38b) A theorem of calculus of probability and some applications

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§ 1. The theorem we want to deal with concerns the properties of sums of many uncorrelated terms having a known statistical distribution. The fundamental theorem on these sums is due to Laplace.[1] We announce the theorem together with an outline of its demonstration towards which we must take the first steps in establishing this new theorem. Let n be a very large number and let $y_1, y_2 \ldots, y_n$ represent n unknowns, of which we know their statistical distribution; that is, we know that the probability that y_i has a value ranging between y_i and $y_i + dy_i$ is $\varphi_i(y_i)dy_i$, where φ_i is a known function for which, obviously

$$\int_{-\infty}^{\infty} \varphi_i(y)dy = 1, \tag{1}$$

which means that y_i certainly has a value between $-\infty$ and $+\infty$. In addition we will assume that the statistical distribution of y_i is not affected by the values that the other y's can assume, that is, we assume the y_i's are completely uncorrelated. Then we take y_i to have a vanishing average, that is:

$$\bar{y}_i = \int_{-\infty}^{\infty} y\varphi_i(y)dy = 0. \tag{2}$$

Finally the average of the squared y_i is

$$\bar{y}_i^2 = \int_{-\infty}^{\infty} y^2\varphi_i(y)dy = k_i^2 \tag{3}$$

and assume that, for any i, k_i^2 is negligible with respect to $\sum_1^n k_i^2$. Under these assumptions, Laplace's theorem holds which says that: The probability that inequalities

$$x \leq \sum_1^n y_i \leq x + dx \tag{4}$$

hold at the same time is given by

$$F(x)dx = \frac{1}{\sqrt{2\pi \sum_1^n k_i^2}} e^{-\frac{x^2}{2\sum_1^n k_i^2}} dx. \tag{5}$$

To demonstrate this, we let r be a number $\leq n$ and let $F(r, x)dx$ be the probability that the inequalities

$$x \leq \sum_1^r y_i \leq x + dx \tag{6}$$

[1] *Théorie analytique des probabilités*, Oeuvres, VII, p. 309.

hold true. Now, if p is any value, let us look for the probability that the inequalities

$$\sum_1^{r-1} y_i < p < \sum_1^r y_i \tag{7}$$

hold simultaneously, that is, that the addition of y_r to $\sum_1^{r-1} y_i$ does not exceed p. This probability is obviously given by

$$\int_0^\infty d\xi F(r-1, p-\xi) \int_\xi^\infty \varphi_r(y) dy.$$

Analogously, the probability that inequalities

$$\sum_1^{r-1} y_i > p > \sum_1^r y_i \tag{8}$$

hold simultaneously is

$$\int_0^\infty d\xi F(r-1, p+\xi) \int_\xi^\infty \varphi_r(y) dy.$$

The difference between these two probabilities is obviously given by the difference between the probability that $\sum_1^r y_i > p$ and the probability that $\sum_1^{r-1} y_i > p$, that is by

$$\int_p^\infty F(r, x) dx - \int_p^\infty F(r-1, x) dx \quad .$$

Then we have

$$\int_p^\infty F(r, x) dx - \int_p^\infty F(r-1, x) dx = \int_0^\infty d\xi F(r-1, p-\xi) \int_\xi^\infty \varphi_r(y) dy$$

$$- \int_0^\infty d\xi F(r-1, p+\xi) \int_\xi^\infty \varphi_r(y) dy\,.$$

In the r.h.s. we can reverse the integrations by the formulas

$$\int_0^\infty d\xi \int_\xi^\infty dy = \int_0^\infty dy \int_0^y d\xi \qquad ; \qquad \int_0^\infty d\xi \int_{-\infty}^{-\xi} dy = \int_{-\infty}^0 dy \int_0^{-y} d\xi$$

and this becomes, also replacing ξ by $-\xi$ in the second term

$$\int_{-\infty}^\infty \varphi_r(y) dy \int_0^y F(r-1, p-\xi) d\xi\,.$$

As an approximation we set

$$F(r-1, p-\xi) = F(r-1, p) - \xi \frac{\partial F(r-1, p)}{\partial p}\,.$$

Thus the above expression becomes

$$F(r-1, p) \int_{-\infty}^\infty \varphi_r(y) dy \int_0^y d\xi - \frac{\partial F(r-1, p)}{\partial p} \int_{-\infty}^\infty \varphi_r(y) dy \int_0^y \xi d\xi$$

$$= F(r-1,p)\int_{-\infty}^{\infty} y\varphi_r(y)dy - \frac{1}{2}\frac{\partial F(r-1,p)}{\partial p}\int_{-\infty}^{\infty} y^2\varphi_r(y)dy$$

i.e., remembering (2) and (3):

$$-\frac{k_r^2}{2}\frac{\partial F(r-1,p)}{\partial p}\,.$$

In this way we obtain the equality

$$\int_p^{\infty} F(r,x)dx - \int_p^{\infty} F(r-1,x)dx = -\frac{k_r^2}{2}\frac{\partial F(r-1,p)}{\partial p}\,. \tag{9}$$

Differentiating it with respect to p we obtain

$$-F(r,p) + F(r-1,p) = -\frac{k_r^2}{2}\frac{\partial^2 F(r-1,p)}{\partial p^2}\,. \tag{10}$$

In this let us replace $r-1$ by r, p by x, and in our approximation, set

$$F(r+1,x) - F(r,x) = \frac{\partial}{\partial r}F(r,x).$$

Then (10) gives for $F(r,x)$ the differential equation

$$\frac{\partial}{\partial r}F(r,x) = -\frac{k_{r+1}^2}{2}\frac{\partial^2}{\partial x}F(r,x)\,. \tag{11}$$

Replacing r by a new variable

$$t = \int_0^{r+1} k_i^2 di\,, \tag{12}$$

(11) becomes

$$\frac{\partial F}{\partial t} = \frac{1}{2}\frac{\partial^2 F}{\partial x^2}\,. \tag{13}$$

Then one obviously has the condition that, for any t

$$\int_{-\infty}^{\infty} F dx = 1 \tag{14}$$

and that for $t = 0$, F has a nonvanishing value only when $|x|$ is infinitesimal. It is known that these conditions are more than sufficient to determine F. They are satisfied by setting

$$F = \frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}}\,.$$

By assigning t to its value, which in our degree of approximation is $\sum_1^r k_i^2$, we find

$$F(r,x) = \frac{1}{\sqrt{2\pi\sum_1^r k_i^2}}e^{-\frac{x^2}{2\sum_1^r k_i^2}}\,. \tag{15}$$

Then one obviously has $F(x) = F(n,x)$, and then

$$F(x) = \frac{1}{\sqrt{2\pi\sum_1^n k_i^2}}e^{-\frac{x^2}{2\sum_1^n k_i^2}} \qquad Q.E.D.$$

§ 2. Let us keep the notation and the assumptions made at the beginning of the previous section and in addition assume that all the $\varphi_i(y)$ are equal (as a consequence we will cancel their index). Then let us indicate by a an arbitrary positive value. Thus we can state the following

Theorem 2.1. *The probability that at least one among the quantities*

$$y_1, y_1 + y_2, y_1 + y_2 + y_3, \ldots, \sum_1^n y_n$$

exceeds a is given by

$$\frac{2}{\sqrt{\pi}} \int_{\frac{a}{\sqrt{2nk^2}}}^{\infty} e^{-x^2} dx$$

provided that a is great enough with respect to k.

In particular, if n tends to infinity, such a probability tends to 1, i.e., to certainty. To demonstrate this, let us indicate by $F(r,x)dx(x < a)$ the probability that the inequalities (6) are satisfied and in addition all r quantities

$$y_1, y_1 + y_2, \ldots, \sum_1^r y_i \tag{16}$$

are less than a. At the same time, the same arguments of the previous section show us that $F(r,x)$ still will satisfy the differential equation (11) which, in this case, can be written as

$$\frac{\partial F}{\partial r} = \frac{k^2}{2} \frac{\partial^2 F}{\partial x^2} \,. \tag{17}$$

The boundary conditions will instead be changed. In fact, we observe that

$$\int_{-\infty}^{a} F(r,x)dx$$

gives the probability that none of quantities (16) exceeds a and then

$$-\int_{-\infty}^{a} F(r+1,x)dx + \int_{-\infty}^{a} F(r,x)dx$$

gives the proability that, because of the addition of y_{r+1}, $\sum_1^r y_i$ then exceeds a. A calculation analogous to the one performed in the previous section shows us that this probability is

$$\int_0^\infty F(r, a-\xi)d\xi \int_\xi^\infty \varphi(y)dy$$

i.e., in our degree of approximation, neglecting ξ with respect to a

$$F(r,a) \int_0^\infty d\xi \int_\xi^\infty \varphi(y)dy$$

that is, by reversing the quadratures

$$F(r,a)\int_0^\infty \varphi(y)dy\int_0^y d\xi = F(r,a)\int_0^\infty y\varphi(y)dy.$$

By now setting

$$h=\int_0^\infty y\varphi(y)dy \tag{18}$$

we find

$$\int_{-\infty}^a \{F(r+1,x)-F(r,x)\}\,dx = -hF(r,a)\,.$$

But, in our usual degree of approximation, we can set

$$F(r+1,x)-F(r,x)=\frac{\partial F(r,x)}{\partial r}$$

and the previous equation becomes

$$\frac{\partial}{\partial r}\int_{-\infty}^a F(r,x)dx = -hF(r,a)\,. \tag{19}$$

After all, our unknown function F must satisfy the differential equation (17) on the interval $-\infty, a$; satisfy equation (19) at the extremity a; then it must vanish together with its derivatives at the extremity $-\infty$ and, for $r=0$, have a nonvanishing value only for very small $|x|$, but with the condition that the area comprised between it and the x axis be $=1$.

It is easy to prove, at least when h is positive as in our case, that these conditions are sufficient to determine F.

Therefore we observe that by multiplying (17) by dx and integrating it between $-\infty$ and a, one finds

$$\frac{k^2}{2}\left(\frac{\partial F}{\partial x}\right)_a = \frac{\partial}{\partial r}\int_{-\infty}^a F(r,x)dx$$

as a consequence, (19) becomes

$$\frac{k^2}{2h}\left(\frac{\partial F(r,x)}{\partial x}\right)_a + F(r,a)=0\,. \tag{19}$$

Then, for our purposes it is evidently sufficient to prove that if a function $\Phi(r,x)$ is $=0$ for $r=0$ and satisfies the equations

$$\frac{\partial \Phi}{\partial r}=\frac{k^2}{2}\frac{\partial^2\Phi}{\partial x^2} \qquad ; \qquad \frac{k^2}{2h}\left(\frac{\partial\Phi}{\partial x}\right)_{x=a}+\phi(r,a)=0 \tag{20}$$

and if for $x=-\infty$ it is always $=0$, then it is certainly identically zero. In fact one has

$$\int_{-\infty}^a\left(\frac{\partial\Phi}{\partial x}\right)^2 dx = \int_{-\infty}^a \frac{\partial}{\partial x}\left(\Phi\frac{\partial\Phi}{\partial x}\right)dx - \int_{-\infty}^a \Phi\frac{\partial^2\Phi}{\partial x^2}dx$$

that is, because of (20)

$$\int_{-\infty}^{a}\left(\frac{\partial\Phi}{\partial x}\right)^2 dx = \left(\Phi\frac{\partial\Phi}{\partial x}\right)_{-\infty}^{a} - \frac{2}{k^2}\int_{-\infty}^{a}\Phi\frac{\partial\Phi}{\partial r}dx$$

$$= \Phi(r,a)\left(\frac{\partial\Phi}{\partial x}\right)_{x=a} - \frac{1}{k^2}\frac{\partial}{\partial r}\int_{-\infty}^{a}\Phi^2 dx = -\frac{2h}{k^2}\Phi^2(r,a) - \frac{1}{k^2}\frac{\partial}{\partial r}\int_{-\infty}^{a}\Phi^2 dx$$

i.e.,

$$\int_{-\infty}^{a}\left(\frac{\partial\Phi}{\partial x}\right)^2 dx + \frac{2h}{k^2}\Phi^2(r,a) + \frac{1}{k^2}\frac{\partial}{\partial r}\int_{-\infty}^{a}\Phi^2(r,x)dx = 0\,. \tag{21}$$

Let us now suppose that, for some value of r and x, Φ could be different from zero; then for some value $\overline{r}$ of r, $\int_{-\infty}^{a}\Phi^2 dx$ would certainly be positive; in addition, since $r = 0$ implies $\phi = 0$ and so $\int_{-\infty}^{a}\Phi^2(0,x)dx = 0$, there will certainly be some value of r between zero and $\overline{r}$ for which $\frac{d}{dr}\int_{-\infty}^{a}\Phi^2(r,x)dx$ is positive.

Now, the first two terms in (21) cannot be negative; the first one is, at least in some cases, positive and this is absurd. Therefore it must certainly be that $\phi(r,x) = 0$.
Q.E.D.

Given this, it will be enough for us to find any solution whatever fulfilling the conditions imposed to be sure that it is the solution we were looking for. Let us try to see if our conditions can be satisfied by setting

$$F(r,x) = \frac{1}{k\sqrt{2\pi r}}e^{-\frac{x^2}{2rk^2}} - \frac{1}{k\sqrt{2\pi}}\int_0^r \frac{u(\rho)e^{-\frac{(a-x)^2}{2(r-\rho)k^2}}}{\sqrt{r-\rho}}d\rho \tag{22}$$

where $u(\rho)$ is a function to be determined. With this position, the differential equation (17) and the limiting conditions for $x = -\infty$ and $r = 0$ are certainly satisfied. Then it remains to determine $u(\rho)$ so that (19) is satisfied too.

Now from (22) we have

$$F(r,a) = \frac{1}{k\sqrt{2\pi r}}e^{-\frac{a^2}{2rk^2}} - \frac{1}{k\sqrt{2\pi}}\int_0^r \frac{u(\rho)d\rho}{\sqrt{r-\rho}}$$

$$\int_{-\infty}^{a}F(r,x)dx = \frac{1}{k\sqrt{2\pi r}}\int_{-\infty}^{a}e^{-\frac{x^2}{2rk^2}}dx - \frac{1}{k\sqrt{2\pi}}\int_0^r\frac{u(\rho)d\rho}{\sqrt{r-\rho}}\int_{-\infty}^{a}e^{-\frac{(a-x)^2}{2(r-\rho)k^2}}dx$$

$$= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\frac{a}{k\sqrt{2r}}}e^{-x^2}dx - \frac{1}{2}\int_0^r u(\rho)d\rho \tag{23}$$

and so

$$\frac{\partial}{\partial r}\int_{-\infty}^{a}F(r,x)dx = -\frac{ae^{-\frac{a^2}{2rk^2}}}{2k\sqrt{2\pi r^3}} - \frac{1}{2}u(r)$$

with which (19) becomes

$$\frac{e^{-\frac{a^2}{2rk^2}}}{k\sqrt{2\pi r}}\left(h - \frac{a}{2r}\right) = \frac{h}{k\sqrt{2\pi}}\int_0^r\frac{u(\rho)d\rho}{\sqrt{r-\rho}} + \frac{u(r)}{2} \tag{24}$$

which is an integral equation of the second kind for the unknown function $u(\rho)$. In spite of all our efforts, we have not succeeded in solving it exactly; we only have an approximate solution. We shall deal with this shortly. We want to prove first, without approximation, that one has

$$\int_0^\infty u(r)dr = 1\ .$$

Therefore, let ϑ be an arbitrary positive quantity and let us multiply both sides of (24) by $\sqrt{\theta}e^{-\theta r}dr$ and then integrate from $r = 0$ to $r = \infty$. One finds

$$\frac{\sqrt{\theta}h}{k\sqrt{2\pi}}\int_0^\infty \frac{e^{-\theta r-\frac{a^2}{2rk^2}}}{\sqrt{r}}dr - \frac{a\sqrt{\theta}}{2k\sqrt{2\pi}}\int_0^\infty \frac{e^{-\theta r-\frac{a^2}{2rk^2}}}{r^{3/2}}dr$$

$$= \frac{h\sqrt{\theta}}{k\sqrt{2\pi}}\int_0^\infty e^{-\theta r}dr\int_0^r \frac{u(\rho)d\rho}{\sqrt{r-\rho}} + \frac{\sqrt{\theta}}{2}\int_0^\infty e^{-\theta r}u(r)dr$$

$$= \frac{h\sqrt{\theta}}{k\sqrt{2\pi}}\int_0^\infty u(\rho)d\rho\int_\rho^\infty \frac{e^{-\theta r}dr}{\sqrt{r-\rho}} + \frac{\sqrt{\theta}}{2}\int_0^\infty e^{-\theta r}u(r)dr$$

$$= \frac{h}{k\sqrt{2}}\int_0^\infty e^{-\theta\rho}u(\rho)d\rho + \frac{\sqrt{\theta}}{2}\int_0^\infty e^{-\theta r}u(r)dr\,.$$

In addition one has

$$\sqrt{\theta}\int_0^\infty \frac{e^{-\theta r-\frac{a^2}{2rk^2}}}{\sqrt{r}}dr = 2\int_0^\infty e^{-x^2-\frac{a^2\theta}{2k^2x^2}}dx = \sqrt{\pi}e^{-\frac{a\sqrt{2\theta}}{k}}\ .$$

Passing to the limit $\theta = 0$ the above equation then becomes

$$\frac{h}{k\sqrt{2}} = \frac{h}{k\sqrt{2}}\int_0^\infty u(\rho)d\rho\,,$$

from which it follows that

$$\int_0^\infty u(\rho)d\rho = 1\,. \qquad Q.E.D. \tag{25}$$

At this point we can already get an interesting result. In fact, from (23) we have

$$\lim_{r=\infty}\int_{-\infty}^a F(r,x)dx = \lim_{r=\infty}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\frac{a}{k\sqrt{2r}}} e^{-x^2}dx - \frac{1}{2}\int_0^\infty u(r)dr = 0\,. \tag{26}$$

If we remember the meaning of $F(r,x)$ this result can be read:

The probability that at least one of the values (16) exceeds a becomes certain when r tends to infinity. We remark that this result holds true independently of the approximation we are going to make to solve (24).

Let us now turn to the approximate solution of (24).

For this we observe that, as one can immediately verify,

$$w(r) = \frac{ae^{-\frac{a^2}{2rk^2}}}{k\sqrt{2\pi r^3}} \tag{27}$$

is a solution of the integral equation of the second kind

$$\frac{e^{-\frac{a^2}{2rk^2}}}{k\sqrt{2\pi r}}\left(h+\frac{a}{2r}\right)=\frac{h}{k\sqrt{2\pi}}\int_0^r \frac{w(\rho)d\rho}{\sqrt{r-\rho}}+\frac{1}{2}w(r) \tag{28}$$

which differs from (24) only in the sign inside the bracket of the left hand side. Now, owing to the assumptions we have made, whenever r is large enough so that $e^{-\frac{a^2}{2rk^2}}$ is not too small, $a/2r$ is negligible with respect to h and then we can assume $w(r)$ to be an approximate solution of (24) by setting

$$u(r)=\frac{ae^{-\frac{a^2}{2rk^2}}}{k\sqrt{2\pi r^3}}\,. \tag{29}$$

It is easy to check that from (29) one has $\int_0^\infty u(r)dr=1$.

Now, from (23) we get

$$\int_{-\infty}^a F(r,x)dx=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\frac{a}{k\sqrt{2r}}} e^{-x^2}dx-\frac{1}{2}\frac{ae^{-\frac{a^2}{2\rho k^2}}}{k\sqrt{2\pi\rho^3}}d\rho$$

$$=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\frac{a}{k\sqrt{2r}}} e^{-x^2}dx-\frac{1}{\sqrt{\pi}}\int_{\frac{a}{k\sqrt{2r}}}^{\infty} e^{-x^2}dx=1-\frac{2}{\sqrt{\pi}}\int_{\frac{a}{k\sqrt{2r}}}^{\infty} e^{-x^2}dx\,.$$

And so

$$1-\int_{-\infty}^a F(r,x)dx=\frac{2}{\sqrt{\pi}}\int_{\frac{a}{k\sqrt{2r}}}^{\infty} e^{-x^2}dx\,. \tag{30}$$

Remembering now the meaning of $F(r,x)$ one immediately realizes that

$$1-\int_{-\infty}^a F(r,x)dx$$

represents the probability that at least one of expressions (16) is greater than a. Therefore (30) completely demonstrates the theorem we have stated.

§ 3. The theorem just proved is susceptible to immediate application to a famous theorem of the calculus of probability: Peter and Paul play a game of chance. In each game each one has probability 1/2 to win; the stake is always of k lire. Now Peter is infinitely rich, on the contrary Paul owns only a lire. If at a certain moment Peter is able to win all the holdings of Paul, the latter is ruined and is obliged to stop the game. So we are in the case considered in the above theorem and we can conclude that, after a sufficient number of games Peter will certainly ruin Paul; moreover, if a is much greater than k the probability that this outcome happens in n games is

$$\frac{2}{\sqrt{\pi}}\int_{\frac{a}{k\sqrt{2n}}}^{\infty} e^{-x^2}dx\,.$$

§ 4. We want now to apply the above theorem to an astronomical problem. Let us consider an elliptic comet which intersects Jupiter's orbit. The cometary orbit will obviously be perturbed by the action of Jupiter, and this happens especially when Jupiter and the comet pass very close. Now it may happen that in these continuous transformations the comet's orbit ends up changing into a parabolic or hyperbolic orbit; then the comet will go away forever escaping from the attraction of Jupiter and the Sun. I wish to study what is the probability that this happens within a certain time. As far as I know the theory of the influence of Jupiter on the cometary orbits has never been studied from this point of view; people only dealt with this matter[2] looking for an explanation of the capture of comets into parabolic orbits when passing by chance close to Jupiter.

We will make the following simplifying assumptions, the same as for the restricted 3-body problem:

The comet has a negligible mass, so that it does not perturb neither Jupiter nor the Sun.

The mass of Jupiter (m) is negligible with respect to the mass of the Sun (M). In this way we are allowed to assume the Sun to be fixed and to consider the orbit of the comet to be appreciably perturbed only when passing in the close neighborhood of Jupiter.

Jupiter's orbit is circular.

Comet's orbit is coplanar with Jupiter's orbit.

We designate the velocity of Jupiter by u, by V the velocity of the comet when it crosses Jupiter's orbit with respect to a reference frame moving along this orbit with velocity u, and by θ the angle between the direction of V and Jupiter's orbit. If v is the absolute velocity of the comet, when it is crossing Jupiter's orbit one will have

$$v^2 = u^2 + \mathrm{V}^2 + 2u\mathrm{V}\cos\theta\,. \tag{31}$$

Let us suppose that once, while the comet is crossing Jupiter's orbit, it passes very close this planet. Then it will be affected by a strong perturbation. Let b be the smallest distance between the two bodies if they were not attracted to one another. According to our assumptions, in order that the perturbation is considerable b must be very small when compared with the curvature radii of the two unperturbed orbits so that, during this "collision", the comet will appreciably describe a Keplerian hyperbolic orbit during its motion around Jupiter.

§ 5. Thus, let us consider this relative motion, referring to polar coordinates (r, φ) having Jupiter as a pole and the polar axis parallel to the direction of the incoming comet. Since the motion is a Keplerian motion, we have

$$\frac{1}{r} = \mathrm{A} - \mathrm{B}\cos(\varphi - \varphi_0) \tag{32}$$

since A, B, φ_0 are constant. Moreover, for $\varphi = 0$, r must be infinite, that is

$$\mathrm{A} - \mathrm{B}\cos\varphi_0 = 0\,. \tag{33}$$

[2]Tisserand, ≪*Traité de mécanique céleste*≫, Vol. IV, pp. 198-216; Callandreau, *Ann. de l'observatoire* **22**; A. Newton, *Mem. of the Nat. Acad. of Sci.*, **6**.

Therefore it must be that

$$b = \lim_{r=\infty} r \sin\varphi = \lim_{\varphi=0} \frac{\sin\varphi}{\mathrm{A} - \mathrm{B}\cos(\varphi - \varphi_0)} = -\frac{1}{\mathrm{B}\sin\varphi_0}\,. \tag{34}$$

The area constant is then evidently Vb and owing to the well known formulas for Keplerian motion one has

$$\mathrm{A} = \frac{m}{\mathrm{V}^2 b^2}\,. \tag{35}$$

From (33) and (34) we can now obtain the other two constants. One finds exactly

$$\tan\varphi_0 = -\frac{\mathrm{V}^2 b}{m} \qquad , \qquad \mathrm{B} = \frac{1}{b}\sqrt{1 + \frac{m^2}{b^2\mathrm{V}^4}}\,. \tag{36}$$

Now, let ψ be the angle between the direction of the comet when approaching and its direction when going away. Obviously one will have:

$$\psi = 2\varphi_0 - \pi$$

and then

$$\tan\frac{\psi}{2} = -\cot\varphi_0 = \frac{m}{\mathrm{V}^2 b}\,. \tag{37}$$

We can conclude that the perturbation consists in keeping V unchanged and in altering θ by the angle ψ given by (37). Now it is convenient to calculate the average of the square of ψ. Therefore we observe that one has:

$$\psi = 2\arctan\frac{m}{\mathrm{V}^2 b}$$

and then

$$\int_{-\infty}^{\infty} \psi^2 db = 4\int_{-\infty}^{\infty} \left(\arctan\frac{m}{\mathrm{V}^2 b}\right)^2 db$$

$$= \frac{4m}{\mathrm{V}^2}\int_{-\infty}^{\infty} \left(\arctan\frac{1}{x}\right)^2 dx = \frac{8m}{\mathrm{V}^2}\int_{0}^{\infty} \left(\arctan\frac{1}{x}\right)^2 dx$$

by putting

$$h = \int_0^{\infty} \left(\arctan\frac{1}{x}\right)^2 dx \approx 2.5$$

so one has

$$\int_{-\infty}^{\infty} \psi^2 db = \frac{8mh}{\mathrm{V}^2}\,. \tag{38}$$

Now, b being very small, the probability that its value lies between b and $b + db$ is obviously

$$\frac{db}{2\pi\mathrm{R}\sin\theta}$$

where R is the radius of Jupiter's orbit. The average of the square of ψ is therefore

$$\bar{\psi^2} = \int_{-\infty}^{\infty} \psi^2 \frac{db}{2\pi \mathrm{R} \sin\theta} = \frac{4mh}{\pi \mathrm{R}\ \mathrm{V}^2 \sin\theta} \,. \tag{39}$$

§ 6. In its motion around the Sun the energy constant of our comet is given by

$$\frac{v^2}{2} - \frac{\mathrm{M}}{\mathrm{R}} = \mathrm{W} \,.$$

As is well known, a Keplerian orbit is elliptic, parabolic or hyperbolic according to whether the energy constant is negative, zero or positive; now, remembering (31) we find for our comet:

$$\mathrm{W} = \frac{1}{2}\left(u^2 + V^2 + 2uV\cos\theta - 2\frac{\mathrm{M}}{\mathrm{R}}\right)$$

but since for Jupiter we have the relation:

$$\frac{u^2}{\mathrm{R}} = \frac{\mathrm{M}}{\mathrm{R}^2}$$

we can write

$$2\mathrm{W} = \mathrm{V}^2 + 2u\mathrm{V}\cos\theta - \frac{\mathrm{M}}{\mathrm{R}} \,.$$

Since in the subsequent perturbations V is not changed and only θ changes, in order that the comet can become hyperbolic it is necessary that W, negative at present, can become positive corresponding to suitable values of θ. Then it must be that

$$\mathrm{V}^2 + 2u\mathrm{V} > \frac{\mathrm{M}}{\mathrm{R}}$$

but we remark that

$$u = \sqrt{\frac{\mathrm{M}}{\mathrm{R}}}$$

so that the above inequality can be written:

$$\left(\mathrm{V} + \sqrt{\frac{\mathrm{M}}{\mathrm{R}}}\right)^2 > \frac{2\mathrm{M}}{\mathrm{R}}$$

from which *

and finally reduces to

$$\mathrm{V} > \left(\sqrt{2} - 1\right)\sqrt{\frac{\mathrm{M}}{\mathrm{R}}} = \left(\sqrt{2} - 1\right)u \,. \tag{40}$$

We will therefore assume this inequality certainly to be satisfied. Moreover, for some values of θ, W must certainly be negative, otherwise the cometary orbit could not be elliptic; so one will have:

$$\mathrm{V}^2 + 2u\mathrm{V} < \frac{\mathrm{M}}{\mathrm{R}}$$

*Editor's Note: At this point, in Fermi's manuscript there is a blank line which, obviously, would have contained the expansion of the square of the last formula.

from which as above

$$V > \left(\sqrt{2}+1\right)\sqrt{\frac{M}{R}} = \left(\sqrt{2}+1\right)u\,. \tag{41}$$

Therefore let us assume that V satisfies (40) and (41) and denote by θ_0 the particular value of θ for which the comet's orbit is hyperbolic, i.e., one has $W = 0$, that is

$$V^2 + 2uV\cos\theta_0 = \frac{M}{R}$$

and so

$$\cos\theta_0 = \frac{\frac{M}{R} - V^2}{2uV} = \frac{u^2 - V^2}{2uV}\,. \tag{42}$$

When θ is greater than θ_0, one has $W < 0$ and then the comet describes an elliptic orbit; on the contrary, when θ is less then θ_0 the orbit is hyperbolic.

Now we will suppose that initially the orbit is elliptical and very stretched, so that θ is very close to θ_0, and precisely slightly larger. We call θ^* this initial value.

Whenever the comet goes beyond Jupiter's orbit θ is changed by an amount ψ; the average of the square of ψ depends indeed on θ, as (39) shows, but since we have assumed that θ remains always very close to θ_0 we can set

$$\bar{\psi}^2 = \frac{4mh}{\pi R\, V^2 \sin\theta_0} \tag{43}$$

if after a certain time θ became $< \theta_0$ the comet would become hyperbolic and would go away forever. Therefore we are can apply the theorem of §2. Then we must put $a = \theta^* - \theta_0$; $k^2 = \frac{4mh}{\pi R\, V^2 \sin\theta_0}$. And the theorem we proved tells us that:

The probability that the comet will be changed into a hyperbolic one after having crossed Jupiter's orbit n times is:

$$\frac{2}{\sqrt{\pi}} \int_{\frac{\theta^* - \theta_0}{\sqrt{\frac{8mhn}{\pi R\, V^2 \sin\theta_0}}}}^{\infty} e^{-x^2} dx \tag{44}$$

and then tends to 1 when n tends to infinity.

In the strict sense one could object that the above calculations would fail if the value of V were such that, when the orbit is parabolic, the comet took the same time as Jupiter to go from A to B, where A is the point where the comet enters Jupiter's orbit, and B the point where it goes out. In Figure 1.3, S is the Sun, AJB Jupiter's orbit, and AKB the orbit of the comet. But it is easy to realize that this case certainly cannot happen if the comet describes its trajectory with direct motion. In fact, if v is the absolute velocity in A of the comet in its parabolic orbit, one has

$$v^2 = u^2 + V^2 + 2uV\cos\theta_0$$

and then from (42)

$$v^2 = 2u^2$$

that is:

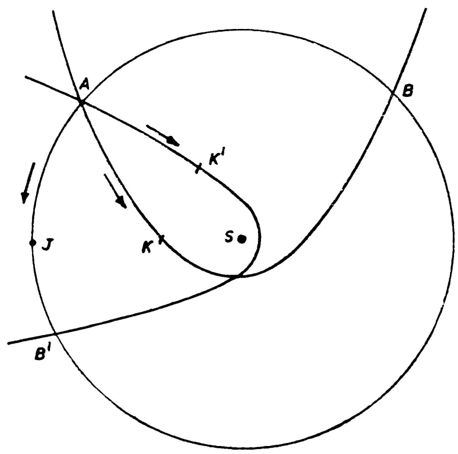
$$v > u \ . \tag{45}$$

Now, the velocity of the comet is not constant, but along the whole tract AKB it is always greater than in the extremes A and B, thus inequality (45) holds true with all the more reason in whole tract AKB. On the other hand, if the motion is direct one has that arc AKB is shorter than arc AJB, and since it is covered with even higher velocity it is certain that the comet will arrive at B before Jupiter.

If on the contrary the motion of the comet were retrograde, and it described for instance the orbit AK'B' in the sense indicated by the arrow one would have

arc AK'B' > arc AJB'

and then, though (45) still holds, it is evident that for a particular value of the parameter of the cometary orbit it can happen that the two heavenly bodies take the same time to go from A to B'; of course this can only happen for a particular value of V.



Now if this happened it could be that the comet's orbit, elliptical at first, crossed Jupiter when passing through A and got changed into a parabolic one; but in this case it would meet Jupiter again when passing through B and could in case have a new perturbation which would change it into an elliptical orbit again. For this reason we consider this particular value of V ruled out from our calculations.

§ 7. At last we want to consider the possibility that before being changed into a hyperbolic orbit the comet can crash into Jupiter and then be destroyed. What is the probability of this event?

For this let us look first for the probability that the comet, crossing once the orbit of Jupiter, it collides with it. For this we indicate by ρ the sum of the radii of Jupiter and the comet. To have a collision it is necessary that the perihelian distance of Jupiter from the comet, as calculated though the formulas for Kepler motion is smaller than ρ. Call δ this perihelian distance; from the formulas of §5 it follows that

$$\frac{1}{\delta} = \mathrm{A} + \mathrm{B}$$

and then from (35) and (36)

$$\frac{1}{\delta} = \frac{m}{\mathrm{V}^2 b^2} + \frac{1}{b}\sqrt{1 + \frac{m}{\mathrm{V}^4 b^2}}\,.$$

If we want the collision to occur it must be that $\delta < \rho$ and then

$$\frac{m}{\mathrm{V}^2 b^2} + \frac{1}{b}\sqrt{1 + \frac{m}{\mathrm{V}^4 b^2}} > \frac{1}{\rho}$$

by multiplying this inequality by the quantity, certainly positive

$$\rho\left(\frac{1}{b}\sqrt{1 + \frac{m}{\mathrm{V}^4 b^2}} > \frac{1}{\rho} - \frac{m}{\mathrm{V}^2 b^2}\right)$$

we find

$$\frac{\rho}{b^2}\frac{1}{b}\sqrt{1 + \frac{m}{\mathrm{V}^4 b^2}} > \frac{1}{\rho} - \frac{m}{\mathrm{V}^2 b^2}$$

and summing the last two inequalities

$$\left(\frac{2m}{\mathrm{V}^2} + \rho\right)\frac{1}{b^2} > \frac{1}{\rho}$$

from which finally

$$|b| < \sqrt{\rho^2 + \frac{2m\rho}{\mathrm{V}^2}}\,. \tag{46}$$

We recall now that the probability that the value of b lies between b and $b + db$ is $\frac{db}{2\pi \mathrm{R}\sin\theta_0}$ and so the probability p that the collision occurs in only one crossing of Jupiter's orbit is given by

$$p = \frac{1}{\pi \mathrm{R}\sin\theta_0}\sqrt{\rho^2 + \frac{2m\rho}{\mathrm{V}^2}}\,. \tag{47}$$

We will assume p to be very small, and this obviously is equivalent to considering Jupiter's radius to be negligible if compared with the radius of its orbit. Let us now look for the probability that a collision occurs at the n-th time the comet crosses Jupiter's orbit. Therefore it is evidently necessary that the collision has not occurred before and the probability of this is obviously $(1 - p)^{n-1}$, that is in our approximation

$$e^{-pn}\ .$$

That the comet has not yet been changed into a hyperbolic orbit; and, having supposed p to be extremely small, remembering (44) and for the sake of brevity setting:

$$\frac{\theta^* - \theta_0}{\sqrt{\frac{8mh}{\pi \mathrm{RV}^2 \sin\theta_0}}} = \mathrm{H}$$

we can hold that the probability of this event is given by

$$1-\frac{2}{\sqrt{\pi}}\int_{\frac{\mathrm{H}}{\sqrt{n}}}^{\infty}e^{-x^2}dx=\frac{2}{\sqrt{\pi}}\int_0^{\frac{\mathrm{H}}{\sqrt{n}}}e^{-x^2}dx\ .$$

And finally that the collision really occurs, for which we have the probability p. After all the probability that the collision occurs the n-th time is

$$\frac{2e^{-pn}p}{\sqrt{\pi}}\int_0^{\frac{\mathrm{H}}{\sqrt{n}}}e^{-x^2}dx$$

and therefore the probability that the collision occurs any time whatsoever will be the sum of the above expression from $n=1$ to $n=\infty$, or replacing the sum by an integral

$$\frac{2p}{\sqrt{\pi}}\int_0^{\infty}e^{-pn}dn\int_0^{\frac{\mathrm{H}}{\sqrt{n}}}e^{-x^2}dx\ .$$

In this expression it is convenient to reverse the integration by the formula

$$\int_0^{\infty}dn\int_0^{\frac{\mathrm{H}}{\sqrt{n}}}dx=\int_0^{\infty}dx\int_0^{\frac{\mathrm{H}}{x^2}}dn$$

and in this way one finds for the sought after probability the expression:

$$\frac{2p}{\sqrt{\pi}}\int_0^{\infty}e^{-x^2}dx\int_0^{\frac{\mathrm{H}}{x^2}}e^{-pn}dn=\frac{2}{\sqrt{\pi}}\int_0^{\infty}e^{-x^2}\left(1-e^{-\frac{p\mathrm{H}}{x^2}}\right)dx$$

$$=1-\frac{2}{\sqrt{\pi}}\int_0^{\infty}e^{-x^2-\frac{p\mathrm{H}}{x^2}}dx=1-e^{-2\sqrt{p\mathrm{H}}}\ .$$

The probability that the collision never occurs is then:

$$e^{-2\sqrt{p\mathrm{H}}}\ .$$