2) On the electrostatics of a homogeneous gravitational field and on the weight of electromagnetic masses

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INTRODUCTION

The aim of this paper is to investigate in the framework of general relativity how a homogeneous gravitational field modifies the electrostatic phenomena occurring in it. Once the differential equation relating the electrostatic potential to the charge density, which corresponds to the Poisson equation in classical electrostatics, is established, one is able to integrate it at least when the gravitational field is weak enough (and certainly the gravitational field of the Earth amply satisfies this condition), obtaining in this way the corrections to Coulomb's law due to the presence of the gravitational field.

In a first application the distribution of the electric charges on a conducting sphere is studied, showing that the sphere polarizes by means of the gravitational field.

The second application is devoted to studying the weight of an electromagnetic mass, that is the force exerted on a fixed system of electric charges (e.g., sustained by a rigid dielectric), as a consequence of the presence of the gravitational field.

One finds that such a weight is given by the acceleration of gravity times u/c^2, where u denotes the electrostatic energy of the charges of the system, and c is the velocity of light. So the gravitational mass, namely the ratio between the weight and the acceleration of gravity, does not coincide in general with the inertial mass for the system under consideration, since the latter is given by $(4/3)u/c^2$ (with the same notation) if the system is endowed with spherical symmetry for example.

Besides it is known how special relativity leads us to take $\Delta u/c^2$ as the increase of the *inertial* mass of a system getting an energy Δu, and this fact can be easily related to the aforementioned result.

Finally, it is shown how to find a point having the same properties, with respect to the weight of the considered system of charges, as the center of gravity with respect to the weight of an ordinary system of material masses.

PART 1

ELECTROSTATICS IN A GRAVITATIONAL FIELD

§ 1.– Let us consider a portion of the spacetime where a homogeneous gravitational field is present, and assume the electrostatic phenomena that we think are taking place in it to be weak enough to neglect the effect they produce on the metric describing the region under consideration. Under this assumption, the line element of the spacetime manifold can be written as [1]

$$ds^2 = a\,dt^2 - dx^2 - dy^2 - dz^2 \ , \tag{1}$$

where a is a function only of z.

The variables t, x, y, z will also be denoted by x_0, x_1, x_2, x_3, and the coefficients of the quadratic form (1) by g_{ik}. Let φ_i be the vector potential, and F_{ik} the electromagnetic field. Then we have

$$\mathrm{F}_{ik} = \varphi_{i,k} - \varphi_{k,i} \ , \tag{2}$$

referring ourselves to the fundamental form (1).

By limiting our considerations to electrostatic fields, we can set $\varphi_1 = \varphi_2 = \varphi_3 = 0$, and, for the sake of brevity, $\varphi_0 = \varphi$. Thus one has:

$$\mathrm{F}_{ik} = \varphi_{i,k} - \varphi_{k,i} = \frac{\partial \varphi_i}{\partial x_k} - \frac{\partial \varphi_k}{\partial x_i} \ ,$$

that is

$$\begin{cases} \mathrm{F}_{01} = \dfrac{\partial\varphi}{\partial x} \ , \qquad \mathrm{F}_{02} = \dfrac{\partial\varphi}{\partial y} \ , \qquad \mathrm{F}_{03} = \dfrac{\partial\varphi}{\partial z} \ , \\ \mathrm{F}_{23} = \mathrm{F}_{31} = \mathrm{F}_{12} = 0 \ , \qquad \mathrm{F}_{ik} = -\mathrm{F}_{ki} \ , \qquad \mathrm{F}_{ij} = 0 \ . \end{cases} \tag{3}$$

In addition one has:

$$\mathrm{F}^{(ik)} = \sum_{hk} g^{(ih)} g^{(jk)} \mathrm{F}_{(hk)} = g^{(ii)} g^{(jj)} \mathrm{F}_{(ij)} \ ,$$

from which by noting that:

$$g^{(00)} = \frac{1}{a} \ , \qquad g^{(11)} = g^{(22)} = g^{(33)} = -1 \ ,$$

one obtains

$$\begin{cases} \mathrm{F}^{(01)} = -\dfrac{1}{a}\dfrac{\partial\varphi}{\partial x} \ , \qquad \mathrm{F}^{(02)} = -\dfrac{1}{a}\dfrac{\partial\varphi}{\partial y} \ , \qquad \mathrm{F}^{(03)} = -\dfrac{1}{a}\dfrac{\partial\varphi}{\partial z} \ , \\ \mathrm{F}^{(23)} = \mathrm{F}^{(31)} = \mathrm{F}^{(12)} = 0 \ , \qquad \mathrm{F}^{(ik)} = -\mathrm{F}^{(ki)} \ , \qquad \mathrm{F}^{(ii)} = 0 \ . \end{cases} \tag{4}$$

In the case under consideration here, the action can be written in the form

$$W = \int_\omega \sum_{ik} \mathrm{F}_{ik}\mathrm{F}^{(ik)}\, d\omega + \int de \int \varphi\, dx_0 \ , \tag{5}$$

where

$$d\omega = \sqrt{-||g_{ik}||}\, dx_0\, dx_1\, dx_2\, dx_3 = \sqrt{a}\, dx\, dy\, dz\, dt$$

[1] T. Levi-Civita, Note II. "Sui ds^2 einsteiniani". *Rend. Acc. Lincei*, **27**, 1° sem. N° 7.

is the hypervolume element of the manifold, and the integration corresponding to $d\omega$ has to be performed over a specific region of the manifold, while the integrations corresponding to de, dx_0 have to be extended to all the elements of electric charge whose world lines cross the region under consideration and to the portions of those world lines lying in it, respectively.

§ 2.– In the variation of W, φ can be arbitrarily varied, under the single condition that $\delta\varphi = 0$ on the boundary of the integration domain.

The variations δx, δy, δz instead, in addition to the condition $\delta x = \delta y = \delta z = 0$ on the boundary, could also be subjected to further conditions to be determined in each particular case. For example, inside a conducting body they will be quite arbitrary, while in a rigid dielectric they will have to represent the components of a rigid virtual displacement, and so on.

By putting the quantities (3), (4) into (5), one obtains:

$$W = -\frac{1}{2}\iiiint \frac{1}{\sqrt{a}}\left\{\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2 + \left(\frac{\partial\varphi}{\partial z}\right)^2\right\} dx\,dy\,dz\,dt + \int de \int \varphi\,dt\ , \tag{6}$$

from which

$$\begin{aligned}\delta W = &\iiiint \delta\varphi\left[\frac{1}{\sqrt{a}}\,\Delta_2\,\varphi + \frac{\partial\varphi}{\partial z}\frac{d(1/\sqrt{a})}{dz} + \rho\right] dx\,dy\,dz\,dt \\ &+ \iiiint \rho\left(\frac{\partial\varphi}{\partial x}\,\delta x + \frac{\partial\varphi}{\partial y}\,\delta y + \frac{\partial\varphi}{\partial z}\,\delta z\right) dx\,dy\,dz\,dt\end{aligned} \tag{7}$$

as immediately follows by noting that $dx = dy = dz = 0$ along a given world line, as a consequence of our assumptions, and $\rho\,dx\,dy\,dz = de$, since ρ is the electric density.

Then in order for δW to vanish identically, since $\delta\varphi$ is arbitrary inside the integration domain, one finds that

$$\Delta_2\,\varphi - \frac{d\,\log\sqrt{a}}{dz}\frac{\partial\varphi}{\partial z} = -\rho\sqrt{a}\ . \tag{8}$$

Moreover, one must also have

$$\iiiint \rho\left(\frac{\partial\varphi}{\partial x}\,\delta x + \frac{\partial\varphi}{\partial y}\,\delta y + \frac{\partial\varphi}{\partial z}\,\delta z\right) dx\,dy\,dz\,dt = 0\ , \tag{9}$$

for every system of values for δx, δy, δz satisfying the assumed constraints.

Equation (8) contains the generalization of Poisson's law, to which (8) reduces if a is constant, that is if the gravitational field is absent.

§ 3.– If we indicate by G the acceleration of gravity of the field under consideration, namely the acceleration with which a free material point begins to move, one has:

$$\mathrm{G} = -\frac{1}{2}\frac{da}{dz}\ . \tag{10}$$

With this (8) becomes:

$$\Delta_2\,\varphi + \frac{\mathrm{G}}{a}\frac{\partial\varphi}{\partial z} = -\rho\sqrt{a}\ . \tag{11}$$

In order to find the solution of (11), given ρ at each point, we imagine the electric charges to be contained in a small region around the origin of the coordinates. Moreover, we will set $a = c^2$ at the origin (with c the velocity of light near the origin), and we will assume gravity to be so weak that those terms which contain the square of the ratio $l\,\mathrm{G}/c^2$ can be neglected, where l represents the maximum length entering into the problem under consideration. Under these assumptions, we can set:

$$\sqrt{a} = c + \frac{1}{2c}\frac{da}{dz}z = c\left(1 - \frac{\mathrm{G}}{c^2}z\right)\ .$$

Therefore (11) can be written as:

$$\Delta_2\,\varphi + \frac{\mathrm{G}}{c^2}\frac{\partial\varphi}{\partial z} = -c\left(1 - \frac{\mathrm{G}}{c^2}z\right)\rho\ . \tag{12}$$

The integral of that equation in this approximation, as can be directly verified, is given by:

$$\begin{aligned}\varphi_{\mathrm{P}} &= \frac{c}{4\pi}\int\left(1 - \frac{\mathrm{G}}{c^2}\right)z_{\mathrm{M}}\,d\tau_{\mathrm{M}}\left(\frac{1}{r} - \frac{\mathrm{G}}{2c^2}\frac{z_{\mathrm{P}} - z_{\mathrm{M}}}{r}\right)\\ &= \frac{c}{4\pi}\int \rho_{\mathrm{M}}\,d\tau_{\mathrm{M}}\left(\frac{1}{r} - \frac{\mathrm{G}}{2c^2}\frac{z_{\mathrm{P}} + z_{\mathrm{M}}}{r}\right)\ ,\end{aligned} \tag{13}$$

where M is the generic point of the region τ_{M} containing the electric charges, P is the point at which the potential φ is evaluated, and r is the distance MP.

Given the linearity of equation (12), any integral of the equation:

$$\Delta_2\,\varphi + \frac{\mathrm{G}}{c^2}\frac{\partial\varphi}{\partial z} = 0\ , \tag{12*}$$

obtained by setting $\rho = 0$ in (12), can be added to (13). This integral will represent the field due to causes external to ρ_{M}. For the application we have in mind it is convenient to consider a particular solution to (12)* given by

$$\varphi = -c\mathrm{E}^*_x x - c\mathrm{E}^*_y y + \frac{c^2}{\mathrm{G}}\mathrm{E}^*_z e^{-\frac{\mathrm{G}}{c^2}z}\ , \tag{14}$$

with $\mathrm{E}^*_x, \mathrm{E}^*_y, \mathrm{E}^*_z$ constants.

At the origin one has

$$\mathrm{E}_x = -\frac{1}{c}\,\mathrm{F}_{01}\ , \qquad \mathrm{E}_y = -\frac{1}{c}\,\mathrm{F}_{02}\ , \qquad \mathrm{E}_z = -\frac{1}{c}\,\mathrm{F}_{03}\ ,$$

since E is the electric force.

From this it follows that the electric force of the external field (14) has components

$$\mathrm{E}^*_x\ , \quad \mathrm{E}^*_y\ , \quad \mathrm{E}^*_z\ .$$

§ 4.– Let us now calculate the electric field due to a charge e concentrated at the origin of the coordinates. From (13) one has:

$$\varphi = \frac{ce}{4\pi}\left(\frac{1}{r} - \frac{\mathrm{G}}{2c^2}\frac{z}{r}\right) , \tag{15}$$

and this formula gives the generalization of Coulomb's law, as immediately follows by setting $\mathrm{G} = 0$. Recalling (3) one gets:

$$\begin{cases} \mathrm{F}_{01} = \dfrac{ce}{4\pi}\left(\dfrac{x}{r^3} - \dfrac{\mathrm{G}}{2c^2}\dfrac{zx}{r^3}\right) , \\[2ex] \mathrm{F}_{02} = \dfrac{ce}{4\pi}\left(\dfrac{y}{r^3} - \dfrac{\mathrm{G}}{2c^2}\dfrac{zy}{r^3}\right) , \\[2ex] \mathrm{F}_{03} = \dfrac{ce}{4\pi}\left(\dfrac{z}{r^3} - \dfrac{\mathrm{G}}{2c^2}\dfrac{z^2}{r^3} + \dfrac{\mathrm{G}}{2c^2}\dfrac{1}{r}\right) . \end{cases} \tag{16}$$

We can summarize all three of the preceding formulas in a single vector formula. In fact by denoting the vector with components $\mathrm{F}_{01}, \mathrm{F}_{02}, \mathrm{F}_{03}$ by F_0, letting $\vec{a}$ be a vector of magnitude 1 and orientation MP, and finally letting $\vec{G}$ be a vector of magnitude G and orientation z, (16) can be written as:

$$\mathrm{F}_0 = \frac{ce}{4\pi}\left\{\frac{\vec{a}}{r^2} + \frac{\vec{\mathrm{G}}\times\vec{a}}{2c^2 r}\,\vec{a} - \frac{1}{2c^2 r}\,\vec{\mathrm{G}}\right\} . \tag{17}$$

It is interesting to compare this formula with the one which gives the electric force exerted by an electric charge e which in the absence of gravitational attraction has acceleration $\vec{\Gamma}$, quasi-stationary motion and velocity negligible with respect to the speed of light. Such a force is expressed by

$$\mathrm{E} = \left\{\frac{\vec{a}}{r^2} + \frac{\vec{\Gamma}\times\vec{a}}{c^2 r}\,\vec{a} - \frac{1}{2c^2 r}\,\vec{\Gamma}\right\} , \tag{18}$$

with the same notation.

From here one sees that, by setting

$$\vec{\Gamma} = -\frac{\vec{\mathrm{G}}}{2} \tag{19}$$

in (18), one obtains

$$\mathrm{F}_0 = c\mathrm{E} .$$

This result can be put into words as follows, noting that $c\mathrm{E}$ is the electric part of the electromagnetic field generated by the charge in accelerated motion:

The electric part $(\mathrm{F}_{01}, \mathrm{F}_{02}, \mathrm{F}_{03})$ of the electromagnetic field (F_{ik}) generated by an electric charge at rest in a homogeneous field of strength G is equal to the electric part of the electromagnetic field which the same charge would produce in the absence of gravitational field if it moved under the conditions indicated above with acceleration G/2 in the direction opposite to the gravitational field.

§ 5.– Now, let us study how the distribution of the electricity over a conductor is modified by the gravitational field. To this end, let us note that since δx, δy, δz are arbitrary inside the conductor, from (9) it follows that $\varphi = \mathit{constant}$ inside, and so $\rho = 0$ by (8). Thus the electricity is completely at the surface. Then let us assume that our conductor is a sphere with center O at the origin of the coordinates and of radius R.

Let us try to satisfy the condition $\varphi = \mathit{constant}$ in the interior by assuming the following expression for the surface electric density at a generic point M of the surface:

$$\frac{e}{4\pi R^2} + \frac{e}{r}\, a\cos\theta \ , \tag{20}$$

where θ represents the angle spanned by the radius vector OM from the z-axis, and a is a constant to be determined, which we assume to be of the order of magnitude G/c^2. From (13), the potential at a point P inside will be given by:

$$\varphi_{\mathrm{P}} = \frac{c}{4\pi}\int_\sigma \left(\frac{e}{4\pi r^2} + \frac{e}{r}\, a\cos\theta\right)\left(\frac{1}{r} - \frac{\mathrm{G}}{2c^2}\frac{z_{\mathrm{P}} + z_{\mathrm{M}}}{r}\right) d\sigma \ ,$$

where the integration must be extended over the whole surface σ of the sphere. By neglecting terms of order greater than G/c^2 one obtains:

$$\begin{aligned}\varphi_{\mathrm{P}} = {} & \frac{ce}{16\pi^2 r^2}\int \frac{d\sigma}{r} + \frac{cea}{4\pi r}\int \frac{\cos\theta\, d\sigma}{r} \\ & - \frac{ce\mathrm{G}z_{\mathrm{P}}}{32\pi^2\mathrm{R}^2c^2}\int\frac{d\sigma}{r} - \frac{ce\mathrm{G}}{22\pi^2 R^2 c^2}\int \frac{z_{\mathrm{M}}d\sigma}{r} \ .\end{aligned} \tag{21}$$

However, since P is inside, one has:

$$\int \frac{d\sigma}{r} = 4\,\pi\,\mathrm{S} \ , \qquad \int \frac{\cos\theta}{r}\, d\sigma = \frac{4}{3}\,\pi\, z_{\mathrm{P}} \ , \qquad \int \frac{z_{\mathrm{M}}}{r}\, d\sigma = \frac{4}{3}\,\pi\,\mathrm{R}\, z_{\mathrm{P}} \ .$$

Thus one finds:

$$\varphi_{\mathrm{P}} = \frac{ce}{4\pi\mathrm{R}} + \frac{c}{3}\left(\frac{e}{\mathrm{R}}\, a - \frac{e}{2\pi\mathrm{R}c^3}\right) z_{\mathrm{P}} \ . \tag{22}$$

So if we require φ_{P} to be constant, we will have to set

$$a = \frac{1}{2\pi}\frac{\mathrm{G}}{c^2} \ .$$

By substituting this value into (20), one finds the following expression for the surface density:

$$\frac{e}{4\pi\mathrm{R}^2} + \left(1 + \frac{2\mathrm{G}}{c^2}\,\mathrm{R}\,\cos\theta\right) \ . \tag{23}$$

Therefore, the fact of being in a gravitational field produces a polarization of the sphere with moment

$$\frac{2}{3}\frac{\mathrm{G}}{c^2}\, e\,\mathrm{R}^2 \ .$$

PART 2

WEIGHT OF ELECTROMAGNETIC MASSES

§ 6.– Suppose we have a system of charges held by a rigid support in such a way that the δx, δy, δz of § 2 have to be given the form corresponding to the components of a rigid displacement. Leaving the rotational displacements till later, let us consider now the translational ones, that is to say assume that δx, δy, δz are arbitrary functions of time, but do not depend on x, y, z.

Then we will try to satisfy (9) by thinking of the potential φ_{P} at a generic point P as the sum of the potential given by (13) and one of the form (14). We will denote these two terms by $\varphi_{\mathrm{P}}{}'$ and $\varphi_{\mathrm{P}}{}''$, and suppose that the ratio between the derivatives of $\varphi_{\mathrm{P}}{}'$ and $\varphi_{\mathrm{P}}{}''$ with respect to any direction whatsoever is of order $l\,\mathrm{G}/c^2$, having decided to neglect the quadratic terms. Hence (9) can be written as:

$$\int dt \Big\{ \int_{\tau_{\mathrm{P}}} \delta x \left(\frac{\partial \varphi'}{\partial x} + \frac{\partial \varphi''}{\partial x} \right) \rho_{\mathrm{P}}\, d\tau_{\mathrm{P}} + \delta y \left(\frac{\partial \varphi'}{\partial y} + \frac{\partial \varphi''}{\partial y} \right) \rho_{\mathrm{P}}\, d\tau_{\mathrm{P}} + \delta z \left(\frac{\partial \varphi'}{\partial z} + \frac{\partial \varphi''}{\partial z} \right) \rho_{\mathrm{P}}\, d\tau_{\mathrm{P}} \Big\} = 0 \; .$$

Given that δx, δy, δz are arbitrary functions of time, independent of each other, this equation gives rise to the three equivalent equations:

$$\int_{\tau_{\mathrm{P}}} \left(\frac{\partial \varphi'}{\partial x} + \frac{\partial \varphi''}{\partial x} \right) \rho_{\mathrm{P}}\, d\tau_{\mathrm{P}} = \int_{\tau_{\mathrm{P}}} \left(\frac{\partial \varphi'}{\partial y} + \frac{\partial \varphi''}{\partial y} \right) \rho_{\mathrm{P}}\, d\tau_{\mathrm{P}} = \int_{\tau_{\mathrm{P}}} \left(\frac{\partial \varphi'}{\partial z} + \frac{\partial \varphi''}{\partial z} \right) \rho_{\mathrm{P}}\, d\tau_{\mathrm{P}} = 0 \; . \tag{24}$$

Now from the expression (13) for $\varphi_{\mathrm{P}}{}'$, by noting that

$$\frac{\partial r}{\partial x_{\mathrm{P}}} = \frac{x_{\mathrm{P}} - x_r}{r} \; ,$$

one immediately obtains:

$$\int_{\tau_{\mathrm{P}}} \frac{\partial \varphi'}{\partial x} \rho_{\mathrm{P}}\, d\tau_{\mathrm{P}} = -\frac{c}{4\pi} \int_{\tau_{\mathrm{P}}}\!\int_{\tau_{\mathrm{M}}} \rho_{\mathrm{P}} \rho_{\mathrm{M}}\, d\tau_{\mathrm{P}} d\tau_{\mathrm{M}} \left\{ \frac{x_{\mathrm{P}} - x_{\mathrm{M}}}{r^3} - \frac{\mathrm{G}}{2c^2} \frac{(x_{\mathrm{P}} - x_{\mathrm{M}})(z_{\mathrm{P}} + z_{\mathrm{M}})}{r^3} \right\} \; ,$$

where both integrals have to be performed over the region occupied by the charges. By interchanging P and M on the right hand side, which changes nothing, one obtains:

$$\int_{\tau_{\mathrm{P}}} \frac{\partial \varphi'}{\partial x} \rho_{\mathrm{P}}\, d\tau_{\mathrm{P}} = -\frac{c}{4\pi} \int_{\tau_{\mathrm{M}}}\!\int_{\tau_{\mathrm{P}}} \rho_{\mathrm{M}} \rho_{\mathrm{P}}\, d\tau_{\mathrm{M}} d\tau_{\mathrm{P}} \left\{ \frac{x_{\mathrm{M}} - x_{\mathrm{P}}}{r^3} - \frac{\mathrm{G}}{2c^2} \frac{(x_{\mathrm{M}} - x_{\mathrm{P}})(z_{\mathrm{M}} + z_{\mathrm{P}})}{r^3} \right\} \; ,$$

from which, by taking half the sum:

$$\int_{\tau_{\mathrm{P}}} \frac{\partial \varphi'}{\partial x} \rho_{\mathrm{P}}\, d\tau_{\mathrm{P}} = 0 \; . \tag{25}$$

In a completely analogous way:

$$\int_{\tau_{\mathrm{P}}} \frac{\partial\varphi'}{\partial y}\rho_{\mathrm{P}}\, d\tau_{\mathrm{P}} = 0 \ . \tag{26}$$

On the other hand, similarly:

$$\int_{\tau_{\mathrm{P}}} \frac{\partial\varphi'}{\partial z}\rho_{\mathrm{P}}\, d\tau_{\mathrm{P}} = -\frac{c}{4\pi}\int_{\tau_{\mathrm{P}}}\int_{\tau_{\mathrm{M}}} \rho_{\mathrm{P}}\rho_{\mathrm{M}}\, d\tau_{\mathrm{P}} d\tau_{\mathrm{M}} \left\{ \frac{z_{\mathrm{P}} - z_{\mathrm{M}}}{r^3} - \frac{\mathrm{G}}{2c^2}\frac{(z_{\mathrm{P}} - z_{\mathrm{M}})(z_{\mathrm{P}} + z_{\mathrm{M}})}{r^3} + \frac{\mathrm{G}}{2c^2}\frac{1}{r} \right\} ,$$

interchanging M and P:

$$\int_{\tau_{\mathrm{P}}} \frac{\partial\varphi'}{\partial z}\rho_{\mathrm{P}}\, d\tau_{\mathrm{P}} = -\frac{c}{4\pi}\int_{\tau_{\mathrm{M}}}\int_{\tau_{\mathrm{P}}} \rho_{\mathrm{M}}\rho_{\mathrm{P}}\, d\tau_{\mathrm{M}} d\tau_{\mathrm{P}} \left\{ \frac{z_{\mathrm{M}} - z_{\mathrm{P}}}{r^3} - \frac{\mathrm{G}}{2c^2}\frac{(z_{\mathrm{M}} - z_{\mathrm{P}})(z_{\mathrm{M}} + z_{\mathrm{P}})}{r^3} + \frac{\mathrm{G}}{2c^2}\frac{1}{r} \right\} ,$$

and by taking half the sum:

$$\int_{\tau_{\mathrm{P}}} \frac{\partial\varphi'}{\partial z}\rho_{\mathrm{P}}\, d\tau_{\mathrm{P}} = -\frac{c}{4\pi}\int_{\tau_{\mathrm{P}}}\int_{\tau_{\mathrm{M}}} \frac{\rho_{\mathrm{P}}\rho_{\mathrm{M}}}{r}\, d\tau_{\mathrm{P}} d\tau_{\mathrm{M}} = -\mathrm{G}\,\frac{\mathrm{U}}{c^2}\, e \ , \tag{27}$$

denoting by U the electrostatic energy of the system (apart from the gravitational correction terms). As a consequence of the assumptions made about the derivatives of φ_{P}'', we can certainly write, with our approximation:

$$\begin{cases} \displaystyle\int_{\tau} \frac{\partial\varphi''}{\partial x}\rho\, d\tau = -c\,\mathrm{E}_x^* e \ , \\ \displaystyle\int_{\tau} \frac{\partial\varphi''}{\partial y}\rho\, d\tau = -c\,\mathrm{E}_y^* e \ , \\ \displaystyle\int_{\tau} \frac{\partial\varphi''}{\partial z}\rho\, d\tau = -c\,\mathrm{E}_z^* e \ , \end{cases}$$

where $e = \int_{\tau} \rho\, d\tau$ indicates the total charge of the system. By substituting the expression just obtained into (24) one finds:

$$\begin{cases} e\,\mathrm{E}_x^* = 0 \ , \\ e\,\mathrm{E}_y^* = 0 \ , \\ e\,\mathrm{E}_z^* = -\mathrm{G}\,\dfrac{\mathrm{U}}{c^2} \ . \end{cases}$$

Our result is contained in these formulas. In fact, they tell us that in order to maintain our system in equilibrium an external field (E^*) is required to exert a force on the system given (in the first approximation) by $e\,\mathrm{E}^*$, which must be understood to balance the weight of the system, which is therefore given by $-e\,\mathrm{E}^*$, and so has components

$$0,\ 0,\ \mathrm{G}\,\frac{\mathrm{U}}{c^2} \ . \tag{28}$$

With this we conclude that *the weight of an electromagnetic mass always has the vertical direction and magnitude equal to the weight of a material mass* u/c^2.

§ 7.– In the preceding section we have taken δx, δy, δz to be the components of a translational displacement. If instead one takes the components of a virtual rotational displacement with the axis passing through the origin of the coordinates, namely setting

$$\delta x = qz - ry \; ; \qquad \delta y = rx - pz \; ; \qquad \delta z = py - qx \; , \tag{29}$$

the integral (9), apart from the contribution due to the external field φ'', becomes

$$\int d\tau \left\{ p \int_\tau \rho \left(y \frac{\partial \varphi}{\partial z} - z \frac{\partial \varphi}{\partial y} \right) d\tau + q \int_\tau \rho \left(z \frac{\partial \varphi}{\partial x} - x \frac{\partial \varphi}{\partial z} \right) d\tau \right.$$
$$\left. + r \int_\tau \rho \left(x \frac{\partial \varphi}{\partial y} - y \frac{\partial \varphi}{\partial x} \right) d\tau \right\} . \tag{30}$$

The integrals between curly brackets are easily evaluated using (13) through methods similar to that used in the previous section. They have the values:

$$-\frac{\mathrm{G}}{8\pi c} \iint \frac{y_\mathrm{P}}{r} \rho_\mathrm{P} \rho_\mathrm{M} \, d\tau_\mathrm{P} d\tau_\mathrm{M} \; ; \quad + \frac{\mathrm{G}}{8\pi c} \iint \frac{x_\mathrm{P}}{r} \rho_\mathrm{P} \rho_\mathrm{M} \, d\tau_\mathrm{P} d\tau_\mathrm{M} \; ; \quad 0 \; . \tag{31}$$

By taking as the origin the point O′ defined by the point O and the vector

$$\mathrm{O}' - \mathrm{O} = \frac{1}{2\mathrm{U}} \iint \frac{\mathrm{P} - \mathrm{O}}{r} \rho_\mathrm{P} \rho_\mathrm{M} \, d\tau_\mathrm{P} d\tau_\mathrm{M} \; ,$$

one sees immediately that the three integrals vanish for *any* orientation of the system about O′. As a consequence, with respect to the new origin, the integral (9) is identically zero, namely the moment of the weight with respect to O′ is zero for any orientation of the system; thus O′ enjoys the properties of the center of gravity.

Pisa, March 1921.