12) The principle of adiabatics and the systems which do not admit angle coordinates

"Il principio delle adiabatiche ed i sistemi che non ammettono coordinate angolari,"
Nuovo Cimento ***25****, 171–175, (1923)*

§ 1. - The importance of the Ehrenfest's principle of adiabatics for the determination of the selection rules for the stationary orbits of a system, in the Bohr theory, is well known. [1] This principle, as we know, can be stated as follows: Let us assume that, in a mechanical system, the forces or the constraints are continuously modified with time but very slowly in comparison with the periods of the system, or according to Ehrenfest's expression, adiabatically; the principle of adiabatics states that, if the system is initially in a quantum preferred orbit, it will still be there at the end of the transformation.

Let us consider, for instance, a pendulum and imagine shortening its string at a very low rate in comparison with the period of the pendulum itself. The frequency ν of the pendulum will then grow slowly, but it is easy to realize that the energy u also will grow and in such a way that the ratio u/ν remains constant. In this way, if this ratio was initially an integer multiple of Planck's constant h, it will always remain the same and so the state of the system will remain quantum preferred during the whole transformation. For further examples we refer Ehrenfest's memoir.

The formal basis for the principle of the adiabatics is provided by Burger's theorem. [2] Let us consider a system that in certain general coordinates $q_1, q_2, \ldots, q_f$ allows the separation of variables [3] Then put

$$I_K = \oint p_K dq_K \qquad (K = 1, 2, ..., f) \tag{1}$$

where p_K are the momenta canonically conjugate to q_K and the integral extends over a complete oscillation of the coordinate q_K according to the rules of quantum theory; in this way the conditions in order that the given orbit of the system be quantum preferred are:

$$I_1 = n_1 h \ ; \ I_2 = n_2 h \ ; ....; \ I_f = n_f h \tag{2}$$

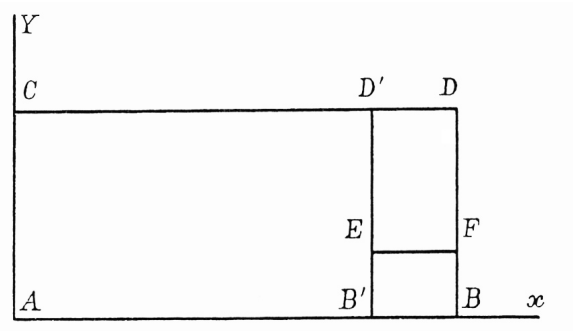
since $n_1, n_2, \ldots, n_f$ are integers. Let us suppose now that we modify our system adiabatically, but in such a way that it allows separation of variables at any instant. Burger's theorem states that in this case the integrals $I_1, I_2, \ldots, I_f$ do not change

[1] Ehrenfest, *Ann. d. Phys.* **51**, 327 (1916).
[2] Burgers, Versl. Akad. van. Wetensch. – Amsterdam 1916, 1917; Ann. d. Phys. **52**, 195 (1917).
[3] For the validity of Burger's conclusions it is sufficient, more generally, that the system admit angle coordinates, i.e., it is possible to introduce in place of q_K, p_K new variables w_K, j_K such that the q_K's, expressed by means of the (w_K, j_K) are periodic with period 1 in the variables w_K, and the energy in the new coordinates becomes a function only of the j's. Then from the Hamilton equations, the j's must be constants and the w's linear functions of the time; the q's can be expanded as Fourier series functions of the time with f indexes.

during the transformation, i.e., that they are adiabatic invariants. Therefore, if conditions (2) are satisfied at the beginning of the transformation, they will also be satisfied at the end; then the principle of the adiabatics is satisfied.

In this Note I intend to show by means of a simple example that if a system adiabatically transforms into another system and the initial and final states both admit separation of variables, but the intermediate states do not, the I_K are no longer adiabatic invariants. In this case the principle of adiabatics loses its validity.

§ 2. - Let us consider a mass point moving on a plane inside a rectangle; we shall assume that no force acts on the point while it is inside the rectangle, but it bounces off the walls when it hits them. Consider sides AB and AC of the rectangle as coordinate axes x, y. Now, it is evident that our system admits separation of variables in these coordinates. Calling a, b the lengths of sides AB, AC, coordinate x in fact oscillates between values 0, a; coordinate y between values $0, b$.



Moreover, if at a certain instant the components of the velocity are u, v, at an instant whatever they will be $\pm u$, $\pm v$, where one must choose the sign + or − according to whether the relative coordinate is increasing or decreasing at the instant in question. The conjugate momenta to x and y will be $\pm mu$, $\pm mv$, where m is the mass of the point; then one will have

$$I_x = \oint (\pm mu)\, dx = \int_0^a mu\, dx + \int_a^0 (-mu)\, dx = 2\, mua \tag{3}$$

and analogously

$$I_y = 2\, mvb\,. \tag{3′}$$

Now we want to study how I_x and I_y change if we transform our system adiabatically. We intend to transform the rectangle ABCD into the other one AB′CD′; such a transformation can be carried out in three ways:

(1) one shifts the segment BD parallel to itself until it arrives at B′D′;
(2) one shifts the segment BB′ parallel to itself until it arrives at DD′, so that at an intermediate instant, the mass point can move inside the concave polygon AB′EFDC;

(3) one arbitrarily deforms the broken line B′BDD′ until it coincides with the segment B′D′. Excluding from our considerations this last case which is clearly a bit complicated, we shall limit ourselves to discuss the former two.

Concerning the first case, we observe that here at any instant the point can always move inside a rectangle, so separation of variables is always possible in the intermediate times as well; according to Burger's theorem, in this case we must expect that I_x and I_y remain invariant. This is obviously evident for I_x, since neither b nor v change during the transformation and therefore due to (3′), neither does I_y. As for I_x, instead, a decreases during the transformation, being reduced from $a = \mathrm{AB}$ to $a' = \mathrm{AB}'$; but at the same time u increases in consequence of the bounces on the moving wall and an immediate consideration shows that things go just so that the product au and therefore also I_x remain constant, [4] obviously under the condition that the transformation is realized slowly enough.

If we pass on to consider case (2), it is easy to see that now things are different. As for I_x, in fact one immediately sees that the x component of the velocity remains unchanged (except for the sign), since it could change its absolute value only hitting a moving wall parallelly to the x-axis, but the only moving wall, EF, moves parallelly to y; instead a decreases from AB to AB′. In all, therefore I_x reduces in the ratio a'/a and so does not remain constant. Likewise also I_y does not remain constant; in fact b remains unchanged whereas v increases due to the collisions with the moving wall EF. An immediate evaluation shows that v, and then also I_y, increases in the ratio a/a'.

From the above considerations we can conclude that the integrals I_K are adiabatic invariants only if in the intermediate states the system always admits separation of variables or at least, according to Burger's theorems, always admits a system of angular coordinates. On the contrary, at least in general, this is not true if the system does not always exhibit a multiperiodic motion. On the other hand, this fact is easily understandable also from the point of view of quantum theory. In fact one knows, following Bohr, that a well defined quantization is possible only if the motion of the system is multiperiodic. Then one can realize that, if in the intermediate states the system cannot be quantized rigorously, this inaccuracy is transmitted to the final state.

Göttingen, February 1923.

[4] In fact the number of hits on the moving wall BD in the time interval dt is obviously $\frac{u}{2a}dt$; on the other hand, if V is the velocity of wall BD, the velocity of the point will experience an increase of 2V at every hit; then the increase of u in time dt will be:

$$du = 2V\frac{u}{2a}dt = \frac{u}{a}Vdt = -\frac{u}{a}da$$

since, obviously, $-da = Vdt$. By integrating the preceding equation, we find exactly $ua = const.$, as stated above.

13) Some theorems of analytical mechanics of great importance for quantum theory

"Alcuni teoremi di Meccanica Analitica importanti per la Teoria dei Quanti,"
Nuovo Cimento ***25****, 271–285, (1923)*

§ 1. - Ehrenfest's principle of adiabatics,[1] as is known, states that if a mechanical system is in a quantum orbit and its mechanism, forces or constraints is changed in an infinitely slow way, the system remains in a quantum selected orbit during the whole transformation.

In order that this principle have a definite sense, it is obviously necessary that the final orbit of the system only depend on the final mechanism and not on the particular sequence of intermediate mechanisms followed during the transformation. Burgers[2] has shown that this really is the case, at least for that kind of system which up to now has only been considered in quantum theory, i.e., for systems which either admit a complete separation of variables or at least can be represented by means of angular coordinates.[3] In this case, their motion can always be considered as resulting from periodic motions, generally having as many periods as there are degrees of freedom, in the case of degeneracy, with a lower number. But exactly now that the study of the simplest atomic structures has been exhausted,[4] which belong to this type, some problems which do not admit angular coordinates have been presenting themselves ever more insistently, above all the three-body problem which occurs in the study of hydrogen molecule, and in the simplified four-body problem which arises in the study of the hydrogen molecule. As is known, all attempts made up to now to reduce the study of these systems to that of systems with angular coordinates have been in vain. It is therefore desirable to investigate if and how far is it possible to attempt an extension of the principle of adiabatics to general systems, hoping that it can give some information which can help in the search for rules suitable to determine the preferred orbits of these more general systems.

§ 2. - First of all we better fix a classification of the systems to be studied. Therefore we turn to the usual representation of the state of the system by means of a point in a $2f$-dimensional space Γ, which has $q_1, q_2, \ldots, q_f$ as general coordinates of the system and $p_1, p_2, \ldots, p_f$ as their conjugate momenta. Through each point of this space we have a trajectory which corresponds to the motion of the system having its initial position and velocity determined by that point. We shall assume the forces and the constraints of the system are time-independent and the forces

[1] P. Ehrenfest, *Ann. d. Phys.* **51**, p. 327, 1916.

[2] Burgers, *Versl. Akad. van Wetensch. Amsterdam*, 25 November 1916; *Ann. d. Phys.* **52**, p. 195, 1917; *Phil. Mag.* **33**, p. 514, 1917.

[3] See for instance Sommerfeld, *Atombau und Spektrallinen*, III ed. Zusatz 7.

[4] They are the hydrogen atom and its various perturbations (Zeeman effect, Stark effect, and fine structure) and the ion of the hydrogen molecule H_2^+, where nucleus rotations are not present.

admit a potential so that there exists an integral representing conservation of energy. We denote by E the hypersurfaces energy = constant; through each point of Γ, one of the E's passes containing the trajectory through that point (because fo the energy integral).

The so called quasi-ergodic[5] mechanical systems enjoy the property that the trajectory generally passes infinitely close to every point of E, so as to densely fill a $(2f-1)$-dimensional manifold.

However, it might be that our system, besides the energy integral, admits some other first integral uniform and independent of time. In this case the manifold filled by the trajectory will obviously have a lower number of dimensions. Then let us assume that our system have on the whole m uniform first integrals independent of time,

$$\Phi_1(p,q) = c_1; \Phi_2 = c_2; \ldots; \Phi_m = c_m$$

where c_i are arbitrary constants. Through each point of Γ, we shall have a $(2f-m)$-dimensional manifold G, the intersection of the m hypersurfaces $\Phi_i = c_i$; and the trajectory passing through that point will be completely contained in G.

In general it will not be possible to find, within G, a submanifold which contains the whole trajectory; on the contrary, on the analogy of quasi-ergodic systems, we shall assume for our systems that in general the whole G be densely filled by the trajectory, i.e. that the trajectory passes infinitely close to all the points of G. In this way, the trajectory will come out characterized, at least in its statistical elements, by the only knowledge of the values $\Phi_1, \Phi_2, \ldots, \Phi_m$ corresponding to it.

Therefore we call these values *characteristics of the trajectory.*

Thus a quasi-ergodic system has only one characteristic, its energy.

A system with energy independent of time, which admits the separation of variables, has in general as many characteristics as degrees of freedom, corresponding to the f constants a of the Jacobi's complete integral; a higher number can only occur in case of degeneracy, i.e. when linear relations with integer coefficients between the fundamental frequencies exist.

Let us consider, for instance, the motion of a point in a plane acted on by a force proportional to the distance from two orthogonal straight lines. If the two attraction coefficients are not commensurate, the point describes an open Lissajous' curve in the plane. And in the four-dimensional space Γ the representative point densely fills a two-dimensional surface G. Therefore the system has two characteristics; for them we can take the energies of the projections of the motion onto the two orthogonal straight lines.

If instead the attraction coefficients are commensurate, the Lissajous' curve degenerates to a closed curve and G becomes one-dimensional; this corresponds to three characteristics.

[5]The author recently demonstrated that ordinary mechanical systems are in general quasi-ergodic, so that this is the most common case.

§ 3. - Now we shall assume to be able to change arbitrarily the forces, or the constraints of the system, i.e., which in totality, with a happy terminology due to P. Hertz, [6] we shall call the *mechanism* of the system. If we change the mechanism in an infinitely slow way, we have what is called an adiabatic transformation; and in § 5, we shall easily find a system of differential equations which shows how the characteristics of the system change when the guiding parameter of the mechanism μ changes adiabatically. But as we have already mentioned, one can apply Ehrenfest's principle to a definite system only if the values that its characteristics take at the end of an adiabatic transformation only depend on the final mechanism and not on the intermediate mechanisms encountered during the transformation. To study this question, we shall assume in what follows that the mechanism, rather than depending on only one parameter, depends on two parameters λ and μ. The dependence of the characteristics on λ and μ, instead of being on a system of ordinary differential equations, will then be obviously expressed by a system of total differential equations; then the conditions for having the final values of the characteristics not depending on the path followed during the transformation in the λ, μ plane coincide with the integrability conditions for this system. We shall demonstrate that these conditions, for the quasi-ergodic system, are actually satisfied. Instead, for systems having more than one characteristic, in general they are not satisfied although important classes of exceptions exist.

§ 4. - Before passing to the study of the adiabatic transformations it is convenient to consider some formulae which are useful for calculating the probability that, at any instant, the representative point is in G. For uniformity of notation, we denote the coordinates of Γ by $x_1, x_2, \ldots, x_{2f}$ rather than $q_1, \ldots, q_f, p_1, \ldots, p_f$. Our problem can now be formulated in this way: calculate the probability that, at a certain instant, $x_1, x_2, \ldots, x_{2f-m}$ have values between x_1 and $x_1 + dx_1$, x_2 and $x_2 + dx_2$,...,x_{2f-m} and $x_{2f-m} + dx_{2f-m}$, while the remaining m x's obviously take the values necessary to maintain the representative point in G.

Statistical mechanics as we know from Liouville's theorem states that the necessary condition for having a stationary distribution of points in the Γ space is that their density in Γ should have a constant value on any G.

A volume element of Γ can be written $dx_1, dx_2, \ldots, dx_{2f}$, but also, taking as new variables $x_1, x_2, \ldots, x_{2f-m}, \Phi_1, \Phi_2, \ldots, \Phi_m$ as $\frac{1}{D} dx_1, dx_2, \ldots, dx_{2f-m}, d\Phi_1, d\Phi_2, \ldots, d\Phi_m$, where D is the functional determinant $\frac{\partial(\Phi_1,\ldots,\Phi_m)}{\partial(x_{2f-m+1},\ldots,x_{2f})}$. And since during the motion $d\Phi_1, d\Phi_2, \ldots, d\Phi_m$ obviously remain constant, the aforesaid volume element comes out to be proportional to $\frac{1}{D} dx_1, \ldots, dx_{2f-m}$.

Therefore also the sought after probability is proportional to this expression; and since the total probability is obviously = 1, we finally find that the desired

[6] P. Hertz, *Ann. d. Phys.* **33**, pp. 225, 537, 1910; Weber, Gans, *Repertorium der Physik* **1**, 2, 1916. We refer to these articles for any explanations regarding the statistical part of the text.

probability is given by

$$\frac{\frac{d\sigma}{D}}{\int \frac{d\sigma}{D}}, \tag{1}$$

where for short we set $d\sigma = dx_1, dx_2, \ldots, dx_{2f-m}$ and the integral is extended to all values of $x_1, x_2, \ldots, x_{2f-m}$ corresponding to the points of G.

Before leaving this discussion, we also want to deduce a formula that will be useful in the case of quasi-ergodic systems. In this case G is a hypersurface, and we assume for the sake of simplicity it should be closed, and such to be intersected in only one point by the radii vectors coming out from a pole within it. This because a more general approach, even though it is not essentially different, would lead to rather complicated calculations. We refer the space Γ to polar coordinates, by characterizing each point by means of its radius vector and the intersection of this one with the unit hypersphere having the pole as center. We call H the only characteristic, i.e., the energy. In accordance with what said above, the probability that at a certain instant the representative point lies within an element of solid angle $d\omega$ is proportional to the hypervolume comprised between the two hypersurfaces $H(x_1, \ldots, x_{2f}) = H$, and $H(x_1, \ldots, x_{2f}) = H + dH$, and the solid angle $d\omega$. This volume, except for the constant factor dH, is evidently $\frac{r^{2f-1}d\omega}{H_r}$, where $H_r = \frac{\partial H}{\partial r}$. Since the total probability must be =1 , we find that the wanted probability is given by

$$\frac{r^{2f-1}\frac{d\omega}{H_r}}{\int r^{2f-1}\frac{d\omega}{H_r}}, \tag{2}$$

where the integral is extended over the entire unit sphere.

§ 5. - In this section we assume the mechanism of our system to be a function of a parameter μ and we aim to study how the characteristics change when this parameter changes adiabatically. Since the mechanism depends on the parameter μ, in general the characteristics $\Phi_1, \Phi_2, \ldots, \Phi_m$ will also depend on μ, besides on the p's and q's. Then, if at a certain instant the parameter μ changes by $d\mu$, the characteristic Φ_i will correspondingly undergo the change $\frac{\partial \Phi_i}{\partial \mu}\partial\mu$. Since we are in the presence of an adiabatic change, to have the effective change of Φ_i, we must consider the average of this expression which, according to the results of the previous section, will be

$$d\mu \frac{\int \frac{\partial \Phi_i}{\partial \mu}\frac{d\sigma}{D}}{\int \frac{d\sigma}{D}} \tag{3}$$

which turns out to be a function only of μ and $\Phi_1, \ldots, \Phi_m$. The dependence of the characteristics on μ in an adiabatic transformation will then be expressed by the system of ordinary differential equations:

$$\frac{d\Phi_1}{d\mu} = \frac{\int \frac{\partial \Phi_1}{\partial \mu}\frac{d\sigma}{D}}{\int \frac{d\sigma}{D}}; \frac{d\Phi_2}{d\mu} = \frac{\int \frac{\partial \Phi_2}{\partial \mu}\frac{d\sigma}{D}}{\int \frac{d\sigma}{D}}; \ldots; \frac{d\Phi_m}{d\mu} = \frac{\int \frac{\partial \Phi_m}{\partial \mu}\frac{d\sigma}{D}}{\int \frac{d\sigma}{D}}. \tag{4}$$

If we know the values of the Φ's, for instance for $\mu = 0$, the integration of this system gives us their values for any μ. In the particular case of quasi-ergodic systems, the system (4) reduces to the single equation:

$$\frac{dH}{d\mu} = \frac{\int \frac{H_\mu}{H_r} r^{2f-1} d\omega}{\int \frac{d\omega}{H_r} r^{2f-1}}\,, \tag{5}$$

where $H_\mu = \frac{\partial H}{\partial \mu}$.

§ 6. - Now we want to study in which cases the final values of the characteristics are independent of the way followed in passing adiabatically from the initial mechanism to the final one. Therefore we shall represent the mechanism of the system as a function of two parameters, λ and μ. If one alters adiabatically these two parameters, $d\lambda$ and $d\mu$ respectively, the same conclusion of the preceding section shows that the corresponding change of the characteristic is:

$$d\Phi_i = \frac{\int \frac{\partial \Phi_i}{\partial \lambda} \frac{d\sigma}{D}}{\int \frac{d\sigma}{D}} d\lambda + \frac{\int \frac{\partial \Phi_i}{\partial \mu} \frac{d\sigma}{D}}{\int \frac{d\sigma}{D}} d\mu, . \qquad (i = 1, 2, \ldots, m) \tag{6}$$

The coefficients of $d\lambda$ and $d\mu$ are clearly functions only of λ and $\Phi_1, \ldots, \Phi_m$, so the m equations (6) represent a system of total differential equations; if it will be completely integrable, the final values of the Φ's will effectively be independent of the way followed during the transformation, or else it will not be so. We want to demonstrate that, in the case of quasi-ergodic systems, the condition of complete integrability is satisfied. In fact, for these systems, the system (6) reduces to a single equation of total differentials analogous to (5)

$$dH = L d\lambda + M d\mu\,, \tag{7}$$

where

$$L = \frac{\int \frac{r^{2f-1} H_\lambda d\omega}{H_r}}{\int \frac{r^{2f-1} d\omega}{H_r}}; \quad M = \frac{\int \frac{r^{2f-1} H_\mu d\omega}{H_r}}{\int \frac{r^{2f-1} d\omega}{H_r}}\,, \tag{8}$$

and then L and M represent two functions of λ, μ and H. As we know, for obtaining the complete integrability of (7), it is necessary and sufficient that the total derivatives of L with respect to μ and of M with respect to λ be equal. Therefore it must be that

$$\frac{\partial L}{\partial \mu} + M \frac{\partial L}{\partial H} = \frac{\partial M}{\partial \lambda} + L \frac{\partial M}{\partial H}\,. \tag{9}$$

To demonstrate that this equality is really satisfied, let us begin to calculate its first term. Therefore, let us imagine giving independent variations δH and $\delta \mu$ to H and μ, leaving λ unchanged; then we will have

$$\delta L = \frac{\partial L}{\partial H} \delta H + \frac{\partial L}{\delta \mu} \delta \mu\,. \tag{10}$$

On the other hand, from the first of (8), we remark that:

$$\delta L = \frac{1}{\left(\int \frac{r^{2f-1}d\omega}{H_r}\right)^2}\Bigg\{ \left(\int \frac{r^{2f-1}d\omega}{H_r}\right)\delta \int \frac{r^{2f-1}H_\lambda d\omega}{H_r} - \\ - \left(\int \frac{r^{2f-1}H_\lambda d\omega}{H_r}\right)\delta \int \frac{r^{2f-1}d\omega}{H_r^2}\Bigg\}. \tag{11}$$

In the calculation of the two variations of the integrals within the curly brackets, we can of course interchange the symbols δ and $\int$, since the limits of the integral do not change because it is extended over the entire unit hypersphere. Then we have:

$$\delta \int \frac{r^{2f-1}d\omega}{H_r} = (2f-1)\int \frac{r^{2f-2}\delta r d\omega}{H_r} - \int \frac{r^{2f-1}\delta H_r d\omega}{H_r^2}\,. \tag{12}$$

On the other hand, from the invariance on the unit sphere, one has:

$$\delta H = H_r \delta r + H_\mu \delta\mu$$

where from

$$\delta r = \frac{\delta H}{H_r} - \frac{H_\mu}{H_r}\delta\mu$$

and also

$$\delta H_r = H_{rr}\delta r + H_{r\mu}\delta\mu = \frac{H_{rr}}{H_r}\delta H + \left(H_{r\mu} - \frac{H_{rr}H_\mu}{H_r}\right)\delta\mu\,.$$

By substituting these expressions of δr, δH_r into (12), one finds:

$$\delta \int \frac{r^{2f-1}d\omega}{H_r} = \delta H\Bigg\{ (2f-1)\int \frac{r^{2f-2}d\omega}{H_r^2} - \int \frac{r^{2f-1}H_{rr}d\omega}{H_r^3}\Bigg\} \\ -\delta\mu\Bigg\{ (2f-1)\int \frac{r^{2f-2}H_\mu d\omega}{H_r^2} + \int \frac{r^{2f-1}d\omega}{H_r^2}\left(H_{\mu r} - \frac{H_\mu H_{rr}}{H_r}\right)\Bigg\}.$$

In a similar way one finds:

$$\delta \int \frac{r^{2f-1}H_\lambda d\omega}{H_r} = \delta H\Bigg\{ (2f-1)\int \frac{r^{2f-2}H_\lambda d\omega}{H_r^2} + \int \frac{r^{2f-1}H_{\lambda r}d\omega}{H_r^2} \\ - \int \frac{r^{2f-1}H_\lambda H_{rr}}{H_r^3}d\omega\Bigg\} + \delta\mu\Bigg\{ -(2f-1)\int \frac{r^{2f-2}H_\lambda H_\mu d\omega}{H_r^2} \\ + \int \frac{r^{2f-1}d\omega}{H_r}\left(H_{\lambda r} - \frac{H_{\lambda r}H_\mu}{H_r}\right) - \int \frac{r^{2f-1}H_\lambda d\omega}{H_r^2}\left(H_{\mu r} - \frac{H_\mu H_{rr}}{H_r}\right)\Bigg\}.$$

By substituting these two last expressions into (11) and comparing with (10), one finally finds:

$$\frac{\partial L}{\partial H} = \frac{1}{\left(\int \frac{r^{2f-1}d\omega}{H_r}\right)^2}\Bigg[\left(\int \frac{r^{2f-1}d\omega}{H_r}\right)\Bigg\{ (2f-1)\int \frac{r^{2f-2}H_\lambda d\omega}{H_r^2} \\ + \int \frac{r^{2f-1}H_{\lambda r}d\omega}{H_r^2} - \int \frac{r^{2f-1}H_\lambda H_{rr}}{H_r^3}d\omega\Bigg\} \\ - \left(\int \frac{r^{2f-1}H_\lambda d\omega}{H_r}\right)\Bigg\{ (2f-1)\int \frac{r^{2f-2}d\omega}{H_r^2} - \int \frac{r^{2f-1}H_{rr}d\omega}{H_r^3}\Bigg\}\Bigg],$$

$$\frac{\partial L}{\partial \mu} = \frac{1}{\left(\int \frac{r^{2f-1}d\omega}{H_r}\right)^2}\left[\left(\int \frac{r^{2f-1}d\omega}{H_r}\right)\left\{-(2f-1)\int \frac{r^{2f-2}H_\lambda H_\mu d\omega}{H_r^2}\right.\right.$$
$$\left. + \int \frac{r^{2f-1}d\omega}{H_r}\left(H_{\lambda\mu} - \frac{H_{\lambda r}H_\mu}{H_r}\right) - \int \frac{r^{2f-1}H_\lambda d\omega}{H_r^2}\left(H_{\mu r} - \frac{H_\mu H_{rr}}{H_r}\right)\right\}$$
$$\left. - \int \frac{r^{2f-1}H_\lambda d\omega}{H_r}\left\{(2f-1)\int \frac{r^{2f-2}H_\mu d\omega}{H_r^2} + \int \frac{r^{2f-1}d\omega}{H_r^2}\left(H_{\mu r} - \frac{H_\mu H_{rr}}{H_r}\right)\right\}\right].$$

These two last equations, together with the second of (8), give us all the elements necessary to calculate the first term of (9). Once it has been calculated, it is immediate to recognize from its explicit expression that λ and μ appear symmetrically; then ((9) is verified.

Therefore we can conclude that, for quasi-ergodic systems, the value assumed by the energy at the end of an adiabatic transformation does not depend at all on the intermediate mechanisms of the transformation.

§ 7. - We return now to consider the systems with more than one characteristic. In order that for these systems as well, the final characteristics be independent of the intermediate mechanisms of the transformation, the conditions of complete integrability of the system (6) should be satisfied. But, if through a calculation, obviously more complicated than that performed in the preceding section but not essentially different from it, we effectively build up these conditions, we find that in general they are not satisfied. Rather than to report here this lengthy calculation, we prefer to show the argument through an example of a system with two characteristics. The example we choose is very similar to another one I have recently used in a note on the principle of adiabatics.

From an origin O, we draw in a plane two orthogonal axes x, y. Then we take in the first quadrant two points P, Q and draw the perpendiculars from them to axes (PA, PB, QC, QD). We shall assume that P be internal to the rectangle OCQD. Now let us suppose that inside concave polygon APBDQCA a mass point is moving not acted on by forces and elastically bouncing off the walls of the polygon. Absolute values u, v of the components of the velocity of the point on axes x, y keep evidently constant during the motion, therefore the system has two characteristics. Let us suppose then to keep point Q (of coordinates a, b) fixed and to move point P (of coordinates λ, μ). In this way we shall have realized a mechanical system with two characteristics u, v and depending on two parameters λ, μ. By easy arguments, analogous to the ones carried out in the note quoted above, one finds that, changing adiabatically the position of point P, u and v change following the rule:

$$d\log u = \frac{2\mu\, d\lambda}{ab - \lambda\mu}; \quad d\log v = \frac{2\lambda\, d\mu}{ab - \lambda\mu}$$

obviously neither of these two equations is completely integrable; therefore the values that u and v take at the end of a transformation also depend on the path

followed by point P. Then, in general, it is not possible to apply Ehrenfest's principle to systems with more characteristics.

§ 8. - However, some important classes of exceptions to this rule exist. We aim to study them in this section. The first one, and also the most important, is that of the systems with angular coordinates. Of these systems, according to Burgers' theorems, we not only know that Ehrenfest's principle can be applied (in the sense that it leads in any case to definite final conditions) but also that for them the aforesaid principle results to be *verified, by experience* as a logical consequence of Sommerfeld's conditions which are supported by all the theory and the experience made on the hydrogen atom. Another remarkable class of exceptions to the conclusions of § 7 is the following: Let us assume that of the m characteristics of our system only one, the energy, depends explicitly on parameters λ, μ of the mechanism. I say that for these systems, at the end of every adiabatic transformation, the energy takes a value independent of the intermediate mechanisms, while the other characteristics even stay unchanged. The fact that all the characteristics, but the energy, stay unchanged comes out evident from the circumstance that, since they do not contain the parameters explicitly, stay unchanged in all the elementary processes of the transformation; the same conclusion can be drawn from system (6) since, if Φ_i is one of these characteristics, one has by hypothesis $\frac{\partial \Phi_i}{\partial \lambda} = \frac{\partial \Phi_i}{\partial \mu} = 0$. For demonstrating that the final value of the energy does not depend on the path followed during the transformation in the plane λ, μ, one could put forward a consideration analogous to that of § 6.

But it is easier to remark that, on the basis of the hypothesis, by means of a canonical transformation independent of the parameters, one can try to transform the characteristics independent of the parameters into coordinates of Γ.

After this, the consideration of § 6 can be repeated word for word and the constant characteristics simply stand for constant parameters.

Systems of this kind occur very frequently in applications; for instance, of this kind are all the systems which have, as only uniform integrals besides the energy (and not dependent on the energy), some integral of the conservation of momentum, or angular momentum, since the latter are always independent of the parameters of the mechanism.

§ 9. - As regards a possible application of this to the theory of quanta, we remark the following: On the basis of our conclusions, the possibility of an extension of Ehrenfest's principle is ruled out, save for the aforementioned exceptions. Instead, for quasi-ergodic systems, or the exceptions studied in § 8, such an application is not a priori ruled out, though obviously it is not possible now to foresee if experience will confirm its results. All the same, one might try if, going on in this way, some useful information on the rules for the determination of the quantum orbits of the systems without angular coordinates could be obtained. Of course, Ehrenfest's principle by itself, even if in case that experience should confirm it in this more general application, is not sufficient for the determination of such rules. It only

allows us, when we know the selected orbits of a certain system, to deduce the orbits for all the systems which can be obtained from it by means of an adiabatic transformation. Therefore perhaps it might be useful, apart from the complexity of calculations, for finding the quantitative relations between the spark spectra, for instance of the alkaline metals, and the arc spectra of the noble gases. In fact, the systems which emit these spectra only differ in the charge of the nucleus and then can be easily transformed one into another.

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