# THE SPECIAL THEORY OF RELATIVITY

BY

J. AHARONI

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# PREFACE TO THE SECOND EDITION

Soon after the publication of the first edition an important paper, entitled 'Self-energy and stability of the classical electron', was published by F. Rohrlich. It clarified the old puzzle of the 4/3 factor appearing in the expression for the electromagnetic mass $m_0$ of an electron when expressed in terms of its electrostatic energy, as derived by M. Abraham and H. A. Lorentz. Rohrlich proved that in a consistent application of the special theory of relativity, the 4/3 factor does not occur and must be replaced by 1. Rohrlich's paper initiated new interest in the problem and it turned out that actually a similar solution had already been proposed by B. Kwal in 1949 and the same result obtained as far back as 1922 by E. Fermi who used a different method. It can now be stated that the abolition of the 4/3 factor is also implicit in Dirac's paper on the classical theory of the electron (1938). It is difficult to explain why all the earlier papers passed unnoticed. Possibly this was due to Poincaré's idea to link the 4/3 factor with the instability of an electric charge on purely electrostatic forces. This new situation required a revision of the part of the book dealing with electrodynamics and to give a clear exposition of this development I found it necessary to enlarge this part considerably and also to include a paragraph on the retarded and advanced potentials as well as a brief account of the Wheeler–Feynman absorber theory of radiation. On revising the book, I also made a number of improvements, particularly in the part dealing with spinors.

To Dr. G. A. Perkins I am indebted for some criticism in connexion with the Proca field.

J. A.

*Imperial College*
*London, S.W. 7*

In the case of photons (or any other particle of zero mass), we cannot use (4.16), because the quotient becomes indefinite. We must revert to the original definition when we get

$$m_{r\text{ph}} = \frac{E_{\text{ph}}}{c^2}. \tag{4.17}$$

## 5. The Energy-Momentum Vector of an Electron

In the preceding sections we were dealing with particles in a primitive sense. They were specified by their mass and kinematic properties. Tacitly, they were also assumed to be very small. In addition no fields were assumed to surround them and contribute to their properties. Such an idealization is useful in many applications and such particles are often referred to as bare particles. However, in general particles are surrounded by some field, and in particular, electrons are surrounded by an electromagnetic field which obeys Maxwell's equations. In this section we shall deal with the question of whether this additional feature interferes with the particle concept. In other words, the question which arises is whether an electron together with the electromagnetic field which accompanies it and which undergoes changes as the state of motion of the electron is altered, can still be treated as a particle obeying relativistic dynamics. This turns out to be the case, although in one respect the dynamics has to be extended. When the electron is accelerated a certain amount of energy is radiated and extra work has to be done to account for this loss of energy.

To begin with, consider an electron at rest. The field is then the electrostatic Coulomb field. Let $W^0$ be the total energy of the field and $\mathbf{P}^0$ its total momentum. For an electron at rest the field has spherical symmetry and consequently the total momentum should be zero. Thus we have

$$W^0 = \tfrac{1}{2}\iiint \mathbf{E}^{02}\, d^3x^0, \qquad \mathbf{P}^0 = 0. \tag{5.1}$$

In accordance with the theory of relativity

$$m_{0e} = \frac{W^0}{c^2} \tag{5.2}$$

will have to be looked upon as the rest mass of the electron or at least as an electromagnetic contribution to the mass. Denoting any other contribution by $m_{0m}$ (the 'mechanical' mass) and the total rest mass by $m_0$, we get

$$m_0 = m_{0e}+m_{0m}. \tag{5.3}$$

$m_0$ would thus be the observed mass of the electron. The problem now



amounts to finding out whether the energy $W$ and momentum $\mathbf{P}$ of the electromagnetic field when the electron is moving with a velocity $\mathbf{u}$, turns out to be equal to

$$W = \frac{m_{0e}\, c^2}{(1-u^2/c^2)^{\frac{1}{2}}}, \qquad \mathbf{P} = \frac{m_{0e}\, \mathbf{u}}{(1-u^2/c^2)^{\frac{1}{2}}}, \tag{5.4}$$

or, to use a more relativistic notation, to find out whether

$$P_\mu = \left(\mathbf{P}, i\frac{W}{c}\right) = m_{0e}\, U_\mu. \tag{5.5}$$

What this really amounts to, is to determine whether a unique and relativistically invariant definition of the total energy and momentum of the field can be given so that (5.5) will hold.

Let $T_{\mu\nu}$ be the energy-momentum tensor of the electromagnetic field of an electron. Then, as was already shown on p. 113, it is possible to form a four-vector $P_\mu(\sigma)$ as follows:

$$P_\mu(\sigma) = -\frac{1}{c}\iiint\limits_{\sigma} T_{\mu\nu}(\sigma)\, d\sigma_\nu, \tag{5.6}$$

where $\sigma$ is a space-like surface to which all observers must refer when transformations are envisaged. The difficulty with this four-vector is, as indicated, its dependence on the hypersurface $\sigma$. $P_\mu(\sigma)$ will be independent of $\sigma$ only in the absence of charges, when $\partial T_{\mu\nu}/\partial x_\nu = 0$. In the presence of charges this is no longer the case and various choices of $\sigma$ will lead to different energy-momentum vectors. Essentially only one choice could satisfy (5.4). This difficulty was resolved independently by Kwal and Rohrlich. They pointed out that when considering an electron in motion it is not sufficient to introduce a relativistic four-vector in accordance with (5.6), but it is also necessary to select a hypersurface $\sigma$ which is linked to the motion of the electron in a relativistically invariant manner. Since $\sigma$ is a space-like surface and its normal is a time-like vector, the only possible choice which results from the above condition is to have the hypersurface orthogonal to the four-velocity at the intersection of the world line with the surface. Since the charge is assumed to be concentrated in a small region or else in a point, the surface can be chosen arbitrarily outside the region of intersection (where $\partial T_{\mu\nu}/\partial x_\nu = 0$). For this reason the hyperplane may be assumed as plane. Let

$$d\sigma_\nu = d\sigma . n_\nu, \tag{5.7}$$

where $n_\nu$ is a unit vector perpendicular to the hyperplane and $d\sigma$ (a scalar) is the magnitude of the surface element $d\sigma_\nu$. Since $n_\nu$ is a time-like unit vector, we have

$$n_\nu^2 = -1. \tag{5.8}$$

$d\sigma_\nu$ and $n_\nu$ must be parallel to $U_\nu$. Consequently, on using $U_\nu^2 = -c^2$ we get the following relations:

$$n_\nu = \frac{U_\nu}{c}, \tag{5.9}$$

$$d\sigma_\nu = d\sigma . n_\nu = d\sigma \frac{U_\nu}{c}. \tag{5.10}$$

(5.10) expresses the invariant link between the four-velocity of the electron and the hyperplane which must be chosen to define the energy-momentum vector of the electromagnetic field which accompanies the electron.

$d\sigma$ is a scalar and may be evaluated in any frame of reference. The simplest is the rest system of the electron ($x_\nu^0$). In this case $U_k = 0$ ($k = 1, 2, 3$), $U_4 = ic$, $n_k = 0$, $n_4 = i$ and it follows that

$$d\sigma = dx^0 dy^0 dz^0 = d^3x^0. \tag{5.11}$$

Moreover, it can now be seen that the hyperplane which has been selected in accordance with (5.10) is given by

$$t^0 = \text{const.} \tag{5.12}$$

This is in agreement with (5.1), where the surface over which the energy density is integrated has already been chosen as $t^0 = \text{const.}$ The world line in the rest system consists of a parallel to the $t^0$-axis and the plane $t^0 = \text{const.}$ is perpendicular to it.

The situation is now as follows: $d\sigma_\nu$ has relative to the rest system the components $d\sigma_1^0 = d\sigma_2^0 = d\sigma_3^0 = 0$, $d\sigma_4^0 = id^3x^0$. On using (5.10), for a system of coordinates relative to which the electron moves along the $x$-axis with a velocity $u_x = u$, we get the following result:

$$d\sigma_1 = \gamma\, d^3x^0 \frac{u}{c}, \quad d\sigma_2 = 0, \quad d\sigma_3 = 0, \quad d\sigma_4 = i\gamma\, d^3x^0. \tag{5.13}$$

This is also obtainable through a Lorentz transformation of $(0, 0, 0, id^3x^0)$, using (9.17), p. 35. Let $T_{\mu\nu}^0$ be the energy-momentum tensor in the rest system. From (4.13), p. 100, this tensor is given by

$$T_{\mu\nu}^0 = \begin{bmatrix} W^0 - E_x^{02} & -E_x^0 E_y^0 & -E_x^0 E_z^0 & 0 \\ -E_x^0 E_y^0 & W^0 - E_y^{02} & -E_y^0 E_z^0 & 0 \\ -E_x^0 E_z^0 & -E_y^0 E_z^0 & W^0 - E_z^{02} & 0 \\ 0 & 0 & 0 & -W^0 \end{bmatrix} = \begin{bmatrix} T_{11}^0 & T_{12}^0 & T_{13}^0 & 0 \\ T_{12}^0 & T_{22}^0 & T_{23}^0 & 0 \\ T_{13}^0 & T_{23}^0 & T_{33}^0 & 0 \\ 0 & 0 & 0 & T_{44}^0 \end{bmatrix}. \tag{5.14}$$



On transforming this tensor to a system which moves relative to the rest system with a velocity $-u$, we get (in the same way as in (3.6), p. 98):

$$T_{\mu\nu} = \begin{bmatrix} \gamma^2(T_{11}^0 - \beta^2 T_{44}^0) & \gamma T_{12}^0 & \gamma T_{13}^0 & i\gamma^2\beta(T_{11}^0 - T_{44}^0) \\ \gamma T_{12}^0 & T_{22}^0 & T_{23}^0 & i\gamma\beta T_{12}^0 \\ \gamma T_{13}^0 & T_{23}^0 & T_{33}^0 & i\gamma\beta T_{13}^0 \\ i\gamma^2\beta(T_{11}^0 - T_{44}^0) & i\gamma\beta T_{12}^0 & i\gamma\beta T_{13}^0 & -\gamma^2\beta^2 T_{11}^0 + \gamma^2 T_{44}^0 \end{bmatrix}, \tag{5.15}$$

where $\beta = uc^{-1}$. (As a check one can verify that the trace of the matrix $T_{\mu\nu}^0$ is equal to that of $T_{\mu\nu}$, since $\sum T_{\mu\mu}$ is an invariant.) From (5.6) and (5.13) follows

$$P_\mu(\sigma) = -\frac{1}{c}\left[\iiint \gamma\left(T_{\mu 1}\frac{u}{c} + iT_{\mu 4}\right) d^3x^0\right]. \tag{5.16}$$

From the spherical symmetry in the rest system it follows that integrals of $T_{12}^0$, $T_{13}^0$, $T_{23}^0$ over the whole space vanish. Also, since $T_{4k}^0$ is zero, on inserting from (5.15) in (5.16) we get:

$$P_1(\sigma) = -\gamma \frac{u}{c^2} \iiint T_{44}^0\, d^3x^0 = \gamma m_{0e} u_x = m_{0e} U_1,$$

$$P_2(\sigma) = 0,$$

$$P_3(\sigma) = 0,$$

$$P_4(\sigma) = -\frac{1}{c} \iiint i\gamma^3 T_{44}^0 (1 - \beta^2)\, d^3x^0 = i\gamma m_{0e} c = m_{0e} U_4, \tag{5.17}$$

and more generally

$$P_\mu(\sigma) = m_{0e} U_\mu. \tag{5.18}$$

In this way (5.5) is proved and the four-vector character of the energy and momentum of the field of an electron is verified.

It must be added that for a long time a great deal of confusion prevailed in connexion with the energy and momentum of the field of an electron. This was caused by not holding to the same hyperplane when transforming from the rest system to the laboratory system in which the electron moves. Thus, instead of using the surface $t^0 = \text{const.}$ in both systems, the surface $t = \text{const.}$ was used in the laboratory system. Naturally, the four-vector character of the energy and momentum of the field did not come out. According to Abraham and Lorentz, the total energy and momentum of the field were to be calculated from

$$P_\mu^{\text{AL}} = -\frac{i}{c} \iiint_{t=\text{const.}} T_{\mu 4}\, d^3x. \tag{5.19}$$

If one commits the further mistake (as also pointed out by Kwal) of using on the one hand the covariant tensor transformation (5.15) to constant over the extension of the electron. Often the passage is also made to a point electron. This passage produces a divergence but it is possible to circumvent this difficulty by a renormalization procedure which leaves all the observables of motion and radiation unaltered without destroying relativistic covariance. It must be mentioned at this stage that in certain theories which treat the electron as a point charge, it is assumed that the electron does not interact with itself at all and that the field of the electron only produces the interaction with other charges. In particular, as already mentioned, Wheeler and Feynman have elaborated on a theory in which radiation from an electron can take place only if other charges are present to absorb the radiation. On the basis of this theory an equation of motion for an electron can be derived only if an adequate absorber is also introduced (the absorber theory). We shall deal with this aspect briefly later on.

The first elaborate theory of an electron of finite size was produced by Abraham who considered the electron to be a rigid spherical particle. Abraham attempted to explain the whole mass of the electron as due to its self-field (the reaction of the field of the electron) and accepted the rigidity and the fact that the electron holds together in spite of the electrostatic repulsion between its parts, in the same way as we deal in ordinary dynamics of rigid bodies with their motion, without inquiring how the rigid body is held together. This separation of the dynamics of the electron from the problem of its internal stability is forced upon us to this day. However, Abraham's theory could not be relativistic because the concept of a rigid body with its six degrees of freedom cannot be reconciled with the theory of relativity. For one reason, a rigid body would enable us to transmit instant signals from one locality to another. Another non-relativistic feature in Abraham's theory was the assumption that the electron remained spherical while in motion. An improvement on Abraham's theory in this respect was achieved by Lorentz who took into account the Lorentz–Fitzgerald contraction, so that a spherical electron became an oblate ellipsoid while in motion. However, this does not reconcile the theory of relativity with the concept of a rigid body which would enable us to attach an invariant significance to simultaneity and hence to a hyperplane $t =$ const., in direct contradiction to the Lorentz transformation. As a consequence of this, in Abraham's theory as well as in that of Lorentz, the calculation led to a result which is not acceptable in a relativistic theory. It led to an electromagnetic mass which is given by $\frac{4}{3}W^0/c^2$, where $W^0$ is the electrostatic energy, instead of $W^0/c^2$. This error is similar to the one which was discussed in



the previous section in connexion with the energy-momentum four-vector. Hence, a relativistic theory of an electron of finite extension cannot be achieved by considering it as a rigid body, and a Lorentz contraction cannot remedy the situation. The only possible relativistic treatment of an electron of finite size can be achieved by considering it as a region of charge of a given shape, not at time $t =$ const. but over a hypersurface $t(x, y, z)$ which bears an invariant relationship to the world-tube of the electron. This relationship must also be maintained in the limit when the electron is taken to be a point charge and the world tube becomes a world line. The first to realize this was Fermi, as far back as 1922, but Fermi's paper went somehow into oblivion. Also it derived only the mass term and not the complete equation of motion including the radiative reaction. Fermi obtained for the electromagnetic mass the value $W^0/c^2$.

A relativistic equation of motion for a point electron was first derived by Dirac in 1938. It involved the calculation of a self-field in the vicinity of the electron. According to Dirac's theory, the electrostatic energy contributes to the mass the amount $W^0/c^2$ without a factor $\frac{4}{3}$, as in Fermi's theory. But somehow this result and its close relation to the energy-momentum vector did not get sufficient attention. Perhaps this was due to the fact that one of Dirac's main objects was to devise a method of subtracting the infinite electrostatic self- mass without violating relativistic covariance and without interference with the observed radiation. In Dirac's theory the remaining mass then is not electromagnetic and is not deduced but taken as observed.

The simplest approach to the problem is to postulate both the required interaction between the field and the electron as well as Maxwell's equations (as was done so far for the latter) without seeking a variational principle. The interaction will be assumed to result in a force which is given as follows:

$$F_\kappa = \frac{e}{c} F_{\kappa\lambda} U_\lambda. \tag{6.1}$$

$F_{\kappa\lambda}$ is the sum of the external field $F^{\text{in}}_{\kappa\lambda}$ and a certain average of the self-field $F^s_{\kappa\lambda}$. For an electron at rest, or when in uniform motion, this average field is zero, as a result of the symmetry of the self-field. When the electron is accelerated the average field is no longer zero.

In (6.1) it is already assumed that the electron is held together by some non-electric force which counteracts the electrostatic repulsion between the various charge elements. We thus assume that the world tube is of constant width.

A more fundamental equation than (6.1) which can be derived from a variational principle (see H. Weyl, *Space, Time, and Matter*) is the following:

$$f_\kappa = \frac{1}{c}\rho_0 F_{\kappa\lambda} U_\lambda, \tag{6.2}$$

where $f_\kappa$ is a force density, $\rho_0$ is the rest density of the charge, and $F_{\kappa\lambda}$ the actual total field. $f_\kappa$ is obtained from a force by dividing by the rest volume. Similarly, to obtain $\rho_0$ we divide total charge by rest volume. $\rho_0$ is a scalar and $f_\kappa$ a four-vector.

If the electron possesses a 'mechanical mass' $m_{0m}$, its acceleration will give rise to an additional inertial force $-m_{0m}\dot{U}$ and (6.1) will give

$$\frac{e}{c} F_{\kappa\lambda} U_\lambda - m_{0m}\dot{U}_\kappa = 0, \tag{6.3}$$

where the dot indicates differentiation with respect to proper time. On the other hand, if one adopts the Thomson–Abraham point of view that the mass of the electron is purely of electromagnetic origin ($m_{0m} = 0$), we have instead of (6.3)

$$\frac{e}{c} F_{\kappa\lambda} U_\lambda = 0. \tag{6.4}$$

Until recently, it was thought that on relativistic grounds Abraham's idea of a purely electromagnetic mass is untenable. However, the situation has altered since the correction of the $\frac{4}{3}$ factor was achieved and the four-vector character of the energy and momentum of the field proved. It only remains to see whether any future solution of the stability problem of the electron will entail an additional mass term. The idea of a point electron and the procedure of renormalization which it requires (such as Dirac's subtraction of the electromagnetic self-energy) is also, in all probability, provisional.

The first step for deriving the equation of motion consists in getting a relativistic expression for the resultant force which arises from the self-field or the average self-field. Each element of charge and current interacts with all the other elements, and because of retardation we must consider the electron not at a given time $t$ but over a space-time extension, so that the averaging will involve a four-dimensional integral over a region in the Minkowski space. Each element of the charge traces a world line and an extended electron will thus produce a world tube (see Fig. 27). The region of integration will be bounded by the invariant wall of the tube and by two end surfaces, $\sigma_1$ and $\sigma_2$. It is essential not only that these surfaces be adopted by all observers, but also, as pointed out first



by Fermi, that they bear an invariant relationship to the world tube. This can be achieved by taking the end pieces $\sigma_1$ and $\sigma_2$ to be orthogonal to the world tube. The mistake committed by Abraham, Lorentz, and many others can be traced to their choosing surfaces $t_1$ = const. and $t_2$ = const. (as indicated in Fig. 27) and taking the different surfaces $t_1'$ = const. and $t_2'$ = const. when a change to another frame of reference is performed. Such a procedure would be justified if simultaneity was absolute and a rigid body could exist. As pointed out by Fermi, the nearest approach to a rigid body can be achieved if at each moment we transform the electron to rest and assume that in the surfaces $t_0$ = const. the electron has the same shape and size. This leads to the surfaces which are perpendicular to the tube and have the same cross-section.

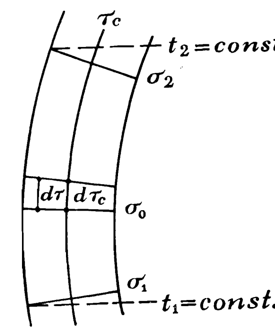


FIG. 27

When dealing with an electron of small size, it is convenient to introduce a central world line $\tau_c$ relative to which all the other lines will be referred. With the help of this central line the resultant self-force may be postulated as follows:

$$F^s_\kappa = \frac{d}{d\tau_c}\int\int\int\int_{\sigma_1}^{\sigma_2} c^{-1} J_\lambda F^s_{\kappa\lambda}\, d\sigma\, d\tau, \tag{6.5}$$

where $d\sigma$ is the invariant magnitude of a surface element on a surface $\sigma$ and $c\,d\tau$ an invariant line element perpendicular to $\sigma$. We have

$$c\,d\sigma . d\tau = d^4 x_\kappa. \tag{6.6}$$

A great simplification in the problem of deriving an equation of motion is achieved if we consider a quasi-stationary motion, in which the acceleration is small and the radiation may be neglected. The situation is parallel to the one encountered in the theory of quasi-stationary currents. To get to the quasi-stationary limit, two ways are open to us. The first, much in line with Fermi, is to work out the integral in (6.5) with the permissible approximations under quasi-stationary conditions. The other way is to make use of the fact that at each moment and corresponding velocity, the field of the electron yields an energy-momentum vector. Under quasi-stationary conditions we may assume that all the external force is doing is to alter the energy-momentum vector at a rate which is proportional to the external force. This method was not available to Fermi in 1922 because at that time the four-vector character of the energy and momentum of the field of an electron was not yet recognized. With this knowledge at our disposal, we get under quasi-stationary conditions:

$$\frac{dP_\kappa}{d\tau} = (m_{0e}+m_{0m})\frac{dU_\kappa}{d\tau} = \frac{e}{c}F^{\text{in}}_{\kappa\lambda}U_\lambda, \tag{6.7}$$

where $m_{0e}$ depends on the electrostatic energy $W^0$ as follows:

$$m_{0e} = \frac{W^0}{c^2} \tag{6.8}$$

and $m_{0m}$ is any mechanical mass which the electron may possess.

Following Fermi,† we evaluate the integral (6.5) to the relevant approximation as follows. We must first express the dependance of $d\tau$ on $d\tau_c$ in terms of the separation between the world lines $\tau$ and $\tau_c$. Let $R$ be the radius of curvature of a world line $\tau$ and let $R_c$ be this radius for the central world line. This radius is closely related to the acceleration

$$\frac{1}{R_c^2} = \frac{(dx_\mu/d\tau)^2(d^2x_\mu/d\tau^2)^2-(dx_\mu/d\tau\;\; d^2x_\mu/d\tau^2)^2}{(\dot{x}_\mu^2)^3} = \frac{U_\mu^2\dot{U}_\mu^2-(U_\mu\dot{U}_\mu)^2}{(U_\mu^2)^3}. \tag{6.9}$$

On using

$$U_\mu^2 = -c^2, \qquad U_\mu\dot{U}_\mu = 0, \tag{6.10}$$

we get

$$\frac{1}{R_c^2} = \frac{\dot{U}_\mu^2}{c^4}, \tag{6.11}$$

$$R_c = \frac{c^2}{A}, \tag{6.12}$$

where

$$A = +(\dot{U}_\mu^2)^{\frac{1}{2}} \tag{6.13}$$

is the magnitude of the acceleration. Let $\kappa_\mu$ be a space-like four-vector along the radius of curvature with magnitude $A/c^2$. Then

$$\kappa_\mu = -\frac{\dot{U}_\mu}{c^2} \tag{6.14}$$

† The average self-force as defined in (6.5) is due to the author and not to Fermi. Fermi based his calculation on an action principle for a continuously distributed charge which retains its shape in the rest system, and argued that variations which consist of shifts of the space $xyz$ along planes $t$ = const. are not consistent with the theory of relativity under the conditions of the problem, and that instead the variations should consist of shifts of $\sigma$-surfaces which leave these perpendicular to the world lines. One consequence of this condition is that the region over which the action is integrated must be bounded by surfaces $\sigma_1$ and $\sigma_2$ as indicated in Fig. 27. If the action is integrated over a region bounded by the surfaces $t_1$ = const. and $t_2$ = const. and the variations are along surfaces $t$ = const., one gets the factor $\frac{4}{3}$. In the correct method the factor is 1. (See also Chapter 4, section 1, the variation principle.) In a field theory the choice of $\sigma_1$ and $\sigma_2$ is arbitrary as long as they are space surfaces. In our present case we are dealing with an electron which retains its shape in its rest system and the perpendicularity of the surfaces to the world lines is essential.



is the curvature of the world line. Let $\pi_\mu$ be a space-like vector joining the intersections of $\tau_c$ and $\tau$ with $\sigma_0$. $c^2\kappa_\mu/A$ is a unit vector and as can be seen from Fig. 28,

$$R = R_c - \frac{\kappa_\mu\pi_\mu c^2}{A} \tag{6.15}$$

and to a sufficient approximation

$$\frac{d\tau}{d\tau_c} = \frac{R}{R_c} = 1-\frac{\kappa_\mu\pi_\mu c^2}{AR_c} = 1-\kappa_\mu\pi_\mu \tag{6.16}$$

so that

$$d\tau = d\tau_c(1-\kappa_\mu\pi_\mu) = d\tau_c\left(1+\frac{\dot{U}_\mu\pi_\mu}{c^2}\right). \tag{6.17}$$

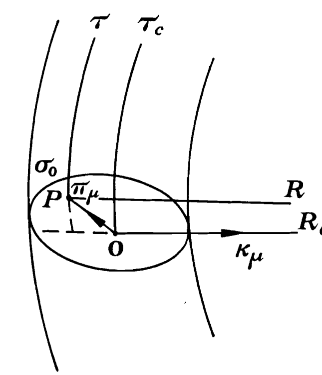


Fig. 28

Consequently, (6.5) becomes

$$F^s_\kappa = \frac{d}{d\tau}\iiiint\limits_{\sigma_0}^{\sigma} c^{-1}J_\lambda F^s_{\kappa\lambda}\left(1+\frac{\dot{U}_\mu\pi_\mu}{c^2}\right)d\sigma d\tau_c, \tag{6.18}$$

$$F^s_\kappa = \iiint\limits_{\sigma} c^{-1}J_\lambda F^s_{\kappa\lambda}\left(1+\frac{\dot{U}_\mu\pi_\mu}{c^2}\right)d\sigma. \tag{6.19}$$

This expression becomes simpler if the rest system of the electron is introduced ($d\tau \to dt$, $\dot{U}_\mu\pi_\mu \to \dot{\mathbf{u}}\boldsymbol{\pi}$, $d\sigma \to dx^0dy^0dz^0$, $J_1 = J_2 = J_3 = 0$, $J_4 = ic\rho$, $F_{k\lambda} \to \mathbf{E}$, $\rho^0dx^0dy^0dz^0 = de$):

$$(\mathbf{F}^s)_{\text{rest system}} = \int \mathbf{E}^s\left(1+\frac{\dot{\mathbf{u}}\boldsymbol{\pi}}{c^2}\right)de. \tag{6.20}$$

Fermi was the first to derive this formula and pointed out that it corrects the result obtained by Lorentz and repeated by many others with regard to the factor $\frac{4}{3}$ for the electromagnetic mass. If instead of the surfaces $\sigma_0$ and $\sigma$ chosen by Fermi one takes $t_0 = $ const. and $t = $ const., we have $d\tau = d\tau_c$ (instead of (6.17)) and the self-force comes out to be

$$(\mathbf{F}^s)_{\text{rest system}} = \int \mathbf{E}^s \, de \quad \text{(according to Lorentz).} \tag{6.21}$$

It is this integral which yields

$$(\mathbf{F}^s)_{\text{rest system}} = \mathbf{F} = -\frac{4}{3}\frac{W^0}{c^2}\dot{\mathbf{u}}. \tag{6.22}$$

Fermi showed that if we restrict ourselves to terms linear in $\dot{\mathbf{u}}$ the additional term appearing in (6.20) is given by

$$\int \mathbf{E}^s \frac{\dot{\mathbf{u}}\boldsymbol{\pi}}{c^2} \, de = \frac{W^0}{3c^2}\dot{\mathbf{u}}, \tag{6.23}$$

so that

$$\mathbf{F}^s = -\frac{W^0}{c^2}\dot{\mathbf{u}} = -m_{0e}\dot{\mathbf{u}}. \tag{6.24}$$

From this result in the rest system it follows further that in an arbitrary inertial frame we shall have

$$F^s_\kappa = -m_{0e}\dot{U}_\kappa \tag{6.25}$$

and (6.8) will hold when an external force is present. The proof of (6.23) proceeds as follows. $\mathbf{E}^s$, which appears in the integral, is the sum of two forces: (1) the Coulomb force

$$\frac{\mathbf{r}}{r^3} \, dede',$$

where $r$ is the distance between the charge elements $de$ and $de'$, and (2) a force which is proportional to the acceleration $\dot{\mathbf{u}}$ (see next section, (7.49)). Hence, when this part is multiplied by $\dot{\mathbf{u}}$ it will give a contribution which is proportional to $\dot{\mathbf{u}}^2$. To obtain the mass term we are only interested in terms with $\dot{\mathbf{u}}$ as a factor, hence we disregard the second part of $\mathbf{E}^s$ and we are left with the evaluation of

$$\mathbf{I} = \frac{1}{c^2}\iint \frac{\mathbf{r}}{r^3}(\dot{\mathbf{u}}\boldsymbol{\pi}) \, dede'. \tag{6.26}$$

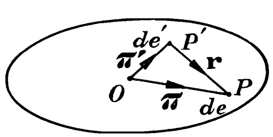


FIG. 29

The value of this integral is not changed if $de$ and $de'$ are exchanged. $\mathbf{r}$ becomes $-\mathbf{r}$ and $\boldsymbol{\pi}$ which points from $O$ to $P$ (see Fig. 29) becomes a vector $\boldsymbol{\pi}'$ which points from $O$ to $P'$. Hence,

$$\mathbf{I} = \frac{1}{c^2}\iint \frac{\mathbf{r}}{r^3}(\dot{\mathbf{u}}\boldsymbol{\pi}) \, dede' = -\frac{1}{c^2}\iint \frac{\mathbf{r}}{r^3}(\dot{\mathbf{u}}\boldsymbol{\pi}') \, dede'. \tag{6.27}$$



Taking the arithmetic mean of these two expressions and noting that $\boldsymbol{\pi}-\boldsymbol{\pi}' = \mathbf{r}$, we get

$$\mathbf{I} = \frac{1}{2c^2}\iint \frac{\mathbf{r}}{r^3}[\dot{\mathbf{u}}(\boldsymbol{\pi}-\boldsymbol{\pi}')] \, dede' = \frac{1}{2c^2}\iint \frac{\mathbf{r}}{r^3}(\mathbf{r}\dot{\mathbf{u}}) \, dede'. \tag{6.28}$$

Consider the $x$-component

$$I_x = \frac{1}{2c^2}\iint \frac{(x-x')}{r^3}[\dot{u}_x(x-x')+\dot{u}_y(y-y')+\dot{u}_z(z-z')] \, dede'. \tag{6.29}$$

Here we may replace $(x-x')^2$, $(x-x')(y-y')$, $(x-x')(z-z')$ by their average in all directions for any given $r$. These averages are $\frac{1}{3}r^2$, 0, 0 respectively. Hence

$$I_x = \frac{1}{6c^2}\iint \frac{dede'}{r} = \frac{W^0}{3c^2}\dot{u}_x \tag{6.30}$$

and (6.23) follows.

Eqn. (6.7) for the quasi-stationary case, which may be expressed as

$$\frac{dP_\kappa}{d\tau} = m_0\frac{dU_\kappa}{d\tau} = \frac{e}{c}F^{\text{in}}_{\kappa\lambda}U_\lambda, \tag{6.31}$$

where $m_0$ is the total rest mass of the electron, is used whenever radiation is negligible, and has many practical applications. In particular it is used in all methods for determining the observable rest mass $m_0$, in which $F_{k\lambda}$ is a static or quasi-static electromagnetic field.

Expressed in terms of a Newton force (6.31) becomes

$$m_0\frac{dU_\kappa}{dt} = \left(1-\frac{u^2}{c^2}\right)^{\frac{1}{2}}\frac{e}{c}F_{\kappa\lambda}U_\lambda, \tag{6.32}$$

where $F_{\kappa\lambda} = F^{\text{in}}_{\kappa\lambda}$. In the remainder of this section the superfix 'in' is dropped.

For the three space components $\kappa = x, y, z$ we thus get

$$m_0\frac{dU_a}{dt} = \frac{e}{c}F_{a\lambda}u_\lambda, \tag{6.33}$$

where

$$u_\lambda = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}, ic\right) = (\mathbf{u}, ic). \tag{6.34}$$

Inserting for $F_{a\lambda}$ the corresponding expressions in terms of $\mathbf{E}$ and $\mathbf{H}$ we get

$$m_0\frac{dU_a}{dt} = \frac{e}{c}[\mathbf{u}, \mathbf{H}]+e\mathbf{E}. \tag{6.35}$$

The fourth component of (6.32) yields

$$\frac{dW}{dt} = e\mathbf{E}.\mathbf{u} = e\left(E_x\frac{dx}{dt}+E_y\frac{dy}{dt}+E_z\frac{dz}{dt}\right). \tag{6.36}$$