of formation or the final equations of motion of the charged insulator.

In summary, a total stress-momentum-energy tensor T^{ij} has been derived for the charged insulator model of the electron. It can be written as the sum of the electromagnetic, binding-force, and bare-mass stress-momentum-energy tensors

$$T^{ij}(\mathbf{r},t) = T^{ij}_\mathrm{em} + \frac{e^2}{32\pi^2\epsilon_0 a^4}h(a-r_0)g^{ij} + \frac{c^2}{4\pi a^2}(m_\mathrm{ins}+M_0)\delta(r_0-a)u^i u^j \quad (6.31)$$

with the bare mass M_0 equal, of course, to $-m_\mathrm{es}/3 = -e^2/(24\pi\epsilon_0 ac^2)$. In (6.31) the right-hand side of (6.19) has replaced ρ in (6.16) and (6.22), and $\mathbf{r}_0$ is given in terms of $(\mathbf{r},t)$ by the general Lorentz transformation (6.20). The four-divergence of T^{ij} produces the time rate of change of the total momentum-energy density, for the charge distribution bound to the insulator, throughout all space and time; specifically

$$\begin{aligned}\frac{\partial T^{ij}}{\partial x^j} &= -f^i_\mathrm{em} - f^i_\mathrm{b} - f^i_0 \\ &= \frac{\rho(\mathbf{r},t)}{\gamma e}\left[(m_\mathrm{es}+m_\mathrm{ins})c^2\frac{\mathrm{d}u^i}{\mathrm{d}s} - \frac{e^2}{6\pi\epsilon_0}\left(\frac{\mathrm{d}^2u^i}{\mathrm{d}s^2} + u^i\frac{\mathrm{d}u_j}{\mathrm{d}s}\frac{\mathrm{d}u^j}{\mathrm{d}s}\right)\right] + O(a)\end{aligned} \quad (6.32)$$

with $\rho(\mathbf{r},t)$ given in (6.19).

The integral over all space of $-\partial T^{ij}/\partial x^j$ produces the sum of the electromagnetic, binding, and bare-mass force-powers given previously in equations (6.1) through (6.3) as well as the radiation reaction and higher order electromagnetic force-power terms. In other words, $\partial T^{ij}/\partial x^j$ integrated over all space yields a four-vector force-power and the consistent equations of motion

(5.12) for the charged insulator when this integral is set equal to the externally applied force. Writing the externally applied force f^i_{ext} as

$$f^i_{\text{ext}} = [\mathbf{f}_{\text{ext}}, \mathbf{f}_{\text{ext}} \cdot \mathbf{u}/c] = \frac{\partial T^{ij}_{\text{ext}}}{\partial x^j} \tag{6.33}$$

we have

$$\frac{\partial}{\partial x^j}(T^{ij} + T^{ij}_{\text{ext}}) = 0. \tag{6.34}$$

Also, the integral of T^{i4} over all space produces the four-vector sum of the electromagnetic, binding-force, and bare-mass momentum-energies given in the equations (6.4) through (6.6), plus the four-vector electromagnetic radiation-reaction momentum-energy. If the velocity of the charge distribution is constant (no external force applied), the right-hand side of (6.32) is zero, or equivalently, the divergence of T^{ij} is zero, and it thereby yields a conserved four-vector momentum-energy.

When the velocity $\mathbf{u}$ is a constant the stress-momentum-energy tensor T^{ij} given in (6.31), together with (6.20), is basically the same as Schwinger's "first stress tensor" [23, eq. (42)]. The difference is due to Schwinger's tensor having its bare-mass portion distributed throughout the oblate spheroid, whereas we have assumed the bare mass and mass of the insulator are distributed with the thin shell of charge. Of course, the stress tensors of Schwinger are not derived from the detailed analysis of the charged insulator model of the electron, but are constructed by subtracting a charge-current stress tensor, for a charge in uniform motion, from the electromagnetic stress-momentum-energy tensor, so that the divergence of the resulting tensor is zero. The stress tensors of Schwinger are discussed further in the following section.

6.2 Redefinition of Electromagnetic Momentum and Energy

A number of authors, beginning apparently with Fermi [33], have suggested that the consideration of specific binding forces and bare masses could be avoided in obtaining the equation of motion (5.12) by redefining the electromagnetic momentum and energy (and associated stress-momentum-energy tensor) used to determine the self electromagnetic force and power [23], [32], [34]. In particular, they replace the original electromagnetic momentum and energy densities, $\epsilon_0 \mathbf{E}\times\mathbf{B}$ and $\epsilon_0(E^2+c^2B^2)/2$, in the second integrals of (3.1) and (3.2) by new momentum and energy densities, $\mathbf{g}_{\text{new}}(\mathbf{r},t)$ and $w_{\text{new}}(\mathbf{r},t)$, such that the total momentum $\mathbf{G}_{\text{new}}$ and energy W_{new}

$$\mathbf{G}_{\text{new}}(t) = \int\limits_{\text{all space}} \mathbf{g}_{\text{new}}(\mathbf{r},t)\mathrm{d}V \tag{6.35a}$$

$$W_{\text{new}}(t) = \int\limits_{\text{all space}} w_{\text{new}}(\mathbf{r},t)\mathrm{d}V \tag{6.35b}$$

transform as a four-vector, at least when the charge has constant velocity, and satisfy the consistency requirements (5.14) and (5.16b). Moreover, $\mathbf{g}_{\text{new}}$ and w_{new} can be chosen to eliminate the 4/3 factor that arises using the conventional definition of electromagnetic momentum and energy.

For example, if the stress-momentum-energy tensor is redefined so that the momentum density $\mathbf{g}_{\text{new}}(\mathbf{r},t)$ equals $\gamma^2\mathbf{u}$ multiplied by any invariant function involving the electromagnetic field, charge-current, or both [15, sec. 1.23] (invariant with respect to all inertial frames moving with constant relative velocities), and the energy density $w_{\text{new}}(\mathbf{r},t)$ equals γ^2c^2 times the same invariant, that is

$$\mathbf{g}_{\text{new}}(\mathbf{r},t) = \gamma^2\mathbf{u}I \tag{6.36a}$$

$$w_{\text{new}}(\mathbf{r},t) = \gamma^2c^2I \tag{6.36b}$$

where $\mathbf{u}$ is the velocity of the charge, and I is the invariant, then the total momentum and energy in (6.35) of a charge distribution moving with constant velocity transform as a four-vector. The total momentum and energy in (6.35) calculated from (6.36) determine a four-vector because $(\gamma\mathbf{u}, \gamma c)$ is a four-vector and $\int I\gamma \mathrm{d}V$ over all space is an invariant, *provided I is calculated for a charge distribution moving with constant velocity.*

Rohrlich et al. [32], [34] redefine the momentum-energy to yield the specific invariant

$$I = \frac{\epsilon_0}{2c^2}\left(E^2 - c^2B^2\right) \tag{6.37}$$

which can be inserted into (6.36) and integrated in (6.35) for a uniformly charged sphere moving with constant velocity $\mathbf{u}$ to give the four-vector

$$\mathbf{G}_{\text{new}}(t) = m_{\text{es}}\gamma\mathbf{u} \tag{6.38a}$$

$$w_{\text{new}}(t) = m_{\text{es}}\gamma c^2 \tag{6.38b}$$

$$m_{\text{es}} = \frac{\epsilon_0}{2c^2}\int\limits_{\text{all space}}(E^2 - c^2B^2)\gamma\mathrm{d}V = \frac{e^2}{8\pi\epsilon_0 ac^2}\,. \tag{6.38c}$$

For a charged sphere moving with arbitrary velocity $\mathbf{u}$, (6.37) still yields (6.38) for the dominant $1/a$ terms of the momentum and energy. Thus when one replaces $\epsilon_0\mathbf{E}\times\mathbf{B}$ and $\epsilon_0(E^2+c^2B^2)/2$ in the self electromagnetic force and power equations (3.1) and (3.2) by $\mathbf{g}_{\text{new}}$ and w_{new} in (6.36a) and (6.36b), with I inserted from (6.37), the $1/a$ terms in the final forms of the force and power equations of motion, (5.12a) and (5.12b), emerge without the explicit introduction of binding forces or a nonzero bare mass. However, for arbitrary velocity $\mathbf{u}$ the invariant (6.37) does not predict the correct radiation reaction terms in the equations of motion (5.12).

Alternative momentum and energy densities to (6.36) can be found that produce consistent results for the $1/a$ terms (consistent with the requirements (5.14) and (5.16b) on the rate of change of linear and angular momentum-energy) and correct radiation reaction terms in the momentum and energy equations of motion. Probably the simplest way to do this is to subtract the momentum-energy density $(\mathbf{g}_s, w_s)$ from the original electromagnetic momentum-energy density $\epsilon_0[\mathbf{E}\times\mathbf{B}, (E^2+c^2B^2)/2]$ to form

$$\mathbf{g}_{\text{new}} = \epsilon_0 \mathbf{E}\times\mathbf{B} - \mathbf{g}_s \tag{6.39a}$$

$$w_{\text{new}} = \frac{\epsilon_0}{2}(E^2+c^2B^2) - w_s \tag{6.39b}$$

such that

$$\mathbf{G}_{\text{new}} = \epsilon_0 \int\limits_{\text{all space}} \mathbf{E}\times\mathbf{B}\,\mathrm{d}V - \int\limits_{\text{all space}} \mathbf{g}_s \mathrm{d}V \tag{6.40a}$$

and

$$W_{\text{new}} = \frac{\epsilon_0}{2} \int\limits_{\text{all space}} (E^2+c^2B^2)\mathrm{d}V - \int\limits_{\text{all space}} w_s \mathrm{d}V \tag{6.40b}$$

will form the four-vector $(m_s\gamma\mathbf{u}, m_s\gamma c^2)$, that is

$$\mathbf{G}_{\text{new}} = m_s\gamma\mathbf{u} \tag{6.41a}$$

$$W_{\text{new}} = m_s\gamma c^2 \tag{6.41b}$$

when the charge has constant velocity, where m_s is an arbitrary constant mass. For a relativistically rigid charged sphere moving with constant velocity, we see from Appendix B or (6.4a)–(6.4b) that

$$\epsilon_0 \int\limits_{\text{all space}} \mathbf{E}\times\mathbf{B}\,\mathrm{d}V = \frac{4}{3} m_{\text{es}}\gamma\mathbf{u} \tag{6.42a}$$

and

$$\frac{\epsilon_0}{2} \int\limits_{\text{all space}} (E^2+c^2B^2)\mathrm{d}V = \frac{4}{3} m_{\text{es}} c^2 \left(\gamma - \frac{1}{4\gamma}\right) \tag{6.42b}$$

which combine with (6.40) and (6.41) to show that $\mathbf{g}_s$ and w_s must satisfy

$$\int\limits_{\text{all space}} \mathbf{g}_s \mathrm{d}V = \left(\frac{4}{3} m_{\text{es}} - m_s\right)\gamma\mathbf{u} \tag{6.43a}$$

$$\int\limits_{\text{all space}} w_s \mathrm{d}V = \left(\frac{4}{3} m_{\text{es}} - m_s\right)\gamma c^2 - \frac{m_{\text{es}}c^2}{3\gamma}\,. \tag{6.43b}$$

Moreover, if $(\mathbf{g}_s, w_s)$ are chosen to satisfy (6.43a) and (6.43b) for arbitrary velocity $\mathbf{u}$, then the time derivative of (6.40a) and (6.40b) for arbitrary velocity

will yield $1/a$ terms consistent with (5.14) and (5.16b), and correct radiation reaction terms (and all higher electromagnetic terms) in the self force and power.

Schwinger [23] has derived divergenceless stress-momentum-energy tensors for constant velocity charge-current distributions, such that a $(\mathbf{g}_s, w_s)$ can satisfy (6.43) for $m_s = m_{\text{em}}$, or $(\mathbf{g}_s, w_s)$ can satisfy (6.43) for m_s equal to the electrostatic mass m_{es}. And, in fact, his method can be immediately generalized to find a $(\mathbf{g}_s, w_s)$ that will satisfy (6.43) for an arbitrary value of the constant mass m_s in the $1/a$ term of the redefined momentum-energy given by (6.40).

Specifically, Schwinger rewrites the electromagnetic force-power density for uniformly moving (constant velocity) charge distributions, that are spherically symmetric in their rest frames, as the divergence of a tensor that depends only on the charge-current distribution. When this force-power tensor, which is not unique, is subtracted from the electromagnetic stress-momentum-energy tensor, a new divergenceless stress-momentum-energy tensor results for which the total momentum-energy is a four-vector. In particular, he finds the two stress-momentum-energy tensors

$$T_1^{ij} = T_{\text{em}}^{ij} + (g^{ij} - u^i u^j)\mathcal{T} \tag{6.44a}$$

and

$$T_2^{ij} = T_{\text{em}}^{ij} + g^{ij}\mathcal{T} \tag{6.44b}$$

where $\mathcal{T}$ is a scalar that depends on the spherical charge distribution. (The first is found by subtracting the tensor $u^i u^j \mathcal{T}$, which is divergenceless at constant velocity, from the second.) For the uniformly moving shell of charge

$$\mathcal{T} = \frac{e^2}{32\pi^2\epsilon_0 a^4} h(a - r_0) \tag{6.44c}$$

with r_0 given in terms of $(\mathbf{r}, t)$ through the Lorentz transformation. Thus, the first tensor (6.44a) is essentially the same as the stress-momentum-energy tensor (6.31) derived for the charged insulator model when the mass of the insulator m_{ins} is zero. Its mass, determined by the integral of the energy or momentum over all space, equals the electrostatic mass. (As mentioned in Section 6.1, the slight difference between (6.44a) and (6.31) with m_{ins} zero is the result of the bare-mass portion of Schwinger's tensor being distributed throughout the oblate spheroid rather than in the thin shell of charge.) The mass associated with the second tensor (6.44b) equals the electromagnetic mass. It would correspond to a charged insulator with the mass of the insulator material equal to 1/3 the electrostatic mass. *Basically, the method of Schwinger is a mathematical way to include the effect of Poincaré stresses without relating them to the physics of the charged particle.*

Of course, there are drawbacks to redefining the electromagnetic momentum and energy. If the momentum and energy densities are changed in the second integrals of (3.1) and (3.2), so as to also change the values of the time derivatives of these integrals, these new values of self electromagnetic force and power will no longer equal the Lorentz force and power (the first integrals in (3.1) and (3.2)) for the shell of charge. This implies one or more of the following alternatives:

1. the definition of the Lorentz force must change
2. Maxwell's equations must change
3. the charge-current distribution must change
4. unknown forces (electromagnetic or nonelectromagnetic) are present that contribute to the total self force and power of the charge distribution.

None of these alternatives seem very attractive because they each involve introducing extra unknowns unnecessarily into the simple, deterministic model of the electron as an insulator that remains spherical and uniformly charged in every proper inertial frame of reference. Also, none of the redefined stress-momentum-energy tensors predict the second and higher order binding forces on the right-hand side of (4.22) that are necessary to hold the accelerating charge to the insulator.

7

Momentum and Energy Relations

The equations of motion (5.12) for the charged insulating sphere of radius a moving with arbitrary center velocity $\mathbf{u}(t)$ can be rewritten in four-vector notation [13] as

$$F^i_{\text{ext}} = mc^2 \frac{\mathrm{d}u^i}{\mathrm{d}s} - \frac{e^2}{6\pi\epsilon_0}\left(\frac{\mathrm{d}^2u^i}{\mathrm{d}s^2} + u^i \frac{\mathrm{d}u_j}{\mathrm{d}s}\frac{\mathrm{d}u^j}{\mathrm{d}s}\right) + O(a) \tag{7.1}$$

with

$$F^i_{\text{ext}} \equiv \gamma\left(\mathbf{F}_{\text{ext}}, \mathbf{F}_{\text{ext}} \cdot \mathbf{u}/c\right) \tag{7.2a}$$

$$u^i \equiv \gamma\left(\mathbf{u}/c, 1\right) \tag{7.2b}$$

$$u_i \equiv \gamma\left(-\mathbf{u}/c, 1\right) \tag{7.2c}$$

$$\mathrm{d}s \equiv c\mathrm{d}t/\gamma\,. \tag{7.2d}$$

The measured rest mass m of the charged insulator equals $(m_{\text{es}} + m_{\text{ins}})$ with $m_{\text{es}} = e^2/(8\pi\epsilon_0 ac^2)$.

The total momentum $\mathbf{G}_{12}$ and energy W_{12} supplied by the external force to the charge between the times t_1 and t_2 are given by

$$\mathbf{G}_{12} = \int_{t_1}^{t_2} \mathbf{F}_{\text{ext}}(t)\mathrm{d}t \tag{7.3a}$$

and

$$W_{12} = \int_{t_1}^{t_2} \mathbf{F}_{\text{ext}}(t) \cdot \mathbf{u}(t)\mathrm{d}t \tag{7.3b}$$

or in four-vector notation

$$G^i_{12} = (c\mathbf{G}_{12}, W_{12}) = \int_{s_1}^{s_2} F^i_{\text{ext}}\mathrm{d}s\,. \tag{7.4}$$

Substituting F^i_{ext} from (7.1) into (7.4) we obtain

$$G^i_{12} = mc^2\left[u^i(s_2) - u^i(s_1)\right] - \frac{e^2}{6\pi\epsilon_0}\left[\frac{du^i}{ds}(s_2) - \frac{du^i}{ds}(s_1)\right] - \frac{e^2}{6\pi\epsilon_0}\int_{s_1}^{s_2} u^i \frac{du_j}{ds}\frac{du^j}{ds} ds + O(a) . \quad (7.5)$$

If the velocity and acceleration of the particle are the same at times t_1 and t_2, that is, at s_1 and s_2, the momentum-energy in (7.5) reduces to

$$G^i_{12} = -\frac{e^2}{6\pi\epsilon_0}\int_{s_1}^{s_2} u^i \frac{du_j}{ds}\frac{du^j}{ds} ds + O(a) . \quad (7.6)$$

In three-vector notation

$$u^i \frac{du_j}{ds}\frac{du^j}{ds} = -\gamma\left[\frac{\gamma^4}{c^5}|\dot{\mathbf{u}}|^2 + \frac{\gamma^6}{c^7}(\mathbf{u}\cdot\dot{\mathbf{u}})^2\right](\mathbf{u}, c) \quad (7.7)$$

so that (7.6) becomes

$$G^i_{12} = (c\mathbf{G}_{12}, W_{12}) = \frac{e^2}{6\pi\epsilon_0 c^4}\int_{t_1}^{t_2}\left[\gamma^4|\dot{\mathbf{u}}|^2 + \frac{\gamma^6}{c^2}(\mathbf{u}\cdot\dot{\mathbf{u}})^2\right](\mathbf{u}, c) dt + O(a) . \quad (7.8)$$

The integrand in (7.8) is just the momentum and energy radiated per unit time by an accelerating point charge [35], [3, sec. 15]. Thus, (7.8) says that the momentum and energy imparted to the charge by the externally applied force during any time interval is equal to the momentum and energy radiated by that charge, provided the initial and final velocities and accelerations are the same. In other words, the du^i/ds and d^2u^i/ds^2 terms in the equation of motion (7.1) represent reversible rates of change of momentum-energy, while the $u^i du_j/ds\, du^j/ds$ term represents the irreversible rate of change of momentum-energy that radiates to the far field.

The reversible du^i/ds term is, of course, the usual rate of change of momentum-energy four-vector in the relativistic version of Newton's second law of motion

$$mc^2\frac{du^i}{ds} = m\gamma\frac{d}{dt}(\gamma\mathbf{u}, \gamma c) . \quad (7.9)$$

Its integral over a proper time interval determines the reversible change in kinetic momentum-energy of the particle during that time interval.

The reversible d^2u^i/ds^2 term can be written in three-vector form as

$$-\frac{e^2}{6\pi\epsilon_0}\frac{d^2u^i}{ds^2} = -\frac{e^2\gamma}{6\pi\epsilon_0 c^3}\frac{d}{dt}\left\{\left[\gamma^2\dot{\mathbf{u}} + \frac{\gamma^4}{c^2}(\mathbf{u}\cdot\dot{\mathbf{u}})\mathbf{u}\right], \frac{1}{c}(\gamma^4\mathbf{u}\cdot\dot{\mathbf{u}})\right\} . \quad (7.10)$$

When this perfect differential is integrated over proper time it yields a reversible change in momentum-energy that cannot be classified as either a change in kinetic momentum-energy or a change in radiated momentum-energy (which is irreversibly lost to the far field). Schott [36] called the energy portion of (7.10), that is

$$-\frac{e^2\gamma^4}{6\pi\epsilon_0 c^4}(\mathbf{u}\cdot\dot{\mathbf{u}}) \quad (7.11)$$

the "acceleration energy" because it "must be regarded as work stored in the electron in virtue of its acceleration." Therefore, this part of (7.10) is sometimes referred to as the Schott energy term, although Abraham [3, sec. 15] had previously separated the reversible momentum as well as the reversible energy of (7.10) in his derivation of the radiation reaction for a charge moving with arbitrary velocity.

Before and after the external force is applied, the acceleration of the charge is zero (ignoring the pre-acceleration that will be discussed in Chapter 8) so that the Schott acceleration momentum-energy is zero and, as expected, the momentum-energy that has been supplied by the external force has been converted entirely to kinetic and radiated momentum-energy. However, while the external force is being applied, the charge is accelerating and the momentum-energy supplied by the external force is converted to "Schott acceleration momentum-energy," as well as kinetic and radiated momentum-energy.

A physically intuitive understanding of the "acceleration" momentum-energy can be gained by looking at (7.1) for time harmonic motion. With the help of (7.7), (7.9) and (7.10), the momentum and energy equations of motion in (7.1) may be written separately in three-vector notation as

$$\mathbf{F}_{\text{ext}} = m\frac{d(\gamma\mathbf{u})}{dt} - \frac{e^2}{6\pi\epsilon_0 c^3}\left\{\frac{d}{dt}\left[\gamma^2\dot{\mathbf{u}} + \frac{\gamma^4}{c^2}(\mathbf{u}\cdot\dot{\mathbf{u}})\mathbf{u}\right] - \frac{\gamma^4}{c^2}\left[|\dot{\mathbf{u}}|^2 + \frac{\gamma^2}{c^2}(\mathbf{u}\cdot\dot{\mathbf{u}})^2\right]\mathbf{u}\right\} + \mathbf{O}(a) \quad (7.12a)$$

and

$$\mathbf{F}_{\text{ext}}\cdot\mathbf{u} = mc^2\frac{d\gamma}{dt} - \frac{e^2}{6\pi\epsilon_0 c^3}\left\{\frac{d}{dt}(\gamma^4\mathbf{u}\cdot\dot{\mathbf{u}}) - \gamma^4\left[|\dot{\mathbf{u}}|^2 + \frac{\gamma^2}{c^2}(\mathbf{u}\cdot\dot{\mathbf{u}})^2\right]\right\} + O(a). \quad (7.12b)$$

The first terms on the right-hand sides of (7.12) can be interpreted simply as the rates of change of kinetic momentum and energy required to accelerate the energy of formation of the moving charge. To understand the second terms on the right-hand sides of (7.12), consider a charge oscillating rectilinearly with sinusoidal frequency ω, so that the velocity is given by

$$u(t) = U_0 \sin(\omega t) \quad (7.13)$$

and the radiation reaction terms in the energy equation of motion (7.12b) become

$$-\frac{\mathrm{d}}{\mathrm{d}t}(\gamma^4 \mathbf{u}\cdot\dot{\mathbf{u}}) = -U_0^2\omega^2\cos(2\omega t) \tag{7.14}$$

$$\gamma^4\left[|\dot{\mathbf{u}}|^2 + \frac{\gamma^2}{c^2}(\mathbf{u}\cdot\dot{\mathbf{u}})^2\right] = U_0^2\omega^2\cos^2(\omega t) \tag{7.15}$$

when $(u/c)^2 \ll 1$.

The irreversible reaction term (7.15) behaves as a time-harmonic radiated power, that is, it has the time dependence of the Poynting vector integrated over a closed surface in the far field. Its average over time has the positive value $U_0^2\omega^2/2$. The reversible "acceleration" reaction term (7.14) behaves as a reactive power whose average over time is zero. In other words, if the oscillating charge were an antenna fed by a single-frequency input voltage and current, (7.14) and (7.15) would contribute to the reactive and resistive (radiation resistance) parts, respectively, of the input impedance of the antenna [37]. (The time rate of change of the kinetic energy of the oscillating charge has a time dependence proportional to

$$U_0^2\omega\sin(2\omega t)\,, \quad (u/c)^2 \ll 1 \tag{7.16}$$

and thus it would also contribute to the reactive part of the input impedance of the antenna.)

For a charge whose velocity and acceleration are continually increasing with time, rather than oscillating, the reversible kinetic energy continually increases, the irreversible radiated power increases, and more and more reversible Schott acceleration energy is taken from the electromagnetic fields of the charge. A similar unlimited increase in the irreversible (radiated) and reversible (reactive kinetic and Schott) energies occurs when the frequency of an oscillating charge or electric dipole is continually increased, as one can see from (7.14)–(7.16). However, the reactive energy taken from the oscillating charge, although it can increase without limit by increasing the frequency, is always returned to zero and supplied to the charge in an equal amount during each half period of oscillation.

Before leaving this section, note that the quantity $\gamma^4\left[|\dot{\mathbf{u}}|^2 + \gamma^2(\mathbf{u}\cdot\dot{\mathbf{u}})^2/c^2\right]$ can be substituted from (7.12b) to simplify the form of the equation of motion (7.12a) to

$$\gamma^2[\mathbf{F}_{\mathrm{ext}} - (\mathbf{F}_{\mathrm{ext}}\cdot\mathbf{u})\mathbf{u}/c^2] = m\gamma^3\dot{\mathbf{u}} - \frac{e^2\gamma}{6\pi\epsilon_0 c^3}\frac{\mathrm{d}}{\mathrm{d}t}(\gamma^3\dot{\mathbf{u}}) + \mathbf{O}(a). \tag{7.17a}$$

This form of the force equation of motion comes in handy in the next section of this chapter and in Chapter 8. The power equation of motion (7.12b) can be recast into the form

$$\mathbf{F}_{\mathrm{ext}}\cdot\mathbf{u} = mc^2\frac{\mathrm{d}\gamma}{\mathrm{d}t} - \frac{e^2\gamma}{6\pi\epsilon_0 c}\left(\frac{\mathrm{d}^2\gamma}{\mathrm{d}t^2} - \frac{\gamma^3}{c^2}|\dot{\mathbf{u}}|^2\right) + O(a). \tag{7.17b}$$

7.1 Hyperbolic Motion

For relativistically uniform acceleration, which is defined as [32, secs. 5-3 and 6-11]

$$\frac{d^2u^i}{ds^2} + u^i\frac{du_j}{ds}\frac{du^j}{ds} = 0 \tag{7.18}$$

the reversible Schott reactive power cancels the radiated power and the equation of motion (7.1) reduces to that of an uncharged particle

$$F^i_{\mathrm{ext}} = mc^2\frac{du^i}{ds} + O(a) \tag{7.19}$$

that is, the time rate of change of kinetic momentum-energy equals the applied force minus the $O(a)$ terms. Because in the proper inertial reference frame (7.18) reduces to simply $\ddot{\mathbf{u}} = 0$, that is, $\dot{\mathbf{u}} =$ constant, it is appropriate to refer to the motion defined by (7.18) as relativistically uniform acceleration. The charged particle radiates by drawing energy from the reactive fields of the charge, the reactive fields continually being replenished by the increasing acceleration of the charge.

Neglecting terms of $O(a)$, equation (7.19) reduces to the Newtonian equation of motion in four-vector form

$$F^i_{\mathrm{ext}} = mc^2\frac{du^i}{ds}. \tag{7.20}$$

Relativistically uniform acceleration or "hyperbolic motion" is described in numerous physics texts [32]. Here we show that the equations (7.18) and (7.20) cannot both be satisfied unless there is an inertial frame in which the motion is rectilinear and the externally applied force is constant, such as when a charge is placed in a uniform electrostatic field.

To prove this result, use the procedure for deriving (7.17a) to express the three vector part of (7.18) as simply

$$\frac{\mathrm{d}}{\mathrm{d}t}(\gamma^3\dot{\mathbf{u}}) = 0 \tag{7.21}$$

and write the three-vector part of (7.20) as

$$\mathbf{F}_{\mathrm{ext}} = m\frac{\mathrm{d}}{\mathrm{d}t}(\gamma\mathbf{u}). \tag{7.22}$$

The solution to (7.21) is

$$\gamma^3\dot{\mathbf{u}} = \mathbf{A}_0 \tag{7.23}$$

where $\mathbf{A}_0$ is a constant vector. The result in (7.23) implies that the acceleration of the charge perpendicular to the direction of $\mathbf{A}_0$ is zero. Consequently, the velocity of the charge is constant in the direction perpendicular to the direction of $\mathbf{A}_0$. In the inertial reference frame moving with this constant velocity, the motion is rectilinear. Moreover, (7.18) and thus (7.21) still hold in this inertial reference frame in which the motion is rectilinear because (7.18) is a Lorentz invariant expression. Therefore, in this rectilinear reference frame, (7.22) and (7.23) become

$$F_{\text{ext}} = m\frac{\mathrm{d}}{\mathrm{d}t}(\gamma u) \tag{7.24}$$

and

$$\gamma^3 \dot{u} = \frac{\mathrm{d}}{\mathrm{d}t}(\gamma u) = A_0 . \tag{7.25}$$

A comparison of (7.24) and (7.25) immediately reveals that

$$F_{\text{ext}} = mA_0 . \tag{7.26}$$

In other words, relativistically uniform motion can only be achieved if there exists an inertial reference frame in which the motion is rectilinear and the externally applied force is equal to a constant, for example, an electron accelerated by an electrostatic field E_0 between two oppositely charged parallel plates that are normal to the direction of motion of the charge. Except during the short transition intervals when the electron enters and leaves the parallel plates, the motion of the electron is determined entirely by the relativistic Newtonian equation of motion

$$eE_0 = m\frac{\mathrm{d}}{\mathrm{d}t}(\gamma u). \tag{7.27}$$

The problem of the motion of a charge between two parallel plates, including its behavior during the entrance and exit transition intervals, is considered in Section 8.2.4 after the equation of motion is modified to eliminate pre-acceleration and pre-deceleration.

7.2 Runaway Motion

"Runaway solutions" are homogeneous solutions to (7.1). In other words, the Schott reactive power cancels both the radiated power and the kinetic power so that (7.1) is satisfied without an applied external force

$$\frac{\mathrm{d}^2u^i}{\mathrm{d}s^2} = -u^i\frac{\mathrm{d}u_j}{\mathrm{d}s}\frac{\mathrm{d}u^j}{\mathrm{d}s} + \frac{6\pi\epsilon_0 mc^2}{e^2}\frac{\mathrm{d}u^i}{\mathrm{d}s} \tag{7.28}$$

(neglecting $O(a)$ terms), or in three-vector notation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\gamma^2\dot{\mathbf{u}} + \frac{\gamma^4}{c^2}(\mathbf{u}\cdot\dot{\mathbf{u}})\mathbf{u}\right] = \frac{\gamma^4}{c^2}\left[|\dot{\mathbf{u}}|^2 + \frac{\gamma^2}{c^2}(\mathbf{u}\cdot\dot{\mathbf{u}})^2\right]\mathbf{u} + \frac{6\pi\epsilon_0 mc^3}{e^2}\frac{\mathrm{d}(\gamma\mathbf{u})}{\mathrm{d}t} \tag{7.29a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\gamma^4\mathbf{u}\cdot\dot{\mathbf{u}}) = \gamma^4\left[|\dot{\mathbf{u}}|^2 + \frac{\gamma^2}{c^2}(\mathbf{u}\cdot\dot{\mathbf{u}})^2\right] + \frac{6\pi\epsilon_0 mc^5}{e^2}\frac{\mathrm{d}\gamma}{\mathrm{d}t} . \tag{7.29b}$$

A formal, general solution to the homogeneous equation of motion (7.29a) can be found by rewriting this equation in the shortened form given in (7.17a), namely

$$\frac{\mathrm{d}}{\mathrm{d}t}(\gamma^3\dot{\mathbf{u}}) - \frac{1}{\tau_e}\gamma^2\dot{\mathbf{u}} = 0 , \qquad \tau_e = \frac{e^2}{6\pi\epsilon_0 mc^3} \tag{7.30}$$

Recasting this equation as

$$\frac{\mathrm{d}}{\mathrm{d}t}(\gamma^3\dot{\mathbf{u}}) - \frac{1}{\tau_e\gamma}(\gamma^3\dot{\mathbf{u}}) = 0 \tag{7.31}$$

allows us to solve for $\gamma^3\dot{\mathbf{u}}$ as

$$\gamma^3\dot{\mathbf{u}} = \mathbf{A}_1 \exp\left(\frac{1}{\tau_e}\int_0^t \frac{\mathrm{d}t'}{\gamma(t')}\right) + \mathbf{A}_2 \tag{7.32}$$

wherein $\mathbf{A}_1$ and $\mathbf{A}_2$ are arbitrary constants. Since $1/\gamma(t) = \sqrt{1-u^2(t)/c^2}$, the integral in (7.32) approaches a positive infinite value unless $u(t) \to c$. Therefore, as $t \to \infty$, the speed of the particle "runs away" toward the speed of light, even though there is no external force applied.

Schott derived the runaway motion as part of his solution to the equation of rectilinear motion in his 1915 paper [36, eq. (21)]; see Section 8.1. He called this motion "exponential motion" and argued "that the exponential motion is not realizable experimentally, at any rate not with electric fields at our command."

Although these runaway solutions are presumably not physically realizable, they are mathematically valid homogeneous solutions to the differential equation of motion that do not violate conservation of momentum-energy. Both the increasing reversible kinetic momentum-energy and the increasing irreversible radiated momentum-energy are taken entirely from the reservoir of reversible Schott reactive momentum-energy that is continually being supplied by the increasing acceleration of the charge. It is emphasized that the unlimited supply of Schott reactive momentum-energy for the runaway modes is produced by the unlimited increase in the four-acceleration of the particle, and is not dependent upon the radius of the charge approaching zero or mass of the particle approaching infinity.

Despite the fact that the homogeneous runaway solutions do not violate the conservation of momentum-energy, it is shown in Chapter 8 that the runaway behavior is eliminated from the complete solution of (7.1) by invoking the asymptotic condition of zero acceleration as t approaches infinity.

In an attempt to get an equation of motion that involves only the kinetic and radiated momentum-energy of a charged particle, one may be tempted to simply discard the reactive momentum-energy term in (7.1) or its three-vector equivalent in (7.12). Unfortunately, the resulting simplified equation of motion would no longer be consistent with $F^i_{\text{ext}}u_i$ equal to zero. In terms of the three-vector equations, the scalar product of **u** with (7.12a) would no longer equal (7.12b).

It seems quite remarkable that without the insight and transformations of special relativity, Abraham was able to determine the reversible (reactive) parts of the radiation reaction force and power in (7.12) from a knowledge of the radiated momentum and energy of an accelerating point charge; then prove that the solution was unique [3, sec. 15]. (In the four-vector notation of (7.1) and with the transformations of special relativity, the determination of the reversible part of the radiation reaction from the radiated part is an elementary exercise. Uniqueness of solution follows from the fact that a four-vector which reduces to zero in the proper inertial frame must be zero in an arbitrary inertial frame.)

8

Solutions to the Equation of Motion

As a preliminary to solving the equation of motion (7.1) for the uniformly charged sphere of radius a and total charge e, write the magnitude of the four acceleration in (7.1) as

$$\frac{\mathrm{d}u_j}{\mathrm{d}s}\frac{\mathrm{d}u^j}{\mathrm{d}s} = \frac{(\mathbf{w}\cdot\mathbf{w}')^2}{\gamma^2 c^6} - \frac{w'^2}{c^4} \tag{8.1a}$$

where $\mathbf{w}$ is defined in terms of the velocity of the center of the shell by

$$\mathbf{w} = \gamma\mathbf{u}, \quad \gamma = (1-u^2/c^2)^{-1/2} = (1+w^2/c^2)^{1/2} \tag{8.1b}$$

and the primes denote derivatives with respect to the proper time

$$\mathrm{d}\tau = \mathrm{d}t/\gamma\ . \tag{8.1c}$$

Insertion of (8.1) into (7.1) yields the three-vector equation for $\mathbf{w}$

$$\gamma\mathbf{F}_{\text{ext}}(\tau) = m\mathbf{w}' - \frac{e^2}{6\pi\epsilon_0 c^3}\left[\mathbf{w}'' - \frac{1}{c^2}\left(w'^2 - \frac{(\mathbf{w}\cdot\mathbf{w}')^2}{c^2\gamma^2}\right)\mathbf{w}\right] + \mathbf{O}(a). \tag{8.2}$$

For rectilinear motion in the x direction

$$\mathbf{F}_{\text{ext}}(\tau) = F_{\text{ext}}(\tau)\hat{\mathbf{x}} \tag{8.3a}$$

$$\mathbf{w} = w\hat{\mathbf{x}} \tag{8.3b}$$

and (8.2) becomes

$$\gamma F_{\text{ext}}(\tau) = mw' - \frac{e^2}{6\pi\epsilon_0 c^3}\left[w'' - \frac{1}{c^2}\frac{w'^2 w}{(1+w^2/c^2)}\right] + O(a)\ . \tag{8.4}$$

(In (8.2)–(8.4) and throughout the rest of this chapter, the vector and scalar functions $\mathbf{F}_{\text{ext}}(\tau)$ and $F_{\text{ext}}(\tau)$ are compact notations for $\mathbf{F}_{\text{ext}}[t(\tau)]$ and $F_{\text{ext}}[t(\tau)]$, respectively.) Following Schott [36] we see that the substitution

$$w/c = \sinh(\mathcal{V}/c) \tag{8.5}$$

reduces this equation for rectilinear motion to the simpler differential equation

$$\frac{F_{\text{ext}}(\tau)}{m} = \mathcal{V}'(\tau) - \tau_e \mathcal{V}''(\tau) + \frac{O(a)}{m} \tag{8.6}$$

where

$$\tau_e = \frac{e^2}{6\pi\epsilon_0 mc^3} \,. \tag{8.7}$$

If m and e equal the mass and charge of an electron, the time interval τ_e is equal to the amount of time it takes light to travel four thirds the classical radius of the electron.

8.1 Solution to the Equation of Rectilinear Motion

If the $O(a)/m$ terms in (8.6) are neglected, the most general solution to the resulting equation of rectilinear motion can be written as

$$\mathcal{V}'(\tau) = -\exp(\tau/\tau_e)\left[\frac{1}{m\tau_e}\int_0^\tau F_{\text{ext}}(\tau')\exp(-\tau'/\tau_e)\mathrm{d}\tau' + A\right],$$
$$-\infty < \tau < \infty \tag{8.8}$$

where the external force is applied at $\tau = 0$ $(t = 0)$ and is assumed zero for all time $\tau < 0$ $(t < 0)$. Integration of (8.8) with respect to the proper time τ gives the general solution for $\mathcal{V}$ as

$$\mathcal{V}(\tau) = B + \frac{1}{m}\int_0^\tau F_{\text{ext}}(\tau')\mathrm{d}\tau'$$
$$-\exp(\tau/\tau_e)\left[\frac{1}{m}\int_0^\tau F_{\text{ext}}(\tau')\exp(-\tau'/\tau_e)\mathrm{d}\tau' + \tau_e A\right],$$
$$-\infty < \tau < \infty \,. \tag{8.9}$$

Integrating (8.9) with respect to the proper time, one could also obtain the position of the center of the shell. This would introduce a third arbitrary constant (A and B being the other two) that can be determined by specifying the position of the particle at a certain time, or in the remote past.

To determine the two remaining constants, A and B, two other boundary conditions are required. This is one more constant and boundary condition than is required by Newton's second law of motion for uncharged particles, which involves only the first derivative of velocity, rather than the first and second derivatives in (8.6). At first thought, since the external force is not applied until $\tau = 0$, one might set the velocity and acceleration equal to zero at $\tau = 0$ to obtain zero for both the constants A and B. Then (8.9) would become

$$\mathcal{V}(\tau) = \frac{1}{m}\int_0^\tau F_{\text{ext}}(\tau')\mathrm{d}\tau' - \frac{\exp(\tau/\tau_e)}{m}\int_0^\tau F_{\text{ext}}(\tau')\exp(-\tau'/\tau_e)\mathrm{d}\tau',$$
$$-\infty < \tau < \infty \,. \tag{8.10}$$

Unfortunately, there is a serious problem with the solution (8.10). The velocity function $\mathcal{V}(\tau)$ and all its derivatives approach infinity $[u(t) \to c]$ as $\tau \to \infty$, even when the external force is applied for a finite time.

Returning to (8.8) or (8.9) we see that these "runaway solutions" (called "exponential motion" by Schott [36]; see Section 7.2) are eliminated as $\tau \to \infty$ if and only if the constant A is given by

$$A = -\frac{1}{m\tau_e}\int_0^\infty F_{\text{ext}}(\tau')\exp(-\tau'/\tau_e)\mathrm{d}\tau'. \tag{8.11}$$

Equation (8.11) ensures that the acceleration in (8.8) approaches zero as $\tau \to \infty$, if the external force approaches zero as $\tau \to \infty$; and thus (8.11) can be considered a result of the "asymptotic condition" [12], [32]

$$\lim_{t\to\infty} \dot{\mathbf{u}}(t) = 0 \tag{8.12a}$$

when

$$\lim_{t\to\infty} \mathbf{F}_{\text{ext}}(t) = 0 \,. \tag{8.12b}$$

(Rohrlich [32, sec. 8-2] points out that the asymptotic condition can be based on a fundamental "principle of undetectability of small charge," which asserts that the motion of a charged particle must approach that of a neutral particle in the limit as the charge approaches zero.) After insertion of A from (8.11), equations (8.8) and (8.9) can be written as

$$\mathcal{V}'(\tau) = \frac{1}{m\tau_e}\int_\tau^\infty F_{\text{ext}}(\tau')\exp[-(\tau'-\tau)/\tau_e]\mathrm{d}\tau',$$
$$-\infty < \tau < \infty \tag{8.13a}$$

and

$$\mathcal{V}(\tau) = B + \frac{1}{m}\int_\tau^\infty F_{\text{ext}}(\tau')\exp[-(\tau'-\tau)/\tau_e]\mathrm{d}\tau' + \frac{1}{m}\int_0^\tau F_{\text{ext}}(\tau')\mathrm{d}\tau',$$
$$-\infty < \tau < \infty \tag{8.13b}$$

or

$$\mathcal{V}(\tau) = B + \frac{1}{m}\int_0^\infty F_{\text{ext}}(\tau+\tau')\exp(-\tau'/\tau_e)\mathrm{d}\tau' + \frac{1}{m}\int_0^\tau F_{\text{ext}}(\tau')\mathrm{d}\tau' ,$$

$$-\infty < \tau < \infty . \qquad (8.13\text{c})$$

A final boundary condition is needed to evaluate the constant B in (8.13b)–(8.13c). One can evaluate B by specifying the initial velocity, but this procedure leads to a velocity in the remote past ($\tau \to -\infty$) that depends on the external force, which we have assumed is applied at $\tau = 0$. Specifically, if one enforces the initial condition $\mathcal{V}(0) = 0$ in (8.13b)–(8.13c), then both the constant B and the velocity function in the remote past are given by

$$B = \mathcal{V}(-\infty) = -\frac{1}{m}\int_0^\infty F_{\text{ext}}(\tau')\exp(-\tau'/\tau_e)\mathrm{d}\tau' . \qquad (8.14)$$

Physically, it is much more appealing to demand that in the remote past the velocity be zero or a constant that is independent of the applied force. Thus, if the final boundary condition on the motion of the charge is an asymptotic condition on the velocity in the remote past; in particular, for zero velocity in the remote past

$$\lim_{t\to-\infty} \mathbf{u}(t) = 0 \qquad (8.15)$$

then $B = 0$ and (8.13c) becomes

$$\begin{aligned}
\mathcal{V}(\tau) &= \frac{1}{m}\int_0^\infty F_{\text{ext}}(\tau+\tau')\exp(-\tau'/\tau_e)\mathrm{d}\tau' + \frac{1}{m}\int_0^\tau F_{\text{ext}}(\tau')\mathrm{d}\tau' \\
&= \frac{1}{m}\int_\tau^\infty F_{\text{ext}}(\tau')\exp[-(\tau'-\tau)/\tau_e]\mathrm{d}\tau' + \frac{1}{m}\int_0^\tau F_{\text{ext}}(\tau')\mathrm{d}\tau' \\
&= \tau_e\mathcal{V}'(\tau) + \frac{1}{m}\int_0^\tau F_{\text{ext}}(\tau')\mathrm{d}\tau' , \quad -\infty < \tau < \infty . \qquad (8.16)
\end{aligned}$$

Equation (8.16), combined with the definitions (8.5) and (8.1b), is the general solution for the rectilinear velocity $\mathbf{u}$ of the center of the shell of charge for all time under the two asymptotic conditions that the acceleration approaches zero in the distant future (for zero external force in the distant future) and the velocity approaches zero in the remote past. Of course, the external force must be well-behaved enough for the integrals in (8.16) to exist, and the solution was obtained under the assumption that the terms of $O(a)/m$ in (8.6) could be neglected.

The solution (8.16) exhibits a number of peculiarities. The most unsettling one, pre-acceleration, or acceleration before the external force is applied at $\tau = 0$, is considered in Section 8.2. A second peculiarity, namely that the velocity at any instant of time depends on the externally applied force at all future times, is discussed near the end of Section 8.2.2.

A third peculiarity with the solution (8.16) is that if the force is zero after it is applied over a finite time interval, $0 \le \tau < \tau_0$, the velocity function reduces to

$$\mathcal{V}(\tau) = \mathcal{V}(\tau_0) = \frac{1}{m}\int_0^{\tau_0} F_{\text{ext}}(\tau')\mathrm{d}\tau' , \quad \tau > \tau_0 \qquad (8.17\text{a})$$

the final velocity function one would obtain if the radiation term $\mathcal{V}''$ in (8.6) (as well as the $O(a)/m$ terms) were ignored entirely. This result in (8.17a) is not so objectionable, if one realizes from (8.5) and (8.1b) that

$$\begin{aligned}
\gamma(t)u(t) = \gamma(t_0)u(t_0) &= c\sinh\left[\frac{1}{mc}\int_0^{\tau_0} F_{\text{ext}}(\tau')\mathrm{d}\tau'\right] \\
&\ne \frac{1}{m}\int_0^{t_0} F_{\text{ext}}(t')\mathrm{d}t' , \quad t > t_0 \qquad (8.17\text{b})
\end{aligned}$$

so it does not imply that the radiated momentum-energy is zero, or that the impulse and work supplied to the charged sphere by the external force is converted to kinetic energy alone. (Note that $\int_0^{\tau_0} F_{\text{ext}}(\tau')\mathrm{d}\tau' \ne \int_0^{t_0} F_{\text{ext}}(t')\mathrm{d}t'$.) To see this, integrate (7.12) over all time that the velocity is changing ($-\infty < t < t_0$) to get (for rectilinear motion)

$$\int_{-\infty}^{t_0} F_{\text{ext}}\mathrm{d}t = \int_0^{t_0} F_{\text{ext}}\mathrm{d}t = m\gamma(t_0)u(t_0) + \frac{e^2}{6\pi\epsilon_0 c^5}\int_{-\infty}^{t_0}\gamma^6\dot{u}^2 u(t)\mathrm{d}t \qquad (8.18\text{a})$$

$$\int_{-\infty}^{t_0} F_{\text{ext}}u\mathrm{d}t = \int_0^{t_0} F_{\text{ext}}u\mathrm{d}t = mc^2[\gamma(t_0) - 1] + \frac{e^2}{6\pi\epsilon_0 c^3}\int_{-\infty}^{t_0}\gamma^6\dot{u}^2(t)\mathrm{d}t . \qquad (8.18\text{b})$$

The reversible Schott acceleration momentum-energy in (7.12) does not contribute to (8.18) because the final acceleration and the acceleration in the remote past are both zero. The first terms on the right-hand sides of (8.18) give the total change in the kinetic momentum-energy of the charged sphere, while the second terms give the total momentum-energy radiated by the charged sphere. During pre-acceleration ($-\infty < t < 0$) only the runaway solution is present, and, as explained in Chapter 7, the Schott momentum-energy cancels both the kinetic and radiated momentum-energy. If the final velocity of the charge also equals zero ($u(t_0) = 0$) the change in the kinetic momentum-energy is zero and (8.18) confirms that the entire impulse and work delivered by the external force is converted to radiated momentum-energy. Note that even when the final velocity (as well as velocity in the remote past) is zero, we have the inequalities

$$\int_0^{t_0} F_{ext} dt \neq \frac{e^2}{6\pi\epsilon_0 c^5} \int_0^{t_0} \gamma^6 \dot{u}^2 u(t) dt \tag{8.19a}$$

and

$$\int_0^{t_0} F_{ext} u dt \neq \frac{e^2}{6\pi\epsilon_0 c^3} \int_0^{t_0} \gamma^6 \dot{u}^2(t) dt \,. \tag{8.19b}$$

That is, in order for the total momentum and energy radiated to equal the impulse and work delivered by the externally applied force when the final velocity (and velocity in the remote past) are zero, the integration of the radiated time rate of change of momentum and energy must include the pre-acceleration, because the initial velocity $u(0)$ is not zero in the pre-acceleration solution (8.16).

8.1.1 Formal Solution to the General Equation of Motion

A formal solution to the general equation of motion can be found from its form given in (7.17a), which can be rewritten as

$$\frac{\gamma^2}{m}\left[\mathbf{F}_{ext} - (\mathbf{F}_{ext}\cdot\mathbf{u})\mathbf{u}/c^2\right] = \gamma^3\dot{\mathbf{u}} - \tau_e\gamma\frac{d}{dt}(\gamma^3\dot{\mathbf{u}}) \tag{8.20}$$

with the $\mathbf{O}(a)/m$ terms neglected. The solution to this first order differential equation for $(\gamma^3\dot{\mathbf{u}})$, under the asymptotic condition in (8.12) is given for all t as

$$\gamma^3(t)\dot{\mathbf{u}}(t) = \frac{1}{m\tau_e}\int_t^\infty \gamma(t')\left\{\mathbf{F}_{ext}(t') - [\mathbf{F}_{ext}(t')\cdot\mathbf{u}(t')]\,\mathbf{u}(t')/c^2\right\} \cdot \exp\left[-(1/\tau_e)\int_t^{t'}\gamma^{-1}(t'')dt''\right]dt' \tag{8.21a}$$

which can be rewritten as

$$\gamma^3(\tau)\dot{\mathbf{u}}(\tau) = \frac{1}{m\tau_e}\int_\tau^\infty \gamma(\tau')\left\{\mathbf{F}_{ext}(\tau') - [\mathbf{F}_{ext}(\tau')\cdot\mathbf{u}(\tau')]\,\mathbf{u}(\tau')/c^2\right\} \cdot \exp[-(\tau'-\tau)/\tau_e]d\tau' \tag{8.21b}$$

after making the change of variable $d\tau = dt/\gamma$.

The equations in (8.21) represent a rigorous solution for the acceleration $\dot{\mathbf{u}}(t)$ in terms of both $\mathbf{u}(t)$ and $\mathbf{F}_{ext}(t)$. However, because $\mathbf{u}(t)$ is unknown, (8.21) is not a useful solution for calculating $\dot{\mathbf{u}}(t)$ or $\mathbf{u}(t)$. Nonetheless, the solution (8.21) reveals that the acceleration $\dot{\mathbf{u}}(t)$ of the charged sphere does not vanish before the external force is applied (say at $t = \tau = 0$) and thus the

solution to the general equation of motion, like the solution to the rectilinear equation of motion derived in Section 8.1, exhibits pre-acceleration. If the external force $\mathbf{F}_{ext}(t)$ applied to the charged sphere is zero for $t > t_0$, (8.21) also reveals that the acceleration $\dot{\mathbf{u}}(t) = 0$ for $t > t_0$, but that the solution anticipates the shutting off of the external force by displaying pre-deceleration.

Both the rectilinear solution in (8.16) and the general solution in (8.21b) also reveal that the pre-acceleration (and pre-deceleration) diminishes as the externally applied force varies more slowly on the scale of τ_e [38].

8.2 Cause and Elimination of the Pre-Acceleration

The solution (8.21) to the general equation of motion and the solution (8.16) to the rectilinear equation of the motion both predict nonzero acceleration before the external force is applied at $t = 0$ or $\tau = 0$. One may be tempted to simply set the acceleration or velocity equal to zero for $\tau < 0$ to eliminate the pre-acceleration in (8.16), for example. However, the resulting solution does not satisfy the original differential equation (8.6) (with $O(a)/m$ terms neglected) because delta functions and derivatives of the delta functions are introduced into the derivatives of the velocity at $\tau = 0$. For example, if the external force is a step function applied at $\tau = 0$

$$F_{ext}(\tau) = \begin{cases} 0 \;, & \tau < 0 \\ F_0 \;, & \tau \geq 0 \end{cases} \tag{8.22}$$

then the solution (8.16) becomes simply

$$\mathcal{V}(\tau) = \frac{F_0\tau_e}{m}\begin{cases} \exp(\tau/\tau_e) \;, & \tau \leq 0 \\ 1 + \tau/\tau_e \;, & \tau \geq 0 \,. \end{cases} \tag{8.23}$$

We see that (8.23) satisfies the equation of motion (8.6) (with the $O(a)/m$ terms neglected) for all τ, whereas setting $\mathcal{V}(\tau) = 0$ for $\tau < 0$ in (8.23) violates the equation of motion by introducing delta and doublet functions in $\mathcal{V}'(\tau)$ and $\mathcal{V}''(\tau)$ at $\tau = 0$. Similarly, a spurious delta function is introduced into $\mathcal{V}''(\tau)$ by differentiating (8.23) and setting the acceleration zero for $\tau < 0$, regardless of the initial velocity.[1] (Note that the condition $\lim_{\tau\to\infty} F_{ext}(\tau) = m\lim_{\tau\to\infty}\mathcal{V}'(\tau) = F_0$ for this step-function external force in (8.16) replaces the asymptotic condition in (8.12).)

Although the noncausal pre-acceleration decays in the past at the rapid rate of $1/(2.718\cdots)$ in the proper time interval τ_e that light takes to travel 4/3 the classical radius of the electron if m and e are the mass and charge of the

[1] In Section 8.2.2 we show that a rigorous derivation of the self force near the nonanalytic points in time of the external force (here at $\tau = 0$) allows for delta-like functions that eliminate the noncausal pre-acceleration to be added to the original equation of motion immediately after $\tau = 0$.

electron, it is disconcerting that the pre-acceleration appears in the solution to the equation of motion because the equation of motion is based on causal (retarded-potential) solutions to Maxwell's equations. It is not surprising that the equation of motion of a charged particle allows homogeneous solutions like the runaway modes, which are not present in Newton's second law of motion for uncharged particles, because the radiation reaction introduces time derivatives of acceleration into the equation of motion. The disturbing feature of the equation of motion is that when the asymptotic condition in (8.12) is applied to eliminate the runaway modes from the inhomogeneous solution, noncausal pre-acceleration cannot be avoided for a solution that remains well-behaved at $t = \tau = 0$, the time the external force is first applied.

The cause of the pre-acceleration solution will be determined by returning to the derivation of the equation of motion of the extended model of the electron. Before doing so, however, let us show that the pre-acceleration is not eliminated by including the higher order terms in the equation of motion ($O(a)/m$ terms in the equation of rectilinear motion (8.6)).

The pre-acceleration solution (8.16) is a solution to (8.6) when the $O(a)/m$ terms are negligible. If the mass m is replaced by $m_{\text{es}} = e^2/(8\pi\epsilon_0 ac^2)$, then one can obtain sufficient conditions for the $O(a)/m_{\text{es}}$ terms to be negligible in the proper frame ($\mathbf{u} = 0$) by returning to the series expansion for the self electromagnetic force (see (D.17) in Appendix D) and noting that the linear terms of $O(a)/m_{\text{es}}$ in the proper-frame version of (8.6) are negligible if the sum

$$\frac{2c}{3a}\sum_{n=3}^{\infty}\left(\frac{-2a}{c}\right)^n \frac{1}{n!}\frac{\mathrm{d}^n\mathbf{u}(t)}{\mathrm{d}t^n} \tag{8.24a}$$

in the proper frame of reference of the charge is negligible compared to the first two terms on the right-hand side of (8.6). For this sum to be negligible, it is sufficient that

$$\frac{2a}{c}\left|\frac{\mathrm{d}^{n+1}\mathbf{u}}{\mathrm{d}t^{n+1}}\right| \ll (n+1)\left|\frac{\mathrm{d}^n\mathbf{u}}{\mathrm{d}t^n}\right|, \quad n = 2, 3, \ldots\ . \tag{8.24b}$$

Moreover, it can be shown that the nonlinear $O(a)/m_{\text{es}}$ terms are negligible compared to the first two terms on the right-hand side of the proper-frame version of (8.6) if, in addition to the conditions in (8.24b), we also have that in the proper reference frame

$$\frac{a}{c}\left|\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t}\right| \ll c \tag{8.24c}$$

and

$$\frac{a}{c}\left|\frac{\mathrm{d}^2\mathbf{u}}{\mathrm{d}t^2}\right| \ll \left|\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t}\right|. \tag{8.24d}$$

The conditions (8.24b)–(8.24d) are sufficient conditions for neglecting the terms in the proper-frame equation of motion beyond the radiation reaction

term. The conditions in (8.24b) and (8.24d) say that the fractional change in the first and higher time derivatives of velocity of the charge in the proper frame is small during the time interval it takes light to traverse the charge distribution. The condition (8.24c) says that the velocity of the charge changes by a small fraction of the speed of light in the time interval light takes to traverse the charge distribution.

The pre-acceleration solution (8.16) (with m_{es} replacing m) behaves as $\exp[3c\tau/(4a)]$ for $\tau < 0$ and thus does not satisfy the condition that (8.24a) is negligible compared to the radiation reaction term because

$$\frac{\mathrm{d}^{n+1}}{\mathrm{d}\tau^{n+1}}\exp[3c\tau/(4a)] = \frac{3c}{4a}\frac{\mathrm{d}^n}{\mathrm{d}\tau^n}\exp[3c\tau/(4a)]. \tag{8.25}$$

(For the sake of argument, we are assuming that the conditions on the derivatives of the velocity function $\mathcal{V}$ for neglecting the $O(a)$ terms in (8.6) are at least as strong as those on the derivatives of the proper-frame velocity in (8.24b)–(8.24d).) Moreover, the nonlinear $O(a)/m_{\text{es}}$ terms in (8.6) for this pre-acceleration solution have $(\exp[3c\tau/(4a)])^n$ dependence with $n \geq 2$ and thus the nonlinear $O(a)/m_{\text{es}}$ terms cannot cancel the linear $O(a)/m_{\text{es}}$ terms in this pre-acceleration solution. Thus, one could initially conclude that the pre-acceleration solution in (8.16) may not be a valid solution to the equation of motion (8.6) for the charged insulator of radius a when the $O(a)/m_{\text{es}}$ terms are retained.

Unfortunately, when the $O(a)/m_{\text{es}}$ terms are retained, the pre-acceleration (runaway solution for $\tau < 0$) is not eliminated, just the time dependence of the pre-acceleration is altered. Specifically, the analyses of Herglotz [39] and Wildermuth [40] show that runaway solutions to the linearized, homogeneous form of our equation of motion (7.1) exist for all time, so that pre-acceleration exists for $t < 0$, regardless of how many linear higher order terms are included in the linearized equation of motion [41]. (These results of Herglotz and Wildermuth apply to the charged insulator even when $m_{\text{ins}} \neq 0$ as long as the sum of the bare mass and material mass of the insulator, $M_0 + m_{\text{ins}}$, is less than zero. This condition is met as $a \to 0$ because the bare mass has the negative value of $M_0 = -m_{\text{es}}/3$ and even when the mass of the insulator is not zero, the value of the sum $m_{\text{ins}} - m_{\text{es}}/3 \to -m_{\text{es}}/3$ as $a \to 0$.)

The analyses of Herglotz and Wildermuth are approximate in that they neglect all $O(a)/m_{\text{es}}$ terms involving nonlinear products of the time derivatives of velocity in the proper-frame equation of motion (see Section 8.4). However, the analysis of motion of the two-charge (dumbbell) problem [42], although it neglects the self force of each individual charge, includes nonlinear terms and also exhibits the existence of runaway solutions. Thus, in general, the inclusion of higher order terms in the equation of motion fails to eliminate the pre-acceleration, whether or not the mass is kept at a fixed value as $a \to 0$. This conclusion is also confirmed by the work of Bauer and Dürr [43] on the nonrelativistically rigid model of the extended electron.

8.2.1 Cause of the Pre-Acceleration

The cause of the pre-acceleration can be found by examining the assumptions involved in the derivation of the equation of motion. In Chapters 2 through 5 and the Appendices, the equation of motion was obtained for the extended model of the electron as a spherical charged insulator of radius a. To simplify the discussion, concentrate on the force equation of motion (7.12a) in the proper frame of reference of the charged insulator

$$\mathbf{F}_{\text{ext}}(t) = m\dot{\mathbf{u}} - \frac{e^2}{6\pi\epsilon_0 c^3}\ddot{\mathbf{u}} + \mathbf{O}(a)\,. \tag{8.26}$$

As explained in Section 5.1, the rest mass, or coefficient of the $\dot{\mathbf{u}}$ term in (8.26), is determined ultimately, not from the electromagnetic self force, but from the relativistic generalization of Newton's second law of motion and the Einstein mass-energy relation. In particular, the rest mass must equal the total energy of formation of the charged insulator divided by c^2.

The $\ddot{\mathbf{u}}$ and higher order reaction terms in the equation of motion (8.26) are determined from the derivation of the self electromagnetic force. This derivation, outlined in Appendix A, depends upon expanding the position, velocity, and acceleration of each element of charge at the retarded time ($t' = t - R'(t')/c$) in a Taylor series about the present time (t). For example, the velocity of the element of charge at $\mathbf{r}'$ in the proper frame is expanded as

$$\mathbf{u}(\mathbf{r}',t') = \mathbf{u}\left(\mathbf{r}', t - \frac{R'(t')}{c}\right) = -\dot{\mathbf{u}}(\mathbf{r}',t)\frac{R'(t')}{c} + \ddot{\mathbf{u}}(\mathbf{r}',t)\frac{R'^2(t')}{2c^2} + \cdots \tag{8.27a}$$

and similarly for $\dot{\mathbf{u}}(\mathbf{r}',t')$, where the distance $R'(t')$ has the Taylor series expansion

$$R'(t') = R(t) - \frac{R(t)\mathbf{R}\cdot\dot{\mathbf{u}}(\mathbf{r}',t)}{2c^2} + \cdots\,. \tag{8.27b}$$

These Taylor series expansions are valid *provided the velocity function* $\mathbf{u}(\mathbf{r}',t')$ *is an analytic function of complex time* t' *for*

$$|t'-t| \le [R'(t')/c]_{\max}\,. \tag{8.28}$$

(Analyticity of $\mathbf{u}(\mathbf{r}',t')$ implies the analyticity of $R'(t')$ and $\dot{\mathbf{u}}(\mathbf{r}',t')$ through integration and differentiation, respectively.) For the self-force calculation in the proper frame of reference, $R'(t')$ does not exceed a value of about $2a$ (assuming the velocity does not change rapidly between t' and t; in other words, assuming the velocity change is a small fraction of the speed of light during the time it takes light to traverse the charge distribution — a condition implied by (8.24c)), and thus (8.28) can be rewritten as

$$|t'-t| \le \Delta t_a \tag{8.29a}$$

where

$$\Delta t_a \approx \frac{2a}{c}\,. \tag{8.29b}$$

Even if the magnitude of the velocity change (Δu) during the time $2a/c$ is a significant fraction of the speed of light, (8.29b) can be replaced by

$$\Delta t_a \approx \frac{2a}{c}\frac{1}{1-\Delta u/c} \tag{8.29c}$$

which still approaches zero as $a \to 0$ for $\Delta u < c$.

Assume that the external force $\mathbf{F}_{\text{ext}}(t)$ *has at most a finite jump discontinuity*[2] *across* $t = 0$ *and that after the external force is applied at* $t = 0$, *the external force* $\mathbf{F}_{\text{ext}}(t)$, *and the velocity of the charge* $\mathbf{u}(\mathbf{r},t)$, *is an analytic function of* t *in a* Δt_a*-wide strip of the complex* t *plane about the real* t *axis for* $t > 0$. Then the Taylor series expansions hold for $t > \Delta t_a$, the equation of motion in (8.26) is valid for $t > \Delta t_a$, and with the $\mathbf{O}(a)$ terms neglected it implies that $\mathbf{u}(t)$ is an analytic function of time for $t > \Delta t_a$ if and only if $\mathbf{F}_{\text{ext}}(t)$ is an analytic function of time for $t > \Delta t_a$.[3] However, since the external force and velocity are zero for $t < 0$, the velocity cannot be an analytic function of t' for t' in the interval of (8.29a) if t is less than Δt_a (and $t \ge 0$). Also, note that because $t' \le t$ and $\mathbf{u}(\mathbf{r},t)$ with all its time derivatives are zero for $t < 0$, the left- and right-hand sides of the equations in (8.27) are equal for $t < 0$. ***In all then, the Taylor series expansions in*** (8.27) ***are invalid during (and only during) the time interval***

$$0 \le t \le \Delta t_a\,. \tag{8.30}$$

Consequently, the following expression obtained by inserting equation (A.10) into (A.1) of Appendix A for the self electromagnetic force in the proper frame is ***not valid*** during this short transition time interval (8.30) in which the external force is first applied

$$\mathbf{F}_{\text{em}}(t) = \frac{1}{4\pi\epsilon_0}\iint\limits_{\text{charge}}\left\{\frac{\hat{\mathbf{R}}}{R^2} + \frac{1}{2c^2R}\left[\frac{\mathbf{r}'\cdot\dot{\mathbf{u}}}{c^2} - 1\right]\left[(\hat{\mathbf{R}}\cdot\dot{\mathbf{u}})\hat{\mathbf{R}} + \dot{\mathbf{u}}\right]\right.$$
$$\left. + \frac{3}{8}\frac{\hat{\mathbf{R}}}{c^4}\left[(\hat{\mathbf{R}}\cdot\dot{\mathbf{u}})^2 - |\dot{\mathbf{u}}|^2\right] + \frac{3(\hat{\mathbf{R}}\cdot\dot{\mathbf{u}})\dot{\mathbf{u}}}{4c^4} + \frac{2\ddot{\mathbf{u}}}{3c^3} + \mathbf{O}(R)\right\} de'de, \quad u = 0. \tag{8.31}$$

[2] A finite external force ensures that the external force is integrable in time and that the acceleration outside the transition interval in (8.30) is finite; see the last paragraph of Section 8.2.2.

[3] If a solution to the differential equation $\mathbf{F}_{\text{ext}}(t) = m\dot{\mathbf{u}}(t) - [e^2/(6\pi\epsilon_0 c^3)]\ddot{\mathbf{u}}(t)$ (which holds for $t > \Delta t_a$) exists in a complex neighborhood of the real time t, then $\dot{\mathbf{u}}(t)$ and $\ddot{\mathbf{u}}(t)$ exist in this complex neighborhood of real t, and thus $\mathbf{u}(t)$ is an analytic function of time if $\mathbf{F}_{\text{ext}}(t)$ is an analytic function of time at time t. If $\mathbf{F}_{\text{ext}}(t)$ is not analytic at some time t, then $\mathbf{u}(t)$ cannot be analytic at time t because if it were the right-hand side of the differential equation would be analytic at time t while the left-hand side would not.

One can see directly from equation (A.2) of Appendix A how the integral in (8.31) should be modified for $t < R'(t')/c$. Specifically, for $t < R'(t')/c$, the functions $\mathbf{u}(\mathbf{r}', t')$ and $\dot{\mathbf{u}}(\mathbf{r}', t')$ are identically zero so that $\mathbf{dE}(\mathbf{r}, t)$ in (A.2) reduces to $de'\hat{\mathbf{R}}'/(4\pi\epsilon_0 R'^2)$. Thus, a simple modification to the integrand of (8.31), shown in the following equation (8.32), produces an expression for the self electromagnetic force that is valid for all time in the proper frame

$$\mathbf{F}_{\rm em}(t) = \frac{1}{4\pi\epsilon_0}\iint_{\rm charge}\left\{\frac{\hat{\mathbf{R}}'}{R'^2} + h\left(t - \frac{R'(t')}{c}\right)\left[\frac{\hat{\mathbf{R}}}{R^2} - \frac{\hat{\mathbf{R}}'}{R'^2} + \frac{1}{2c^2R}\right.\right.$$
$$\cdot\left[\frac{\mathbf{r}'\cdot\dot{\mathbf{u}}}{c^2} - 1\right]\left[(\hat{\mathbf{R}}\cdot\dot{\mathbf{u}})\hat{\mathbf{R}} + \dot{\mathbf{u}}\right] + \frac{3}{8}\frac{\hat{\mathbf{R}}}{c^4}\left[(\hat{\mathbf{R}}\cdot\dot{\mathbf{u}})^2 - |\dot{\mathbf{u}}|^2\right]$$
$$\left.\left.+\frac{3(\hat{\mathbf{R}}\cdot\dot{\mathbf{u}})\dot{\mathbf{u}}}{4c^4} + \frac{2\ddot{\mathbf{u}}}{3c^3} + \mathbf{O}(R)\right]\right\} de'de, \quad u = 0 \tag{8.32}$$

where $h(t)$ in (8.32) is the unit step function defined by

$$h(t) = \begin{cases} 0\,, & t < 0 \\ 1\,, & t \geq 0\,. \end{cases} \tag{8.33}$$

Although the step function appearing in (8.32) represents a minor modification, it prevents the closed-form evaluation of the double integration in (8.32) during the time interval (8.30). Nonetheless, for $t < 0$, the step function $h(t - R'(t')/c) = 0$ and $\mathbf{R}'(t') = \mathbf{R}$. Therefore, since $\int\int_{\rm charge}\mathbf{R}/R^2 de'de = 0$, the value of the integral in (8.32) is zero for $t < 0$, and for $t > \Delta t_a$, the integral yields the usual expression (A.11) for the self electromagnetic force in the proper frame.

During the transition time interval in (8.30) after the finite external force is first applied, there appears to be no simple way to evaluate the double integral in (8.32) to determine the self electromagnetic force because the time dependence of $\mathbf{u}(\mathbf{r}', t')$, and thus $R'(t')$ in $h[t - R'(t')/c]$, is unknown a priori.[4] Nonetheless, the difference between the correct expression for the self electromagnetic force in (8.32) and the original expression in (8.31) implies that a transition force must be added to the original self electromagnetic force in (A.11) during the time interval in (8.30). In the following section, we show that such a transition force can be chosen to eliminate the pre-acceleration from the equation of motion.

Abraham also realized that the traditional series representation of the self electromagnetic force became invalid for "discontinuous movements" of the charge. In [3, sec. 23] he states, "These two forces [electromagnetic mass term plus radiation reaction] are basically nothing other than the first two terms of a progression which increases in accordance with increasing powers of the electron's radius a. ... Because the internal force is determined by the velocity and acceleration existing in a finite interval preceding the affected point in time, such a progression is always possible when the movement is continuous and its velocity is less than the speed of light. ... The series will converge more poorly the closer the movement approaches a discontinuous movement and the velocity approaches the speed of light . ... It fails completely for discontinuous movements. ... Here, other methods must be employed when computing the internal force." Abraham goes on to derive the radiated energy and momentum of a charged sphere with discontinuous velocity [3, sec. 25], [44]. He also derives Sommerfeld's general integral formulas for the internal electromagnetic force [45]. Neither he nor Sommerfeld, however, evaluates or interprets these general integrals except to show they yield a null result for a charged sphere moving with constant velocity.

Schott [46], [16, p. 283] also concludes that "the approximation [used to obtain the Lorentz-Abraham equation of motion] fails during an interval of time, which is comparable with the time required by an electromagnetic wave to pass across the electron and includes the instant at which the discontinuity occurs." More recently, Valentini [47] observes that "the usual derivations of the Lorentz-[Abraham-]Dirac equation are only valid at times such that [the position of and force applied to the particle] are analytic functions [of time]," and that nonanalyticity of these functions is responsible for the noncausal pre-acceleration in the solution to the Lorentz-Abraham-Dirac equation of motion.

[4] It was argued in the first edition of the book that the self electromagnetic force in (8.32) approached zero as $\mathbf{O}(t^2)$ from positive values of t, but this argument assumed that $\mathbf{u}(\mathbf{r}', t')$ was slowly varying in the transition interval (8.30), and this is not necessarily the case.

8.2.2 Elimination of the Pre-Acceleration

We have shown that (8.32) rather than (8.31) is the formal expression for the self electromagnetic force on the spherical shell of charge that holds for all time t in the proper inertial reference frame and reduces to (8.31) for $t > \Delta t_a$ and $t < 0$ under the assumptions that the externally applied force is zero for $t < 0$, has at most a finite jump discontinuity across $t = 0$, and is an analytic function of complex t about the real t axis for $t > 0$. As pointed out in the previous subsection, however, the double integration over the charge in (8.32) cannot, in general, be evaluated in closed form during the time interval $0 \leq t \leq \Delta t_a$ because of the presence of the step function $h(t - R'(t')/c)$ in the integrand of (8.32) and because of the unknown behavior of $R'(t')$ during this time interval. Nonetheless, (8.32) can still be re-expressed as a differential equation that leads to an equation of motion free of pre-acceleration.

To do this, begin with the evaluation of (8.32) for $t > \Delta t_a$ (or $t < 0$). Then (8.32) equals (8.31) which is simply the original self force evaluated in (A.11), that is

$$\mathbf{F}_{\rm em}(t) = -\frac{e^2}{6\pi\epsilon_0 ac^2}\dot{\mathbf{u}} + \frac{e^2}{6\pi\epsilon_0 c^3}\ddot{\mathbf{u}} + \mathbf{O}(a)\,, \quad t > \Delta t_a,\ t < 0\,. \tag{8.34}$$

Although (8.32) cannot be evaluated in closed form in the interval $0 \le t \le \Delta t_a$, we can see that the step function in (8.32) changes the coefficients of $\dot{\mathbf{u}}$ and $\ddot{\mathbf{u}}$ in the equation of motion during this time interval. Thus, we can formally express the self electromagnetic force for $t \ge 0$ as

$$\mathbf{F}_{\rm em}(t) = -\frac{e^2}{6\pi\epsilon_0 a c^2}\eta_1(t)\dot{\mathbf{u}} + \frac{e^2}{6\pi\epsilon_0 c^3}\eta_2(t)\ddot{\mathbf{u}} + \mathbf{O}(a) \tag{8.35}$$

where the functions $\eta_1(t)$ and $\eta_2(t)$ equal zero at $t = 0$, equal 1 for $t > \Delta t_a$ and $t < 0$, and have some unknown, possibly rapid variation that depends on a and the behavior of $\mathbf{u}(\mathbf{r}', t')$ (and thus $R'(t')$) in the interval $0 \le t \le \Delta t_a$. Following the procedure used in Section 5.1, we assume the relativistic version of Newton's second law of motion requires that the external force applied to a charged particle should equal, apart from the radiation reaction and the forces of order a, the time derivative of momentum of the particle. That is,

$$\mathbf{F}_{\rm ext}(t) = m\dot{\mathbf{u}}(t) - \eta_2(t)\frac{e^2}{6\pi\epsilon_0 c^3}\ddot{\mathbf{u}}(t) + \mathbf{O}(a) \tag{8.36}$$

where $m = m_{\rm es} + m_{\rm ins}$ is the rest mass of the particle (charge plus insulator) and the required bare mass term is no longer $(m_{\rm es} - m_{\rm em})\dot{\mathbf{u}}$ but $[m_{\rm es} - \eta_1(t) m_{\rm em}]\dot{\mathbf{u}}$.[5] Since the functional time dependence of $\eta_2(t)$ is unknown, divide the equation in (8.36) by $\eta_2(t)$ to get[6]

$$\frac{\mathbf{F}_{\rm ext}(t)}{\eta_2(t)} = \frac{m}{\eta_2(t)}\dot{\mathbf{u}}(t) - \frac{e^2}{6\pi\epsilon_0 c^3}\ddot{\mathbf{u}}(t) + \mathbf{O}(a)\,. \tag{8.39}$$

We can rewrite this equation as

$$\mathbf{F}_{\rm ext}(t) + \mathbf{f}_a(t) = m\dot{\mathbf{u}}(t) - \frac{e^2}{6\pi\epsilon_0 c^3}\ddot{\mathbf{u}}(t) + \mathbf{O}(a) \tag{8.40}$$

where

$$\mathbf{f}_a(t) = \begin{cases} 0 & , \quad t < 0 \\ [\mathbf{F}_{\rm ext}(t) - m\dot{\mathbf{u}}(t)]\,[1/\eta_2(t) - 1]\,, & \quad t \ge 0\,. \end{cases} \tag{8.41}$$

Although the exact time dependence of the transition force $\mathbf{f}_a(t)$ is unknown, it vanishes outside the interval given in (8.30) and it may be a complicated function of time whose contribution to the equation of motion in (8.40) does not necessarily approach zero as $a \to 0$. In other words, $\mathbf{f}_a(t)$ may contain delta functions and their time derivatives as $a \to 0$. (For example, if the actual $\mathbf{u}(t)$ has a jump across $t = \Delta t_a$ as $a \to 0$ and $\mathbf{F}_{\rm ext}(t)$ is finite, equation (8.40) implies that $\mathbf{f}_a(t) + \mathbf{O}(a)$ contains a doublet function across $t = \Delta t_a$.) In brief, a causal equation of motion with the correct rest mass of the charged particle requires a transition force $\mathbf{f}_a(t)$ to remove the pre-acceleration and a modified bare mass term $[m_{\rm es} - \eta_1(t) m_{\rm em}]\dot{\mathbf{u}}$ to produce the correct rest mass.

The modified proper-frame equation of motion in (8.40) can be rewritten

$$\frac{\mathbf{F}_{\rm ext}(t) + \mathbf{f}_a(t)}{m} = \dot{\mathbf{u}} - \tau_e\ddot{\mathbf{u}} + \frac{\mathbf{O}(a)}{m} \tag{8.42}$$

where the time constant τ_e is the same as in (8.7). In an arbitrary inertial reference frame, and in four-vector notation, (8.42) generalizes to

$$\frac{F^i_{\rm ext} + f^i_a}{mc^2} = \frac{{\rm d}u^i}{{\rm d}s} - \tau_e c\left(\frac{{\rm d}^2u^i}{{\rm d}s^2} + u^i\frac{{\rm d}u_j}{{\rm d}s}\frac{{\rm d}u^j}{{\rm d}s}\right) + \frac{O(a)}{m} \tag{8.43}$$

with the four-vector force given by

$$f^i_a = \gamma\,(\mathbf{f}_a, \mathbf{f}_a\cdot\mathbf{u}/c) \tag{8.44}$$

[5] The corrected equation of motion (8.36) can thus be re-expressed as

$$\mathbf{F}_{\rm ext}(t) + \mathbf{F}_{\rm em}(t) = [M_0 + m_{\rm ins} + (1 - \eta_1(t))\, m_{\rm em}]\dot{\mathbf{u}} \tag{8.37}$$

where, as in Chapter 5, $M_0 = m_{\rm es} - m_{\rm em}$. If one postulates a priori an equation of motion for a classical charged particle with specified rest mass m given in the proper frame by $\mathbf{F}_{\rm ext}(t) + \mathbf{F}_{\rm em}(t) = (M_0 + m_{\rm ins})\dot{\mathbf{u}} = M\dot{\mathbf{u}}$ without the $(1 - \eta_1(t))\, m_{\rm em}\dot{\mathbf{u}}$ term, an initial-value solution free of pre-acceleration and runaway behavior does not generally exist for small values of a [43]. In other words, for small enough a and a given rest mass m, causality is violated if one assumes in an arbitrary inertial frame that the equation of motion has the form

$$\mathbf{F}_{\rm ext}(t) + \int_{\rm charge} \rho(\mathbf{r}, t)[\mathbf{E}(\mathbf{r}, t) + \mathbf{u}(\mathbf{r}, t)\times\mathbf{B}(\mathbf{r}, t)]{\rm d}V = M\frac{{\rm d}(\gamma\mathbf{u})}{{\rm d}t} \tag{8.38}$$

regardless of the value chosen for the constant "material mass" M or the velocity $\mathbf{u}(\mathbf{r}, t)$ [38], [49]. This result does not imply that Maxwell's equations with Lorentz forces are inconsistent, only that the postulated equation of motion in (8.38) (which does not include the $(1 - \eta_1(t))\, m_{\rm em}\dot{\mathbf{u}}$ term) is flawed for the classical model of an accelerating charged particle with a given charge e and rest mass m. *The more fundamental equation of motion that is used in* (8.36) *and its relativistic generalization* (8.43) *equates the sum of the externally applied force and the radiation reaction part of the self force to the rest mass of the particle times the acceleration.* As will be shown, this more fundamental equation of motion, unlike (8.38), allows for a well-behaved causal initial-value solution free of runaway motion.

[6] In the first edition of this book, we left the corrected equation of motion in the form of (8.36), in which the function $\eta_2(t)$ was assumed to vary monotonically from a value of zero at $t = 0$ to a value of unity at $t = \Delta t_a$. Although this simplified approximation to $\eta_2(t)$ eliminated the pre-acceleration, a similar approximation to the corrected equation of motion to eliminate pre-deceleration can lead to a violation of energy conservation, as shown by Baylis and Huschilt [48]. The general form (8.40) of the corrected equation of motion allows conservation of momentum and energy while eliminating both pre-acceleration and pre-deceleration for the extended charged insulator model, as demonstrated in Section 8.2.5.

such that f_a^i is zero outside the interval $(0 \le s \le \Delta s_a \approx c\Delta t_a)$ and $f_a^i u_i = 0$. (Note that $\Delta s_a \approx c\Delta t_a$ even if the initial velocity is not zero because Δs_a is an invariant and it equals $c\Delta t_a$ in the proper inertial reference frame.)

The modified equation of motion (8.43) can be rewritten in three-vector notation from (7.12) as

$$\frac{\mathbf{F}_{\text{ext}} + \mathbf{f}_a}{m} = \frac{\mathrm{d}(\gamma \mathbf{u})}{\mathrm{d}t} - \tau_e \left\{ \frac{\mathrm{d}}{\mathrm{d}t}\left[\gamma \frac{\mathrm{d}}{\mathrm{d}t}(\gamma \mathbf{u})\right] - \frac{\gamma^4}{c^2}\left[|\dot{\mathbf{u}}|^2 + \frac{\gamma^2}{c^2}(\mathbf{u}\cdot\dot{\mathbf{u}})^2\right]\mathbf{u}\right\} + \frac{\mathbf{O}(a)}{m} \tag{8.45a}$$

and

$$\frac{(\mathbf{F}_{\text{ext}} + \mathbf{f}_a)\cdot \mathbf{u}}{mc^2} = \frac{\mathrm{d}\gamma}{\mathrm{d}t} - \tau_e \left\{ \frac{\mathrm{d}}{\mathrm{d}t}\left(\gamma \frac{\mathrm{d}\gamma}{\mathrm{d}t}\right) - \frac{\gamma^4}{c^2}\left[|\dot{\mathbf{u}}|^2 + \frac{\gamma^2}{c^2}(\mathbf{u}\cdot\dot{\mathbf{u}})^2\right]\right\} + \frac{O(a)}{m} \tag{8.45b}$$

in which we have used the relations $\gamma \mathrm{d}(\gamma \mathbf{u})/\mathrm{d}t = \gamma^2\dot{\mathbf{u}} + \gamma^4(\mathbf{u}\cdot\dot{\mathbf{u}})\mathbf{u}/c^2$ and $\gamma \mathrm{d}\gamma/\mathrm{d}t = \gamma^4 \mathbf{u}\cdot\dot{\mathbf{u}}/c^2$.

An extremely simple way to arrive at (8.43) (or (8.45)) is to observe that outside the transition interval this equation holds rigorously without the transition function (under the assumption that the sum of the external force and the radiation reaction part of the self force equals the rest mass times the relativistic acceleration). Within the transition interval, $\mathrm{d}u^i/\mathrm{d}s - \tau_e c[\mathrm{d}^2u^i/\mathrm{d}s^2 + u^i(\mathrm{d}u_j/\mathrm{d}s)(\mathrm{d}u^j/\mathrm{d}s)] + O(a)/m - F^i_{\text{ext}}/(mc^2)$ has to equal some unknown function of time that can simply be denoted by $f_a^i/(mc^2)$.

A formal solution to (8.45) with the $\mathbf{O}(a)/m$ terms neglected can be obtained for $\dot{\mathbf{u}}(t)$ by replacing $\mathbf{F}_{\text{ext}}$ in (8.21) with $(\mathbf{F}_{\text{ext}} + \mathbf{f}_a)$ to get

$$\dot{\mathbf{u}}(t) = \frac{1}{m\tau_e\gamma^3(t)} \int_t^\infty \gamma(t')\Big\{\mathbf{F}_{\text{ext}}(t') + \mathbf{f}_a(t') - [(\mathbf{F}_{\text{ext}}(t') + \mathbf{f}_a(t'))\cdot\mathbf{u}(t')]\,\mathbf{u}(t')/c^2\Big\} \exp\left[-(1/\tau_e)\int_t^{t'} \gamma^{-1}(t'')\mathrm{d}t''\right]\mathrm{d}t' \tag{8.46a}$$

or, equivalently, with the change of integration variables $\mathrm{d}t'/\gamma = \mathrm{d}\tau'$ and $\mathrm{d}t''/\gamma = \mathrm{d}\tau''$

$$\dot{\mathbf{u}}(\tau) = \frac{1}{m\tau_e\gamma^3(\tau)} \int_\tau^\infty \gamma^2(\tau')\Big\{\mathbf{F}_{\text{ext}}(\tau') + \mathbf{f}_a(\tau') - [(\mathbf{F}_{\text{ext}}(\tau') + \mathbf{f}_a(\tau'))\cdot\mathbf{u}(\tau')]\,\mathbf{u}(\tau')/c^2\Big\} \exp[-(\tau'-\tau)/\tau_e]\mathrm{d}\tau' \tag{8.46b}$$

which has to be zero for $t < 0$ under the assumption that the exact classical solution is causal and the externally applied force $\mathbf{F}_{\text{ext}}$ is zero for $t < 0$. As is customarily done with $\mathbf{u}$ and $\mathbf{F}_{\text{ext}}$, the same symbol is used for $\mathbf{f}_a$ in (8.42)–(8.46) independently of its inertial reference frame or its dependent variable s, t, or τ.

The necessity of a transition force $\mathbf{f}_a(t)$, which contributes only during the short time it takes light to travel across the charged sphere, can be understood physically by considering two differential elements of charge at either end of the charge distribution. These two elements are at rest separated by a distance $2a$. When the external force is first applied, each of these charge elements accelerates and radiates. However, each element of charge does not experience the radiation from the other until approximately the time it takes light to travel between them . Thus, there will be a time delay in the radiation reaction force of about $2a/c$ between these two elements of charge separated by $2a$ (assuming here that the velocity of the charge does not change by an appreciable fraction of c during this time interval). For the other combinations of charge elements separated by a distance less than $2a$, the time delay of the radiation will be proportionately less. The double integration over the entire sphere of charge elements $\mathrm{d}e$ and $\mathrm{d}e'$ produces a continuous addition of radiation forces with time delays varying from zero to about $2a/c$. Moreover, the trailing end of the charge first experiences the change in field from the leading end a short time before the leading end of the charge first experiences the change in field from the trailing end. In all, the transition force $\mathbf{f}_a(t)$ appears between the time the external force is first applied and the time $\Delta t_a \approx 2a/c$ after which the self electromagnetic force can be expressed entirely in terms of the present velocity and its time derivatives. The functional time dependence of this transition force must also account for the addition of the bare mass in order to maintain causality in the solution to the equation of motion.

The transition force $\mathbf{f}_a(t)$ in the equation of motion allows solutions to the equation of motion that satisfy initial conditions on velocity and that are free of pre-acceleration. Although the exact dependence on time of the function $\mathbf{f}_a(t)$ is unknown, it vanishes outside the interval in (8.30), that is

$$\mathbf{f}_a(t) = 0\,, \qquad t \notin [0, \Delta t_a]. \tag{8.47}$$

The requirement that the function $\mathbf{f}_a(t)$ eliminates the pre-acceleration further restricts the time dependence of $\mathbf{f}_a(t)$. The necessary and sufficient condition for there to be no pre-acceleration can be expressed formally by demanding that $\dot{\mathbf{u}}(t) = 0$ for $t < 0$ in the solution (8.46) to obtain from (8.46b)

$$\int_0^{\Delta t_a} \gamma^2(\tau')\{\mathbf{f}_a(\tau') - [\mathbf{f}_a(\tau')\cdot\mathbf{u}(\tau')]\,\mathbf{u}(\tau')\}\mathrm{e}^{-\tau'/\tau_e}\mathrm{d}\tau' = -\int_0^\infty \gamma^2(\tau')\{\mathbf{F}_{\text{ext}}(\tau') - [\mathbf{F}_{\text{ext}}(\tau')\cdot\mathbf{u}(\tau')]\,\mathbf{u}(\tau')\}\mathrm{e}^{-\tau'/\tau_e}\mathrm{d}\tau'. \tag{8.48}$$

The important feature of (8.48) that allows this condition to be satisfied is that the τ dependence $(\mathrm{e}^{\tau/\tau_e})$ has canceled to leave the left- and right-hand sides of (8.48) independent of τ (and thus independent of t). Moreover, if

$\mathbf{u}(\tau)$ and $\mathbf{F}_{\text{ext}}(\tau)$ are expandable in a power series for $\tau > 0$, the function $\gamma^2(\tau')\{\mathbf{F}_{\text{ext}}(\tau') - [\mathbf{F}_{\text{ext}}(\tau') \cdot \mathbf{u}(\tau')]\mathbf{u}(\tau')\}$ in the integrand on the right-hand side of (8.48) can be expanded in a power series about $\tau' = 0^+$. Term by term integration of this power series then shows that the right-hand side of (8.48) can be expressed solely in terms of the values of the external force and velocity and their time derivatives at $\tau' = 0^+$. This re-expression of the right-hand side of (8.48) is philosophically appealing because it removes the apparent dependence of the right-hand side of (8.48) on values of the external force later than $\tau' = 0^+$ ($t = 0^+$) and thus later than $\tau' = \Delta t_a$ after which time $\mathbf{f}_a(\tau')$ is zero.

To determine a more explicit form of the condition in (8.48) that applies to rectilinear motion, rewrite (8.43) for rectilinear motion in the form of (8.6) by means of the change of variables defined at the beginning of Chapter 8 to get

$$\frac{F_{\text{ext}}(\tau) + f_a(\tau)}{m} = \mathcal{V}'(\tau) - \tau_e \mathcal{V}''(\tau) + \frac{O(a)}{m}. \tag{8.49}$$

As usual, $F_{\text{ext}}(\tau)$, and now $f_a(\tau)$, are compact notations for $F_{\text{ext}}[t(\tau)]$ and $f_a[t(\tau)]$, and it is assumed that $\tau = 0$ when $t = 0$. As in (8.13) and (8.16), the solution to (8.49) for all τ under the asymptotic condition in (8.12) and with the $O(a)/m$ terms neglected is given by

$$\mathcal{V}'(\tau) = \frac{1}{m\tau_e} \int_\tau^\infty [F_{\text{ext}}(\tau') + f_a(\tau')] \exp[-(\tau' - \tau)/\tau_e] \mathrm{d}\tau' \tag{8.50a}$$

$$\begin{aligned} \mathcal{V}(\tau) &= \frac{1}{m} \int_\tau^\infty [F_{\text{ext}}(\tau') + f_a(\tau')] \exp[-(\tau' - \tau)/\tau_e] \mathrm{d}\tau' \\ &\qquad + \frac{1}{m} \int_0^\tau [F_{\text{ext}}(\tau') + f_a(\tau')] \mathrm{d}\tau' \\ &= \tau_e \mathcal{V}'(\tau) + \frac{1}{m} \int_0^\tau [F_{\text{ext}}(\tau') + f_a(\tau')] \mathrm{d}\tau' \end{aligned} \tag{8.50b}$$

where in (8.50b) it is assumed that the velocity is zero before the external force is applied. Since $F_{\text{ext}}(\tau)$ and velocity are zero for $\tau < 0$, the velocity and acceleration functions must be zero for $\tau < 0$. Thus, (8.50) imply

$$\int_0^\infty [F_{\text{ext}}(\tau') + f_a(\tau')] \mathrm{e}^{-\tau'/\tau_e} \mathrm{d}\tau' = 0 \tag{8.51a}$$

or

$$\int_0^{\Delta t_a} f_a(\tau') \mathrm{e}^{-\tau'/\tau_e} \mathrm{d}\tau' = -\int_0^\infty F_{\text{ext}}(\tau') \mathrm{e}^{-\tau'/\tau_e} \mathrm{d}\tau' \tag{8.51b}$$

a result that can also be gotten directly from (8.48) by noting that the integrands in (8.48) contain the factor $(1 - u^2/c^2) = 1/\gamma^2$ for rectilinear motion. The integrals of $f_a(\tau)$ have the finite limits given in (8.48) and (8.51b) because $f_a(\tau)$ is zero outside the interval $0 \le \tau \le \Delta t_a$.

If there are more than one nonanalytic point in the otherwise analytic external force function $F_{\text{ext}}(\tau)$, for example, at $\tau = \tau_1 = 0$ and at $\tau = \tau_2$, rewrite $F_{\text{ext}}(\tau)$ as

$$F_{\text{ext}}(\tau) = F_1(\tau) + h(\tau - \tau_2)[F_{\text{ext}}(\tau) - F_1(\tau)] \tag{8.52}$$

in which $h(\tau)$ is the unit step function defined in (8.33). The function $F_1(\tau)$ is defined as follows: $F_1(\tau) = F_{\text{ext}}(\tau)$ for $\tau < \tau_2$ ($F_{\text{ext}}(\tau) = 0$, $\tau < 0$) and, in addition, $F_1(\tau)$ is the analytic continuation in the complex τ plane about the real τ axis of $F_{\text{ext}}(\tau)$ from $\tau < \tau_2$ to $\tau \ge \tau_2$. Thus, $F_1(\tau)$ is an analytic function of complex τ about the real τ axis for $\tau > \tau_1 = 0$. Moreover, the function in the square bracket, $F_{\text{ext}}(\tau) - F_1(\tau)$, is an analytic function of complex τ about the real τ axis for all $\tau > \tau_2$.

There will now be two transition functions, that is, $f_a(\tau) = f_{a1}(\tau) + f_{a2}(\tau)$, where the first, $f_{a1}(\tau)$, is nonzero only in the interval $[0, \Delta t_a]$ and the second, $f_{a2}(\tau)$, is nonzero only in the interval $[\tau_2, \tau_2 + \Delta t_a]$. Then the solutions for $\mathcal{V}'(\tau)$ and $\mathcal{V}(\tau)$ in (8.50) become

$$\begin{aligned} \mathcal{V}'(\tau) &= \frac{1}{m\tau_e} \int_\tau^\infty [F_1(\tau') + f_{a1}(\tau')] \exp[-(\tau' - \tau)/\tau_e] \mathrm{d}\tau' \\ &+ \frac{1}{m\tau_e} \int_\tau^\infty h(\tau' - \tau_2)[F_{\text{ext}}(\tau') - F_1(\tau') + f_{a2}(\tau')] \exp[-(\tau' - \tau)/\tau_e] \mathrm{d}\tau' \end{aligned} \tag{8.53a}$$

$$\mathcal{V}(\tau) = \tau_e \mathcal{V}'(\tau) + \frac{1}{m} \int_0^\tau [F_{\text{ext}}(\tau') + f_a(\tau')] \mathrm{d}\tau', \quad \mathcal{V}(0) = 0. \tag{8.53b}$$

For there to be no pre-acceleration before $\tau = \tau_1 = 0$ and no pre-acceleration (or pre-deceleration) before $\tau = \tau_2$, the transition functions $f_{a1}(\tau)$ and $f_{a2}(\tau)$ must satisfy

$$\int_0^\infty [F_1(\tau') + f_{a1}(\tau')] \mathrm{e}^{-\tau'/\tau_e} \mathrm{d}\tau' = 0 \tag{8.54a}$$

and

$$\int_{\tau_2}^\infty [F_{\text{ext}}(\tau') - F_1(\tau') + f_{a2}(\tau')] \mathrm{e}^{-\tau'/\tau_e} \mathrm{d}\tau' = 0 \tag{8.54b}$$

or, equivalently

$$\int_0^{\Delta t_a} f_{a1}(\tau)e^{-\tau/\tau_e}d\tau = -\int_0^{\infty} F_1(\tau)e^{-\tau/\tau_e}d\tau \tag{8.55a}$$

and

$$\int_0^{\Delta t_a} f_{a2}(\tau+\tau_2)e^{-\tau/\tau_e}d\tau = -\int_0^{\infty}[F_{\rm ext}(\tau+\tau_2) - F_1(\tau+\tau_2)]e^{-\tau/\tau_e}d\tau. \tag{8.55b}$$

With the help of (8.54) or (8.55), the solution for $\mathcal{V}'(\tau)$ in (8.53a) can be written for τ outside the transition intervals as

$$\mathcal{V}'(\tau) = \frac{1}{m\tau_e}\begin{cases} 0 & , \ \tau < 0 \\ \int_\tau^\infty F_1(\tau')\exp[-(\tau'-\tau)/\tau_e]d\tau' & , \ \Delta t_a < \tau < \tau_2 \\ \int_\tau^\infty F_{\rm ext}(\tau')\exp[-(\tau'-\tau)/\tau_e]d\tau' & , \ \tau_2 + \Delta t_a < \tau\,. \end{cases} \tag{8.56}$$

For N nonanalytic points in time τ_n of the otherwise analytic external force function $F_{\rm ext}(\tau)$, the first occurring at $\tau = \tau_1 = 0$ when the externally applied force is first applied to the charged sphere, there are transition functions $f_{an}(\tau)$ nonzero only in the interval $[\tau_n, \tau_n + \Delta t_a]$ such that $f_a(\tau) = \sum_{n=1}^N f_{an}(\tau)$. The external force can be expressed in terms of N functions $[F_1(\tau), F_2(\tau), \cdots, F_N(\tau)]$ such that $F_n(\tau)$ is the analytic continuation of $F_{\rm ext}(\tau)$ about the real τ axis from $\tau_n < \tau < \tau_{n+1}$ to $\tau \geq \tau_{n+1}$ for $n = 1, 2, \cdots, N-1$, with $F_N(\tau) = F_{\rm ext}(\tau)$ for $\tau_N < \tau < \infty$, and $F_1(\tau) = F_{\rm ext}(\tau) = 0$ for $\tau < 0$:

$$\begin{aligned} F_{\rm ext}(\tau) = F_1(\tau) &+ h(\tau-\tau_2)[F_2(\tau) - F_1(\tau)] + \cdots \\ &+ h(\tau-\tau_N)[F_N(\tau) - F_{N-1}(\tau)]. \end{aligned} \tag{8.57}$$

Therefore, $F_1(\tau)$ is an analytic function of complex τ about the real τ axis for $\tau > \tau_1 = 0$, and each of the functions in the square brackets, $F_{n+1}(\tau) - F_n(\tau)$, $n = 1, 2, \cdots, N-1$, is an analytic function of complex τ about the real τ axis for $\tau > \tau_n$.

The acceleration function $\mathcal{V}'(\tau)$ satisfies the differential equation

$$\frac{1}{m}\left[F_{\rm ext}(\tau) + \sum_{n=1}^N f_{an}(\tau)\right] = \mathcal{V}'(\tau) - \tau_e\mathcal{V}''(\tau) + \frac{O(a)}{m} \tag{8.58}$$

which has the exact solution (neglecting the $O(a)/m$ terms) given by

$$\begin{aligned} \mathcal{V}'(\tau) &= \frac{1}{m\tau_e}\int_\tau^\infty [F_1(\tau') + f_{a1}(\tau')]\exp[-(\tau'-\tau)/\tau_e]d\tau' \\ &+ \frac{1}{m\tau_e}\int_\tau^\infty h(\tau'-\tau_2)[F_2(\tau') - F_1(\tau') + f_{a2}(\tau')]\exp[-(\tau'-\tau)/\tau_e]d\tau' \\ &\ \vdots \\ &+ \frac{1}{m\tau_e}\int_\tau^\infty h(\tau'-\tau_N)[F_N(\tau') - F_{N-1}(\tau') + f_{aN}(\tau')]\exp[-(\tau'-\tau)/\tau_e]d\tau' \end{aligned} \tag{8.59a}$$

and

$$\mathcal{V}(\tau) = \tau_e\mathcal{V}'(\tau) + \frac{1}{m}\int_0^\tau \left[F_{\rm ext}(\tau') + \sum_{n=1}^N f_{an}(\tau')\right]d\tau'\,, \quad \mathcal{V}(0) = 0 \tag{8.59b}$$

with the transition functions $f_{an}(\tau)$ eliminating all pre-acceleration and pre-deceleration by satisfying the conditions

$$\int_0^{\Delta t_a} f_{an}(\tau+\tau_n)e^{-\tau/\tau_e}d\tau = -\int_0^\infty [F_n(\tau+\tau_n) - F_{n-1}(\tau+\tau_n)]e^{-\tau/\tau_e}d\tau\,, \quad n = 1, 2, \cdots, N \tag{8.60}$$

with $\tau_1 = 0$ and $F_0(\tau) \equiv 0$. The conditions (8.60) on the transition functions $f_{an}(\tau)$ allow $\mathcal{V}'(\tau)$ in (8.59a) to be expressed outside the transition intervals $[\tau_n, \tau_n + \Delta t_a]$ as simply

$$\mathcal{V}'(\tau) = \frac{1}{m\tau_e}\begin{cases} 0 & , \ \tau < \tau_1 = 0 \\ \int_\tau^\infty F_n(\tau')\exp[-(\tau'-\tau)/\tau_e]d\tau', & \tau_n + \Delta t_a < \tau < \tau_{n+1}\,, \end{cases} \quad n = 1, 2, \cdots, N \tag{8.61}$$

where $\tau_{N+1} = \infty$. This result says that the acceleration function $\mathcal{V}'(\tau)$ is determined at a time τ outside the transition intervals by the values of the externally applied force at the time τ and its analytic continuation to $\tau = \infty$. In each domain $\tau_n + \Delta t_a < \tau < \tau_{n+1}$, the acceleration function $\mathcal{V}'(\tau)$ is the particular solution to the differential equation of motion

$$\frac{F_n(\tau)}{m} = \mathcal{V}'(\tau) - \tau_e\mathcal{V}''(\tau)\,, \quad \tau_n < \tau < \infty \tag{8.62}$$

that converges as $\tau \to \infty$.

The transition functions $f_{an}(\tau)$, which are nonzero only in the transition intervals $[\tau_n, \tau_n + \Delta t_a]$, may contain delta functions and their derivatives as

$\Delta t_a \to 0$. That is, even as $a \to 0$, the acceleration function $\mathcal{V}'(\tau)$ may produce a finite jump in the velocity function $\mathcal{V}(\tau)$ across the transition intervals. Since $\mathcal{V}'(\tau)$ in (8.61) is not given for τ in the transition intervals, (8.61) *cannot* be used to determine these possible jumps in velocity across the transition intervals. The differential equation for $\mathcal{V}'(\tau)$ valid for all τ is given by (8.58). Its solution for $\mathcal{V}'(\tau)$ and $\mathcal{V}(\tau)$ given in (8.59) can also be expressed as

$$\mathcal{V}'(\tau) = \sum_{n=1}^{N} \mathcal{V}_n'(\tau) \tag{8.63a}$$

$$\mathcal{V}(\tau) = \sum_{n=1}^{N} \mathcal{V}_n(\tau)\,, \quad \mathcal{V}_n(\tau_n) = 0 \tag{8.63b}$$

where

$$\mathcal{V}_n'(\tau) = \frac{1}{m\tau_e}\int_{\tau}^{\infty} h(\tau'-\tau_n)[F_n(\tau') - F_{n-1}(\tau') + f_{an}(\tau')]\exp[-(\tau'-\tau)/\tau_e]\mathrm{d}\tau' \tag{8.63c}$$

and

$$\mathcal{V}_n(\tau) = \tau_e\mathcal{V}_n'(\tau) + \frac{1}{m}\int_{\tau_n}^{\tau} h(\tau'-\tau_n)[F_n(\tau') - F_{n-1}(\tau') + f_{an}(\tau')]\mathrm{d}\tau' \tag{8.63d}$$

for all τ, so that $\mathcal{V}_n'(\tau)$ satisfies

$$h(\tau-\tau_n)[F_n(\tau)-F_{n-1}(\tau)+f_{an}(\tau)]/m = \mathcal{V}_n'(\tau) - \tau_e\mathcal{V}_n''(\tau), \; -\infty < \tau < \infty \tag{8.63e}$$

with $\mathcal{V}_n(\tau_n) = 0$. The jumps in the acceleration functions across the nth transition interval $[\tau_n, \tau_n + \Delta t_a]$ are independent of $f_{an}(\tau)$, whereas the jumps in the velocity functions are determined by $f_{an}(\tau)$. These jumps are given by $\Delta\mathcal{V}' = \Delta\mathcal{V}_n'$ and $\Delta\mathcal{V} = \Delta\mathcal{V}_n$. Note, however, that, in general, $\mathcal{V}'(\tau_n) \neq \mathcal{V}_n'(\tau_n) = 0$, $\mathcal{V}(\tau_n) \neq \mathcal{V}_n(\tau_n) = 0$, $\mathcal{V}'(\tau_n + \Delta t_a) \neq \mathcal{V}_n'(\tau_n + \Delta t_a) = \Delta\mathcal{V}_n'$, and $\mathcal{V}(\tau_n + \Delta t_a) \neq \mathcal{V}_n(\tau_n + \Delta t_a) = \Delta\mathcal{V}_n$.

The modified equation of motion (8.43) or (8.45), and its rectilinear version (8.58), still admit a homogenous runaway solution. However, this runaway solution is eliminated from the modified equation of motion by the asymptotic condition in (8.12) without introducing noncausal motion into the solution.

Although the solution to the modified equation of motion (8.45) is free of pre-acceleration, it may be bothersome that for $\tau > \tau_n + \Delta t_a$ the solution in (8.61) to the modified (and original) equation of motion depends on the values of the external force at all future times. This result becomes understandable if it is remembered that (8.61) is the solution to an equation of motion obtained under the restriction that the externally applied force function $F_n(\tau)$ be an analytic function of time about the real τ axis for all $\tau > \tau_n$, because the

values of an analytic function on an interval of a singly connected domain of analyticity determine uniquely the function over the rest of the domain. For example, assume that for $\tau > \tau_n$ the external force $F_n(\tau')$ in (8.61) can be expanded in a power series about τ to recast (8.61) in the form

$$\mathcal{V}'(\tau) = \frac{1}{m}\sum_{k=0}^{\infty}(\tau_e)^k\frac{\mathrm{d}^k F_n(\tau)}{\mathrm{d}\tau^k}, \quad \tau_n + \Delta t_a < \tau < \tau_{n+1} \tag{8.64}$$

which simply states that the acceleration at any one time ($\tau_n + \Delta t_a < \tau < \tau_{n+1}$) depends on the time derivatives of the applied force as well as the applied force itself at that time. (Note that (8.64) is not a valid representation for $\mathcal{V}'(\tau)$ in the transition interval $\tau_n < \tau < \tau_n + \Delta t_a$ containing the transition force in addition to the externally applied force.)

If the restriction that the external force $F_n(\tau)$ be an analytic function of τ for all $\tau > \tau_n$ is ignored, and $F_n(\tau)$ is allowed to attain a strong enough infinite singularity at some future point in time, as in the case of the charged sphere being attracted to the center of a Coulomb field ($1/r^2$ singularity), the integration in (8.61) may not converge for all values of τ before the sphere reaches the singularity [50].

8.2.3 Determination of the Transition Force for Rectilinear Motion

Although the exact variation with τ of each $f_{an}(\tau)$ is unknown, they can be represented operationally from equations (8.63) in terms of delta functions as the radius a of the charge approaches zero so that $\Delta t_a \to 0$ and the mass m is renormalized to a prescribed value. From (8.63c)–(8.63d) we see that $\mathcal{V}_n(\tau) = 0$ for $\tau < \tau_n$ and it is a differentiable function of τ for $\tau > \tau_n + \Delta t_a$, that is, for $\tau \geq \tau_n^+$ as $\Delta t_a \to 0$, where τ_n^+ indicates a value infinitesimally larger than τ_n. Thus, we can represent $\mathcal{V}_n(\tau)$ for all τ as

$$\mathcal{V}_n(\tau) = h(\tau - \tau_n^+)\mathcal{V}_n^+(\tau) \tag{8.65}$$

with $h(\tau)$ denoting the unit step function and the $\mathcal{V}_n^+(\tau)$ on the right-hand side of this equation is a differentiable function of τ equal to $\mathcal{V}_n(\tau)$ for $\tau \geq \tau_n^+$. Differentiating this equation produces

$$\mathcal{V}_n'(\tau) = \delta(\tau - \tau_n^+)\mathcal{V}_n^+(\tau) + h(\tau - \tau_n^+)\mathcal{V}_n^{+\prime}(\tau) \tag{8.66a}$$

$$\mathcal{V}_n''(\tau) = \delta'(\tau - \tau_n^+)\mathcal{V}_n^+(\tau) + 2\delta(\tau - \tau_n^+)\mathcal{V}_n^{+\prime}(\tau) + h(\tau - \tau_n^+)\mathcal{V}_n^{+\prime\prime}(\tau) \tag{8.66b}$$

where $\delta(\tau)$ is the Dirac delta function. Combining $\mathcal{V}_n'$ and $\mathcal{V}_n''$, we have

$$\begin{aligned}\mathcal{V}_n' - \tau_e\mathcal{V}_n'' = \delta(\tau - \tau_n^+)[\mathcal{V}_n(\tau_n^+) - \tau_e\mathcal{V}_n'(\tau_n^+)] - \tau_e\delta'(\tau - \tau_n^+)\mathcal{V}_n(\tau_n^+)\\ + h(\tau - \tau_n^+)[F_n(\tau) - F_{n-1}(\tau)]/m\end{aligned} \tag{8.67}$$

in which we have used the equation of motion (8.63e) in the domain $\tau \geq \tau_n^+$, namely, $[F_n(\tau) - F_{n-1}(\tau)]/m = \mathcal{V}_n'(\tau) - \tau_e \mathcal{V}_n''(\tau)$, as well as the operational relationships $\delta(\tau - \tau_n^+)\mathcal{V}_n^+(\tau) = \delta(\tau - \tau_n^+)\mathcal{V}_n(\tau_n^+)$ and $\delta'(\tau - \tau_n^+)\mathcal{V}_n^+(\tau) = \delta'(\tau - \tau_n^+)\mathcal{V}_n(\tau_n^+) - \delta(\tau - \tau_n^+)\mathcal{V}_n'(\tau_n^+)$.

Comparing (8.67) with the equation of motion (8.63e) shows that the transition force can be expressed as

$$\frac{f_{an}(\tau)}{m} = [\Delta\mathcal{V}_n - \tau_e \Delta\mathcal{V}_n']\,\delta(\tau - \tau_n^+) - \tau_e \Delta\mathcal{V}_n\,\delta'(\tau - \tau_n^+) \tag{8.68}$$

where $\Delta\mathcal{V}_n$ has replaced $\mathcal{V}_n(\tau_n^+)$, and $\Delta\mathcal{V}_n'$ has replaced $\mathcal{V}_n'(\tau_n^+)$. Since $f_{an}(\tau)$ alone determines $\Delta\mathcal{V}_n$, we are free to decide the value of $\Delta\mathcal{V}_n$ in (8.68). However, $\Delta\mathcal{V}_n'$ is determined solely by the externally applied force and is independent of $f_{an}(\tau)$. Thus, it is a parameter whose value cannot be changed in (8.68). Choosing $\Delta\mathcal{V}_n = 0$ leaves only the delta function in (8.68) and makes the velocity function continuous. Choosing $\Delta\mathcal{V}_n = \tau_e \Delta\mathcal{V}_n'$ leaves only the doublet function in (8.68) and produces a jump in velocity approximately equal to the change in velocity produced by the original pre-acceleration or in the original equation of motion (see Sections 8.2.4 and 8.2.5).

We can verify that the transition functions in (8.68) remove the pre-acceleration (or pre-deceleration) by confirming that they satisfy the conditions in (8.60). From (8.68) we see that

$$\int_0^{\Delta t_a} f_{an}(\tau + \tau_n) e^{-\tau/\tau_e} d\tau = -m\tau_e \Delta\mathcal{V}_n' \tag{8.69}$$

and from (8.63c)

$$\mathcal{V}_n'(\tau_n^+) = \Delta\mathcal{V}_n' = \frac{1}{m\tau_e} \int_{\tau_n^+}^{\infty} [F_n(\tau') - F_{n-1}(\tau')] \exp[-(\tau' - \tau_n^+)/\tau_e] d\tau'$$

$$= \frac{1}{m\tau_e} \int_0^{\infty} [F_n(\tau + \tau_n) - F_{n-1}(\tau + \tau_n)] e^{-\tau/\tau_e} d\tau \tag{8.70}$$

so that the conditions in (8.60) are satisfied and the pre-acceleration (or pre-deceleration) is eliminated.

Inserting $f_{an}(\tau)$ from (8.68) into (8.63c) and (8.63d), we can write as $a \to 0$

$$\mathcal{V}_n'(\tau) = h(\tau - \tau_n) \frac{1}{m\tau_e} \int_0^{\infty} [F_n(\tau' + \tau) - F_{n-1}(\tau' + \tau)] e^{-\tau'/\tau_e} d\tau' \tag{8.71a}$$

$$\mathcal{V}_n(\tau) = \tau_e \mathcal{V}_n'(\tau) + h(\tau - \tau_n)\left(\Delta\mathcal{V}_n - \tau_e \Delta\mathcal{V}_n' + \frac{1}{m} \int_{\tau_n}^{\tau} [F_n(\tau') - F_{n-1}(\tau')] d\tau'\right) \tag{8.71b}$$

which when summed in (8.63a) and (8.63b) yield

$$\mathcal{V}'(\tau) = \frac{1}{m\tau_e} \begin{cases} 0 & , \quad \tau < \tau_1 = 0 \\ \int_0^{\infty} F_n(\tau' + \tau) e^{-\tau'/\tau_e} d\tau' , & \tau_n^+ < \tau < \tau_{n+1} , \end{cases}$$
$$n = 1, 2, \cdots, N \tag{8.72a}$$

$$\mathcal{V}(\tau) = \tau_e \mathcal{V}'(\tau) + \sum_{n=1}^{N} h(\tau - \tau_n)(\Delta\mathcal{V}_n - \tau_e \Delta\mathcal{V}_n') + \frac{1}{m} \int_0^{\tau} F_{\text{ext}}(\tau') d\tau' \tag{8.72b}$$

or by substituting $\Delta\mathcal{V}_n'$ from (8.70) and $\mathcal{V}_n'(\tau)$ from (8.71a) into (8.63a)

$$\mathcal{V}(\tau) = \sum_{n=1}^{N} h(\tau - \tau_n)\Bigg(\Delta\mathcal{V}_n + \frac{1}{m} \int_0^{\infty} [F_n(\tau' + \tau) - F_n(\tau' + \tau_n) \tag{8.72c}$$
$$- F_{n-1}(\tau' + \tau) + F_{n-1}(\tau' + \tau_n)] e^{-\tau'/\tau_e} d\tau'\Bigg) + \frac{1}{m} \int_0^{\tau} F_{\text{ext}}(\tau') d\tau'$$

expressions that are free of pre-acceleration and pre-deceleration.

The equations in (8.71) and (8.72) satisfy the jump conditions

$$\Delta\mathcal{V}_n' = \mathcal{V}'(\tau_n^+) - \mathcal{V}'(\tau_n) = \frac{1}{m\tau_e} \int_0^{\infty} [F_n(\tau + \tau_n) - F_{n-1}(\tau + \tau_n)] e^{-\tau/\tau_e} d\tau \tag{8.73a}$$

$$\Delta\mathcal{V}_n = \mathcal{V}(\tau_n^+) - \mathcal{V}(\tau_n) . \tag{8.73b}$$

Although the jumps in the acceleration function are determined by the integrations of the externally applied force in (8.73a), the jumps in the velocity function in (8.73b) remain undetermined. The drawback of the corrected equation of motion, as discussed in Section 8.2.2, is that the exact time dependences of the transition functions $f_{an}(\tau)$ required by causality at each nonanalytic point in time τ_n of $F_{\text{ext}}(\tau)$ cannot be determined before the velocity of the charge is known. We have, however, been able to reduce this ambiguity in the $f_{an}(\tau)$ to the jumps $\Delta\mathcal{V}_n$ in the velocity function across the transition intervals. In Section 8.2.5, it is shown that reasonable values for the $\Delta\mathcal{V}_n$ can be chosen to conserve momentum and energy while maintaining a non-negative value for the energy radiated during the transition intervals.

8.2.4 Motion of Charge in a Uniform Electric Field for a Finite Time

Assume a uniform electrostatic field E_0 is applied to a charge e with mass m initially at rest at $t_1 = \tau_1 = 0$ and that the electric field is turned off at time $t = t_2$ $(\tau = \tau_2)$. For example, the charge could be accelerated between two infinitesimally thin plates of a parallel-plate capacitor charged to produce the electric field E_0. It could be released at time $t = 0$ from one plate of the capacitor and leave through a small hole in the second plate at time $t = t_2$. With the $O(a)/m$ terms neglected, the acceleration and velocity as a function of time can be found for the original equation of motion from (8.13a) and (8.16) (or from (8.59) without the transition forces) as

$$\mathcal{V}'_{\text{pre}}(\tau) = \frac{eE_0}{m}\begin{cases} \left(1 - e^{-\tau_2/\tau_e}\right) e^{\tau/\tau_e}, & \tau \le 0 \\ \left(1 - e^{(\tau-\tau_2)/\tau_e}\right), & 0 \le \tau \le \tau_2 \\ 0, & \tau_2 \le \tau \end{cases} \tag{8.74a}$$

$$\mathcal{V}_{\text{pre}}(\tau) = \frac{eE_0}{m}\begin{cases} \tau_e\left(1 - e^{-\tau_2/\tau_e}\right) e^{\tau/\tau_e}, & \tau \le 0 \\ \tau_e\left(1 - e^{(\tau-\tau_2)/\tau_e}\right) + \tau, & 0 \le \tau \le \tau_2 \\ \tau_2, & \tau_2 \le \tau \end{cases} \tag{8.74b}$$

where the subscripts "pre" indicate the solution to the original equation of motion that exhibits pre-acceleration and pre-deceleration. Note that the change in the velocity function due to the pre-acceleration before $\tau = 0$ and the pre-deceleration before $\tau = \tau_2$ is approximately equal to $eE_0\tau_e/m$ for $\tau_2 \gg \tau_e$. Also, the pre-acceleration involves τ_2 and thus anticipates when the externally applied force turns off (at $\tau = \tau_2$) as well as when it is first applied (at $\tau = 0$).

The solution to the modified equation of motion that is free of noncausal pre-acceleration and pre-deceleration can be found by evaluating (8.72) for $N = 2$ to get (as $a \to 0$ with renormalized mass m)

$$\mathcal{V}'(\tau) = \frac{eE_0}{m}\begin{cases} 0, & \tau < 0 \\ 1, & 0^+ < \tau < \tau_2 \\ 0, & \tau_2^+ < \tau \end{cases} \tag{8.75a}$$

$$\mathcal{V}(\tau) = \begin{cases} 0, & \tau < 0 \\ \Delta\mathcal{V}_1 + eE_0\tau/m, & 0^+ < \tau < \tau_2 \\ \Delta\mathcal{V}_{21} + eE_0\tau_2/m, & \tau_2^+ < \tau \end{cases} \tag{8.75b}$$

with $\Delta\mathcal{V}_{21} = \Delta\mathcal{V}_2 + \Delta\mathcal{V}_1$. Except for the homogeneous exponential (runaway) solution in (8.74) and the discontinuities across the transition intervals at $\tau = 0$ and $\tau = \tau_2$ in (8.75), the solutions in both (8.74) and (8.75) are identical to the solution one obtains by solving the equation of motion without the radiation reaction terms, that is, by solving the relativistic version of Newton's second law of motion. This behavior, which depends uniquely on the external force being constant while it is applied, was discussed in Section 7.1 on hyperbolic (relativistically uniform) motion.

We can make use of this behavior of the motion of a charge in a parallel-plate capacitor to solve the rectilinear version of (8.45) with the $O(a)/m$ terms neglected

$$\frac{F_{\text{ext}}(t)+f_{a1}(t)+f_{a2}(t)}{m} = \frac{d(\gamma u)}{dt} - \tau_e\left\{\frac{d}{dt}\left[\gamma\frac{d}{dt}(\gamma u)\right] - \frac{\gamma^6}{c^2}(\dot{u})^2 u\right\} \tag{8.76a}$$

$$\frac{[F_{\text{ext}}(t) + f_{a1}(t) + f_{a2}(t)]u}{mc^2} = \frac{d\gamma}{dt} - \tau_e\left[\frac{d}{dt}\left(\gamma\frac{d\gamma}{dt}\right) - \frac{\gamma^6}{c^2}(\dot{u})^2\right] \tag{8.76b}$$

for $\gamma(t)u(t)$ (and thus $u(t)$ and $\dot{u}(t)$) outside the transition regions where only the first term (relativistic Newtonian acceleration) remains. Alternatively, the substitutions at the beginning of this chapter can be used to find $\gamma(t)u(t)$ from $\mathcal{V}(\tau)$ given in (8.75b). Either method yields for γu with $F_{\text{ext}}(t) = eE_0$ (as $a \to 0$ with renormalized mass m)

$$\frac{d(\gamma u)}{dt} = \gamma^3(t)\dot{u}(t) = \frac{eE_0}{m}\begin{cases} 0, & t < 0 \\ 1, & 0^+ < t < t_2 \\ 0, & t_2^+ < t \end{cases} \tag{8.77a}$$

$$\gamma(t)u(t) = \begin{cases} 0, & t < 0 \\ \Delta(\gamma u)_1 + eE_0 t/m, & 0^+ < t < t_2 \\ \Delta(\gamma u)_{21} + eE_0 t_2/m, & t_2^+ < t \end{cases} \tag{8.77b}$$

with

$$\gamma(t) = \left\{1 + [\gamma(t)u(t)/c]^2\right\}^{1/2} \tag{8.77c}$$

and $\Delta(\gamma u)_{21} = \Delta(\gamma u)_2 + \Delta(\gamma u)_1$, where $\Delta(\gamma u)_1$ and $\Delta(\gamma u)_2$ are the jumps in $\gamma(t)u(t)$ across the transition intervals at $t = 0$ and $t = t_2$. (If desired, these jumps can be expressed in terms of $\Delta\mathcal{V}_1$ and $\Delta\mathcal{V}_2$; see (8.79) and (8.82) below.)

One sees from this example of the motion of a charge through a parallel-plate capacitor that the transition forces $f_{an}(t)$, which are nonzero only during the short time intervals following the points in time where the externally applied force is discontinuous, remove both the noncausal pre-acceleration and pre-deceleration from the solution to the equation of motion. However, the transition forces $f_{an}(t)$ in the equation of motion change, in general, the momentum and energy of the charged sphere. The next section determines conditions under which this change in momentum-energy is consistent with the conservation of momentum-energy and a non-negative radiated energy during the transition intervals.

8.2.5 Conservation of Momentum-Energy in the Causal Equation of Motion

The transition forces $\mathbf{f}_a(t) = \sum_{n=1}^{N} \mathbf{f}_{an}(t)$ ensure that the solution to the modified equation of motion in (8.45) obeys causality while remaining free of runaway motion. However, these transition forces, in general, add to the momentum and energy of the charged particle. In particular, if the modified power equation of motion (8.45b) with the $O(a)/m$ terms neglected, or its rectilinear version in (8.76b), is integrated over time from before the external force is first applied to a $\gamma\Delta t_a$ after the external force is turned off, the total work done by the external force should equal the total change in kinetic energy of the particle plus the total energy radiated by the charged particle. Since the total change in the reversible reactive Schott acceleration energy is zero and the change in kinetic energy is given by the change in γ, we have to assume that the remaining energy change across all the transition intervals (call it $W_{\rm TI}$) equals the energy radiated during all the transition intervals, an energy that must be equal to or greater than zero. Before the external force turns off, the transition forces may contribute to reversible reactive acceleration energy as well as radiated energy and thus the remaining energy change across any one transition interval (call it $W_{{\rm TI},n}$) need not be equal to or greater than zero. Nonetheless, we shall first determine the conditions under which this energy change $W_{{\rm TI},n}$ across a single transition interval is equal to or greater than zero.

We see from (8.76b) that $W_{{\rm TI},n}$ across the nth transition interval from $t = t_n$ to $t = t_n + \gamma\Delta t_a = t_n^+$ as $\Delta t_a \to 0$ is given by

$$\frac{W_{{\rm TI},n}}{mc^2} = \frac{1}{mc^2}\int\limits_{t_n}^{t_n^+}\left[m\tau_e\gamma^6(\dot u)^2 - f_{an}u\right]dt = \tau_e[\gamma(t_n^+)\dot\gamma(t_n^+) - \gamma(t_n)\dot\gamma(t_n)] - [\gamma(t_n^+) - \gamma(t_n)]\,. \qquad (8.78)$$

The integral of the external force does not appear in the second equality of (8.78) because the work done by the finite external force during the transition interval approaches zero as $\Delta t_a \to 0$. Since the acceleration of the extended charge can contain delta functions in the transition intervals as $a \to 0$, the radiation from the extended charge during the transition intervals is no longer given by just the integration of $m\tau_e\gamma^6(\dot u)^2$ but must include the integrations of $f_{an}u$ as well.[7] The right-hand side of the second equality in (8.78) contains

[7] One may object to the presence of the $f_{an}u$ in the radiation integral of (8.78) on the grounds that Maxwell's equations predict that the energy radiated by an accelerating point charge is given without this term [13], that is, by the $m\tau_e\gamma^6(\dot u)^2$ term alone. However, as (8.32) shows, this derivation from Maxwell's equations does not hold for an extended charge during the short time that it takes light to travel across the charge distribution, even as the radius of the charge approaches zero, because the velocity may become discontinuous as $a \to 0$. (Abraham [3, sec. 25] and Hertz [44] were well aware that the energy radiated during a jump in velocity of an extended model of the electron was not given by the integral of the $m\tau_e\gamma^6(\dot u)^2$ term in (8.78).) Even if the change in velocity across the transition interval is negligible, the textbook derivation and expression for the radiated energy from an accelerated point charge can become invalid for an extended charge distribution. To see this, consider a charge e with an acceleration given by $\sqrt{\delta_\alpha(t)}$, where the pulse width α of the finite-size delta function $\delta_\alpha(t)$ is

the change in Schott acceleration energy minus the change in kinetic energy across the transition interval. Evaluating the right-hand side of (8.78) in terms of the proper time τ and velocity function $\mathcal{V}$ gives

$$\frac{W_{{\rm TI},n}}{mc^2} = \frac{\tau_e}{c}\{\mathcal{V}'(\tau_n^+)\sinh[\mathcal{V}(\tau_n)/c + \Delta\mathcal{V}_n/c] - \mathcal{V}'(\tau_n)\sinh[\mathcal{V}(\tau_n)/c]\} - \cosh[\mathcal{V}(\tau_n)/c + \Delta\mathcal{V}_n/c] + \cosh[\mathcal{V}(\tau_n)/c] \qquad (8.79)$$

where $\Delta\mathcal{V}_n$ is the jump in $\mathcal{V}$ across the transition interval. The values of $\mathcal{V}'(\tau_n)$ and $\mathcal{V}'(\tau_n^+)$ are determined solely by the externally applied force, whereas the jump $\Delta\mathcal{V}_n$ in the velocity function is determined solely by the transition force $f_{an}(\tau)$. The velocity function $\mathcal{V}(\tau_n)$ at the beginning of the transition interval can have any value from $-\infty$ to $+\infty$.

Numerical evaluation of (8.79) reveals that for a value of the velocity function $|\mathcal{V}(\tau_n)/c| \gtrsim 3$ at the beginning of the transition interval, a $\Delta\mathcal{V}_n/c$ across the transition interval can be found to make $W_{{\rm TI},n} \ge 0$ for all values of $\mathcal{V}'(\tau_n^+)$ if the value of acceleration at the beginning of the transition interval is restricted to $|\tau_e\mathcal{V}'(\tau_n)/c| \lesssim .9$. As the value of $|\mathcal{V}(\tau_n)/c|$ becomes smaller than 3, the value of $|\tau_e\mathcal{V}'(\tau_n)/c|$ must become smaller than .9 to keep $W_{{\rm TI},n} \ge 0$ for all values of $\mathcal{V}'(\tau_n^+)$. And if $|\mathcal{V}(\tau_n)/c| \ll 1$, then $|\tau_e\mathcal{V}'(\tau_n)/c|$ must be $\lesssim |\mathcal{V}(\tau_n)/c|/2$; that is, if $|\mathcal{V}(\tau_n)/c| \ll 1$ then $|\tau_e\mathcal{V}'(\tau_n)/c| \ll 1$ to find a $\Delta\mathcal{V}_n/c$ that makes $W_{{\rm TI},n} \ge 0$ for all values of $\mathcal{V}'(\tau_n^+)$. Therefore, for N transition intervals between the time the external force is first applied to when it is shut off, the total change in energy $W_{\rm TI} = \sum_{n=1}^{N} W_{{\rm TI},n}$, which equals the total energy radiated during the transition intervals and thus must be ≥ 0, can indeed be made ≥ 0 by choosing appropriate values for the jumps $\Delta\mathcal{V}_n$ across the transition intervals if the acceleration is restricted to values

$$\frac{|\mathcal{V}'(\tau)|\,\tau_e}{c} \ll 1\,, \quad \tau \notin \text{transition intervals}\,. \qquad (8.80)$$

It is easily proven, for example, from (8.61), that this inequality is satisfied if

$$\frac{|F_{\rm ext}|\tau_e}{mc} \ll 1. \qquad (8.81)$$

greater than zero. The textbook expression for the total energy radiated by a point charge predicts a radiated energy proportional to $e^2\int\delta_\alpha(t)dt = e^2$. Now, distribute the charge e uniformly over the surface of a sphere of radius a. Each element de of this charge distribution will radiate a pulse with the pulse width α of the function $\sqrt{\delta_\alpha(t)}$. Therefore, for $\alpha \ll a/c$, the change in velocity across the transition interval is negligible, yet the fields radiated by different elements of the extended charge distribution will hardly interfere, and the total radiated energy is approximately equal to the integrated sum of the energy radiated by each element (which is proportional to $({\rm d}e)^2$), so that the total energy radiated by the extended charge is proportional to $\int({\rm d}e)^2 \to 0 \ne e^2$. Moreover, the value of the total energy radiated by the extended charge of radius a can vary between 0 and the textbook value for a point charge as the pulse width α of $\delta_\alpha(t)$ varies from $\alpha \ll a/c$ to $\alpha \gg a/c$.

In the proper frame of the moving charged insulator, the inequality in (8.80) is identical to the condition in (8.24c) sufficient to neglect the $O(a)/m$ terms in the proper-frame equation of motion because $\tau_e = 4a/(3c)$. Unfortunately, however, even if a is allowed to approach zero while renormalizing the mass m to a finite value (so that the conditions in (8.24) are all satisfied yet τ_e remains finite and equal to $e^2/(6\pi\epsilon_0 mc^3)$), the condition in (8.80) is still required to keep the energy radiated during the transition intervals equal to or greater than zero; see Section 8.5. Nonetheless, to within the approximation that the unrenormalized equation of motion in (8.76) and its solution are valid, $W_{\rm TI}$ can always be made equal to or greater than zero by choosing the appropriate jumps in velocity across the transition intervals. Also, it can be shown that $W_{\rm TI}$ is a small fraction of the total energy radiated if the amount of time that the external force is applied is several times greater than τ_e. Thus, there appears to be no inconsistency in assuming that $W_{\rm TI}$ represents the energy radiated during all the transition intervals and that the unrenormalized causal equation of motion modified by the transition functions f_{an} does not violate conservation of momentum-energy.

For the example of the parallel-plate capacitor problem discussed in the previous section, we have for the total energy radiated across the two transition intervals

$$\begin{aligned}\frac{W_{\rm TI}}{mc^2} &= \frac{1}{mc^2}\int\limits_0^{\gamma\Delta t_a=0^+}\left[m\tau_e\gamma^6(\dot u)^2 - f_{a1}u\right]{\rm d}t + \frac{1}{mc^2}\int\limits_{t_2}^{t_2+\gamma\Delta t_a=t_2^+}\left[m\tau_e\gamma^6(\dot u)^2 - f_{a2}u\right]{\rm d}t\\ &= \cosh\left(\frac{eE_0\tau_2}{mc}+\frac{\Delta\mathcal{V}_1}{c}\right) - \cosh\left(\frac{\Delta\mathcal{V}_1}{c}\right)\\ &\quad - \left[\cosh\left(\frac{eE_0\tau_2}{mc}+\frac{\Delta\mathcal{V}_1}{c}+\frac{\Delta\mathcal{V}_2}{c}\right)-1\right]\\ &\quad - \frac{eE_0\tau_e}{mc}\left[\sinh\left(\frac{eE_0\tau_2}{mc}+\frac{\Delta\mathcal{V}_1}{c}\right)-\sinh\left(\frac{\Delta\mathcal{V}_1}{c}\right)\right]\end{aligned} \tag{8.82}$$

where the values of $\mathcal{V}'(\tau)$ and $\mathcal{V}(\tau)$ have been taken from (8.75). The first line after the last equality sign in (8.82) gives the total work done on the charged particle by the external force. The second line after the last equality sign subtracts the total change in the kinetic energy of the charged particle. The third line after the last equality sign in (8.82) subtracts the energy radiated by the charged particle while it is outside the transition intervals.

Using values of $\Delta\mathcal{V}_1$ and $\Delta\mathcal{V}_2$ given by

$$\Delta\mathcal{V}_1/c = -\Delta\mathcal{V}_2/c = -C_0\,{\rm sign}(eE_0)\ln[1-|eE_0\tau_e/(mc)|] \tag{8.83}$$

with the constant $C_0 > 1$, the right-hand side of (8.82) is equal to or greater than zero for all values of the time τ_2 that the constant external force eE_0 is applied to the charged sphere, provided $|eE_0\tau_e/(mc)| < 1$; that is[8]

$$W_{\rm TI} \ge 0 \quad \text{if} \quad \left|\frac{eE_0\tau_e}{mc}\right| < 1\,. \tag{8.84}$$

Moreover, (8.84) continues to hold if the initial value of the velocity is changed from zero to an arbitrary value. (If instead of the choice in (8.83), one chooses $\Delta\mathcal{V}_1/c = -\Delta\mathcal{V}_2/c = eE_0\tau_e/(mc)$, then $W_{\rm TI} \ge 0$ if $|eE_0\tau_e/(mc)| \ll 1$, an inequality that conforms to the general criterion in (8.81) for ensuring that $W_{\rm TI} \ge 0$.)

Lastly, we determine the total momentum ($G_{\rm TI}$) radiated by the charged particle across the two transition intervals by evaluating the difference between the total impulse imparted by the external force and the sum of the total change in kinetic momentum of the particle and the momentum it radiates outside the transition intervals. From (8.76a) one finds that for two transition intervals, $G_{\rm TI} = \sum_{n=1}^{n=2} G_{{\rm TI},n}$, where

$$\begin{aligned}\frac{G_{{\rm TI},n}}{mc} &= \frac{1}{mc}\int\limits_{t_n}^{t_n^+}\left[m\tau_e\gamma^6(\dot u)^2 u/c^2 - f_{an}\right]{\rm d}t\\ &= \frac{\tau_e}{c}\left\{\gamma(t_n^+)\frac{\rm d}{{\rm d}t}[\gamma(t_n^+)u(t_n^+)] - \gamma(t_n)\frac{\rm d}{{\rm d}t}[\gamma(t_n)u(t_n)]\right\}\\ &\quad - \frac{1}{c}[\gamma(t_n^+)u(t_n^+) - \gamma(t_n)u(t_n)]\end{aligned} \tag{8.85}$$

which gives for the parallel-plate capacitor problem

$$\begin{aligned}\frac{G_{\rm TI}}{mc} &= \sinh\left(\frac{eE_0\tau_2}{mc}+\frac{\Delta\mathcal{V}_1}{c}\right) - \sinh\left(\frac{\Delta\mathcal{V}_1}{c}\right)\\ &\quad - \sinh\left(\frac{eE_0\tau_2}{mc}+\frac{\Delta\mathcal{V}_1}{c}+\frac{\Delta\mathcal{V}_2}{c}\right)\\ &\quad - \frac{eE_0\tau_e}{mc}\left[\cosh\left(\frac{eE_0\tau_2}{mc}+\frac{\Delta\mathcal{V}_1}{c}\right) - \cosh\left(\frac{\Delta\mathcal{V}_1}{c}\right)\right].\end{aligned} \tag{8.86}$$

[8] If $|eE_0\tau_e/(mc)| > 1$ and $\Delta\mathcal{V}_1$ has a finite value independent of τ_2, no values of $\Delta\mathcal{V}_1$ and $\Delta\mathcal{V}_2$ keep $W_{\rm TI} \ge 0$ for all τ_2. If $\Delta\mathcal{V}_1$ is allowed to depend on τ_2, then $\Delta\mathcal{V}_1$ and $\Delta\mathcal{V}_2$ can be chosen to keep $W_{\rm TI} \ge 0$ for $|eE_0\tau_e/(mc)| > 1$. This would violate causality, however, because the value $\Delta\mathcal{V}_1$ for the jump in velocity when the external force is first applied would have to anticipate the value of the time τ_2 at which the force would be turned off. Another way to avoid the restriction in (8.84) on the magnitude of the external force is to allow the energy change $W_{\rm TI}$ during a transition interval to become negative if the external force is large enough to not satisfy the second inequality in (8.84); for example, by postulating that the charged particle absorbs the energy that it radiated during a short time just prior to the transition interval. The physical mechanism for this re-absorption of radiated energy would remain unexplained, however, within the realm of classical physics.

The first line after the equality sign in (8.86) is the total impulse imparted to the charged particle by the external force. The second line after the equality sign subtracts the total change in the kinetic momentum of the charged particle. The third line after the equality sign in (8.86) subtracts the momentum radiated by the charged particle while it is outside the transition intervals.

For a physically realizable radiated momentum-energy, the absolute value of the radiated momentum multiplied by the speed of light c must be less than or equal to the radiated energy, that is

$$\left|\frac{G_{\mathrm{TI}}}{mc}\right| \leq \frac{W_{\mathrm{TI}}}{mc^2}\,. \tag{8.87}$$

Insertion of $G_{\mathrm{TI}}/(mc)$ and $W_{\mathrm{TI}}/(mc^2)$ from (8.86) and (8.82), respectively, shows that the inequality in (8.87) is satisfied for all $\tau_2 \geq 0$ with $\Delta\mathcal{V}_1$ and $\Delta\mathcal{V}_2$ given in (8.83) under the restriction on the value of the external force given in (8.84). Also, for $eE_0\tau_2/(mc) \gg 1$ the radiated momentum is greater than zero under the restriction on the external force in (8.84). If $C_0 \geq 2$ then the radiated momentum is greater than zero for all $\tau_2 \geq 0$ under the restriction on the external force in (8.84). These results further confirm the interpretation of W_{TI} and G_{TI} as energy and momentum, respectively, radiated by the charged sphere during the transition intervals.

In the next section, the behavior of the solution to the equation of motion in the transition intervals is ignored and the method of successive substitutions is used to derive a power series solution to the rectilinear equation of motion and to the general equation of motion — the first two terms of the latter power series equaling the Landau-Lifshitz approximation [51, sec. 76].

8.3 Power Series Solution to the Equation of Motion

The pre-acceleration solution in (8.16) to the uncorrected rectilinear equation of motion (8.6) was derived in Section 8.1, and the solution (8.72) to the corrected rectilinear equation of motion (8.58) was derived in Section 8.2.2, under the assumption that the $O(a)/m$ terms in (8.6) and (8.58) were negligible. (Of course, the $O(a)/m$ terms vanish and τ_e remains a fixed value if $a \to 0$ while m is renormalized to a finite value.) In this section we shall use the method of successive substitutions to derive a power series solution to the uncorrected rectilinear equation of motion (8.6) and the uncorrected general equation of motion (7.1) or (7.12). (The uncorrected equations of motion omit the correction force $\mathbf{f}_a(t)$ that exists only in the transition intervals following each nonanalytic point in time of the external force $\mathbf{F}_{\mathrm{ext}}(t)$ and that eliminates the noncausality from the exact solution to the uncorrected equations of motion; see Section 8.2. If the external force were an analytic function of complex time about the real time axis for all time from $-\infty < t < +\infty$, no correction transition force would be required.) We begin with the derivation

of a power series solution to the proper-frame equation of motion because of its relative simplicity.

The equation of motion of the charged spherical insulator in its proper inertial frame of reference can be written from (5.12a) as

$$\dot{\mathbf{u}} = \frac{\mathbf{F}_{\mathrm{ext}}}{m_{\mathrm{es}}} + \frac{e^2}{6\pi\epsilon_0 m_{\mathrm{es}} c^3}\ddot{\mathbf{u}} + \mathbf{O}(a^2) \tag{8.88}$$

wherein it is assumed that a is small enough that the mass m_{ins} of the insulator is negligible compared to the electrostatic mass $m_{\mathrm{es}} = e^2/(8\pi\epsilon_0 ac^2)$, which is of order $1/a$. Take the $\dot{\mathbf{u}}$ given in (8.88) and substitute it into $\ddot{\mathbf{u}}$ and all the other derivatives of $\mathbf{u}$ on the right-hand side of (8.88). Repeat this substitution process successively to obtain the following power series solution for $\dot{\mathbf{u}}$ in the proper reference frame

$$\dot{\mathbf{u}} = \frac{1}{m_{\mathrm{es}}}\sum_{n=0}^{\infty}\left(\frac{e^2}{6\pi\epsilon_0 m_{\mathrm{es}} c^3}\right)^n \frac{\mathrm{d}^n\mathbf{F}_{\mathrm{ext}}}{\mathrm{d}t^n} + \mathbf{O}(a^3)\,. \tag{8.89}$$

In view of (8.24b)–(8.24d), the $\mathbf{O}(a^3)$ terms in (8.89) are negligible if

$$\frac{2a}{c}\left|\frac{\mathrm{d}^n\mathbf{F}_{\mathrm{ext}}}{\mathrm{d}t^n}\right| \ll (n+1)\left|\frac{\mathrm{d}^{n-1}\mathbf{F}_{\mathrm{ext}}}{\mathrm{d}t^{n-1}}\right|, \quad n = 1, 2, \ldots \tag{8.90a}$$

and

$$\frac{a}{c}\frac{|\mathbf{F}_{\mathrm{ext}}|}{m_{\mathrm{es}}} \ll c\,. \tag{8.90b}$$

The conditions in (8.90a) state that the fractional changes in the externally applied force and its first and higher order time derivatives are small during the time interval it takes light to traverse the radius of the charged sphere. Condition (8.90b) implies that the externally applied force is not large enough to change the velocity by a significant fraction of c in the time interval light takes to traverse the charged sphere. If the externally applied force satisfies the conditions (8.90) so that the $\mathbf{O}(a^3)$ terms are negligible, (8.89) can just as well be rewritten as

$$\dot{\mathbf{u}} = \frac{1}{m_{\mathrm{es}}}\left(\mathbf{F}_{\mathrm{ext}} + \frac{e^2}{6\pi\epsilon_0 m_{\mathrm{es}} c^3}\frac{\mathrm{d}\mathbf{F}_{\mathrm{ext}}}{\mathrm{d}t}\right) + \mathbf{O}(a^3) \tag{8.91}$$

with the $\mathbf{O}(a^3)$ terms in (8.91) also negligible under the conditions in (8.90). In general, $\mathrm{d}^n\mathbf{F}_{\mathrm{ext}}(t)/\mathrm{d}t^n$ will contain delta functions and their derivatives at $t = 0$ when the external force is first applied and thus the power series solutions in (8.89) and throughout this section will not generally converge at $t = 0$.

If the radius a of the charged sphere is allowed to approach zero in the equation of motion and the mass m_{es} is renormalized to a finite value m, equations (8.88) and (8.89) reduce to

$$\dot{\mathbf{u}} = \frac{\mathbf{F}_{\text{ext}}}{m} + \tau_e \ddot{\mathbf{u}} \tag{8.92}$$

and

$$\dot{\mathbf{u}} = \frac{1}{m} \sum_{n=0}^{\infty} \tau_e^n \frac{\mathrm{d}^n \mathbf{F}_{\text{ext}}}{\mathrm{d}t^n} \tag{8.93}$$

with τ_e given in (8.7) as $e^2/(6\pi\epsilon_0 mc^3)$. Moreover, if the terms of higher order than $n = 1$ in (8.93) are negligible, that is

$$\left| \sum_{n=2}^{\infty} \tau_e^n \frac{\mathrm{d}^n \mathbf{F}_{\text{ext}}}{\mathrm{d}t^n} \right| \ll \left(|\mathbf{F}_{\text{ext}}|, \tau_e \left| \frac{\mathrm{d}\mathbf{F}_{\text{ext}}}{\mathrm{d}t} \right| \right) \tag{8.94}$$

then (8.93) can be approximated by

$$\dot{\mathbf{u}} \approx \frac{1}{m} \left(\mathbf{F}_{\text{ext}} + \tau_e \frac{\mathrm{d}\mathbf{F}_{\text{ext}}}{\mathrm{d}t} \right). \tag{8.95}$$

In other words, just the first of the successive substitutions into (8.92) gives an accurate approximate solution to the proper-frame Lorentz-Abraham-Dirac equation of motion. For an electron, the conditions in (8.94) are satisfied except for changes in the external force that are so rapid that the effect of these changes may not be accurately described by classical physics.

8.3.1 Power Series Solution to Rectilinear Equation of Motion

The uncorrected equation for rectilinear motion in an arbitrary inertial reference frame for the charged insulator with $m = m_{\text{es}}$ can be expressed from (8.6) in the form

$$\mathcal{V}'(\tau) = \frac{F_{\text{ext}}(\tau)}{m_{\text{es}}} + \frac{e^2}{6\pi\epsilon_0 m_{\text{es}} c^3} \mathcal{V}''(\tau) + O(a^2). \tag{8.96}$$

Applying the method of successive substitutions to (8.96), as we did to (8.88), we find the power series solution for $\mathcal{V}'(\tau)$, namely

$$\mathcal{V}'(\tau) = \frac{1}{m_{\text{es}}} \sum_{n=0}^{\infty} \left(\frac{e^2}{6\pi\epsilon_0 m_{\text{es}} c^3} \right)^n \frac{\mathrm{d}^n F_{\text{ext}}(\tau)}{\mathrm{d}\tau^n} + O(a^3) \tag{8.97}$$

which can be rewritten as

$$\mathcal{V}'(\tau) = \frac{1}{m_{\text{es}}} \left(F_{\text{ext}}(\tau) + \frac{e^2}{6\pi\epsilon_0 m_{\text{es}} c^3} \frac{\mathrm{d}F_{\text{ext}}(\tau)}{\mathrm{d}\tau} \right) + O(a^3). \tag{8.98}$$

The power series solution in (8.98) can be integrated with respect to the proper time to find $\mathcal{V}(\tau)$ as

$$\mathcal{V}(\tau) = \frac{1}{m_{\text{es}}} \left[\int_0^\tau F_{\text{ext}}(\tau')\mathrm{d}\tau' + \left(\frac{e^2}{6\pi\epsilon_0 m_{\text{es}} c^3} \right) F_{\text{ext}}(\tau) \right] + O(a^3). \tag{8.99}$$

The solution in (8.98)–(8.99) for the rectilinear velocity function of the charged insulator of radius a contains no runaway solutions, no pre-acceleration, and is obtained using the single initial condition of zero velocity immediately before the external force is applied. Of course, an arbitrary constant velocity could be added to the right-hand side of (8.99) if the velocity were not zero before the external force were applied. However, regardless of the initial conditions the series solution in (8.97) contains spurious delta functions and their time derivatives at $\tau = 0$ that violate the conditions (8.90) and do not satisfy the uncorrected equation of rectilinear motion (8.96) (or the corrected equation of motion (8.49) with the transition force given by (8.68)).

The two terms in the brackets of (8.99) can also be found from the pre-acceleration solution (8.16) (with m_{es} replacing m) by expanding the external force $F_{\text{ext}}(\tau + \tau')$ in a Taylor series about the present time τ (assuming this expansion exists for $\tau > 0$), so that integrating term by term yields [34], [52]

$$\mathcal{V}(\tau) = \frac{1}{m_{\text{es}}} \left[\int_0^\tau F_{\text{ext}}(\tau')\mathrm{d}\tau' + \sum_{n=0}^{\infty} \left(\frac{e^2}{6\pi\epsilon_0 m_{\text{es}} c^3} \right)^{n+1} \frac{\mathrm{d}^n F_{\text{ext}}(\tau)}{\mathrm{d}\tau^n} \right]. \tag{8.100}$$

However, this expansion (8.100) of the pre-acceleration integral in (8.16) does not, in general, yield a valid asymptotic series solution to (8.6) beyond the first term in the summation of (8.100) because (8.16) was derived from the equation of motion (8.6) by neglecting self-force terms of order a^2. In other words, the $O(a^3)$ terms in (8.100) are not equal to the $O(a^3)$ terms in (8.99). (Recall that $1/m_{\text{es}} = O(a)$.)

It should also be noted that the power series solution (8.100) converges to the pre-acceleration solution (8.16) for $\tau > 0$ but not for $\tau < 0$. The reason for this discrepancy between the series solution (8.100) and the exact solution (8.16) to (8.6) with the $O(a)/m_{\text{es}} = O(a^2)$ terms omitted is that $F_{\text{ext}}(\tau + \tau')$ cannot be expanded in a Taylor series about $\tau \le 0$ for all $\tau' \ge 0$ because $F_{\text{ext}}(\tau)$ is identically zero for $\tau < 0$. Also, to within a constant velocity, for $\tau > \Delta t_a$ the solutions to the corrected and uncorrected equations of motion are the same; compare (8.16) with (8.50) for $\tau > \Delta t_a$.

When the external force becomes zero after it is applied for a finite time interval, the power series solution (8.99), like the pre-acceleration solution (8.16), produces the same final velocity that would be produced if the radiation reaction, the $\mathcal{V}''$ term in (8.6), and all higher order terms were neglected. Also, like the pre-acceleration solution, the effect of the radiation reaction on the power series solution for the velocity function $\mathcal{V}$, during the time the external force is applied, approaches zero as $aF_{\text{ext}}/m_{\text{es}}$ as the radius a of the charged sphere approaches zero. Indeed, the motion of the charged insulator should be determined solely by the conventional momentum, $m_{\text{es}}\mathrm{d}(\gamma\mathbf{u})/dt$, as the radius of the shell approaches zero, since the mass m_{es} becomes infinite while the radiation reaction term remains finite as the radius a approaches zero. As long as F_{ext}/m_{es} remains nonzero, however, these results do not imply that the momentum and energy radiated after $t = 0$

$$\frac{e^2}{6\pi\epsilon_0 c^5}\int_{0+}^{t}\gamma^6\dot{u}^2 u(t)\mathrm{d}t \quad \text{and} \quad \frac{e^2}{6\pi\epsilon_0 c^3}\int_{0+}^{t}\gamma^6\dot{u}^2(t)\mathrm{d}t \tag{8.101}$$

respectively, for the power series solution of the charged insulator in rectilinear motion, approach zero as the radius a approaches zero.

If the radius a of the charged sphere is allowed to approach zero and the mass m_{es} is renormalized to a finite value m, (8.96) and (8.97) reduce to

$$\mathcal{V}'(\tau) = \frac{F_{ext}(\tau)}{m} + \tau_e \mathcal{V}''(\tau) \tag{8.102}$$

and

$$\mathcal{V}'(\tau) = \frac{1}{m}\sum_{n=0}^{\infty}\tau_e^n \frac{\mathrm{d}^n F_{ext}(\tau)}{\mathrm{d}\tau^n}. \tag{8.103}$$

The terms of higher order than $n = 1$ in (8.103) can be neglected if

$$\left|\sum_{n=2}^{\infty}\tau_e^n \frac{\mathrm{d}^n F_{ext}(\tau)}{\mathrm{d}\tau^n}\right| \ll \left(|F_{ext}(\tau)|, \tau_e\left|\frac{\mathrm{d}F_{ext}(\tau)}{\mathrm{d}\tau}\right|\right) \tag{8.104}$$

and (8.103) can be approximated by

$$\mathcal{V}'(\tau) \approx \frac{1}{m}\left(F_{ext}(\tau) + \tau_e\frac{\mathrm{d}F_{ext}(\tau)}{\mathrm{d}\tau}\right). \tag{8.105}$$

That is, just the first of the successive substitutions into (8.102) gives an accurate approximate solution to the rectilinear Lorentz-Abraham-Dirac equation of motion.

8.3.2 Power Series Solution to General Equation of Motion

The general equation of motion in an arbitrary inertial reference frame without the correction function $\mathbf{f}_a(t)$ that eliminates the pre-acceleration can be written from (7.1) (with m replaced by m_{es}) as

$$\frac{\mathrm{d}u^i}{\mathrm{d}s} = \frac{F^i_{ext}}{m_{es}c^2} + \frac{e^2}{6\pi\epsilon_0 m_{es}c^2}\left(\frac{\mathrm{d}^2u^i}{\mathrm{d}s^2} + u^i\frac{\mathrm{d}u_j}{\mathrm{d}s}\frac{\mathrm{d}u^j}{\mathrm{d}s}\right) + O(a^2). \tag{8.106}$$

The method of successive substitutions applied to (8.106) produces the power series solution

$$\frac{\mathrm{d}u^i}{\mathrm{d}s} = \frac{1}{m_{es}c^2}\left[F^i_{ext} + \frac{e^2}{6\pi\epsilon_0 m_{es}c^2}\left(\frac{\mathrm{d}F^i_{ext}}{\mathrm{d}s} + \frac{u^i F_{ext,j}F^j_{ext}}{m_{es}c^2}\right)\right] + O(a^3) \tag{8.107}$$

where

$$F_{ext,j} \equiv \gamma(-\mathbf{F}_{ext}, \mathbf{F}_{ext}\cdot\mathbf{u}/c). \tag{8.108}$$

If the radius a of the charged sphere is allowed to approach zero while the mass m_{es} is renormalized to a finite value m, the general equation of motion (8.106) reduces to the Lorentz-Abraham-Dirac (LAD) equation of motion

$$\begin{aligned}\frac{\mathrm{d}u^i}{\mathrm{d}s} &= \frac{F^i_{ext}}{mc^2} + c\tau_e\left(\frac{\mathrm{d}^2u^i}{\mathrm{d}s^2} + u^i\frac{\mathrm{d}u_j}{\mathrm{d}s}\frac{\mathrm{d}u^j}{\mathrm{d}s}\right)\\ &= \frac{F^i_{ext}}{mc^2} + c\tau_e\left(\frac{\mathrm{d}^2u^i}{\mathrm{d}s^2} - u^i u_j\frac{\mathrm{d}^2u^j}{\mathrm{d}s^2}\right)\\ &= \frac{F^i_{ext}}{mc^2} + c\tau_e L^i_j\frac{\mathrm{d}u^j}{\mathrm{d}s}\end{aligned} \tag{8.109}$$

where $L^i_j = (I^i_j - u^i u_j)(\mathrm{d}/\mathrm{d}s)$ and I^i_j is the four-vector identity matrix such that $I^i_j A^j = A^i$ for any four vector A^i. The power series solution obtained by successive substitutions of $\mathrm{d}u^i/\mathrm{d}s$ then becomes for the LAD equation of motion

$$\frac{\mathrm{d}u^i}{\mathrm{d}s} = \frac{1}{mc^2}\left(F^i_{ext} + c\tau_e L^i_j F^j_{ext} + (c\tau_e)^2 L^i_j L^j_k F^k_{ext} + (c\tau_e)^3 L^i_j L^j_k L^k_l F^l_{ext} + \cdots\right). \tag{8.110}$$

If the third and higher order terms in (8.110) are negligible, that is, if

$$\left|(c\tau_e)^2 L^\beta_j L^j_k F^k_{ext} + (c\tau_e)^3 L^\beta_j L^j_k L^k_l F^l_{ext} + \cdots\right| \ll \left(\left|F^\beta_{ext}\right|, c\tau_e\left|L^\beta_j F^j_{ext}\right|\right) \tag{8.111}$$

where the superscript $\beta = (1, 2, 3)$ designates the three-vector part of a four vector, then (8.110) becomes

$$\begin{aligned}\frac{\mathrm{d}u^i}{\mathrm{d}s} &\approx \frac{1}{mc^2}\left(F^i_{ext} + c\tau_e L^i_j F^j_{ext}\right)\\ &= \frac{1}{mc^2}\left[F^i_{ext} + c\tau_e\left(\frac{\mathrm{d}F^i_{ext}}{\mathrm{d}s} - u^i u_j\frac{\mathrm{d}F^j_{ext}}{\mathrm{d}s}\right)\right].\end{aligned} \tag{8.112}$$

With the help of the relationship

$$\begin{aligned}u_j\frac{\mathrm{d}F^j_{ext}}{\mathrm{d}s} &= -\frac{\mathrm{d}u_j}{\mathrm{d}s}F^j_{ext} \approx -\frac{1}{mc^2}\left[F_{ext,j} + c\tau_e\left(\frac{\mathrm{d}F_{ext,j}}{\mathrm{d}s} - u_j u_k\frac{\mathrm{d}F^k_{ext}}{\mathrm{d}s}\right)\right]F^j_{ext}\\ &= -\frac{1}{mc^2}\left[F_{ext,j} + c\tau_e\frac{\mathrm{d}F_{ext,j}}{\mathrm{d}s}\right]F^j_{ext}\end{aligned} \tag{8.113}$$

(8.112) can be re-expressed in the form

$$\frac{du^i}{ds} \approx \frac{1}{mc^2}\left[F^i_{\text{ext}} + c\tau_e\left(\frac{dF^i_{\text{ext}}}{ds} + \frac{u^i F_{\text{ext},j}F^j_{\text{ext}}}{mc^2}\right)\right] \tag{8.114}$$

if

$$c\tau_e\left|\frac{dF^\beta_{\text{ext}}}{ds}\right| \ll \left|F^\beta_{\text{ext}}\right| \tag{8.115a}$$

or, equivalently

$$\tau_e\left|\frac{d(\gamma\mathbf{F}_{\text{ext}})}{dt}\right| \ll |\mathbf{F}_{\text{ext}}| \, . \tag{8.115b}$$

The approximate solution in (8.114) is not always easy to apply in its present form because the external force on a charged particle is usually exerted by an external electromagnetic field such that

$$\mathbf{F}_{\text{ext}}(t) = e[\mathbf{E}_0(\mathbf{r},t) + \mathbf{u}(t)\times\mathbf{B}_0(\mathbf{r},t)] \tag{8.116}$$

the right-hand side of which involves $\mathbf{u}(t)$ and is a function not only of t but also of $\mathbf{r}$, the position of the particle, so that $d\mathbf{r}/dt = \mathbf{u}(t)$. Landau and Lifshitz [51, sec. 76] have partially alleviated this shortcoming of (8.114) by writing the external force in the four-dimensional form [13, eq. (18-32)]

$$F^i_{\text{ext}} = eF^{ij}(\mathbf{r},t)u_j \tag{8.117a}$$

$$F_{\text{ext},i} = eF_i^j(\mathbf{r},t)u_j \tag{8.117b}$$

$$F^{ij}(\mathbf{r},t) = \begin{Bmatrix} 0 & -cB_{0z} & cB_{0y} & E_{0x} \\ cB_{0z} & 0 & -cB_{0x} & E_{0y} \\ -cB_{0y} & cB_{0x} & 0 & E_{0z} \\ -E_{0x} & -E_{0y} & -E_{0z} & 0 \end{Bmatrix} \tag{8.118}$$

$$F_i^j(\mathbf{r},t) = \begin{Bmatrix} 0 & cB_{0z} & -cB_{0y} & -E_{0x} \\ -cB_{0z} & 0 & cB_{0x} & -E_{0y} \\ cB_{0y} & -cB_{0x} & 0 & -E_{0z} \\ -E_{0x} & -E_{0y} & -E_{0z} & 0 \end{Bmatrix} . \tag{8.119}$$

Use of (8.117a) converts (8.114) to

$$\frac{du^i}{ds} \approx \frac{1}{mc^2}\left[F^i_{\text{ext}} + c\tau_e e\left(\frac{d(F^{ij}u_j)}{ds} + \frac{u^i F_{\text{ext},j}F^j_{\text{ext}}}{mc^2 e}\right)\right] . \tag{8.120}$$

With $d(F^{ij}u_j)/ds$ expressed as

$$\frac{d(F^{ij}u_j)}{ds} = \frac{dF^{ij}}{ds}u_j + F^{ij}\frac{du_j}{ds} = \frac{\partial F^{ij}}{\partial x^k}u^k u_j + F^{ij}\frac{du_j}{ds} \tag{8.121a}$$

and

$$F^{ij}\frac{du_j}{ds} \approx \frac{1}{mc^2}F^{ij}\left[F_{\text{ext},j} + c\tau_e\left(\frac{dF_{\text{ext},j}}{ds} + \frac{u_j F_{\text{ext},k}F^k_{\text{ext}}}{mc^2}\right)\right] \tag{8.121b}$$

which, in view of (8.115), can be approximated by

$$F^{ij}\frac{du_j}{ds} \approx \frac{1}{mc^2}\left(F^{ij}F_{\text{ext},j} + \frac{\tau_e}{mce}F^i_{\text{ext}}F_{\text{ext},k}F^k_{\text{ext}}\right) \tag{8.121c}$$

(8.120) becomes

$$\frac{du^i}{ds} \approx \frac{1}{mc^2}\left[F^i_{\text{ext}}\left(1 + \frac{\tau_e^2}{m^2c^2}F_{\text{ext},j}F^j_{\text{ext}}\right) + \tau_e c\left(e\frac{\partial F^{ij}}{\partial x^k}u^k u_j + \frac{1}{mc^2}(eF^{ij} + u^i F^j_{\text{ext}})F_{\text{ext},j}\right)\right] . \tag{8.122}$$

Under the additional assumption that

$$\left|\frac{\tau_e^2}{m^2c^2}F_{\text{ext},j}F^j_{\text{ext}}\right| \ll 1 \tag{8.123a}$$

which is equivalent to

$$\frac{|\mathbf{F}_{\text{ext}}|\tau_e}{mc} \ll \frac{1}{\gamma} \tag{8.123b}$$

(8.120) becomes equal to the approximate solution derived by Landau and Lifshitz [51, sec. 76]

$$\frac{du^i}{ds} \approx \frac{1}{mc^2}\left[F^i_{\text{ext}} + \tau_e c\left(e\frac{\partial F^{ij}}{\partial x^k}u^k u_j + \frac{1}{mc^2}(eF^{ij} + u^i F^j_{\text{ext}})F_{\text{ext},j}\right)\right] . \tag{8.124}$$

This solution is an accurate approximation to the solution of the LAD equation of motion if the inequalities in (8.111), (8.115), and (8.123) are satisfied. Spohn [38] also concludes that the Landau-Lifshitz solution in (8.124) is effectively the solution to the LAD equation of motion if the external force varies slowly on the scale of the parameter τ_e, that is, if the inequalities in (8.111), (8.115), and (8.123) are satisfied. However, it is emphasized that the Landau-Lifshitz solution in (8.124) is an approximate solution to the LAD equation of motion (8.109), as one can readily see by noting that the radiation reaction terms in (8.124) are not generally equal to the radiation reaction term in the LAD equation of motion (8.109) found by inserting du^i/ds from (8.124) into the radiation reaction term of (8.109). Rohrlich [53] argues, nonetheless, that the Landau-Lifshitz solution in (8.124) is an accurate solution to the LAD equation of motion to within the limitations of classical physics imposed by quantum mechanics [51, p. 208], [54]–[55]; see last paragraph of 8.3.3.

The three-vector part of equation (8.124) can be expressed as [51, p. 213]

$$\begin{aligned}\frac{m}{e}\frac{\mathrm{d}(\gamma\mathbf{u})}{\mathrm{d}t} \approx{}& (\mathbf{E}_0+\mathbf{u}\times\mathbf{B}_0)+\tau_e\gamma\left[\left(\frac{\partial}{\partial t}+\mathbf{u}\cdot\nabla\right)\mathbf{E}_0+\mathbf{u}\times\left(\frac{\partial}{\partial t}+\mathbf{u}\cdot\nabla\right)\mathbf{B}_0\right]\\ &+\frac{e\tau_e}{mc}\left[\frac{(\mathbf{u}\cdot\mathbf{E}_0)}{c}\mathbf{E}_0+c(\mathbf{E}_0+\mathbf{u}\times\mathbf{B}_0)\times\mathbf{B}_0\right]\\ &+\frac{e\tau_e\gamma^2}{mc^2}\left[\frac{(\mathbf{u}\cdot\mathbf{E}_0)^2}{c^2}-|\mathbf{E}_0+\mathbf{u}\times\mathbf{B}_0|^2\right]\mathbf{u}\end{aligned}\tag{8.125a}$$

and the fourth component as

$$\begin{aligned}\frac{mc^2}{e}\frac{\mathrm{d}\gamma}{\mathrm{d}t} \approx{}& \mathbf{u}\cdot\mathbf{E}_0+\tau_e\gamma\left[\mathbf{u}\cdot\left(\frac{\partial}{\partial t}+\mathbf{u}\cdot\nabla\right)\mathbf{E}_0\right]\\ &+\frac{e\tau_e}{m}\left[|\mathbf{E}_0|^2+(\mathbf{u}\times\mathbf{B}_0)\cdot\mathbf{E}_0\right]\\ &+\frac{e\tau_e\gamma^2}{m}\left[\frac{(\mathbf{u}\cdot\mathbf{E}_0)^2}{c^2}-|\mathbf{E}_0+\mathbf{u}\times\mathbf{B}_0|^2\right]\end{aligned}\tag{8.125b}$$

after making use of the relations obtainable from (8.116)–(8.119), namely

$$F^i_{\text{ext}}=e\gamma\left(\mathbf{E}_0+\mathbf{u}\times\mathbf{B}_0\,,\,\frac{\mathbf{u}\cdot\mathbf{E}_0}{c}\right)\tag{8.126a}$$

$$\begin{aligned}\frac{\partial F^{ij}}{\partial x^k}u^k u_j ={}& \frac{\gamma^2}{c}\left[\mathbf{u}\cdot\nabla\left(\mathbf{E}_0+\mathbf{u}\times\mathbf{B}_0\,,\,\frac{\mathbf{u}\cdot\mathbf{E}_0}{c}\right)\right.\\ &\left.+\left(\frac{\partial\mathbf{E}_0}{\partial t}+\mathbf{u}\times\frac{\partial\mathbf{B}_0}{\partial t}\,,\,\frac{1}{c}\mathbf{u}\cdot\frac{\partial\mathbf{E}_0}{\partial t}\right)\right]\end{aligned}\tag{8.126b}$$

$$F^{ij}F_{\text{ext},j}=e\gamma\left[\frac{\mathbf{u}\cdot\mathbf{E}_0}{c}\mathbf{E}_0+c(\mathbf{E}_0+\mathbf{u}\times\mathbf{B}_0)\times\mathbf{B}_0\,,\,(\mathbf{E}_0+\mathbf{u}\times\mathbf{B}_0)\cdot\mathbf{E}_0\right]\tag{8.126c}$$

$$F^j_{\text{ext}}F_{\text{ext},j}=e^2\gamma^2\left[\frac{(\mathbf{u}\cdot\mathbf{E}_0)^2}{c^2}-|\mathbf{E}_0+\mathbf{u}\times\mathbf{B}_0|^2\right]\,.\tag{8.126d}$$

As a check, the power equation (8.125b) can also be obtained by taking the dot product of $\mathbf{u}$ with the force equation (8.125a).

The approximate solution (8.125) to the LAD equation of motion can be derived quite easily from the three-vector equations of motion (7.12a) and (7.17a), which become as $a\to 0$ and the mass is renormalized to a finite value m

$$\begin{aligned}m\frac{\mathrm{d}(\gamma\mathbf{u})}{\mathrm{d}t}={}&\mathbf{F}_{\text{ext}}+\frac{e^2}{6\pi\epsilon_0c^3}\left\{\frac{\mathrm{d}}{\mathrm{d}t}\left[\gamma^2\dot{\mathbf{u}}+\frac{\gamma^4}{c^2}(\mathbf{u}\cdot\dot{\mathbf{u}})\mathbf{u}\right]\right.\\ &\left.-\frac{\gamma^4}{c^2}\left[|\dot{\mathbf{u}}|^2+\frac{\gamma^2}{c^2}(\mathbf{u}\cdot\dot{\mathbf{u}})^2\right]\mathbf{u}\right\}\end{aligned}\tag{8.127a}$$

$$m\gamma\dot{\mathbf{u}}=\mathbf{F}_{\text{ext}}-(\mathbf{F}_{\text{ext}}\cdot\mathbf{u})\frac{\mathbf{u}}{c^2}+\frac{e^2}{6\pi\epsilon_0c^3\gamma}\frac{\mathrm{d}}{\mathrm{d}t}(\gamma^3\dot{\mathbf{u}})\,.\tag{8.127b}$$

Substitute $\gamma\dot{\mathbf{u}}$ from (8.127b) into the right-hand side of (8.127a) to obtain the approximate solution

$$m\frac{\mathrm{d}(\gamma\mathbf{u})}{\mathrm{d}t}\approx\mathbf{F}_{\text{ext}}+\tau_e\left[\frac{\mathrm{d}}{\mathrm{d}t}(\gamma\mathbf{F}_{\text{ext}})-\frac{\gamma^2}{mc^2}\left(|\mathbf{F}_{\text{ext}}|^2-\frac{(\mathbf{u}\cdot\mathbf{F}_{\text{ext}})^2}{c^2}\right)\mathbf{u}\right]\,.\tag{8.128}$$

The relationship

$$\frac{\mathrm{d}}{\mathrm{d}t}(\gamma\mathbf{F}_{\text{ext}})=\frac{\mathrm{d}\gamma}{\mathrm{d}t}\mathbf{F}_{\text{ext}}+\gamma\frac{\mathrm{d}\mathbf{F}_{\text{ext}}}{\mathrm{d}t}\approx\frac{(\mathbf{u}\cdot\mathbf{F}_{\text{ext}})\mathbf{F}_{\text{ext}}}{mc^2}+\gamma\frac{\mathrm{d}\mathbf{F}_{\text{ext}}}{\mathrm{d}t}\tag{8.129}$$

converts (8.128) to

$$\begin{aligned}m\frac{\mathrm{d}(\gamma\mathbf{u})}{\mathrm{d}t}\approx{}&\mathbf{F}_{\text{ext}}+\tau_e\left[\frac{(\mathbf{u}\cdot\mathbf{F}_{\text{ext}})\mathbf{F}_{\text{ext}}}{mc^2}+\gamma\frac{\mathrm{d}\mathbf{F}_{\text{ext}}}{\mathrm{d}t}\right.\\ &\left.-\frac{\gamma^2}{mc^2}\left(|\mathbf{F}_{\text{ext}}|^2-\frac{(\mathbf{u}\cdot\mathbf{F}_{\text{ext}})^2}{c^2}\right)\mathbf{u}\right]\end{aligned}\tag{8.130a}$$

which possesses the corresponding power equation of motion

$$mc^2\frac{\mathrm{d}\gamma}{\mathrm{d}t}\approx\mathbf{u}\cdot\mathbf{F}_{\text{ext}}+\tau_e\left\{\frac{\gamma^2}{mc^2}\left[(\mathbf{u}\cdot\mathbf{F}_{\text{ext}})^2-|\mathbf{F}_{\text{ext}}|^2u^2\right]+\gamma\mathbf{u}\cdot\frac{\mathrm{d}\mathbf{F}_{\text{ext}}}{\mathrm{d}t}\right\}.\tag{8.130b}$$

With $\mathbf{F}_{\text{ext}}=e(\mathbf{E}_0+\mathbf{u}\times\mathbf{B}_0)$ and

$$\begin{aligned}\frac{\mathrm{d}\mathbf{F}_{\text{ext}}}{\mathrm{d}t}&=e\left(\frac{\mathrm{d}\mathbf{E}_0}{\mathrm{d}t}+\mathbf{u}\times\frac{\mathrm{d}\mathbf{B}_0}{\mathrm{d}t}+\dot{\mathbf{u}}\times\mathbf{B}_0\right)\\ &\approx e\left[\frac{\mathrm{d}\mathbf{E}_0}{\mathrm{d}t}+\mathbf{u}\times\frac{\mathrm{d}\mathbf{B}_0}{\mathrm{d}t}+\frac{1}{\gamma}\left(\mathbf{E}_0+\mathbf{u}\times\mathbf{B}_0-(\mathbf{u}\cdot\mathbf{E}_0)\frac{\mathbf{u}}{c^2}\right)\times\mathbf{B}_0\right]\end{aligned}\tag{8.131}$$

(8.130a) becomes

$$\begin{aligned}\frac{m}{e}\frac{\mathrm{d}(\gamma\mathbf{u})}{\mathrm{d}t}\approx{}&(\mathbf{E}_0+\mathbf{u}\times\mathbf{B}_0)+\tau_e\gamma\left[\frac{\mathrm{d}\mathbf{E}_0}{\mathrm{d}t}+\mathbf{u}\times\frac{\mathrm{d}\mathbf{B}_0}{\mathrm{d}t}\right]\\ &+\frac{e\tau_e}{mc}\left[\frac{(\mathbf{u}\cdot\mathbf{E}_0)}{c}\mathbf{E}_0+c(\mathbf{E}_0+\mathbf{u}\times\mathbf{B}_0)\times\mathbf{B}_0\right]\\ &+\frac{e\tau_e\gamma^2}{mc^2}\left[\frac{(\mathbf{u}\cdot\mathbf{E}_0)^2}{c^2}-|\mathbf{E}_0+\mathbf{u}\times\mathbf{B}_0|^2\right]\mathbf{u}\,.\end{aligned}\tag{8.132a}$$

Taking the dot product of $\mathbf{u}$ with (8.132a) produces the power equation of motion

$$\begin{aligned}\frac{mc^2}{e}\frac{\mathrm{d}\gamma}{\mathrm{d}t}\approx{}&\mathbf{u}\cdot\mathbf{E}_0+\tau_e\gamma\mathbf{u}\cdot\frac{\mathrm{d}\mathbf{E}_0}{\mathrm{d}t}\\ &+\frac{e\tau_e}{m}\left[|\mathbf{E}_0|^2+(\mathbf{u}\times\mathbf{B}_0)\cdot\mathbf{E}_0\right]\\ &+\frac{e\tau_e\gamma^2}{m}\left[\frac{(\mathbf{u}\cdot\mathbf{E}_0)^2}{c^2}-|\mathbf{E}_0+\mathbf{u}\times\mathbf{B}_0|^2\right].\end{aligned}\tag{8.132b}$$

These two equations in (8.132) become identical to the ones in (8.125) when the total time derivatives of $\mathbf{E}_0$ and $\mathbf{B}_0$ are rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}_0(\mathbf{r},t) = \left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right)\mathbf{E}_0(\mathbf{r},t), \; \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{B}_0(\mathbf{r},t) = \left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right)\mathbf{B}_0(\mathbf{r},t). \tag{8.133}$$

If desired, the partial derivatives with respect to time in (8.133) and (8.125) can be written from Maxwell's equations as

$$\frac{\partial}{\partial t}\mathbf{E}_0(\mathbf{r},t) = c^2\nabla\times\mathbf{B}_0(\mathbf{r},t), \quad \frac{\partial}{\partial t}\mathbf{B}_0(\mathbf{r},t) = -\nabla\times\mathbf{E}_0(\mathbf{r},t). \tag{8.134}$$

For rectilinear motion the equations in (8.130) reduce to

$$m\gamma^2\dot{u} \approx \frac{F_{\text{ext}}}{\gamma} + \tau_e\frac{\mathrm{d}F_{\text{ext}}}{\mathrm{d}t} \tag{8.135}$$

which agrees with (8.105). Note that for hyperbolic motion discussed in Section 7.1, the external force is constant and the Landau-Lifshitz solution in (8.135) is identical to the exact solution in (7.25)–(7.27).

In the following section the Landau-Lifshitz approximation in (8.125) is solved explicitly for a charged particle moving in a uniform magnetic field.

8.3.3 Charge Moving in a Uniform Magnetic Field

The Landau-Lifshitz approximation (8.124) or (8.125) to the solution to the LAD equation of motion has been determined explicitly by Spohn [38] for a number of cases including that of a charge moving in a uniform magnetic field

$$\mathbf{B}_0 = B_0\hat{\mathbf{z}} \tag{8.136}$$

where B_0 is the constant magnitude of the magnetic induction. To simplify the derivation, assume that the velocity of the charge is zero in the z direction. With $\mathbf{E}_0 = 0$ and $\mathbf{B}_0$ given in (8.136), the approximate equations of motion in (8.125) reduce to

$$m\frac{\mathrm{d}(\gamma\mathbf{u})}{\mathrm{d}t} \approx eB_0\mathbf{u}\times\hat{\mathbf{z}} - \frac{e^2B_0^2\tau_e\gamma^2}{m}\mathbf{u} \tag{8.137a}$$

and

$$mc^2\frac{\mathrm{d}\gamma}{\mathrm{d}t} \approx -\frac{e^2B_0^2\tau_e\gamma^2}{m}u^2. \tag{8.137b}$$

The power equation (8.137b) can also be found by dotting $\mathbf{u}$ into (8.137a).

Next, cross $\mathbf{u}$ into (8.137a) to get

$$m\gamma\mathbf{u}\times\dot{\mathbf{u}} \approx -eB_0u^2\hat{\mathbf{z}}. \tag{8.138a}$$

Taking the magnitude of both sides of this equation further gives

$$m\gamma|\mathbf{u}\times\dot{\mathbf{u}}| \approx eB_0u^2. \tag{8.138b}$$

Since the instantaneous radius of curvature of the trajectory of a particle moving in a plane is given by

$$\mathcal{R}(t) = \frac{u^3}{|\mathbf{u}\times\dot{\mathbf{u}}|} \tag{8.139}$$

(8.138b) combines with (8.139) to yield the instantaneous radius of curvature as

$$\mathcal{R}(t) \approx \frac{m\gamma u}{eB_0}. \tag{8.140}$$

Writing u^2 in terms of γ^2 in (8.137b) results in the first order differential for γ

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} \approx -\frac{e^2B_0^2\tau_e}{m^2}\left(\gamma^2 - 1\right) \tag{8.141}$$

which has the solution

$$\frac{\gamma-1}{\gamma+1} \approx \frac{\gamma_0-1}{\gamma_0+1}\mathrm{e}^{-2\alpha t} \tag{8.142}$$

or

$$\gamma(t) \approx \frac{\gamma_0 + \tanh(\alpha t)}{1 + \gamma_0\tanh(\alpha t)} \tag{8.143}$$

where

$$\alpha = \frac{e^2B_0^2\tau_e}{m^2} \tag{8.144}$$

and $\gamma_0 = \gamma(0)$ is the initial value of $\gamma(t)$ related to the initial speed $u_0 = u(0)$ of the particle by

$$\gamma_0 = \frac{1}{\sqrt{1 - u_0^2/c^2}}. \tag{8.145}$$

From the expression for $\gamma(t)$ in (8.143) we find that the speed $u(t)$ of the charged particle is given by

$$u(t) \approx \frac{\gamma_0 u_0\,\mathrm{sech}(\alpha t)}{\gamma_0 + \tanh(\alpha t)}. \tag{8.146}$$

Therefore, the instantaneous radius of curvature of the particle given in (8.140) becomes

$$\mathcal{R}(t) \approx \frac{\mathcal{R}_0\,\mathrm{sech}(\alpha t)}{1 + \gamma_0\tanh(\alpha t)} \tag{8.147}$$

with the initial radius of curvature given by $\mathcal{R}_0 = m\gamma_0u_0/(eB_0)$.

The kinetic energy ($\mathcal{W}$) of the particle is given by

$$\mathcal{W}(t) = mc^2\gamma(t) \approx \mathcal{W}_0\frac{\gamma_0 + \tanh(\alpha t)}{\gamma_0\left[1 + \gamma_0\tanh(\alpha t)\right]} \tag{8.148}$$

with the initial kinetic energy given by $\mathcal{W}_0 = mc^2\gamma_0$.

For $t \gg 1/\alpha$ the speed and instantaneous radius of curvature decay exponentially as

$$u(t) \sim \frac{2\gamma_0 u_0}{\gamma_0 + 1} e^{-\alpha t} \tag{8.149}$$

and

$$\mathcal{R}(t) \sim \frac{2\mathcal{R}_0}{\gamma_0 + 1} e^{-\alpha t} \sim \frac{m}{eB_0} u(t) \,. \tag{8.150}$$

For $t \ll 1/\alpha$ the speed, instantaneous radius of curvature, and kinetic energy of the particle are given approximately as

$$u(t) \sim u_0 \left(1 - \frac{\alpha}{\gamma_0} t\right) \tag{8.151}$$

$$\mathcal{R}(t) \sim \mathcal{R}_0 \left(1 - \alpha\gamma_0 \, t\right) \tag{8.152}$$

$$\mathcal{W}(t) \sim \mathcal{W}_0 \left(1 - \frac{\alpha\gamma_0 u_0^2}{c^2} t\right) . \tag{8.153}$$

Note from (8.152) and (8.153) that the fractional change in radius of curvature and kinetic energy per unit time are approximately equal when the speed of the charge is approximately equal to the speed of light — in agreement with Shen's results [56]. The expression (8.153) predicts a kinetic energy loss per revolution (with the period of revolution approximated by $\mathcal{T}_0 = 2\pi\mathcal{R}_0/u_0$) that agrees exactly with Plass's result [52, eq. 147], and approximately with Schwinger's result [57, eq. I.10] when the speed of the charge equals approximately the speed of light.

The solution (8.124) for the speed, instantaneous radius of curvature, and kinetic energy given in (8.146)–(8.148) for a charge moving perpendicular to a uniform magnetic field is still an approximate solution because (8.124) is an approximate solution to the Lorentz-Abraham-Dirac equation of motion (8.109). The approximation will be accurate if the inequalities in (8.111), (8.115), and (8.123) are satisfied. In particular, the inequality in (8.115b) becomes

$$e\tau_e B_0 \left| \frac{d(\gamma \mathbf{u})}{dt} \times \hat{\mathbf{z}} \right| \approx \frac{e^2 \tau_e B_0^2 u}{m} \ll eB_0 u \tag{8.154a}$$

or, in terms of B_0

$$B_0 \ll \frac{m}{e\tau_e} \tag{8.154b}$$

and the inequality in (8.123b) gives

$$\frac{e\tau_e B_0 u}{mc} \ll \frac{1}{\gamma} \tag{8.155a}$$

or

$$B_0 \ll \frac{m}{e\tau_e}\sqrt{1 - \frac{u_0^2}{c^2}} \tag{8.155b}$$

since $u/c < 1$ and $u \le u_0$.

This last inequality, which is more restrictive than the inequality in (8.154b), can be proven to be a necessary as well as sufficient condition for the Landau-Lifshitz solution to be a good approximation to the exact LAD solution by comparing the approximate $d\gamma/dt$ in (8.141) with the exact $d\gamma/dt$ given from (7.17b) as

$$\frac{d\gamma}{dt} = \tau_e \gamma \left(\frac{d^2\gamma}{dt^2} - \frac{\gamma^3}{c^2} |\dot{\mathbf{u}}|^2 \right) . \tag{8.156}$$

Note that $\mathbf{F}_{ext} \cdot \mathbf{u} = eB_0(\mathbf{u} \times \hat{\mathbf{z}}) \cdot \mathbf{u} = 0$ in (7.17b). If the approximate solution in (8.137) is accurate, it can be inserted into the right-hand side of (8.156) to get a quantity that is approximately equal to the right-hand side of (8.141). From the approximate solution in (8.137) we find

$$\frac{d^2\gamma}{dt^2} \approx -2\alpha\gamma \frac{d\gamma}{dt} = 2\alpha^2 \gamma(\gamma^2 - 1) \tag{8.157a}$$

$$|\dot{\mathbf{u}}|^2 \approx \frac{\alpha c^2}{\tau_e \gamma^4} (1 + \alpha\tau_e)(\gamma^2 - 1) \tag{8.157b}$$

which, when substituted into (8.156), yields

$$\frac{d\gamma}{dt} \approx -\alpha(\gamma^2 - 1)[1 - \alpha\tau_e(2\gamma^2 - 1)] . \tag{8.158}$$

Comparing (8.158) with (8.141), we see that the Landau-Lifshitz solution is a good approximation to the exact solution of the LAD equation of motion if and only if

$$\alpha\tau_e\gamma^2 \ll 1 \tag{8.159}$$

which can be re-expressed as

$$B_0 \ll \frac{m}{e\tau_e}\sqrt{1 - \frac{u_0^2}{c^2}} . \tag{8.160}$$

For an electron (8.160) becomes approximately

$$B_0 \ll 10^{12}\sqrt{1 - \frac{u_0^2}{c^2}} \ \text{Tesla}. \tag{8.161}$$

This means that the solution given in (8.146)–(8.148) for the motion of a slowly moving electron ($u_0^2/c^2 \ll 1$) in a uniform magnetic field becomes an inaccurate approximation to the exact solution of the LAD equation of motion (8.109) if the value of the magnitude of the uniform magnetic field is greater than about 10^{11} Tesla (10^{15} Gauss). This enormous value is on the order of the estimated magnetic fields in neutron stars called magnetars, which have the highest known magnetic fields in the universe. For extremely high energy electrons, (8.161) implies that the approximate solution in (8.146)–(8.148) becomes inaccurate for much smaller values of the applied magnetic field. Thus, the trajectory of extremely high energy charged particles in observable magnetic fields may reveal differences between the Landau-Lifshitz and exact solutions as well as limitations on the applicability of the LAD equation of motion.

For the slowly moving charge in a uniform magnetic field, the condition in (8.160) for the Landau-Lifshitz solution (8.124) to be an accurate solution to the LAD equation of motion (8.109) is equivalent to the condition in (8.81) (and (8.170) of Section 8.5 below) sufficient for the corrected LAD equation of motion (8.168) to satisfy causality while maintaining a non-negative radiated energy during the transition intervals when the external force is applied for a finite time. These comparable limiting conditions on the corrected LAD equation of motion and the Landau-Lifshitz approximate solution to the original (uncorrected) LAD equation of motion is compatible with Rohrlich's contention [53] that the difference between the Landau-Lifshitz approximate solution and the exact solution to the LAD equation of motion lies within the limitations of classical physics imposed by quantum mechanics.

8.4 The Finite Difference Equation of Motion

It was shown in Section 8.2 that if the evaluation of the self electromagnetic force is done properly near the time the external force is first applied, a correction force $\mathbf{f}_a(t)$ must be added to the original equation of motion during the short time interval $0 \le t \le \Delta t$. With the proper choice of $\mathbf{f}_a(t)$, this slight modification removes the pre-acceleration from the solution to the uncorrected equation of motion. Power series solutions obtained in Section 8.3 to the original uncorrected equation of motion also eliminate the pre-acceleration, but at the expense of introducing spurious delta functions and their derivatives that do not satisfy either the uncorrected or corrected equation of motion near the time the external force is first applied.

Through the years a number of other methods have been proposed to eliminate the pre-acceleration that arises in the solution to the original uncorrected equation of motion (7.1) [58]–[62]. However, none of these alternative methods have been entirely successful because they either eliminate a priori all derivatives of acceleration [60]–[62], [63] or they sum infinite series expansions that neglect nonlinear terms [58]–[59]. These latter methods [58]–[59] that have been proposed to eliminate the pre-acceleration or runaway solutions from the equation of motion involve determining explicitly the infinite series of $\mathbf{O}(a)$ terms in the self electromagnetic force in (3.3) of the moving charged insulator of radius a. Specifically, Page [17] wrote down, without showing the derivation, this infinite series and summed it in closed form to reveal that the self electromagnetic force in the proper frame of reference of the charge can be expressed as

$$\mathbf{F}_{\text{em}}(t) = \frac{e^2}{12\pi\epsilon_0 a^2 c}\mathbf{u}(t-2a/c), \quad u=0 \tag{8.162a}$$

or, in an inertial frame in which the charge is moving with nonzero velocity much less than the speed of light, as

$$\mathbf{F}_{\text{em}}(t) = \frac{e^2}{12\pi\epsilon_0 a^2 c}[\mathbf{u}(t-2a/c) - \mathbf{u}(t)], \quad (u/c)^2 \ll 1 \tag{8.162b}$$

provided all nonlinear terms involving products of the time derivatives of the velocity are neglected and the correction force $\mathbf{f}_a(t)$*, explained in Section 8.2, is ignored.*

Equations (8.162) can also be found by discarding all but the first series in the double infinite series that Schott [64] derived for the self electromagnetic force on the noncontracting sphere (Abraham's nonrelativistically rigid model rather than Lorentz's relativistically rigid model of the electron). The infinite number of discarded series involve nonlinear products in Schott's expression that would change for the relativistically rigid model of the electron; however, the linear first series is the same for both relativistically and nonrelativistically rigid models of the electron. (A simple proof of (8.162) is given in Appendix D.)

When the self electromagnetic force (8.162b) is used in the derivation of the equation of motion given in Chapter 5, we obtain

$$\mathbf{F}_{\text{ext}}(t) = (m_{\text{ins}} + M_0)\dot{\mathbf{u}}(t) - \frac{e^2}{12\pi\epsilon_0 a^2 c}[\mathbf{u}(t-2a/c) - \mathbf{u}(t)], \quad (u/c)^2 \ll 1 \tag{8.163}$$

for the nonrelativistic equation of motion. Again, the nonlinear product terms have been neglected in (8.163), and the negative bare mass M_0 is given as $-m_{\text{es}}/3$ in (5.11). (If the bare mass were omitted in (8.163), the rest mass of the charged shell would not equal $m_{\text{ins}} + m_{\text{es}}$.) A relativistic generalization of the finite difference equation (8.162b) has been obtained by Caldirola [59], [65]. Similarly, a relativistic generalization of (8.163) to an arbitrary inertial reference frame can be found by replacing $\mathbf{F}_{\text{ext}}(t)$ with $F^i_{\text{ext}}(s)$, $\dot{\mathbf{u}}(t)$ with $du^i(s)/ds$, $\mathbf{u}(t-2a/c)$ with $u^i(s-2a)$,[9] and $\mathbf{u}(t)$ with $g(s)u^i(s)$, the scalar factor $g(s)$ inserted to make $F^i_{\text{ext}}u_i(s) = 0$. We then have

$$F^i_{\text{ext}}(s) = (m_{\text{ins}} + M_0)\, c^2 \frac{du^i(s)}{ds} - \frac{e^2}{12\pi\epsilon_0 a^2}\left[u^i(s-2a) - g(s)u^i(s)\right] \tag{8.164a}$$

with

[9] In the proper frame ($\mathbf{u}(t)=0$), the three-vector part of the relativistic generalization $u^i(s-2a)$ reduces to $\mathbf{u}(t-2a/c)/\sqrt{1-u^2(t-2a/c)/c^2}$ rather than just $\mathbf{u}(t-2a/c)$. However, to within the approximation that the nonlinear terms involving products of time derivatives of velocity are neglected in the proper frame, $\mathbf{u}(t-2a/c)/\sqrt{1-u^2(t-2a/c)/c^2}$ can be replaced by $\mathbf{u}(t-2a/c)$.

$$F^i_{\rm ext}(s)u_i(s) = 0 = u_i(s)u^i(s-2a) - g(s)\,. \tag{8.164b}$$

so that $g(s) = u_i(s)u^i(s-2a) = u_j(s-2a)u^j(s)$ and

$$F^i_{\rm ext} = (m_{\rm ins} + M_0)\,c^2\frac{{\rm d}u^i}{{\rm d}s} - \frac{e^2}{12\pi\epsilon_0 a^2}\left[u^i(s-2a) - u^i(s)u_j(s)u^j(s-2a)\right]\,. \tag{8.165}$$

Notwithstanding the appealing simplicity of the finite difference equation (8.163) and its relativistic generalization (8.165), there is little justification to accept them as valid equations of motion that are accurate beyond the usual radiation reaction terms, since (8.162) and (8.163) neglect all nonlinear product terms (involving derivatives of velocity), which are not necessarily negligible for the Lorentz model of the electron beyond the $\ddot{\mathbf{u}}$ proper-frame radiation reaction term.

It can be shown that the nonlinear and linear parts of the self electromagnetic force are both zero for certain radiationless motion of a *nonrelativistically* rigid spherical shell, namely, when the shell oscillates with an amplitude smaller than its radius and a period equal to $2a/c$ [66]–[68]. These radiationless oscillations with the self electromagnetic force (8.162) equal to zero would not, in general, be self sustaining, that is, $\mathbf{F}_{\rm ext}(t)$ would not equal zero in (8.163) except for the special case of $m_{\rm ins}+M_0$ equal to zero. (For Lorentz's relativistically rigid model of the electron, Pearle [68] has shown that bounded radiationless motions do not exist.)

The work of Herglotz [39] and Wildermuth [40], discussed in Section 8.2, would suggest that the finite difference (linearized) equation of motion (8.163) does not, in general, eliminate the pre-acceleration, that is, runaway solutions for $t < 0$. This can be demonstrated for rectilinear motion by letting the velocity in (8.163) have $\exp(qt)$ time dependence when $\mathbf{F}_{\rm ext}(t)$ is set equal to zero. The equation that results for q, when the material mass of the insulator is negligible, is then

$$e^{-2aq/c} = 1 - aq/(2c) \tag{8.166}$$

which has the positive real solution

$$q \approx \frac{2(1-5e^{-4})}{1-4e^{-4}}\frac{c}{a} \approx 1.96\frac{c}{a}\,. \tag{8.167}$$

(If the mass of the insulator is not negligible, the equation for q also has a real positive solution provided a is small enough for the value of $m_{\rm ins}+M_0$ to be negative.) This failure of the finite difference equation of motion (8.163) to eliminate the homogeneous runaway solutions (so that pre-acceleration will still arise when the asymptotic condition in (8.12) is applied), coupled with the fact that the finite difference equation (8.163) neglects all nonlinear terms involving products of the time derivatives of velocity, leaves little reason to prefer (8.163) or its relativistic generalization (8.165) to the equation of motion that simply neglects the $O(a)$ terms in (7.1). Moreover, like (7.1) the finite difference equation of motion (8.163) neglects the transition force f^i_a in the corrected equation of motion (8.43). And, as Section 8.2 shows, it is this small but important correction to the conventional equation of motion that eliminates the noncausal pre-acceleration.

8.5 Renormalization of the Equation of Motion

The four-vector form of the Lorentz-Abraham equation of motion for a charged insulating sphere of radius a, charge e, and mass m equal to $m_{\rm es}+m_{\rm ins}$ is given in (7.1) and in (8.43) with the transition forces $f^i_a = \sum_{n=1}^N f^i_{an}$ that remove the pre-acceleration (and pre-deceleration) from the original equation of motion. If the radius a of the charged sphere is allowed to approach zero, while keeping the mass m at a fixed value (mass renormalization), the contribution of the $O(a)$ terms goes to zero (except possibly at nonanalytic transition points in time of the externally applied force) and the original equation of motion in (7.1) becomes identical to the Lorentz-Abraham-Dirac (LAD) equation of motion [12]. The equation of motion in (8.43) then becomes equal to the LAD equation of motion modified by transition forces in the proper time intervals $\Delta t_a \to 0$ following the points in times where the externally applied force is not locally expandable in a power series; for example, at the time $t=0$ when the external force is first applied. This modified LAD equation of motion can be written from (8.43) as

$$F^i_{\rm ext} + \sum_{n=1}^N f^i_{an} = mc^2\frac{{\rm d}u^i}{{\rm d}s} - \frac{e^2}{6\pi\epsilon_0}\left(\frac{{\rm d}^2u^i}{{\rm d}s^2} + u^i\frac{{\rm d}u_j}{{\rm d}s}\frac{{\rm d}u^j}{{\rm d}s}\right)\,. \tag{8.168}$$

The transition forces f^i_{an} in (8.168), which approach delta functions and their derivatives as $a \to 0$, remove the noncausal pre-acceleration and pre-deceleration from the original LAD equation of motion. Moreover, the analysis of rectilinear motion in Section 8.2.5 shows that the jumps in velocity[10] associated with the transition forces $f^i_{an}(t)$ can be chosen to conserve momentum-energy in the equation of motion while maintaining a non-negative radiated energy during the transition intervals, provided the proper-frame acceleration outside the transition intervals is bounded by

$$\frac{|\dot{\mathbf{u}}|\tau_e}{c} \ll 1\,, \qquad \tau_e = \frac{e^2}{6\pi\epsilon_0 mc^3} \tag{8.169}$$

the same condition in (8.24c) for neglecting the $O(a)$ terms in the equation of motion of the extended charged sphere before the mass is renormalized (and thus $\tau_e = 4a/(3c)$). This condition is satisfied if the external force obeys the inequality

[10] A consequence of letting $a \to 0$, and thus $\Delta t_a \to 0$, while renormalizing the mass to a finite value is that changes in velocity across the transition intervals Δt_a become abrupt.

$$\frac{|\mathbf{F}_{\text{ext}}|\tau_e}{mc} \ll 1\,. \tag{8.170}$$

There is some justification, even in classical physics, for renormalizing the mass $m_{\text{es}} + m_{\text{ins}}$ to a finite value m as $a \to 0$ and $m_{\text{es}} = e^2/(8\pi\epsilon_0 ac^2) \to \infty$ to obtain the equation of motion of a point charge. It was mentioned in Chapter 5 that m_{ins} may be negative because it can include gravitational and other attractive formation energies [20], [21]. Thus, as $a \to 0$ it is conceivable that $m_{\text{ins}} \to -\infty$ and that $\lim_{a\to\infty}(m_{\text{es}} + m_{\text{ins}}) = m$, the measured rest mass. It is especially disconcerting, therefore, that for the renormalized causal equation of motion (8.168), the restriction in (8.169) on the magnitude of the acceleration, or in (8.170) on the magnitude of the externally applied force, is needed to ensure this equation of motion satisfies conservation of momentum-energy while keeping the value of the energy radiated during the transition intervals equal to or greater than zero. For the extended charged sphere, the condition (8.169) is understandable because it is the same as the condition in (8.24c) needed, in general, to neglect the $O(a)$ terms in the proper-frame equation of motion (and to ensure that the infinite series comprising $O(a)$ converges). This condition merely implies that the $O(a)$ terms may not be negligible if the speed of the charged sphere changes by an appreciable fraction of the speed of light in the time it takes light to cross the sphere. As $a \to 0$ and the mass is renormalized to a finite value m, however, the conditions in (8.24) are all satisfied except possibly at the nonanalytic transition points in time of the external force where the first and higher order derivatives of the velocity can become singular. One would hope that accounting for these singularities at the transition intervals as $a \to 0$ by the transition forces f^i_{an} in the modified LAD equation of motion (8.168) would rectify the equation of motion regardless of the magnitude of the externally applied force. This is not the case, however, if the external force is large enough to disobey (8.170) because then we have shown that the value of the energy radiated during the transition intervals can become negative, that is, the momentum-energy of the charged particle is not conserved by the equation of motion while maintaining a non-negative radiated energy during the transition intervals; see Footnote 8.

For an electron in an external electric field E, the inequality in (8.170) is satisfied unless $E \not\ll mc/(e\tau_e) = 6\pi\epsilon_0 m^2c^4/e^3 = 2.7 \times 10^{20}$ Volts/meter, an enormously high electric field. Nonetheless, an equation of motion of a mass-renormalized point charge that is both causal and conserves momentum-energy while avoiding a negative radiated energy during the transition intervals no matter how large the value of the external force does not result by simply equating the sum of the point-charge radiation reaction force and the externally applied force to the relativistic Newtonian acceleration force (measured rest mass times relativistic acceleration) and inserting delta-function transition forces at the nonanalytic points in time of the external force to obtain (8.168). A causal classical equation of motion of a mass-renormalized point charge that also conserves momentum-energy with a non-negative radiated energy during the transition intervals for arbitrarily large values of the

external force, if it exists, must involve a more complicated combining of the Newtonian and radiation reaction forces with the externally applied force than just a summation.[11] It seems prudent, therefore, to simply accept (8.168) as the classical causal equation of motion of a mass-renormalized point charge under the restriction in (8.170) on the magnitude of the externally applied force, or to tolerate the noncausality in the original LAD equation of motion given by (8.168) without the transition forces f^i_{an}, especially since these transition forces lead to jumps in velocity across the transition intervals that are not uniquely determined by the externally applied force.

Ultimately, a fully satisfactory equation of motion of a mass-renormalized point charge may require a unified theory of inertial and electromagnetic forces as well as the introduction of quantum effects. Renormalization of the mass of the charged sphere as its radius shrinks to zero is an attempt to extract the equation of motion of the point electron from the classical self electromagnetic forces of an extended charge distribution. Such attempts, as Dirac wrote [69], "bring one up against the problem of the structure of the electron, which has not yet received any satisfactory solution."

[11] To recapitulate the essence of the argument, if the external force $\mathbf{F}_{\text{ext}}(t)$ is an analytic function of complex t in a neighborhood of the real t axis except at the time $t = 0$ when the external force is first applied to the charge e, we have proven from the Maxwell-Lorentz equations that as $a \to 0$ the radiation reaction four-force equals $[e^2/(6\pi\epsilon_0)](d^2u^i/ds^2 + u^i du_j/ds\, du^j/ds)$ for all $t > 2\gamma a/c = 0^+$. Therefore, if the external four-force plus the radiation reaction four-force equals $mc^2 du^i/ds$, where m is the measured rest mass of the charge, then

$$F^i_{\text{ext}} + \frac{e^2}{6\pi\epsilon_0}\left(\frac{d^2u^i}{ds^2} + u^i\frac{du_j}{ds}\frac{du^j}{ds}\right) - mc^2\frac{du^i}{ds} = 0 \quad \underline{\text{during } t > 0^+} \tag{8.171}$$

for a point charge e with fixed rest mass m. Between $t = 0$ and $t = 0^+$, the left-hand side of (8.171) is not equal to zero and its values during this short time interval that approaches zero as $a \to 0$ cannot generally be determined by the Maxwell-Lorentz equations (because of the unknown time dependence of the velocity during this interval). Yet, if a solution exists, the left-hand side of (8.171) must equal some function (or generalized function such as a delta function) between $t = 0$ and $t = 0^+$ that can be denoted by $-f^i_{a1}(s)$ and that is needed to preserve causality. If the external force has N nonanalytic points in time, there are N transition distribution functions $f^i_{an}(s)$ needed to preserve causality and the resulting equation of motion is given in (8.168). Since we have proven that causal solutions to (8.168) can violate the requirement of momentum-energy conservation with non-negative radiated energy during the transition intervals if the magnitude of the external force is not restricted by the inequality in (8.170), it follows that (8.171) cannot hold exactly for extremely large external forces. In other words, *the generalization of Newton's second law of motion to classical point charges with renormalized mass is incompatible with the Maxwell-Lorentz equations and conservation of energy if the magnitude of the externally applied force becomes too large.* The renormalized causal classical equation of motion of a *point* charge encounters a "high acceleration catastrophe."

Appendices

A

Derivation and Transformation of Small-Velocity Force and Power

In this appendix, we derive the proper-frame force equation of motion (3.3) and the small-velocity Lorentz power equation of motion (3.4) directly from the self electromagnetic force and power integrals of the spherical shell of charge. We then transform (3.3) relativistically to obtain the force equation of motion (2.1) for arbitrary velocity. A relativistic transformation of (3.4), however, leads to the erroneous result (3.5) for the power equation of motion rather than the power equation of motion (2.4). We also show that (2.4) does not transform covariantly, thereby confirming that the general power equation of motion (2.4) is not produced by a relativistic transformation of the small-velocity power equation of motion (3.4); see Section 3.1.

Lorentz [4] and numerous modern physics texts, for example [13], [18], [34], have derived the Lorentz force equation of motion (3.3) in the proper (instantaneous rest) frame. But none, as far as I am aware, have directly derived the small-velocity power equation of motion (3.4), because it requires taking into account the variation of the velocity over the charge distribution. Of course, (3.4) could be obtained by letting u/c become much less than unity in the general power equation of motion (2.4), which was rigorously derived by Schott [16]. (As discussed in Chapter 3, Schott's impressive derivation is so involved and lengthy that it discourages a detailed re-examination. Thus we provide an alternative, simpler, yet rigorous derivation of the general force and power equations of motion, (2.1) and (2.4), in Appendix B.)

A.1 Derivation of the Small-Velocity Force and Power

A.1.1 Derivation of the Proper-Frame Force

The self electromagnetic force on the spherical shell of charge in its proper (instantaneous rest) inertial frame of reference can be expressed by the Lorentz force integral in (3.1) with $\mathbf{u}(\mathbf{r},t) = 0$, that is

$$\mathbf{F}_{\mathrm{em}}(t) = \int\limits_{\text{charge}} \mathbf{E}(\mathbf{r},t)\,\mathrm{d}e\,, \quad \mathbf{u}(\mathbf{r},t) = 0 \tag{A.1}$$

where the element of charge $\rho(\mathbf{r},t)\mathrm{d}V$ in (3.1) is relabeled $\mathrm{d}e$ in (A.1). The self electric field $\mathbf{E}(\mathbf{r},t)$ on the charge $\mathrm{d}e$ at position $\mathbf{r}$ is produced by the remainder of the charge in the spherical shell. Specifically, the charge $\mathrm{d}e'$ at the position $\mathbf{r}'(t')$ produces an electric field $\mathrm{d}\mathbf{E}(\mathbf{r},t)$ given by [13]

$$\mathrm{d}\mathbf{E}(\mathbf{r},t) = \frac{\mathrm{d}e'}{4\pi\epsilon_0\left[1-\hat{\mathbf{R}}'\cdot\mathbf{u}/c\right]^3}\left\{\frac{\hat{\mathbf{R}}'}{R'c^2}\times\left[\left(\hat{\mathbf{R}}'-\frac{\mathbf{u}(\mathbf{r}',t')}{c}\right)\times\dot{\mathbf{u}}(\mathbf{r}',t')\right]\right.$$
$$\left.+\frac{1}{R'^2}\left[1-\frac{u^2(\mathbf{r}',t')}{c^2}\right]\left[\hat{\mathbf{R}}'-\frac{\mathbf{u}(\mathbf{r}',t')}{c}\right]\right\} \tag{A.2}$$

where $\mathbf{u}(\mathbf{r}',t')$ and $\dot{\mathbf{u}}(\mathbf{r}',t')$ refer to the velocity and acceleration of the charge $\mathrm{d}e'$ at the retarded time

$$t' = t - \frac{R'}{c} \tag{A.3}$$

that is

$$\mathbf{u}(\mathbf{r}',t') = \frac{\mathrm{d}\mathbf{r}'(t')}{\mathrm{d}t'} \tag{A.4}$$

$$\dot{\mathbf{u}}(\mathbf{r}',t') = \frac{\mathrm{d}^2\mathbf{r}'(t')}{\mathrm{d}t'^2}\,. \tag{A.5}$$

The vector $\mathbf{R}'$ is defined as the difference between the position $\mathbf{r}$ of $\mathrm{d}e$ and the position $\mathbf{r}'(t')$ of $\mathrm{d}e'$ *at the retarded time* t'

$$\mathbf{R}' = \mathbf{r} - \mathbf{r}'(t')\,. \tag{A.6}$$

When one expands $\mathbf{R}'$, $\mathbf{u}(\mathbf{r}',t')$, and $\dot{\mathbf{u}}(\mathbf{r}',t')$ about the present time t, as the radius of the charge shell becomes small, one obtains the following power series expansion of $\mathrm{d}\mathbf{E}(\mathbf{r},t)$ in (A.2)

$$\mathrm{d}\mathbf{E}(\mathbf{r},t) = \frac{\mathrm{d}e'}{4\pi\epsilon_0}\left\{\frac{\hat{\mathbf{R}}}{R^2} - \frac{1}{2c^2R}\left[\hat{\mathbf{R}}\cdot\dot{\mathbf{u}}(\mathbf{r}',t)\hat{\mathbf{R}}+\dot{\mathbf{u}}(\mathbf{r}',t)\right] - \frac{3}{8}\frac{|\dot{\mathbf{u}}(\mathbf{r}',t)|^2}{c^4}\hat{\mathbf{R}}\right.$$
$$\left.+\frac{3}{4}\frac{\hat{\mathbf{R}}\cdot\dot{\mathbf{u}}(\mathbf{r}',t)}{c^4}\dot{\mathbf{u}}(\mathbf{r}',t)+\frac{3}{8}\frac{[\hat{\mathbf{R}}\cdot\dot{\mathbf{u}}(\mathbf{r}',t)]^2}{c^4}\hat{\mathbf{R}}+\frac{2}{3}\frac{\ddot{\mathbf{u}}(\mathbf{r}',t)}{c^3}+\mathbf{O}(R)\right\} \tag{A.7}$$

with $\mathbf{R} = \mathbf{r} - \mathbf{r}'(t)$, and $\mathbf{u}(\mathbf{r}',t) = 0$. Equation (A.7) differs from the corresponding expression in [13] where the dependence of $\mathbf{R}'$ in (A.6) upon the retarded time is ignored. Also, (A.7) differs from the corresponding equation in [18] and [34] as well as [13] by including the spatial dependence of the acceleration and its time derivative over the charge distribution. Both of these differences vanish, as we shall see below, when (A.7) is integrated over $\mathrm{d}e'$ to get $\mathbf{E}(\mathbf{r},t)$ and then $\mathbf{E}(\mathbf{r},t)$ is integrated over $\mathrm{d}e$ in (A.1) to get the self electromagnetic force and the Lorentz force equation of motion. *These differences do not vanish in the subsequent derivation of the self electromagnetic power and thus cannot be ignored in the derivation of the power equation of motion.*

The acceleration $\dot{\mathbf{u}}(\mathbf{r}',t)$ of the charge $\mathrm{d}e'$ at the position $\mathbf{r}'(t)$ can be written in terms of the acceleration $\dot{\mathbf{u}}(t)$ of the center of the shell by using the requirement of special relativity that the spherical shell contracts to an oblate spheroid (to order R^2) as the speed increases. Specifically, we find for $\mathbf{u}(\mathbf{r}',t) = 0$

$$\dot{\mathbf{u}}(\mathbf{r}',t) = \dot{\mathbf{u}}(t) - \frac{\mathbf{r}'\cdot\dot{\mathbf{u}}(t)}{c^2}\dot{\mathbf{u}}(t) + \mathbf{O}(R^2) \tag{A.8}$$

and

$$\ddot{\mathbf{u}}(\mathbf{r}',t) = \ddot{\mathbf{u}}(t) + \mathbf{O}(R)\,. \tag{A.9}$$

Substituting (A.8) and (A.9) into (A.7) and integrating over $\mathrm{d}e'$ gives the final form for the self electric field at $(\mathbf{r},t)$ in terms of the acceleration $(\dot{\mathbf{u}})$ and the time derivative of acceleration $(\ddot{\mathbf{u}})$ of the center of the shell of charge[1]

$$\mathbf{E}(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int\limits_{\text{charge}}\left\{\frac{\hat{\mathbf{R}}}{R^2}+\frac{1}{2c^2R}\left[\frac{\mathbf{r}'\cdot\dot{\mathbf{u}}}{c^2}-1\right]\left[(\hat{\mathbf{R}}\cdot\dot{\mathbf{u}})\hat{\mathbf{R}}+\dot{\mathbf{u}}\right]\right.$$
$$\left.+\frac{3}{8}\frac{\hat{\mathbf{R}}}{c^4}\left[(\hat{\mathbf{R}}\cdot\dot{\mathbf{u}})^2-|\dot{\mathbf{u}}|^2\right]+\frac{3(\hat{\mathbf{R}}\cdot\dot{\mathbf{u}})\dot{\mathbf{u}}}{4c^4}+\frac{2\ddot{\mathbf{u}}}{3c^3}+\mathbf{O}(R)\right\}\mathrm{d}e', \quad u = 0. \tag{A.10}$$

Next insert the self electric field from (A.10) into (A.1) and perform the double integration over the shell of charge. All the terms with an odd number of products of $\hat{\mathbf{R}}$ or $\mathbf{r}'$ integrate to zero and the remaining even product terms integrate to give the familiar expression for the self electromagnetic force in the proper frame of reference

$$\mathbf{F}_{\mathrm{em}}(t) = -\frac{e^2}{6\pi\epsilon_0 ac^2}\dot{\mathbf{u}} + \frac{e^2}{6\pi\epsilon_0 c^3}\ddot{\mathbf{u}} + \mathbf{O}(a), \quad u = 0\,. \tag{A.11}$$

Equating the sum of the externally applied force and the self electromagnetic force to zero, as Lorentz did in his original work [4], one obtains the Lorentz force equation of motion (3.3) in the proper frame of the spherical shell of charge.

[1] Actually, the $\mathrm{d}e'$ integration should be over all charge except the charge $\mathrm{d}e = \rho\mathrm{d}V$ on which we are calculating the self force. However, the field produced by $\rho\mathrm{d}V$ within its own volume $\mathrm{d}V$ approaches zero as $\mathrm{d}V \to 0$ for both stationary and moving charge [73, sec. 2.1.10]. Thus, the value of the integration over all charge is the same as its value with $\mathrm{d}e$ omitted as $\mathrm{d}e \to 0$.

A.1.2 Derivation of the Small-Velocity Power

The power delivered to the moving charge by the self electromagnetic forces within the charge distribution is given by the charge integral in (3.2), namely

$$P_{\rm em}(t) = \int_{\rm charge} \mathbf{u}(\mathbf{r},t)\cdot\mathbf{E}(\mathbf{r},t)\mathrm{d}e \qquad \text{(A.12)}$$

where again the element of charge $\rho(\mathbf{r},t)\mathrm{d}V$ in (3.2) is relabeled as $\mathrm{d}e$ in (A.12). The velocity $\mathbf{u}(\mathbf{r},t)$ of the charge distribution in (A.12) is arbitrary. For small velocity, $\mathbf{u}(\mathbf{r},t)$ can be written in terms of the velocity and acceleration of the center of the shell by using the information that the spherical shell contracts to an oblate spheroid (to order R^2) as the speed of the charge increases; specifically

$$\mathbf{u}(\mathbf{r},t) = \mathbf{u}(t) - \frac{\mathbf{r}\cdot\mathbf{u}(t)}{c^2}\dot{\mathbf{u}}(t) + \mathbf{O}\left(\frac{u^2}{c^2}, R^2\right). \qquad \text{(A.13)}$$

Repeating the derivation that led to (A.10), with small-velocity instead of zero velocity, shows that (A.10) also remains valid to order u^2/c^2, that is

$$\mathbf{E}(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int_{\rm charge}\left\{\frac{\hat{\mathbf{R}}}{R^2} + \frac{1}{2c^2R}\left[\frac{\mathbf{r}'\cdot\dot{\mathbf{u}}}{c^2} - 1\right]\left[(\hat{\mathbf{R}}\cdot\dot{\mathbf{u}})\hat{\mathbf{R}} + \dot{\mathbf{u}}\right]\right.$$
$$\left. + \frac{3}{8}\frac{\hat{\mathbf{R}}}{c^4}\left[(\hat{\mathbf{R}}\cdot\dot{\mathbf{u}})^2 - |\dot{\mathbf{u}}|^2\right] + \frac{3(\hat{\mathbf{R}}\cdot\dot{\mathbf{u}})\dot{\mathbf{u}}}{4c^4} + \frac{2\ddot{\mathbf{u}}}{3c^3} + \mathbf{O}\left(\frac{u^2}{c^2}, R\right)\right\}\mathrm{d}e'. \qquad \text{(A.14)}$$

Substitution of $\mathbf{E}(\mathbf{r},t)$ from (A.14) and $\mathbf{u}(\mathbf{r},t)$ from (A.13) into (A.12) allows $P_{\rm em}(t)$ to be written as

$$P_{\rm em}(t) = \mathbf{u}(t)\cdot\int_{\rm charge}\mathbf{E}(\mathbf{r},t)\mathrm{d}e - \dot{\mathbf{u}}(t)\cdot\iint_{\rm charge}\frac{\hat{\mathbf{R}}}{R^2}\frac{[\mathbf{r}\cdot\mathbf{u}(t)]}{4\pi\epsilon_0c^2}\mathrm{d}e'\mathrm{d}e + O\left(\frac{u^2}{c^2}, a\right). \qquad \text{(A.15)}$$

The integral of the electric field in (A.15) is just the self electromagnetic force given in (A.11). The second integral in (A.15) is the extra term that arises because the velocity of the charge distribution varies with position around the shell. It evaluates to

$$-\dot{\mathbf{u}}(t)\cdot\iint_{\rm charge}\frac{\hat{\mathbf{R}}(\mathbf{r}\cdot\mathbf{u}(t))}{4\pi\epsilon_0c^2R^2}\mathrm{d}e'\mathrm{d}e = -\frac{e^2}{24\pi\epsilon_0ac^2}\mathbf{u}\cdot\dot{\mathbf{u}}. \qquad \text{(A.16)}$$

The self electromagnetic power can thus be written as

$$P_{\rm em}(t) = \mathbf{u}\cdot\mathbf{F}_{\rm em}(t) - \frac{e^2}{24\pi\epsilon_0ac^2}\mathbf{u}\cdot\dot{\mathbf{u}} + O\left(\frac{u^2}{c^2}, a\right) \qquad \text{(A.17)}$$

or

$$P_{\rm em}(t) = -\frac{5e^2}{24\pi\epsilon_0ac^2}\mathbf{u}\cdot\dot{\mathbf{u}} + \frac{e^2}{6\pi\epsilon_0c^3}\mathbf{u}\cdot\ddot{\mathbf{u}} + O(a), \quad \frac{u^2}{c^2} \ll 1. \qquad \text{(A.18)}$$

Setting the sum of the power delivered by the externally applied force $\mathbf{F}_{\rm ext}\cdot\mathbf{u}$ and the self electromagnetic power $P_{\rm em}(t)$ equal to zero, as Lorentz did in his original work [4], one obtains the power equation of motion (3.4) for charge shells with small velocity ($u^2/c^2 \ll 1$).

A.2 Relativistic Transformation of the Small-Velocity Force and Power

As explained in Section 3.1 the point relativistic transformations do not necessarily apply to the integrated force and power that comprise the right-hand sides of the Lorentz force and power equations of motion, (3.3) and (3.4), respectively. Thus, it is not mathematically rigorous to transform the small-velocity equations of motion, (3.3) and (3.4), to obtain the corresponding equations of motion, (2.1) and (2.4), for an arbitrary center velocity of the charge distribution. Nevertheless, a relativistic transformation of the proper-frame force equation of motion (3.3) does yield the general force equation of motion (2.1); whereas, a relativistic transformation of the small-velocity power equation of motion (3.4) does not yield the general power equation of motion (2.4). The proofs of these results follow.

A.2.1 Relativistic Transformation of the Proper-Frame Force

Let K be the proper inertial reference frame in which equation (3.3) is derived, and K' be the arbitrary inertial frame in which the velocity of the center of the charged shell is $\mathbf{u}'$. Thus K has velocity $\mathbf{u}'$ with respect to K'. Equation (3.3) can be divided into components parallel and perpendicular to the velocity $\mathbf{u}'$

$$\mathbf{F}^{\parallel}_{\rm ext} = \frac{e^2}{6\pi\epsilon_0c^2}\left[\frac{\dot{\mathbf{u}}_{\parallel}}{a} - \frac{\ddot{\mathbf{u}}_{\parallel}}{c}\right] + \mathbf{O}(a) \qquad \text{(A.19a)}$$

$$\mathbf{F}^{\perp}_{\rm ext} = \frac{e^2}{6\pi\epsilon_0c^2}\left[\frac{\dot{\mathbf{u}}_{\perp}}{a} - \frac{\ddot{\mathbf{u}}_{\perp}}{c}\right] + \mathbf{O}(a). \qquad \text{(A.19b)}$$

From the relativistic transformation of force

$$\mathbf{F}'^{\parallel}_{\rm ext} = \mathbf{F}^{\parallel}_{\rm ext} = \frac{e^2}{6\pi\epsilon_0c^2}\left[\frac{\dot{\mathbf{u}}_{\parallel}}{a} - \frac{\ddot{\mathbf{u}}_{\parallel}}{c}\right] + \mathbf{O}(a) \qquad \text{(A.20a)}$$

$$\mathbf{F}'^{\perp}_{\rm ext} = \mathbf{F}^{\perp}_{\rm ext}/\gamma' = \frac{e^2}{6\pi\epsilon_0c^2\gamma'}\left[\frac{\dot{\mathbf{u}}_{\perp}}{a} - \frac{\ddot{\mathbf{u}}_{\perp}}{c}\right] + \mathbf{O}(a) \qquad \text{(A.20b)}$$

with $\gamma' = \left(1 - u'^2/c^2\right)^{-1/2}$. The relativistic transformation of acceleration and its time derivative

$$\dot{\mathbf{u}}_{\|} = \gamma'^3 \dot{\mathbf{u}}'_{\|} \tag{A.21a}$$

$$\dot{\mathbf{u}}_{\perp} = \gamma'^2 \dot{\mathbf{u}}'_{\perp} \tag{A.21b}$$

$$\ddot{\mathbf{u}}_{\|} = \gamma'^4 \ddot{\mathbf{u}}'_{\|} + \frac{3\gamma'^6}{c^2}|\dot{\mathbf{u}}'_{\|}|^2 \mathbf{u}' \tag{A.22a}$$

$$\ddot{\mathbf{u}}_{\perp} = \gamma'^3 \ddot{\mathbf{u}}'_{\perp} + \frac{3\gamma'^5}{c^2}(\mathbf{u}' \cdot \dot{\mathbf{u}}')\dot{\mathbf{u}}'_{\perp} \tag{A.22b}$$

substituted into (A.20) produce the equations in the arbitrary K' system

$$\mathbf{F}'^{\|}_{\text{ext}} = \frac{e^2}{6\pi\epsilon_0 c^2}\left[\frac{\gamma'^3 \dot{\mathbf{u}}'_{\|}}{a} - \frac{\gamma'^4 \ddot{\mathbf{u}}'_{\|}}{c} - \frac{3\gamma'^6}{c^3}|\dot{\mathbf{u}}'_{\|}|^2 \mathbf{u}'\right] + \mathbf{O}(a) \tag{A.23a}$$

$$\mathbf{F}'^{\perp}_{\text{ext}} = \frac{e^2}{6\pi\epsilon_0 c^2}\left[\frac{\gamma' \dot{\mathbf{u}}'_{\perp}}{a} - \frac{\gamma'^2}{c}\ddot{\mathbf{u}}'_{\perp} - \frac{3\gamma'^4}{c^3}(\mathbf{u}' \cdot \dot{\mathbf{u}}')\dot{\mathbf{u}}'_{\perp}\right] + \mathbf{O}(a)\,. \tag{A.23b}$$

Adding (A.23a) to (A.23b), combining terms and removing the primes, results in the transformed equation of motion

$$\mathbf{F}_{\text{ext}} = \frac{e^2}{6\pi\epsilon_0 a c^2}\frac{\mathrm{d}}{\mathrm{d}t}(\gamma\mathbf{u}) - \frac{e^2\gamma^2}{6\pi\epsilon_0 c^3}\left\{\ddot{\mathbf{u}} + \frac{3\gamma^2}{c^2}(\mathbf{u}\cdot\dot{\mathbf{u}})\dot{\mathbf{u}} + \frac{\gamma^2}{c^2}\left[\mathbf{u}\cdot\ddot{\mathbf{u}} + \frac{3\gamma^2}{c^2}(\mathbf{u}\cdot\dot{\mathbf{u}})^2\right]\mathbf{u}\right\} + \mathbf{O}(a) \tag{A.24}$$

which is identical to the general equation of motion (2.1) obtained from the self electromagnetic force calculated directly in an inertial frame in which the charge has arbitrary center velocity $\mathbf{u}$.

A.2.2 Relativistic Transformation of the Small-Velocity Power

In an inertial frame K in which the charge has an infinitesimally small center velocity $\mathbf{u}$, we have from equation (3.4)

$$\mathbf{F}_{\text{ext}} \cdot \mathbf{u} = \frac{e^2}{6\pi\epsilon_0 c^2}\left[\frac{5\dot{\mathbf{u}}}{4a} - \frac{\ddot{\mathbf{u}}}{c}\right]\cdot \mathbf{u} + O(a), \quad u \to 0\,. \tag{A.25}$$

In the K' frame, moving with velocity $-\mathbf{u}'$ with respect to K (as u approaches zero), the velocity of the particle is $\mathbf{u}'$. Thus, in the K' frame (A.25) becomes

$$\mathbf{F}_{\text{ext}} \cdot \mathbf{u}' = \frac{e^2}{6\pi\epsilon_0 c^2}\left[\frac{5\dot{\mathbf{u}}}{4a} - \frac{\ddot{\mathbf{u}}}{c}\right]\cdot \mathbf{u}' + O(a)\,. \tag{A.26}$$

From the relativistic transformations of $\mathbf{F}_{\text{ext}}$, $\dot{\mathbf{u}}$, and $\ddot{\mathbf{u}}$ in (A.20a), (A.21a) and (A.22a), we find

$$\mathbf{F}_{\text{ext}} \cdot \mathbf{u}' = \mathbf{F}'_{\text{ext}} \cdot \mathbf{u}' \tag{A.27}$$

and

$$\left[\frac{5\dot{\mathbf{u}}}{4a} - \frac{\ddot{\mathbf{u}}}{c}\right]\cdot \mathbf{u}' = \left[\frac{5\gamma'^3\dot{\mathbf{u}}'}{4a} - \frac{\gamma'^4\ddot{\mathbf{u}}'}{c} - \frac{3\gamma'^6}{c^3}|\dot{\mathbf{u}}'_{\|}|^2\mathbf{u}'\right]\cdot \mathbf{u}'\,. \tag{A.28}$$

Substituting (A.27) and (A.28) into (A.26) and removing the primes, we obtain the general power equation of motion (3.5)

$$\mathbf{F}_{\text{ext}} \cdot \mathbf{u} = \frac{5e^2}{24\pi\epsilon_0 a}\frac{\mathrm{d}\gamma}{\mathrm{d}t} - \frac{e^2\gamma^4}{6\pi\epsilon_0 c^3}\left[\mathbf{u}\cdot\ddot{\mathbf{u}} + \frac{3\gamma^2}{c^2}(\mathbf{u}\cdot\dot{\mathbf{u}})^2\right] + O(a) \tag{A.29}$$

corresponding to the small-velocity power equation of motion (3.4). Unlike the case of the transformed force equation of motion, (A.29) is not identical to the power equation of motion (2.4) obtained from the self electromagnetic power calculated directly in an inertial frame in which the charge has arbitrary center velocity $\mathbf{u}$. As explained in Section 3.1, we cannot rigorously apply the point relativistic transformations to the small velocity self electromagnetic force and power expressions to find the self electromagnetic force and power of an arbitrarily moving charge, because the charge is distributed over an extended region of space and not concentrated at a single point moving with a uniform velocity. The distributed charge motion does not change the final result of the self electromagnetic force calculation, but does change the $1/a$ term in the self electromagnetic power calculation, and the transformation properties of the self electromagnetic power. Indeed, the next section of this Appendix A demonstrates that the power equation of motion (2.4) does not transform covariantly.

A.3 Noncovariance of the Power Equation

Begin with the power equation of motion (2.4) in an arbitrary inertial frame K_a

$$\mathbf{F}_{\text{ext}} \cdot \mathbf{u} = \frac{e^2}{6\pi\epsilon_0 a}\frac{\mathrm{d}}{\mathrm{d}t}\left(\gamma - \frac{1}{4\gamma}\right) - \frac{e^2\gamma^4}{6\pi\epsilon_0 c^3}\left[\mathbf{u}\cdot\ddot{\mathbf{u}} + \frac{3\gamma^2}{c^2}(\mathbf{u}\cdot\dot{\mathbf{u}})^2\right] + O(a). \tag{A.30}$$

In an inertial frame K'_w moving with velocity $\mathbf{w}$ with respect to K_a, the relativistic transformations of $\mathbf{F}_{\text{ext}}$, $\mathbf{u}$, $\dot{\mathbf{u}}$, $\ddot{\mathbf{u}}$, and γ in terms of the corresponding primed variables in the K'_w frame recast (A.30) in the form

$$0 = (\mathbf{u}' + \mathbf{w})\cdot\left\{\mathbf{F}'_{\text{ext}} - \frac{e^2}{6\pi\epsilon_0 a c^2}\left(1 - \frac{1}{4\gamma'^2\gamma_w^2(1 + \mathbf{u}'\cdot\mathbf{w}/c^2)^2}\right)\frac{\mathrm{d}(\gamma'\mathbf{u}')}{\mathrm{d}t'} + \frac{e^2\gamma^2}{6\pi\epsilon_0 c^3}\left[\ddot{\mathbf{u}}' + \frac{3\gamma'^2}{c^2}(\mathbf{u}'\cdot\dot{\mathbf{u}}')\dot{\mathbf{u}}' + \frac{\gamma'^2}{c^2}\left(\mathbf{u}'\cdot\ddot{\mathbf{u}}' + \frac{3\gamma'^2}{c^2}(\mathbf{u}'\cdot\dot{\mathbf{u}}')^2\right)\mathbf{u}'\right] + \mathbf{O}(a)\right\}. \tag{A.31}$$

If (A.31) is to be independent of $\mathbf{w}$ and hold for all $\mathbf{w}$ ($w < c$), then the terms in the curly brackets of (A.31) must be zero, that is

$$\mathbf{F}'_{\text{ext}} = \frac{e^2}{6\pi\epsilon_0 a c^2}\left[1 - \frac{1}{4\gamma'^2\gamma_w^2(1+\mathbf{u}'\cdot\mathbf{w}/c^2)^2}\right]\frac{\mathrm{d}(\gamma'\mathbf{u}')}{\mathrm{d}t'} - \frac{e^2\gamma^2}{6\pi\epsilon_0 c^3}\left[\ddot{\mathbf{u}}' + \frac{3\gamma'^2}{c^2}(\mathbf{u}'\cdot\dot{\mathbf{u}}')\dot{\mathbf{u}}' + \frac{\gamma'^2}{c^2}\left(\mathbf{u}'\cdot\ddot{\mathbf{u}}' + \frac{3\gamma'^2}{c^2}(\mathbf{u}'\cdot\dot{\mathbf{u}}')^2\right)\mathbf{u}'\right] + \mathrm{O}(a). \tag{A.32}$$

Because of the 1/4 term in (A.32), the form of this equation (A.32) depends explicitly on the velocity $\mathbf{w}$ of the K'_w inertial frame. Thus the form of (A.32) is not relativistically invariant with respect to a change of inertial frames, that is, the left- and right-hand sides of the power equation of motion (2.4) do not transform covariantly because of the $-[e^2/(24\pi\epsilon_0 a)]\mathrm{d}(1/\gamma)/dt$ term. Of course, it is this very term that the internal binding forces eliminate from the power equation of motion (2.4); see Chapter 4.

B

Derivation of Force and Power at Arbitrary Velocity

In this appendix the self electromagnetic force and power are derived from equations (3.1) and (3.2) for the shell of charge moving with arbitrary velocity. The $1/a$ terms are derived from the space integrals in (3.1) and (3.2) evaluated for arbitrary, time-varying velocity (unlike the traditional heuristic derivation which assumes a constant velocity charge). The radiation reaction terms are found from the charge (rather than the space) integrals in (3.1) and (3.2) evaluated for a shell of charge moving with arbitrary, time-varying velocity.

B.1 The $1/a$ Terms of Self Electromagnetic Force and Power

The self electromagnetic force and power of the moving shell of charge can be written as space integrals of the electromagnetic fields of the moving charge [15, sec. 2.5, eq. (25) and sec. 2.19, eq. (6)]

$$\mathbf{F}'_{\text{em}}(t') = -\epsilon_0\frac{\mathrm{d}}{\mathrm{d}t'}\int\limits_V \mathbf{E}'(\mathbf{r}',t')\times\mathbf{B}'(\mathbf{r}',t')\mathrm{d}V' + \int\limits_S \bar{\mathbf{T}}'\cdot\hat{\mathbf{n}}'\mathrm{d}S' \tag{B.1}$$

$$P'_{\text{em}}(t') = -\frac{\epsilon_0}{2}\frac{\mathrm{d}}{\mathrm{d}t'}\int\limits_V (E'^2 + c^2B'^2)\mathrm{d}V' - \epsilon_0 c^2\int\limits_S (\mathbf{E}'\times\mathbf{B}')\cdot\hat{\mathbf{n}}'\mathrm{d}S' \tag{B.2}$$

where $\bar{\mathbf{T}}'$ is Maxwell's stress tensor (dyadic) and the primes denote quantities in a K' inertial frame in which the charge shell has arbitrary center velocity $\mathbf{u}'(t')$. The volume V is enclosed by the surface S, which encloses the moving charge distribution.

The force on any part of the charged oblate spheroid (with major axis $2a$ and minor axis $2a(1-u'^2/c^2)^{1/2}$ in the K' frame will be caused by the position of the rest of the charge at an earlier time. In particular, the force field on the leading end of the particle will have left the trailing end of the particle in a time Δt given approximately for small radius a by

$$(c-u')\Delta t = 2a\sqrt{1-u'^2/c^2} \quad \text{or} \quad \Delta t = \frac{2a}{c}\sqrt{\frac{1+u'/c}{1-u'/c}}\,. \tag{B.3}$$

In this time interval Δt the charge will have traveled a distance Δd given approximately by

$$\Delta d = u'\Delta t = \frac{2u'a}{c}\sqrt{\frac{1+u'/c}{1-u'/c}}\,. \tag{B.4}$$

Equation (B.4) says that the motion of the charge, when the charge is farther away from its present position than some finite number times the radius a, will not affect the self electromagnetic force calculation. Thus, we can assume, with no loss of generality in the derivation, that the charge had uniform velocity when evaluating the fields for r' greater than La where L is an indefinitely large but finite number. In other words, if we choose the radius of the surface S larger by a factor L then the major radius of the oblate spheroidal charge distribution, the stress tensor $\bar{\mathbf{T}}'$ and the Poynting vector $\mathbf{E}'\times\mathbf{B}'$ in the surface integrals of (B.1) and (B.2) can be assumed those of a charge distribution moving with constant velocity. Because each of these surface integrals is zero for a constant velocity charge distribution, (B.1) and (B.2) can be written in terms of the volume integrals alone

$$\mathbf{F}'_{\rm em}(t') = -\epsilon_0 \frac{\mathrm{d}}{\mathrm{d}t'}\int\limits_{V_a} \mathbf{E}'(\mathbf{r}',t')\times\mathbf{B}'(\mathbf{r}',t')\,\mathrm{d}V' \tag{B.5}$$

$$P'_{\rm em}(t') = -\frac{\epsilon_0}{2}\frac{\mathrm{d}}{\mathrm{d}t'}\int\limits_{V_a} (E'^2 + c^2B'^2)\,\mathrm{d}V' \tag{B.6}$$

with V_a denoting a finite volume that encloses the charge distribution and having a radius La proportional to the dimension a of the charged shell. The fact that the radius La of the volume V_a approaches zero as a approaches zero, and yet L is an indefinitely large number, is used in the following evaluations of the $1/a$ terms of self force and power.

B.1.1 Evaluation of $1/a$ Term of Self Electromagnetic Force

We want to evaluate the space integral in (B.5) at each instant of time t'. To begin, let this instant of time be $t'=0$, in order to simplify the integral in (B.5) to

$$\mathbf{I}_F = \int\limits_{V_a} \mathbf{E}'(\mathbf{r}',0)\times\mathbf{B}'(\mathbf{r}',0)\,\mathrm{d}V'\,. \tag{B.7}$$

Next write the fields, $\mathbf{E}'(\mathbf{r}',0)$ and $\mathbf{B}'(\mathbf{r}',0)$ in the K' frame in terms of the fields in a proper inertial frame K at rest instantaneously with the center of the charge distribution at $t'=0$. Assume that the origins of the K and K'

frames coincide at $t=t'=0$. Then the relativistic transformations of the fields are given by

$$\mathbf{E}'(\mathbf{r}',0) = \bar{\alpha}'\cdot[\mathbf{E}(\mathbf{r},t) - \mathbf{u}'\times\mathbf{B}(\mathbf{r},t)] \tag{B.8a}$$

$$\mathbf{B}'(\mathbf{r}',0) = \bar{\alpha}'\cdot[\mathbf{B}(\mathbf{r},t) + \mathbf{u}'\times\mathbf{E}(\mathbf{r},t)/c^2] \tag{B.8b}$$

$$\bar{\alpha}' = \gamma'\bar{\mathbf{I}} + (1-\gamma')\hat{\mathbf{u}}'\hat{\mathbf{u}}', \quad \gamma' = (1-u'^2/c^2)^{-1/2} \tag{B.8c}$$

with

$$\mathbf{r}_\perp = \mathbf{r}'_\perp \tag{B.9a}$$

$$\mathbf{r}_\parallel = \gamma'\mathbf{r}'_\parallel \tag{B.9b}$$

$$t = -\gamma'\mathbf{u}'\cdot\mathbf{r}'/c^2 \tag{B.9c}$$

where the subscripts $\perp$ and $\parallel$ mean perpendicular and parallel to the center velocity $\mathbf{u}'$.

Substitute (B.8) and (B.9) into (B.7) and make the change of integration variable

$$\mathbf{r} = \mathbf{r}'_\perp + \gamma'\mathbf{r}'_\parallel \tag{B.10a}$$

so that

$$\mathrm{d}V = \gamma'\mathrm{d}V' \tag{B.10b}$$

and (B.7) becomes

$$\mathbf{I}_F = \frac{1}{\gamma'}\int\limits_{V_a} \bar{\alpha}'\cdot\left[\mathbf{E}(\mathbf{r},t=-\mathbf{u}'\cdot\mathbf{r}/c^2) - \mathbf{u}'\times\mathbf{B}(\mathbf{r},t=-\mathbf{u}'\cdot\mathbf{r}/c^2)\right]$$
$$\times\,\bar{\alpha}'\cdot\left[\mathbf{B}+\mathbf{u}'\times\mathbf{E}/c^2\right]\mathrm{d}V\,. \tag{B.11}$$

Since we have determined in Appendix C the proper-frame electric and magnetic fields, $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$, at a fixed time t, the integral of the fields in (B.11) could be evaluated if it weren't for the fact that $t=-\mathbf{u}'\cdot\mathbf{r}/c^2$ is not fixed but varies with the integration variable $\mathbf{r}$. Fortunately, this difficulty can be overcome, when evaluating the $1/a$ term, by expanding $\mathbf{E}(\mathbf{r},t=-\mathbf{u}'\cdot\mathbf{r}/c^2)$ and $\mathbf{B}(\mathbf{r},t=-\mathbf{u}'\cdot\mathbf{r}/c^2)$ about the fixed time $t=0$; specifically

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}(\mathbf{r},0) + \frac{\partial\mathbf{E}(\mathbf{r},0)}{\partial t}t + \cdots \tag{B.12a}$$

$$\mathbf{B}(\mathbf{r},t) = \mathbf{B}(\mathbf{r},0) + \frac{\partial\mathbf{B}(\mathbf{r},0)}{\partial t}t + \cdots\,. \tag{B.12b}$$

From Maxwell's equations all the time derivatives of $\mathbf{E}(\mathbf{r},0)$ and $\mathbf{B}(\mathbf{r},0)$ can be written in terms of the spatial derivatives

$$\frac{\partial\mathbf{B}(\mathbf{r},0)}{\partial t} = -\nabla\times\mathbf{E}(\mathbf{r},0) \tag{B.13a}$$

$$\frac{\partial \mathbf{E}(\mathbf{r},0)}{\partial t} = c^2\nabla\times\mathbf{B}(\mathbf{r},0) \tag{B.13b}$$

$$\frac{\partial^2 \mathbf{B}(\mathbf{r},0)}{\partial t^2} = -\nabla\times\frac{\partial \mathbf{E}(\mathbf{r},0)}{\partial t} = -c^2\nabla\times\nabla\times\mathbf{B}(\mathbf{r},0) \tag{B.13c}$$

and so on. Substitution of the time derivatives from (B.13) converts (B.12) to

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}(\mathbf{r},0) + c^2\nabla\times\mathbf{B}(\mathbf{r},0)t + \cdots \tag{B.14a}$$

$$\mathbf{B}(\mathbf{r},t) = \mathbf{B}(\mathbf{r},0) - \nabla\times\mathbf{E}(\mathbf{r},0)t + \cdots . \tag{B.14b}$$

When the proper-frame electric and magnetic fields, $\mathbf{E}(\mathbf{r},0), \mathbf{B}(\mathbf{r},0)$ and their curls, are inserted from (C.1) and (C.5) of Appendix C into the right-hand sides of (B.14), and the resulting fields, $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$, are inserted into the integrand of (B.11) with $t = -\mathbf{u}'\cdot\mathbf{r}/c^2$, one finds that the $\mathbf{B}$ field in the integrand of (B.11) does not contribute to the $1/a$ term of the integral and that only the static part of the $\mathbf{E}$ field contributes to the $1/a$ term. In more detail

$$\mathbf{I}_F = \frac{1}{\gamma'}\int\limits_{V_a(a\to 0)} [\bar{\alpha}'\cdot\mathbf{E}(\mathbf{r},0)]\times[\bar{\alpha}'\cdot(\mathbf{u}'\times\mathbf{E}(\mathbf{r},0))/c^2]\,\mathrm{d}V + \mathbf{O}(1) \tag{B.15}$$

or since $\bar{\alpha}'\cdot(\mathbf{u}'\times\mathbf{E}) = \gamma'\mathbf{u}'\times\mathbf{E}$ and $\bar{\alpha}'\cdot\mathbf{E} = \gamma'\mathbf{E} + (1-\gamma')(\hat{\mathbf{u}}'\cdot\mathbf{E})\hat{\mathbf{u}}'$

$$\mathbf{I}_F = \int\limits_{V_a(a\to 0)} \{\mathbf{u}'\left[\gamma' E^2 + (1-\gamma')(\hat{\mathbf{u}}'\cdot\mathbf{E})^2\right] - (\mathbf{u}'\cdot\mathbf{E})\mathbf{E}\}\,\mathrm{d}V + \mathbf{O}(1). \tag{B.16}$$

The electric field $\mathbf{E}(\mathbf{r},0)$ is found by integrating expression (C.1) to get

$$\mathbf{E}(\mathbf{r},0) = \begin{cases} \dfrac{e}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}} + \mathbf{O}(1/r), & r > a \\ \mathbf{O}(1/a), & r < a\,. \end{cases} \tag{B.17}$$

Because $V_a \to 0$ as $a \to 0$, the integration variable $r \to 0$ as $a \to 0$ and we are allowed to use this small r approximation of (C.1) for $\mathbf{E}(\mathbf{r},0)$. With $\mathbf{E}(\mathbf{r},0)$ from (B.17) substituted into the integrand, the integral in (B.16) can be evaluated for large L to give

$$\mathbf{I}_F = \frac{e^2\mathbf{u}'}{4\pi\epsilon_0^2 ac^2}\left[\gamma' + \frac{1-\gamma'}{3} - \frac{1}{3}\right] + \mathbf{O}(1) = \frac{e^2\gamma'\mathbf{u}'}{6\pi\epsilon_0^2 ac^2} + \mathbf{O}(1)\,. \tag{B.18}$$

For the sake of simplifying the relativistic transformations, (B.18) was derived for a specific instant of time $t' = 0$. This instant of time could be any instant of time. Thus (B.18) holds for arbitrary time t', and (B.18) can be substituted into (B.5) to give the $1/a$ term of the self electromagnetic force

$$\mathbf{F}'_{\text{em}}(t') = -\frac{e^2}{6\pi\epsilon_0 ac^2}\frac{\mathrm{d}}{\mathrm{d}t'}(\gamma'\mathbf{u}') + \mathbf{O}(1) \tag{B.19}$$

in the arbitrary K' frame.

B.1.2 Evaluation of $1/a$ Term of Self Electromagnetic Power

Proceeding with the evaluation of the self power integral in (B.6)

$$I_P = \int\limits_{V_a} (E'^2 + c^2 B'^2)\mathrm{d}V' \tag{B.20}$$

in the same manner as in the previous section for the self force integral, one gets

$$I_P = \frac{1}{\gamma'}\int\limits_{V_a(a\to 0)} \left[|\bar{\alpha}'\cdot\mathbf{E}(\mathbf{r},0)|^2 + \gamma'|\mathbf{u}'\times\mathbf{E}'|^2/c^2\right]\mathrm{d}V + O(1)\,. \tag{B.21}$$

With $\mathbf{E}(\mathbf{r},0)$ inserted from (B.17), (B.21) integrates for large L to

$$\begin{aligned} I_P &= \frac{e^2}{4\pi\epsilon_0^2 a}\left[\gamma' + \frac{1}{3}\left(\frac{1}{\gamma'} - \gamma'\right) + \frac{2u'^2}{2c^2}\gamma'\right] + O(1) \\ &= \frac{e^2\gamma'}{4\pi\epsilon_0^2 a}\left(1 + \frac{u'^2}{3c^2}\right) + O(1) \end{aligned} \tag{B.22}$$

which, when inserted into (B.6), gives

$$P'_{\text{em}}(t') = -\frac{e^2}{8\pi\epsilon_0 a}\frac{\mathrm{d}}{\mathrm{d}t'}\left[\gamma'\left(1 + \frac{u'^2}{3c^2}\right)\right] + O(1) \tag{B.23a}$$

or equivalently

$$P'_{\text{em}}(t') = -\frac{e^2}{6\pi\epsilon_0 a}\frac{\mathrm{d}}{\mathrm{d}t'}\left(\gamma' - \frac{1}{4\gamma'}\right) + O(1) \tag{B.23b}$$

for the $1/a$ term of the self electromagnetic power in the arbitrary K' frame.

B.2 Radiation Reaction of Self Electromagnetic Force and Power

The above derivation for the $1/a$ terms of the self electromagnetic force and power in an arbitrary inertial frame from the momentum and energy integrals in (B.5) and (B.6) does not extend easily to finding the radiation reaction ($O(1)$) terms of the self force and power because an infinite number of terms in the series expansion (B.14) of $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$ contribute to the $O(1)$ terms of the momentum and energy integrals. Fortunately, we can find the radiation reaction terms of the self force and power from the charge integrals of (3.1) and (3.2).

B.2.1 Evaluation of the Radiation Reaction Force

To determine the $\mathbf{O}(1)$ terms of the self electromagnetic force in a K' inertial frame in which the shell of charge is moving with arbitrary velocity, we shall evaluate the charge integral in (3.1) at an arbitrary instant of time t'. To reduce the algebra, let this arbitrary time be chosen as $t'=0$, initially, so the self electromagnetic force in the K' frame can be written

$$\mathbf{F}_{\text{em}} = \int_{\text{charge}} \rho'(\mathbf{r}',0)[\mathbf{E}'(\mathbf{r}',0) + \mathbf{u}'(\mathbf{r}',0)\times\mathbf{B}'(\mathbf{r}',0)]\,dV' . \tag{B.24}$$

The electric and magnetic fields, $\mathbf{E}'(\mathbf{r}',0)$ and $\mathbf{B}'(\mathbf{r}',0)$, in (B.24) can be expressed by means of the relativistic transformations (B.8) and (B.9), in terms of the fields in the proper frame K at rest instantaneously with the center of the charge distribution at $t'=0$. Since (C.7) of Appendix C can be used to show that $\mathbf{B}(\mathbf{r},t)$ in (B.8) contributes only to terms of higher order than $\mathbf{O}(1)$, $\mathbf{E}'$ and $\mathbf{B}'$ in (B.24) can be written simply from (B.8) as

$$\mathbf{E}'(\mathbf{r}',0) = \bar{\alpha}'\cdot\mathbf{E}(\mathbf{r},t) \tag{B.25a}$$

$$\mathbf{B}'(\mathbf{r}',0) = \gamma'\mathbf{u}'\times\mathbf{E}(\mathbf{r},t)/c^2 \tag{B.25b}$$

where $\bar{\alpha}'$ is defined in (B.8c), $\mathbf{r}$ and t are given in (B.9), and $\mathbf{u}'$ is the velocity of the center of the charge distribution.

The velocity $\mathbf{u}'(\mathbf{r}',0)$ of the charge distribution in the K' frame can be written in terms of the velocity $\mathbf{u}(\mathbf{r},t)$ in the proper frame by means of the relativistic transformation

$$\mathbf{u}'(\mathbf{r}',0) = \frac{\mathbf{u}(\mathbf{r},t)/\gamma' + \mathbf{u}'\left[\mathbf{u}(\mathbf{r},t)\cdot\mathbf{u}'(1-1/\gamma')/u'^2 + 1\right]}{1+\mathbf{u}(\mathbf{r},t)\cdot\mathbf{u}'/c^2} . \tag{B.26}$$

Similarly, the charge density $\rho'(\mathbf{r}',0)$ in (B.24) transforms relativistically to the proper K frame as

$$\rho'(\mathbf{r}',0) = \gamma'\rho(\mathbf{r},t)\left[1+\mathbf{u}(\mathbf{r},t)\cdot\mathbf{u}'/c^2\right] \tag{B.27}$$

with $\mathbf{r}$ and t again given in (B.9). The velocity $\mathbf{u}(\mathbf{r},t)$ and the charge density $\rho(\mathbf{r},t)$ of the charge distribution at $t = -\gamma'\mathbf{u}'\cdot\mathbf{r}'/c^2 = -\mathbf{u}'\cdot\mathbf{r}/c^2$ in the K frame can be expanded about $t=0$ to give

$$\mathbf{u}(\mathbf{r},t) = -\dot{\mathbf{u}}(\mathbf{r},0)(\mathbf{u}'\cdot\mathbf{r})/c^2 + \mathbf{O}(r^2) \tag{B.28}$$

$$\rho(\mathbf{r},t) = \rho(\mathbf{r},0) - \frac{\partial\rho(\mathbf{r},0)}{\partial t}\frac{\mathbf{u}'\cdot\mathbf{r}}{c^2} + O(r^2) . \tag{B.29}$$

Because $\mathbf{u}(\mathbf{r},0)$ equals zero for a relativistically rigid, nonrotating charge distribution, $\partial\rho(\mathbf{r},0)/\partial t = -\nabla\cdot[\rho(\mathbf{r},0)\mathbf{u}(\mathbf{r},0)] = 0$, and (B.29) becomes simply

$$\rho(\mathbf{r},t) = \rho(\mathbf{r},0) + O(r^2) . \tag{B.30}$$

Substituting $\mathbf{u}(\mathbf{r},t)$ from (B.28) and $\rho(\mathbf{r},t)$ from (B.30) into (B.27) gives

$$\rho'(\mathbf{r}',0) = \gamma'\rho(\mathbf{r},0)\left[1-\frac{\mathbf{u}'\cdot\dot{\mathbf{u}}(\mathbf{r},0)(\mathbf{u}'\cdot\mathbf{r})}{c^4}\right] + O(r^2) \tag{B.31}$$

for the charge density in the K' frame. Similarly, substituting (B.28) into (B.26) and expanding in powers of r gives

$$\mathbf{u}'(\mathbf{r}',0) = \mathbf{u}' - \left[\bar{\mathbf{I}}\gamma' + \frac{\mathbf{u}'\mathbf{u}'}{u'^2}(1-\gamma')\right]\cdot\frac{\dot{\mathbf{u}}(\mathbf{r},0)(\mathbf{u}'\cdot\mathbf{r})}{\gamma'^2c^2} + \mathbf{O}(r^2) . \tag{B.32}$$

The acceleration $\dot{\mathbf{u}}(\mathbf{r},0)$ of the charge distribution in the proper frame was given previously in (A.8) in terms of its center acceleration $\dot{\mathbf{u}}$; thus (A.8) shows that (B.31) and (B.32) remain valid to $O(r^2)$ when the acceleration $\dot{\mathbf{u}}(\mathbf{r},0)$ is replaced by the center acceleration $\dot{\mathbf{u}}$.

Substitute into (B.24) the expressions (B.25) for $\mathbf{E}'(\mathbf{r}',0)$ and $\mathbf{B}'(\mathbf{r}',0)$, (B.31) for $\rho'(\mathbf{r}',0)$, (B.32) for $\mathbf{u}'(\mathbf{r}',0)$ (all with $\mathbf{r}$ and t replaced from (B.9) and the center acceleration $\dot{\mathbf{u}}$ replacing $\dot{\mathbf{u}}(\mathbf{r},0)$ in (B.31) and (B.32)); then make the change of integration variable from $\mathbf{r}'$ to $\mathbf{r}_\perp + \mathbf{r}_\parallel/\gamma'$ to obtain

$$\mathbf{F}'_{\text{em}}(0) = \int_{\text{charge}} \rho(\mathbf{r},0)\left[1-\frac{(\mathbf{u}'\cdot\dot{\mathbf{u}})(\mathbf{u}'\cdot\mathbf{r})}{c^4} + \mathbf{O}(r^2)\right]\left[\bar{\alpha}'\cdot\mathbf{E}(\mathbf{r},t) + \frac{\gamma'}{c^2}\left\{\mathbf{u}'\right.\right.$$
$$\left.\left. - \left[\bar{\mathbf{I}}\gamma' + \frac{\mathbf{u}'\mathbf{u}'}{u'^2}(1-\gamma')\right]\cdot\frac{\dot{\mathbf{u}}(\mathbf{u}'\cdot\mathbf{r})}{\gamma'^2c^2} + \mathbf{O}(r^2)\right\}\times(\mathbf{u}'\times\mathbf{E}(\mathbf{r},t))\right]dV \tag{B.33}$$

with $t = -\mathbf{u}'\cdot\mathbf{r}/c^2$. We want to insert $\mathbf{E}(\mathbf{r},t)$ from (B.14a) into the integrand of (B.33); specifically

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}(\mathbf{r},0) + c^2\nabla\times\mathbf{B}(\mathbf{r},0)t - c^2\nabla\times\nabla\times\mathbf{E}(\mathbf{r},0)\frac{t^2}{2} + \cdots \tag{B.34}$$

with $t = -\mathbf{u}'\cdot\mathbf{r}/c^2$. When one replaces $\mathbf{E}(\mathbf{r},0)$ and $\mathbf{B}(\mathbf{r},0)$ in (B.34) by their integral values given in (C.1) and (C.5), one finds

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}(\mathbf{r},0) + \text{ terms odd in } \hat{\mathbf{r}} + \frac{1}{a}(\text{terms even in } \hat{\mathbf{r}}) . \tag{B.35}$$

As the radius a of the charged sphere approaches zero, the terms odd in $\hat{\mathbf{r}}$ in (B.35) integrate to zero in (B.33). The $1/a$ terms in (B.35) integrate to give $1/a$ terms when multiplied by the terms of order unity in the integrand of (B.33), and zero when multiplied by the terms of order r and higher in (B.33). Also, as a approaches zero, the $\mathbf{O}(r^2)$ terms in (B.33) integrate to zero. In all, (B.33) becomes

$$\mathbf{F}'_{\text{em}}(0) = \left(\frac{1}{a}\right) + \int_{\text{charge}} \rho(\mathbf{r},0)\left[1-\frac{(\mathbf{u}'\cdot\dot{\mathbf{u}})(\mathbf{u}'\cdot\mathbf{r})}{c^4}\right]\left[\bar{\alpha}'\cdot\mathbf{E}(\mathbf{r},0)\right.$$
$$\left. + \frac{\gamma'}{c^2}\left\{\mathbf{u}' - \left[\bar{\mathbf{I}}\gamma' + \frac{\mathbf{u}'\mathbf{u}'}{u'^2}(1-\gamma')\right]\cdot\frac{\dot{\mathbf{u}}(\mathbf{u}'\cdot\mathbf{r})}{\gamma'^2c^2}\right\}\times(\mathbf{u}'\times\mathbf{E}(\mathbf{r},0))\right]dV + \mathbf{O}(a) \tag{B.36}$$

as a approaches zero, where $(1/a)$ in (B.36) denotes the $1/a$ terms.

Inserting $\mathbf{E}(\mathbf{r},0)$ from (C.1) into (B.36), noting that all odd terms integrate to zero, and extracting the $1/a$ terms, we find

$$\mathbf{F}'_{\text{em}}(0) = \left(\frac{1}{a}\right) + \frac{1}{4\pi\epsilon_0} \iint\limits_{\text{charge}} \left[\bar{\alpha}' \cdot \frac{2\ddot{\mathbf{u}}}{3c^3} + \frac{\gamma'}{c^2}\mathbf{u}' \times \left(\mathbf{u}' \times \frac{2\ddot{\mathbf{u}}}{3c^3}\right)\right] \mathrm{d}e'\mathrm{d}e + \mathbf{O}(a)$$
$$= \left(\frac{1}{a}\right) + \frac{e^2}{6\pi\epsilon_0 c^3}\left[\bar{\alpha}' \cdot \ddot{\mathbf{u}} + \frac{\gamma'}{c^2}\mathbf{u}' \times (\mathbf{u}' \times \ddot{\mathbf{u}})\right] + \mathbf{O}(a) \tag{B.37}$$

where we have let $de = \rho(\mathbf{r},0)\mathrm{d}V$ and performed the double integration of the constant integrand over the charge.

With

$$\bar{\alpha}' \cdot \ddot{\mathbf{u}} = \gamma'\ddot{\mathbf{u}} + (1-\gamma')\frac{(\mathbf{u}' \cdot \ddot{\mathbf{u}})\mathbf{u}'}{u'^2} \tag{B.38a}$$

and

$$\frac{\gamma'}{c^2}\mathbf{u}' \times (\mathbf{u}' \times \ddot{\mathbf{u}}) = \frac{-\gamma' u'^2}{c^2}\ddot{\mathbf{u}} + \frac{\gamma'}{c^2}(\mathbf{u}' \cdot \ddot{\mathbf{u}})\mathbf{u}' \tag{B.38b}$$

(B.37) can be written as

$$\mathbf{F}'_{\text{em}}(0) = \left(\frac{1}{a}\right) + \frac{e^2}{6\pi\epsilon_0 c^3}\left[\frac{\ddot{\mathbf{u}}}{\gamma'} + \left(1 - \frac{1}{\gamma'}\right)\frac{(\mathbf{u}' \cdot \ddot{\mathbf{u}})\mathbf{u}'}{u'^2}\right] \tag{B.39a}$$

or

$$\mathbf{F}'_{\text{em}}(0) = \left(\frac{1}{a}\right) + \frac{e^2}{6\pi\epsilon_0 c^3}\left[\ddot{\mathbf{u}}_{\parallel} + \ddot{\mathbf{u}}_{\perp}/\gamma'\right] + \mathbf{O}(a)\,. \tag{B.39b}$$

The derivatives of the acceleration, $\ddot{\mathbf{u}}_{\parallel}$ and $\ddot{\mathbf{u}}_{\perp}$, in the proper K frame can be expressed in terms of the velocity and its derivatives in the arbitrary K' frame by means of the relativistic transformations (A.22). Using these transformations (A.22) converts (B.39b) to

$$\mathbf{F}'_{\text{em}}(t') = \left(\frac{1}{a}\right) + \frac{e^2\gamma'^2}{6\pi\epsilon_0 c^3}\left\{\ddot{\mathbf{u}}' + \frac{3\gamma'^2}{c^2}(\mathbf{u}' \cdot \dot{\mathbf{u}}')\dot{\mathbf{u}}' + \frac{\gamma'^2}{c^2}\left[\mathbf{u}' \cdot \ddot{\mathbf{u}}' + \frac{3\gamma'^2}{c^2}(\mathbf{u}' \cdot \dot{\mathbf{u}}')^2\right]\mathbf{u}'\right\} + \mathbf{O}(a) \tag{B.40}$$

where t' has replaced $t' = 0$ in (B.39) since the time $t' = 0$ could be any instant of time t'.

The order unity term in (B.40) is the radiation reaction part of the self electromagnetic force. Combining the $1/a$ part of the self electromagnetic force in (B.19) with the radiation reaction part in (B.40) produces the total electromagnetic self force to order a in an arbitrary K' inertial reference frame

$$\mathbf{F}'_{\text{em}}(t') = -\frac{e^2}{6\pi\epsilon_0 a c^2}\frac{\mathrm{d}}{\mathrm{d}t}(\gamma'\mathbf{u}') + \frac{e^2\gamma'^2}{6\pi\epsilon_0 c^3}\left\{\ddot{\mathbf{u}}' + \frac{3\gamma'^2}{c^2}(\mathbf{u}' \cdot \dot{\mathbf{u}}')\dot{\mathbf{u}}' + \frac{\gamma'^2}{c^2}\left[\mathbf{u}' \cdot \ddot{\mathbf{u}}' + \frac{3\gamma'^2}{c^2}(\mathbf{u}' \cdot \dot{\mathbf{u}}')^2\right]\mathbf{u}'\right\} + \mathbf{O}(a)\,. \tag{B.41}$$

B.2.2 Evaluation of the Radiation Reaction Power

To determine the $O(1)$ terms of the self electromagnetic power in an arbitrary K' frame, begin with the charge integral in equation (3.2) at an arbitrary instant of time $t' = 0$

$$P'_{\text{em}}(0) = \int\limits_{\text{charge}} \rho'(\mathbf{r}',0)\mathbf{u}'(\mathbf{r}',0) \cdot \mathbf{E}'(\mathbf{r}',0)\,\mathrm{d}V'. \tag{B.42}$$

Applying the same procedure to (B.42) as we applied to (B.24) in the previous section yields instead of (B.39b)

$$P'_{\text{em}}(0) = \left(\frac{1}{a}\right) + \frac{e^2}{6\pi\epsilon_0 c^3}\mathbf{u}' \cdot \ddot{\mathbf{u}}_{\parallel}\,. \tag{B.43}$$

Substituting $\ddot{\mathbf{u}}_{\parallel}$ from (A.22a) into (B.43), rearranging the expression, and replacing the arbitrary time $t' = 0$ with t', results in the radiation reaction power in the arbitrary K' frame

$$P'_{\text{em}}(t) = \left(\frac{1}{a}\right) + \frac{e^2\gamma'^4}{6\pi\epsilon_0 c^3}\left[\mathbf{u}' \cdot \ddot{\mathbf{u}}' + \frac{3\gamma'^2}{c^2}(\mathbf{u}' \cdot \dot{\mathbf{u}}')^2\right] + O(a)\,. \tag{B.44}$$

The $1/a$ part of the self electromagnetic power in (B.23) combines with (B.44) to give the total self electromagnetic power to order a in an arbitrary K' inertial reference frame

$$P'_{\text{em}}(t) = -\frac{e^2}{6\pi\epsilon_0 a}\frac{\mathrm{d}}{\mathrm{d}t'}\left(\gamma' - \frac{1}{4\gamma'}\right) + \frac{e^2\gamma'^4}{6\pi\epsilon_0 c^3}\left[\mathbf{u}' \cdot \ddot{\mathbf{u}}' + \frac{3\gamma'^2}{c^2}(\mathbf{u}' \cdot \dot{\mathbf{u}}')^2\right] + O(a). \tag{B.45}$$

This completes the derivation of the self electromagnetic force and power to order a of Lorentz's model of the electron, that is, a total charge e uniformly distributed on a spherical insulator of radius a moving without rotation with arbitrary center velocity $\mathbf{u}'$. To my knowledge, it is the first rigorous derivation of these results for arbitrary velocity since Schott's [16] rigorous, yet extraordinarily lengthy derivation from the Liénard-Wiechert potentials; see Chapter 3 of the main text.

C

Electric and Magnetic Fields in a Spherical Shell of Charge

Consider the Lorentz model of the electron as a total charge e uniformly distributed within a thin, nonrotating, spherical shell of inner radius a and thickness δ (see Fig. 4.1 of the main text). In a proper inertial reference frame at rest instantaneously with the charge distribution, the velocity $\mathbf{u}(\mathbf{r},t)$ will be zero but the acceleration and higher time derivatives of velocity are, in general, nonzero functions of space and time $[\dot{\mathbf{u}}(\mathbf{r},t), \ddot{\mathbf{u}}(\mathbf{r},t), \ldots]$.

In equation (A.10) of Appendix A the electric field produced by this accelerating charge in its proper frame was found to be

$$\mathbf{E}(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int_{\text{charge}}\left\{\frac{\hat{\mathbf{R}}}{R^2} + \frac{1}{2c^2R}\left[\frac{\mathbf{r}'\cdot\dot{\mathbf{u}}}{c^2} - 1\right]\left[(\hat{\mathbf{R}}\cdot\dot{\mathbf{u}})\hat{\mathbf{R}} + \dot{\mathbf{u}}\right]\right.$$
$$\left. + \frac{3}{8}\frac{\hat{\mathbf{R}}}{c^4}\left[(\hat{\mathbf{R}}\cdot\dot{\mathbf{u}})^2 - |\dot{\mathbf{u}}|^2\right] + \frac{3(\hat{\mathbf{R}}\cdot\dot{\mathbf{u}})\dot{\mathbf{u}}}{4c^4} + \frac{2\ddot{\mathbf{u}}}{3c^3} + \mathbf{O}(R)\right\} de', \quad u = 0 \quad \text{(C.1)}$$

where $\dot{\mathbf{u}}$ and $\ddot{\mathbf{u}}$ in (C.1) refer to the time derivatives of the center velocity of the charged sphere at time t. The position of the charge element de' is designated by $\mathbf{r}'(t)$ and the vector $\mathbf{R}$ is defined as $\mathbf{r} - \mathbf{r}'(t)$.

We can find the magnetic field $\mathbf{B}(\mathbf{r},t)$ from the simple relationship between the electric and magnetic fields of a moving point charge [13]. Letting de' be the moving point charge, and $d\mathbf{E}(\mathbf{r},t)$ and $d\mathbf{B}(\mathbf{r},t)$ be the electric and magnetic fields of this point charge, we have

$$d\mathbf{B}(\mathbf{r},t) = \hat{\mathbf{R}}'(t') \times d\mathbf{E}(\mathbf{r},t)/c \quad \text{(C.2)}$$

where $d\mathbf{E}(\mathbf{r},t)$ is the integrand of (C.1) and $\mathbf{R}'(t')$ is defined as $\mathbf{r} - \mathbf{r}'(t')$, the difference vector between the position $\mathbf{r}$ of the observation point and the position $\mathbf{r}'(t')$ of the element of charge de' at the retarded time $t' = t - R'/c$. Expanding $\hat{\mathbf{R}}'(t')$ in a power series about t and making use of (A.8) gives

$$\hat{\mathbf{R}}'(t') = \hat{\mathbf{R}} - \frac{R}{2c^2}\left[\frac{\mathbf{r}'\cdot\dot{\mathbf{u}}}{c^2} - 1\right]\left[(\hat{\mathbf{R}}\cdot\dot{\mathbf{u}})\hat{\mathbf{R}} - \dot{\mathbf{u}}\right] - R^2\hat{\mathbf{R}}\left[\frac{(\hat{\mathbf{R}}\cdot\dot{\mathbf{u}})^2}{8c^4} + \frac{|\dot{\mathbf{u}}|^2}{8c^4} + \frac{(\hat{\mathbf{R}}\cdot\ddot{\mathbf{u}})}{6c^3}\right] + R^2\left[\frac{(\hat{\mathbf{R}}\cdot\dot{\mathbf{u}})\dot{\mathbf{u}}}{4c^4} + \frac{\ddot{\mathbf{u}}}{6c^3}\right] + \mathbf{O}(R^3). \quad (C.3)$$

Substituting $\hat{\mathbf{R}}'(t')$ from (C.3) and $\mathrm{d}\mathbf{E}(\mathbf{r},t)$ from the integrand of (C.1) into (C.2), one finds that most of the terms cancel leaving merely

$$\mathrm{d}\mathbf{B}(\mathbf{r},t) = \left[\frac{\hat{\mathbf{R}}(t)\times\ddot{\mathbf{u}}}{8\pi\epsilon_0 c^4} + \mathbf{O}(R)\right]\mathrm{d}e' \quad (C.4)$$

or

$$\mathbf{B}(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int_{\text{charge}}\left[\frac{\hat{\mathbf{R}}(t)\times\ddot{\mathbf{u}}}{2c^4} + \mathbf{O}(R)\right]\mathrm{d}e', \quad u=0 \quad (C.5)$$

for the magnetic field in the proper frame.

Equations (C.1) and (C.5) can be integrated in closed form for a uniformly distributed spherical shell of charge with inner radius a and small thickness δ. In particular, the expressions for the fields within the thin shell simplify to

$$\mathbf{E}(\mathbf{r},t) = \frac{e}{4\pi\epsilon_0}\left[\frac{r-a}{\delta a^2}\hat{\mathbf{r}} - \frac{2\dot{\mathbf{u}}}{3ac^2} + \frac{2\ddot{\mathbf{u}}}{3c^3} + \frac{4}{5c^4}\hat{\mathbf{r}}\cdot\left(\dot{\mathbf{u}}\dot{\mathbf{u}} - \frac{\bar{\mathbf{I}}|\dot{\mathbf{u}}|^2}{3}\right)\right] + \mathbf{O}(a) \quad (C.6)$$

$$\mathbf{B}(\mathbf{r},t) = \frac{e}{12\pi\epsilon_0 c^4}\hat{\mathbf{r}}\times\ddot{\mathbf{u}} + \mathbf{O}(a)\,, \quad (C.7)$$

$$u=0\,, \quad (a \le r \le a+\delta)\,.$$

The electric field in (C.6) agrees with the results of Page and Adams [70, secs 56–57] except for the 4/5 term in (C.6), which is missing in their work, because they do not take into account the variation (A.8) in acceleration of the charge with position around the shell. Also Page and Adams do not include the $\ddot{\mathbf{u}}$ term in the magnetic field of (C.7).

D

Derivation of the Linear Terms for the Self Electromagnetic Force

Begin the derivation with the expression (A.2) for the electric field produced by the moving element of charge de' in the shell of charge. Since we want to evaluate this expression (A.2) in a proper reference frame ($u(\mathbf{r},t)=0$) discarding all nonlinear terms in $\dot{\mathbf{u}}, \ddot{\mathbf{u}}, \ldots$, we see, with the help of the expansion (C.3) for $\hat{\mathbf{R}}'(t')$, and (A.8) and (A.13) for $\dot{\mathbf{u}}(\mathbf{r}',t')$ and $\mathbf{u}(\mathbf{r}',t')$, that (A.2) can be simplified immediately to

$$\mathrm{d}\mathbf{E}(\mathbf{r},t) = \frac{\mathrm{d}e'}{4\pi\epsilon_0}\left[\frac{\hat{\mathbf{R}}\times(\hat{\mathbf{R}}\times\dot{\mathbf{u}}(t'))}{Rc^2} + \frac{\hat{\mathbf{R}}' - \mathbf{u}(t')/c}{R'^2(1-\hat{\mathbf{R}}\cdot\mathbf{u}(t')/c)^3}\right] + \text{ nonlinear terms} \quad (D.1)$$

where, of course, $\mathbf{R}'$ is a function of the retarded time $t' = t - R'/c$. Inserting the expansion

$$\left[1 - \frac{\hat{\mathbf{R}}\cdot\mathbf{u}(t')}{c}\right]^{-3} = 1 + \frac{3\hat{\mathbf{R}}\cdot\mathbf{u}(t')}{c} + \text{ nonlinear terms} \quad (D.2)$$

into (D.1) gives

$$\mathrm{d}\mathbf{E}(\mathbf{r},t) = \frac{\mathrm{d}e'}{4\pi\epsilon_0}\left[\frac{\hat{\mathbf{R}}(\hat{\mathbf{R}}\cdot\dot{\mathbf{u}}(t')) - \dot{\mathbf{u}}(t')}{Rc^2} - \frac{\mathbf{u}(t')}{R^2c} + \frac{3\hat{\mathbf{R}}(\hat{\mathbf{R}}\cdot\mathbf{u}(t'))}{R^2c} + \frac{\hat{\mathbf{R}}'(t')}{R'^2}\right] + \text{ nonlinear terms.} \quad (D.3)$$

Now

$$\mathbf{R}'(t') = \mathbf{R}(t) - \frac{\dot{\mathbf{u}}(t)}{2}\left(\frac{R'(t')}{c}\right)^2 + \frac{\ddot{\mathbf{u}}(t)}{6}\left(\frac{R'(t')}{c}\right)^3 + \text{ nonlinear terms} \quad (D.4)$$

or with the insertion of the expansion

$$R'(t') = R\left[1 - \frac{\mathbf{R}\cdot\dot{\mathbf{u}}}{2c^2} + \cdots\right] \tag{D.5}$$

(D.4) becomes

$$\mathbf{R}'(t') = \mathbf{R} - \frac{\dot{\mathbf{u}}(t)}{2}\left(\frac{R}{c}\right)^2 + \frac{\ddot{\mathbf{u}}(t)}{6}\left(\frac{R}{c}\right)^3 + \cdots + \text{ nonlinear terms} \tag{D.6}$$

that is

$$\mathbf{R}'(t') = \mathbf{R}(t - R/c) + \text{ nonlinear terms.} \tag{D.7}$$

Similarly,

$$\mathbf{u}(t') = \mathbf{u}(t - R/c) + \text{ nonlinear terms} \tag{D.8a}$$

$$\dot{\mathbf{u}}(t') = \dot{\mathbf{u}}(t - R/c) + \text{ nonlinear terms} \tag{D.8b}$$

and

$$R'(t') = R(t - R/c) + \text{ nonlinear terms} \tag{D.8c}$$

or

$$R'(t') = R - \dot{R}\frac{R}{c} + \frac{\ddot{R}}{2}\left(\frac{R}{c}\right)^2 - \frac{\dddot{R}}{6}\left(\frac{R}{c}\right)^3 + \cdots + \text{ nonlinear terms.} \tag{D.8d}$$

With

$$\dot{R} = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{R}\cdot\mathbf{R})^{1/2} = \frac{\mathbf{R}}{R}\cdot\frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t} = \hat{\mathbf{R}}\cdot\mathbf{u} = 0 \tag{D.9a}$$

$$\ddot{R} = \hat{\mathbf{R}}\cdot\dot{\mathbf{u}} \tag{D.9b}$$

$$\dddot{R} = \hat{\mathbf{R}}\cdot\ddot{\mathbf{u}} + \text{ nonlinear terms} \tag{D.9c}$$

etc., inserted into (D.8d), $R'(t')$ becomes

$$R'(t') = R + \frac{\hat{\mathbf{R}}\cdot\dot{\mathbf{u}}}{2}\left(\frac{R}{c}\right)^2 + -\frac{\hat{\mathbf{R}}\cdot\ddot{\mathbf{u}}}{6}\left(\frac{R}{c}\right)^3 + \cdots + \text{ nonlinear terms.} \tag{D.10}$$

The vector $\mathbf{R}(t - R/c)$ can also be expanded in the form

$$\mathbf{R}(t - R/c) = \mathbf{R} + \frac{\dot{\mathbf{u}}}{2}\left(\frac{R}{c}\right)^2 - \frac{\ddot{\mathbf{u}}(t)}{6}\left(\frac{R}{c}\right)^3 + \cdots + \text{ nonlinear terms} \tag{D.11}$$

which combines with (D.10) and (D.7) to give

$$\frac{\mathbf{R}'(t')}{R'^3(t')} = \frac{\mathbf{R}(t - R/c)}{R'^3(t')} = \frac{\mathbf{R}}{R^3}\left[1 - \frac{3}{R}\left(\frac{\hat{\mathbf{R}}\cdot\dot{\mathbf{u}}}{2}\left(\frac{R}{c}\right)^2 - \frac{\hat{\mathbf{R}}\cdot\ddot{\mathbf{u}}}{6}\left(\frac{R}{c}\right)^3 + \cdots\right)\right]$$
$$+\frac{1}{R^3}\left[\frac{\dot{\mathbf{u}}}{2}\left(\frac{R}{c}\right)^2 - \frac{\ddot{\mathbf{u}}(t)}{6}\left(\frac{R}{c}\right)^3 + \cdots\right] + \text{ nonlinear terms.} \tag{D.12}$$

When we substitute (D.8a), (D.8b) and (D.12) into (D.3), integrate over $\mathrm{d}e'$, then multiply by $\mathrm{d}e = \rho \mathrm{d}V$ and integrate over $\mathrm{d}e$ to get the total self electromagnetic force, we are left with integrals of the form [17]

$$\iint_{\text{charge}} R^m \mathrm{d}e'\mathrm{d}e = 3\iint_{\text{charge}} \frac{x^2}{R^2} R^m \mathrm{d}e'\mathrm{d}e = \frac{2^{m+1}}{m+2}a^m e^2,$$
$$m = -1, 0, 1, 2, \ldots. \tag{D.13}$$

We see from (D.13) applied to (D.12) that

$$\iint_{\text{charge}} \frac{\mathbf{R}'(t')}{R'^3(t')}\mathrm{d}e'\mathrm{d}e = 0 + \text{ nonlinear terms.} \tag{D.14a}$$

Similarly, from (D.13) applied to the $\mathbf{u}(t')$ part of (D.3)

$$\iint_{\text{charge}} \frac{\mathbf{u}(t')}{R^2}\cdot\left[3\hat{\mathbf{R}}\hat{\mathbf{R}} - \bar{\mathbf{I}}\right]\mathrm{d}e'\mathrm{d}e = 0 + \text{ nonlinear terms} \tag{D.14b}$$

and from (D.13) applied to the $\dot{\mathbf{u}}(t')$ part of (D.3)

$$\iint_{\text{charge}} \frac{\dot{\mathbf{u}}(t')}{R}\cdot\left[\hat{\mathbf{R}}\hat{\mathbf{R}} - \bar{\mathbf{I}}\right]\mathrm{d}e'\mathrm{d}e = -\frac{2}{3}\iint_{\text{charge}} \frac{\dot{\mathbf{u}}(t - R/c)}{R}\mathrm{d}e'\mathrm{d}e$$
$$+ \text{ nonlinear terms.} \tag{D.14c}$$

Thus, integrating (D.3) over $\mathrm{d}e'$ and $\mathrm{d}e$ and using (D.14) shows that the exact expression for the total self electromagnetic force on the charge can be written simply as

$$\mathbf{F}_{\text{em}}(t) = \iint_{\text{charge}} \mathrm{d}\mathbf{E}(\mathbf{r}, t)\mathrm{d}e = -\frac{1}{6\pi\epsilon_0 c^2}\iint_{\text{charge}} \frac{\dot{\mathbf{u}}(t - R/c)}{R}\mathrm{d}e'\mathrm{d}e$$
$$+ \text{ nonlinear terms.} \tag{D.15}$$

Since $\dot{\mathbf{u}}(t - R/c)$ can be expanded in the power series

$$\dot{\mathbf{u}}(t - R/c) = \sum_{n=0}^{\infty}\frac{1}{n!}\frac{\mathrm{d}^{n+1}\mathbf{u}(t)}{\mathrm{d}t^{n+1}}\left(\frac{-R}{c}\right)^n \tag{D.16}$$

substituting (D.16) into (D.15) and applying the integrals (D.13) yields

$$\mathbf{F}_{\text{em}}(t) = \frac{e^2}{12\pi\epsilon_0 a^2 c}\sum_{n=0}^{\infty}\left(\frac{-2a}{c}\right)^{n+1}\frac{1}{(n+1)!}\frac{\mathrm{d}^{n+1}\mathbf{u}(t)}{\mathrm{d}t^{n+1}} + \text{ nonlinear terms} \tag{D.17}$$

or

$$\mathbf{F}_{\mathrm{em}}(t) = \frac{e^2}{12\pi\epsilon_0 a^2 c}\mathbf{u}(t-2a/c) + \text{ nonlinear terms, } u(t) = 0 \qquad \text{(D.18)}$$

or for small velocity

$$\mathbf{F}_{\mathrm{em}}(t) = \frac{e^2}{12\pi\epsilon_0 a^2 c}\left[\mathbf{u}(t-2a/c) - \mathbf{u}(t)\right] + \text{ nonlinear terms, } \frac{u^2}{c^2} \ll 1. \qquad \text{(D.19)}$$

The result (D.18) was stated without proof by Page [17]. It can also be obtained from the first series of a general expression for the self electromagnetic force, on a nonrelativistically rigid charged sphere, that was derived by Schott [64]. The linear part of the self electromagnetic force (D.19) is the same for both relativistically and nonrelativistically rigid spheres.

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