3) On Phenomena Occurring Close to a World Line

"Sopra i fenomeni che avvengono in vicinanza di una linea oraria,"
Rend. Lincei, ***31*** *(I), 21–23, 51–52, 101–103 (1922).* [1]

Note I.

§ 1.− In order to study phenomena which occur close to a world line, i.e., in nonrelativistic language, in a region of space in the spacetime manifold, even varying in time, but always very small compared with the divergences from Euclidean space, it would be convenient to find a particular frame such that close to the line being studied, the spacetime ds^2 will assume a simple form. In order to find such a frame, we must begin with some geometrical considerations.

Let a line L be given in a Riemannian manifold V_n or in a manifold metrically connected in the sense of Weyl.[2] Let us associate at every point P of L a direction y perpendicular to L, with the rule that the direction $y + dy$, corresponding to the point P+dP, will be derived from that y associated to P in the following way: let η be the direction tangent to L at P; let y and η be parallel transported[3] from P to P+dP and let $y+\delta y$ and $\eta+\delta\eta$ be the directions obtained in this way, which because of the fundamental properties of parallel transport will still be orthogonal. If L is not geodesic, $\eta + \delta\eta$ will not coincide with the direction $\eta + d\eta$ of the tangent to L at P+dP, and these two directions at P+dP will define a 2-dimensional subspace. Let us consider at P+dP the element of the S_{n-2} perpendicular to this subspace and let us rigidly rotate around this S_{n-2} all the surrounding particle space until $\eta + \delta\eta$ is superposed on $\eta + d\eta$. Then $y + \delta y$ will be mapped to a position which we will consider to be the direction $y + dy$ relative to the point P+dP. It is clear that, arbitrarily fixing the direction y at a point of L, an integration process will allow it to be obtained at any point of L.

Let us now look for the analytic expressions which translate the indicated operations to a Riemannian manifold, which coincide with those valid for a Weyl metric manifold as long as the "Eichung" is chosen such that the measure of a segment, which moves rigidly around L, will be constant. Let

$$ds^2 = \sum_{ik} g_{ik} dx^i dx^k \qquad (i, k = 1, 2, \ldots n) \tag{1}$$

and let y_i, $y^{(i)}$; η_i, $\eta^{(i)} = dx_i/ds$ be the co- and contravariant systems of the directions y, η. We will then have

$$\frac{\delta\eta^{(i)}}{ds} = -\sum_{hl} \begin{Bmatrix} h\,l \\ i \end{Bmatrix} \eta^{(h)} \frac{dx_l}{ds} = -\sum_{hl} \begin{Bmatrix} h\,l \\ i \end{Bmatrix} \frac{dx_h}{ds}\frac{dx_l}{ds} \ ,$$

[1]Presented by the Correspondent G. Armellini during the session of January 22, 1922.
[2]Weyl, *Space, Time, Matter*, p. 109. Berlin, Springer, 1921.
[3]T. Levi Civita, *Rend. Circ. Palermo*, Vol. XLII, p. 173 (1917).

and moreover $\frac{d\eta^i}{ds} = \frac{d}{ds}\frac{dx_i}{ds} = \frac{d^2x_i}{ds^2}$. Therefore one finds

$$\frac{\delta\eta^{(i)} - d\eta^{(i)}}{ds} = -\left(\frac{d^2x_i}{ds^2} + \sum_{hl}\begin{Bmatrix} h\,l \\ i \end{Bmatrix}\frac{dx_h}{ds}\frac{dx_l}{ds}\right) = -C^i \ .$$

The C^i are the contravariant components of the vector $\mathbf{C}$, the geodetic curvature, namely of a vector having the same orientation as the geodesic principal normal of L and a magnitude equal to its geodesic curvature.

On the other hand, one has

$$\frac{\delta y^{(i)}}{ds} = -\sum_{hk}\begin{Bmatrix} h\,k \\ i \end{Bmatrix} y^{(k)}\frac{dx_k}{ds} \ . \tag{2}$$

Now since y is orthogonal to L, the displacement with which from $y + \delta y$ one gets $y + dy$ will be parallel to the tangent to L and will have magnitude equal to the projection onto the same y of $\delta\eta - d\eta$; that is to say, since y has magnitude 1, equal to the scalar product of $\delta\eta - d\eta$ and y, namely

$$\sum_i(\delta\eta_i - d\eta_i)y^{(i)} = -ds\sum_i C_i y^{(i)} \ .$$

Its contravariant components will therefore be obtained by multiplying its magnitude by the contravariant coordinates of the tangent to L, that is dx_i/ds. These are, in the final analysis, $-dx_i\sum_r C_r y^{(r)}$. From (2) it follows immediately that

$$\frac{dy^{(i)}}{ds} = -\sum_{hk}\begin{Bmatrix} h\,k \\ i \end{Bmatrix} y^{(k)}\frac{dx_k}{ds} - \frac{dx_i}{ds}\sum_h C_h y^h \ . \tag{3}$$

Equation (3), written for $i = 1, 2, \ldots, n$ gives a system of n first order differential equations for the n unknowns $y^{(1)}, y^{(2)}, \ldots, y^{(n)}$, which are therefore determined once the initial data are assigned. It would also be easy to formally verify from (3) that, if the initial values of the $y^{(i)}$ satisfy the condition of perpendicularity to L, such a condition will remain satisfied all along the line.

§ 2.– Let us now assign at a point P_0 of L n mutually orthogonal directions $y_1, y_2, \ldots, y_n$ chosen at will, with the condition that y_n be tangent to L. The directions $y_1, y_2, \ldots, y_{n-1}$ will be perpendicular to L, and we can transport them along L by using the law given in the preceding section, which clearly from its definition preserves their orthogonality. We are then in a position to associate with every point of L n mutually orthogonal directions, the last one of which is tangent to L. Let us now consider our V_n embedded in a Euclidean $\mathrm{S_N}$ with a suitable number of dimensions. We can take as coordinates of a point of V_n the orthogonal Cartesian coordinates of its projection onto the $\mathrm{S_N}$ tangent to V_n at a generic point P of L, having P as the origin and the directions $y_1, y_2, \ldots, y_n$ relative to the point P as directions. In terms of these coordinates, the line element of V_n at P can be written in the form $ds^2_{\mathrm{P}} = dy_1^2 + dy_2^2 + \cdots + dy_n^2$; in addition, they are geodesics at P, as one can immediately see. In other words, for the coordinates y it is possible in a neighborhood of P to set $g_{ii} = 1$, $g_{ik} = 0$ $(i \neq k)$, up to infinitesimals of order greater than the first. Obviously we shall have such a reference frame at every point of L. Let us consider now a point Q_0 of V_n with coordinates $\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_{n-1}, 0$ in the reference frame corresponding to the point P_0 on L. For any other point P of L we can so determine a point Q having in the frame corresponding to P the same coordinates as Q_0 has in the frame corresponding to P_0. The point Q will therefore trace out a line parallel to L. Now we want to find the relation between ds_{Q} and ds_{P}, assuming that the point Q is infinitely close to P. In order to do so, we note that the displacement transporting Q to Q + dQ is composed of the displacements denoted in § 1 by δ and $d-\delta$, and that the first one gives $\delta s_{\mathrm{Q}} = ds_{\mathrm{P}}$ up to infinitesimals of greater order since it is a parallel displacement; the second one is a rotation, which gives $(d-\delta)s_{\mathrm{Q}} = ds_{\mathrm{P}}\, \mathbf{C} \cdot (\mathrm{Q} - \mathrm{P})$, as is seen from § 1 , denoting by $\cdot$ the symbol of the scalar product, and with Q − P the vector with origin at P and endpoint at Q. Moreover, both ds_{Q} and $(d-\delta)s_{\mathrm{Q}}$ have the direction of the tangent to L. Hence, one has $ds_{\mathrm{Q}} = \delta s_{\mathrm{Q}} + (d-\delta)s_{\mathrm{Q}}$; namely

$$ds_{\mathrm{Q}} = ds_{\mathrm{P}}[1 + \mathbf{C} \cdot (\mathrm{Q} - \mathrm{P})] \; . \tag{4}$$

The trajectories of the points Q form a $(n-1)^{ple}$ infinity of lines, so at least with proper limitations through each point M of V_n will pass one of these lines; in this way, we can characterize M through the coordinates of the point Q, $\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_{n-1}$ corresponding to the line passing through M, and the arclength s_{P} of the line L marked off from an arbitrarily chosen origin to that point P corresponding to the Q one coinciding with M.

If M is infinitely close to L, ds_{Q} will be perpendicular to the hypersurface $s_{\mathrm{P}} =$ constant. Thus one will have

$$ds_{\mathrm{M}}^2 = ds_{\mathrm{Q}}^2 + d\bar{\mathrm{y}}_1^2 + d\bar{\mathrm{y}}_2^2 + \cdots + d\bar{\mathrm{y}}_{n-1}^2 \; ;$$

and taking into account (4),

$$ds_{\mathrm{M}}^2 = [1 + \mathbf{C} \cdot (\mathrm{M} - \mathrm{P})]^2 ds_{\mathrm{Q}}^2 + d\bar{\mathrm{y}}_1^2 + d\bar{\mathrm{y}}_2^2 + \cdots + d\bar{\mathrm{y}}_{n-1}^2 \; . \tag{5}$$

As a result, in the neighborhood of L we have found a very simple expression for ds^2.

Note II.

§ 3.– Before passing to the physical application of the results obtained above, we still want to make some geometrical observations. First of all, it is clear that the previous considerations, and so also the formula (5) representing their conclusion, which for any manifold whatsoever are only valid close to L, are instead completely rigorous for Euclidean spaces. So let us associate to the line L of V_n a line L^* in a Euclidean space S_n, in which we indicate the orthogonal cartesian coordinates by

x_i^*. If we indicate with asterisks the symbols referring to the line L*, we can write for S$_n$ the formula analogous to (5):

$$ds^2_{\mathrm{M}^*} = [1 + \mathbf{C}^* \cdot (\mathrm{M}^* - \mathrm{P}^*)]^2\, ds^2_{\mathrm{P}^*} + d\bar{\mathrm{y}}_1^{*2} + d\bar{\mathrm{y}}_2^{*2} + \cdots + d\bar{\mathrm{y}}^{*}_{n-1}{}^2\ ; \qquad (5^*)$$

as in (5) $\mathbf{C}$ is a function of s_P, so in (5*) $\mathbf{C}^*$ is a function of s_{P^*}.

Let $\mathrm{K}^{(1)}$, $\mathrm{K}^{(2)}$, $\cdots$, $\mathrm{K}^{(n-1)}$ be the contravariant components of $\mathbf{C}$ with respect to $\bar{\mathrm{y}}_1$, $\bar{\mathrm{y}}_2$, $\cdots$, $\bar{\mathrm{y}}_{n-1}$, and $\mathrm{K}^{(1)\,*}$, $\mathrm{K}^{(2)\,*}$, $\cdots$, $\mathrm{K}^{(n-1)\,*}$ those of $\mathbf{C}^*$ with respect to $\bar{\mathrm{y}}^*$. Let us try to determine L* in such a way that the functions $\mathrm{K}^{(r)\,*}(s_{\mathrm{P}^*})$ become equal to the $\mathrm{K}^{(r)}(s_\mathrm{P})$. In order to do so, we shall begin by imposing that $s_\mathrm{P} = s_{\mathrm{P}^*}$, i.e., by establishing between the points of L and L* a one-to-one correspondence preserving the arclength. We then note that $\mathrm{K}^{(r)\,*}$ is the projection of $\mathbf{C}^*$ on the r^{th} direction y^*. Namely, one has

$$\mathrm{K}^{(r)\,*} = \sum_{i=1}^{i=n} y^*_{i|r}\, \frac{d^2 x_i^*}{ds_\mathrm{P}^2} \qquad (r = 1, 2, \cdots, n-1). \qquad (6)$$

The $\mathrm{K}^{(r)}$ are then known functions of s_P. The condition $\mathrm{K}^{(r)} = \mathrm{K}^{(r\,*)}$ thus leads to the $(n-1)$ equations

$$\mathrm{K}^{(r)}(s_\mathrm{P}) = \sum_{i=1}^{i=n} y^*_{i|r}\, \frac{d^2 x_i^*}{ds_\mathrm{P}^2} \qquad (r = 1, 2, \cdots, n-1)\ . \qquad (7)$$

On the other hand, (3) once written for the S$_n$, gives us another $n(n-1)$ equations. If we add to these equations the following one

$$ds_\mathrm{P}^2 = dx_1^{*2} + dx_2^{*2} + \cdots + dx_n^{*2}\ , \qquad (8)$$

we obtain a system of $n - 1 + n(n-1) + 1 = n^2$ equations for the n^2 unknowns x_i^*, $y^*_{i|r}$, which can be used to express them in terms of s_P. In this way we can determine the parametric equations $x_i^* = x_i^*(s_\mathrm{P})$ for L*. With that the formula (5*) becomes identical to (5), that is we have represented by applicability the neighborhood of the line L* onto that of L. In addition, since L* is in a Euclidean space, we can say that we have unfolded the neighborhood of L in a Euclidean space, i.e., we have found coordinates which are simultaneously geodesic at each point of L.

Note III.

§ 4.– In order to show the application to the theory of relativity of the results obtained above, we shall assume that V$_n$ is the V$_4$ spacetime and that L is a world line in whose neighborhood we want to study the phenomena. Setting $ds^2_\mathrm{M} = ds^2$ in (5) for the sake of brevity, one finds in this case:

$$ds^2 = [1 + \mathbf{C} \cdot (\mathrm{M} - \mathrm{P})]^2\, ds_\mathrm{P}^2 + d\bar{\mathrm{y}}_1^2 + d\bar{\mathrm{y}}_2^2 + d\bar{\mathrm{y}}_3^2\ .$$

To avoid the appearance of imaginary terms and to restore the homogeneity, it is convenient to make the following change of variables:

$$s_\mathrm{P} = vt\,; \quad \bar{\mathrm{y}}_1 = ix\,; \quad \bar{\mathrm{y}}_2 = iy\,; \quad \bar{\mathrm{y}}_3 = iz\,,$$

where v is a constant with dimensions of a velocity, so that t has the dimensions of time. Thus one obtains

$$ds^2 = a\,dt^2 - dx^2 - dy^2 - dz^2 \ , \tag{9}$$

where

$$a = v^2[1 + \mathbf{C}\cdot(\mathrm{M} - \mathrm{P})]^2 \ . \tag{10}$$

Hereafter, we refer to the space x, y, z using the ordinary symbols of vector calculus. And it is just in this sense that the scalar product which enters in (10) can be understood, provided that $\mathbf{C}$ is considered as the vector whose components are the covariant components of the geodesic curvature of the world line $x = y = z = 0$, and $\mathrm{M} - \mathrm{P}$ is the vector with components x, y, z. We will call x, y, z spatial coordinates, and t time. Sometimes for uniformity we will write x_0, x_1, x_2, x_3 in place of t, x, y, z, and we will also denote the coefficients of the quadratic form (9) by g_{ik}.

§ 5.– Let[4] F_{ik} be the electromagnetic field and $(\varphi_0, \varphi_1, \varphi_2, \varphi_3)$ the first rank tensor "potential" of F_{ik}, such that $\mathrm{F}_{ik} = \varphi_{i,k} - \varphi_{k,i}$. We set $\varphi_0 = \varphi$ and call $\mathbf{u}$ the vector with components $\varphi_1, \varphi_2, \varphi_3$. First of all, we have:

$$\left.\begin{matrix}\mathrm{F}_{01}\\ \mathrm{F}_{02}\\ \mathrm{F}_{03}\end{matrix}\right\} = \mathrm{grad}\,\varphi - \frac{\partial \mathbf{u}}{\partial t} \quad , \quad \left.\begin{matrix}\mathrm{F}_{23}\\ \mathrm{F}_{31}\\ \mathrm{F}_{12}\end{matrix}\right\} = -\mathrm{curl}\,\mathbf{u} \ , \ \mathrm{F}_{ii} = 0 \ , \ \mathrm{F}_{ik} = -\mathrm{F}_{ki} \ ;$$

analogously

$$\left.\begin{matrix}\mathrm{F}^{(01)}\\ \mathrm{F}^{(02)}\\ \mathrm{F}^{(03)}\end{matrix}\right\} = \frac{1}{a}\left(-\mathrm{grad}\,\varphi + \frac{\partial \mathbf{u}}{\partial t}\right) \ , \ \left.\begin{matrix}\mathrm{F}^{(23)}\\ \mathrm{F}^{(31)}\\ \mathrm{F}^{(12)}\end{matrix}\right\} = -\mathrm{curl}\,\mathbf{u} \ , \ \mathrm{F}^{(ii)} = 0 \ , \ \mathrm{F}^{(ik)} = -\mathrm{F}_{(ki)} \ ,$$

so that

$$\frac{1}{4}\sum_{ik} \mathrm{F}_{ik}\mathrm{F}^{(ik)} = \frac{1}{2}\left\{\mathrm{curl}^2\,\mathbf{u} - \frac{1}{a}\left(-\mathrm{grad}\,\varphi + \frac{\partial \mathbf{u}}{\partial t}\right)^2\right\} \ .$$

Let $d\omega$ be the hypervolume element of V_4. We will have

$$d\omega = \sqrt{-||g_{ik}||}\,dx_0\,dx_1\,dx_2\,dx_3 = \sqrt{a}\,dt\,d\tau \ ,$$

where $d\tau = dx\,dy\,dz$ is the volume element of the space.

One also has:

$$\sum \varphi_i dx_i = \varphi dx + \mathbf{u}\cdot d\mathrm{M} \ , \qquad d\mathrm{M} = (dx, dy, dz) \ .$$

[4]See Weyl, op. cit., pp. 186 and 208 for the notation and the Hamiltonian derivation of the laws of physics.

Apart from the action of the metric field, whose variation is zero since we consider it as given *a priori* by (9), the action will assume the following form:

$$W = \frac{1}{4}\int_\omega \sum_{ik} \mathrm{F}_{ik}\mathrm{F}^{(ik)}\, d\omega + \int_e de \int \varphi_i\, dx_i + \int_m dm \int ds \ ,$$

$$\begin{pmatrix} de &= \text{element of electric charge} \\ dm &= \text{element of mass} \end{pmatrix} .$$

By introducing the indicated notation, one finds

$$\begin{aligned} W = \frac{1}{2}\iint &\left\{ \mathrm{curl}^2\, \mathbf{u} - \frac{1}{a}\left(-\mathrm{grad}\,\varphi + \frac{\partial \mathbf{u}}{\partial t}\right)^2 \right\} \sqrt{a}\, dt\, d\tau \\ &+ \iint (\varphi + \mathbf{u}\cdot\mathbf{V}_\mathrm{L})\rho\, d\tau\, dt + \iint \sqrt{a - {\mathbf{V}_\mathrm{M}}^2}k\, d\tau\, dt \ , \end{aligned} \tag{11}$$

where ρ, k are respectively the density of electricity and of matter, so that $de = \rho\, d\tau$, $dm = k\, d\tau$, $\mathbf{V}_\mathrm{L}$ is the velocity of the electric charges, $\mathbf{V}_\mathrm{M}$ that of the masses.

The integrals on the right hand side can be extended to an arbitrary region τ between any two times t_1, t_2. Then one has the constraint that on the boundary of the region τ, and for the two times t_1, t_2, all variations are zero.

Apart from these conditions, the variations of φ and of $\mathbf{u}$ are completely arbitrary. Further conditions can be imposed on the variations of x, y, z thought of as coordinates of an element of charge or mass, expressing the constraints of the specific problem under consideration. Then writing that dW vanishes for any variation $\delta\varphi$ of φ whatsoever, one finds

$$0 = -\iint \left(\mathrm{grad}\,\varphi - \frac{\partial \mathbf{u}}{\partial t}\right)\cdot \delta\,\mathrm{grad}\,\varphi \frac{d\tau\, dt}{\sqrt{a}} + \iint \delta\varphi\rho\, dt\, d\tau \ .$$

Transforming the first integral by a suitable application of Gauss's theorem, and taking into account that $\delta\varphi$ vanishes at the boundary, we find

$$0 = \iint \delta\varphi \left\{ \rho + \mathrm{div}\left[\frac{1}{\sqrt{a}}\left(\mathrm{grad}\,\varphi - \frac{\partial \mathbf{u}}{\partial t}\right)\right]\right\} dt\, d\tau \ ,$$

and since $\delta\varphi$ is arbitrary, we obtain the equation

$$\rho + \mathrm{div}\left[\frac{1}{\sqrt{a}}\left(\mathrm{grad}\,\varphi - \frac{\partial \mathbf{u}}{\partial t}\right)\right] = 0 \ . \tag{12}$$

Analogously, taking the variation of $\mathbf{u}$, one finds

$$\rho\mathbf{V}_\mathrm{L} + \mathrm{curl}(\sqrt{a}\,\mathrm{curl}\,\mathbf{u}) - \frac{\partial}{\partial t}\left[\frac{1}{\sqrt{a}}\left(\mathrm{grad}\,\varphi - \frac{\partial \mathbf{u}}{\partial t}\right)\right] = 0 \ . \tag{13}$$

These last two equations allow us to determine the electromagnetic field, once the charges and their motion are given.

Another set of equations can be obained by varying the trajectories of the charges and masses in W. Let $\delta\mathrm{P}_\mathrm{M}$ be the variation of the trajectory of the masses, $\delta\mathrm{P}_\mathrm{L}$ that of the charges. Moreover, since $\mathbf{u}$ is a vector function of the position and $\mathbf{V}$ a vector, let us denote by $(\partial\mathbf{u}/\partial\mathrm{P})(\mathbf{V})$ the vector with components $\dfrac{\partial u_x}{\partial x}V_x + \dfrac{\partial u_x}{\partial y}V_y + \dfrac{\partial u_x}{\partial z}V_z$, and so on. Now, writing that the variation of W is zero, one finds through the usual methods:

$$\iint \left(\delta\mathrm{P_M}\cdot\operatorname{grad}\varphi - \delta\mathrm{P_L}\left(\frac{\partial\mathbf{u}}{\partial t} + \frac{\partial\mathbf{u}}{\partial\mathrm{P}}(\mathrm{V_L})\right) + \mathrm{V_L}\cdot\frac{\partial\mathbf{u}}{\partial\mathrm{P}}(\delta\mathrm{P_L})\right)\rho\,dt\,d\tau + \iint \delta\mathrm{P_M}\cdot\left\{\frac{dt}{ds}\frac{\operatorname{grad}a}{2} + \frac{d}{dt}\left(\frac{dt}{ds}\mathbf{V}_\mathrm{M}\right)\right\}k\,dt\,d\tau = 0\ . \tag{14}$$

If the δP's at a given time do not depend on their values at other times, the coefficient of dt in (14) must be zero. So one finds:

$$\int \left\{\delta\mathrm{P_M}\cdot\operatorname{grad}\varphi - \delta\mathrm{P_L}\left[\frac{\partial\mathbf{u}}{\partial t} + \frac{\partial\mathbf{u}}{\partial\mathrm{P}}(\mathrm{V_L})\right] + \mathrm{V_L}\cdot\frac{\partial\mathbf{u}}{\partial\mathrm{P}}(\delta\mathrm{P_L})\right\}\rho\,d\tau + \iint \delta\mathrm{P_M}\cdot\left\{\frac{1}{2}\frac{dt}{ds}\operatorname{grad}a + \frac{d}{dt}\left(\frac{dt}{ds}\mathbf{V}_\mathrm{M}\right)\right\}k\,d\tau\ , \tag{15}$$

which has to be satisfied for all systems of δP satisfying the constraints.

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