

### 1) On the dynamics of a rigid system of electric charges on translational motion

*“Sulla dinamica di un sistema rigido di cariche  
elettriche in moto traslatorio,”  
Nuovo Cimento 22, 199–207 (1921).*

§ 1. – When a rigid system of electric charges moves arbitrarily, the electric field it generates is different from that which Coulomb’s law would predict. Now, the electric field produced by the entire system exerts some forces on each element of charge of the system. The resultant of these forces, namely the resultant of the internal electric forces, would of course be identically zero if Coulomb’s law were valid, but it no longer is, however, at least in general, when the system moves, since in such a case that law is no longer valid.

This resultant gives the electromagnetic inertial reaction, and the aim of the present work is precisely its evaluation in the case of an arbitrary system in translational motion. In the case in which the system is a spherical distribution of surface electricity, as it is assumed in most electronic models, it is known that one finds<sup>1</sup> that such a resultant, at least in the first approximation, is given by

$$-\frac{2e^2}{3Rc^3}\mathbf{\Gamma} + \frac{2e^2}{3c^2}\dot{\mathbf{\Gamma}}, \quad (1)$$

where  $e$ ,  $R$  denote the total charge and the radius of the system,  $c$  is the speed of light,  $\mathbf{\Gamma}$  and  $\dot{\mathbf{\Gamma}}$  are the acceleration and its derivative with respect to time. For quasi-stationary motions the second term of (1) becomes negligible, so that (1) reduces to

$$-m\mathbf{\Gamma}, \quad (2)$$

where  $m$  is the electromagnetic mass.

In § 2 one finds the generalization of (1) to the case of any system, referring for example to molecular models, always assuming that the velocity is negligible with respect to the speed of light. If  $F_i$  ( $i = 1, 2, 3$ ) are the components of the resultant in question, one finds

$$F_i = -\sum_k m_{ik}\Gamma_k + \sum_k \sigma_{ik}\dot{\Gamma}_k, \quad (3)$$

where  $m_{ik}$ ,  $\sigma_{ik}$  are some quantities depending on the properties of the system. Therefore one can no longer refer to a scalar electromagnetic mass, but instead in its place one introduces the tensor  $m_{ik}$ .

§ 3 is devoted to the dynamical study of the law for quasi-stationary motions:

$$K_i = \sum_k m_{ik}\Gamma_k, \quad (4)$$

<sup>1</sup>RICHARDSON, *Electron Theory of Matter*, Chapter XIII. The difference between my formulas and those of Richardson is due to the fact that he adopts Heaviside units.

where  $K_i$  are the components of the external force. One shows that with such a law the fundamental kinetic energy theorem and Hamilton's principle continue to hold.

Finally in § 4 the law (4) for quasi-stationary motions, which holds only for small velocities, is generalized to the case of arbitrary velocity using special relativity.

With this the study of electromagnetic masses as inertial masses will be complete. In a forthcoming paper I will consider electromagnetic masses as masses endowed with weight from the point of view of the general theory of relativity.

§ 2. – It is known<sup>2</sup> that the electric force due to a point charge 1 in motion is the sum of two forces, which by assuming the ratio between the velocity  $v$  of the particle and the speed  $c$  of light to be negligible, are: the first one,  $\mathbf{E}_1$ , the force given by Coulomb's law; the second one  $\mathbf{E}_2$  has the form

$$\mathbf{E}_2 = \frac{\mathbf{\Gamma}^* \cdot \mathbf{a}}{c^2 r} \mathbf{a} - \frac{1}{c^2 r} \mathbf{\Gamma}^* . \quad (5)$$

In this formula  $r$  represents the distance between the particle M and the point P at which the force is calculated and  $\mathbf{a}$  is a vector of magnitude 1 and orientation MP. Finally  $\mathbf{\Gamma}^*$  is the acceleration of the particle at the time  $t - (r/c)$ . If instead of the charge 1 at M there is the charge  $\rho d\tau$  ( $\rho$  is the electric density,  $d\tau$  the volume element), the force at P will be  $\rho d\tau (\mathbf{E}_1 + \mathbf{E}_2)$ , so that the force exerted at P by all charges will be  $\int_{\tau} \rho (\mathbf{E}_1 + \mathbf{E}_2) d\tau$ , where the integration must be extended over the whole space  $\tau$  occupied by charges. Now if at the point P there is the charge  $\rho' d\tau'$ , the force  $\rho' d\tau' \int_{\tau} \rho (\mathbf{E}_1 + \mathbf{E}_2) d\tau$  is acting on it.

The force acting on the entire system is therefore

$$\mathbf{F} = \iint \rho \rho' (\mathbf{E}_1 + \mathbf{E}_2) d\tau d\tau' ,$$

where the two integrations must be extended over the same domain. On the other hand, one clearly has

$$\iint \rho \rho' \mathbf{E}_1 d\tau d\tau' = 0 ,$$

so that

$$\mathbf{F} = \iint \rho \rho' \mathbf{E}_2 d\tau d\tau' .$$

If we now denote by  $\mathbf{\Gamma}$  and  $\dot{\mathbf{\Gamma}}$  the acceleration and its derivative with respect to time, at the time  $t$ , if  $r$  is small enough, we can set

$$\mathbf{\Gamma}^* = \mathbf{\Gamma} - \frac{r}{c} \dot{\mathbf{\Gamma}} ,$$

obtaining finally

$$\mathbf{F} = \iint \left( \frac{\mathbf{\Gamma} \cdot \mathbf{a}}{c^2 r} \mathbf{a} - \frac{\mathbf{\Gamma}}{c^2 r} \right) \rho \rho' d\tau d\tau' + \iint \left( \frac{\dot{\mathbf{\Gamma}} \cdot \mathbf{a}}{c^3} \mathbf{a} - \frac{\dot{\mathbf{\Gamma}}}{c^3} \right) \rho \rho' d\tau d\tau' . \quad (6)$$

<sup>2</sup>See, e.g., RICHARDSON, op. cit.

We denote orthogonal Cartesian coordinates by  $x_1, x_2, x_3$ , and let  $(x_i)$  be the coordinates of M,  $(x'_i)$  those of P. The components of  $\mathbf{a}$  are  $a_i = \frac{x'_i - x_i}{r}$ . Writing (6) in scalar form, one thus obtains

$$F_i = - \sum_k m_{ik} \Gamma_k + \sum_k \sigma_{ik} \dot{\Gamma}_k, \quad (7)$$

noting that, under the assumption of translational motion,  $\Gamma_i$  and  $\dot{\Gamma}_i$  are constant when the integration is performed.

Here one has set:

$$\left\{ \begin{array}{l} m_{ii} = \frac{2U}{c^2} - \iint \frac{\rho\rho'(x'_i - x_i)^2}{c^2 r^3} d\tau d\tau', \\ m_{ik} = m_{ki} = - \iint \frac{\rho\rho'(x'_i - x_i)(x'_k - x_k)}{c^2 r^3} d\tau d\tau', \quad i \neq k, \end{array} \right. \quad (8)$$

$$\left\{ \begin{array}{l} \sigma_{ii} = \frac{e^2}{c^3} - \iint \frac{\rho\rho'(x'_i - x_i)^2}{c^3 r^2} d\tau d\tau', \\ \sigma_{ik} = \sigma_{ki} = - \iint \frac{\rho\rho'(x'_i - x_i)(x'_k - x_k)}{c^3 r^2} d\tau d\tau', \quad i \neq k. \end{array} \right. \quad (9)$$

In these formulae  $U$  represents the electrostatic energy of the system =  $\frac{1}{2} \iint \frac{\rho\rho'}{r} d\tau d\tau'$ , and  $e$  the total electric charge =  $\int \rho d\tau = \int \rho' d\tau'$ .

From the expressions (8), (9) it immediately follows that if the axes  $(x_i)$  are substituted by others  $(y_i)$  using the orthogonal substitution

$$y_i = \sum_k \alpha_{ik} x_k,$$

the  $m_{ik}$  and  $\sigma_{ik}$  corresponding to the new axes are given by:

$$m'_{ik} = \sum_{rs} \alpha_{ir} \alpha_{ks} m_{ik},$$

$$\sigma'_{ik} = \sum_{rs} \alpha_{ir} \alpha_{ks} \sigma_{ik}.$$

Hence both  $m_{ik}$  and  $\sigma_{ik}$  are symmetric covariant tensors. Each of them will have three orthogonal principal directions such that, taking the axes to be parallel to them, one has either  $m_{ik} = 0$  or  $\sigma_{ik} = 0$  when  $i \neq k$ .

The principal axes of tensors  $m$ ,  $\sigma$ , however, will be different in general. If the case that the system has spherical symmetry one can do the integrations (8) and (9), since instead of  $\frac{(x'_i - x_i)(x'_k - x_k)}{r^2}$  one can put the average value of this expression over all possible directions MP, since in this case to the two points MP correspond an infinite number of pairs which differ only by orientation. Now, this average value is given by  $\frac{2\pi}{4\pi} \int_0^\pi \cos^2 \theta \sin \theta d\theta$  if  $i = k$  but it is zero if instead  $i \neq k$ .

So one then has

$$m_{11} = m_{22} = m_{33} = \frac{4U}{3c^2} ; \quad m_{23} = m_{31} = m_{12} = 0 ;$$

$$\sigma_{11} = \sigma_{22} = \sigma_{33} = \frac{2e^2}{3c^3} ; \quad \sigma_{23} = \sigma_{31} = \sigma_{12} = 0 .$$

By substituting these values into (7), one obtains the well known formulas if the system consists of a homogeneous spherical layer.

§ 3. – Returning to the general case, we note that for quasi-stationary motions (5) can be replaced by:

$$F_i = - \sum_k m_{ik} \Gamma_k .$$

If one thinks of an external force ( $X_i$ ) acting on the system, the total force will be ( $X_i + F_i$ ). If one now supposes that the system has no material mass one must have  $X_i + F_i = 0$ , and so

$$X_i = \sum_k m_{ik} \Gamma_k . \quad (10)$$

It is easy to show how with the law (10) the principle of the kinetic energy theorem and of Hamilton's principle are preserved. In fact, denoting the velocity by  $V \equiv (V_1, V_2, V_3)$  and multiplying (10) by  $V_i$ , then summing with respect to  $i$  one obtains

$$\sum_i X_i V_i = \sum_{ik} m_{ik} V_k \frac{dV_i}{dt} .$$

Interchanging  $i$  and  $k$  in the second sum, and noting that  $m_{ik} = m_{ki}$

$$\sum_i X_i V_i = \sum_{ik} m_{ik} V_i \frac{dV_k}{dt} ,$$

and summing

$$2 \sum_i X_i V_i = \sum_{ik} m_{ik} \left( V_i \frac{dV_k}{dt} + V_k \frac{dV_i}{dt} \right) = \frac{d}{dt} \sum_{ik} m_{ik} V_i V_k .$$

The first left hand side is twice the potential  $P$  of the external forces. Thus one has

$$P = \frac{dT}{dt} , \quad \text{where} \quad T = \frac{1}{2} \sum_{ik} m_{ik} V_i V_k . \quad (11)$$

Multiplying, instead, the two sides of (10) by  $\delta x$ , and then summing, one similarly gets

$$\begin{aligned} \sum_i X_i \delta x_i &= \frac{1}{2} \sum_{ik} m_{ik} \left( \frac{d^2 x_k}{dt^2} \delta x_i + \frac{d^2 x_i}{dt^2} \delta x_k \right) \\ &= \frac{d}{dt} \left\{ \frac{1}{2} \sum_{ik} m_{ik} (\dot{x}_k \delta x_i + \dot{x}_i \delta x_k) \right\} - \frac{1}{2} \sum_{ik} m_{ik} (\dot{x}_k \delta \dot{x}_i + \dot{x}_i \delta \dot{x}_k) \\ &= \frac{d}{dt} \left\{ \frac{1}{2} \sum_{ik} m_{ik} (\dot{x}_k \delta x_i + \dot{x}_i \delta x_k) \right\} - \delta T . \end{aligned}$$

Multiplying by  $dt$  and integrating between two limits  $t'$ ,  $t''$  at which the variations  $\delta x_i$  are assumed to be zero, one obtains

$$\int_{t'}^{t''} \left( \delta T + \sum_i X_i \delta x_i \right) = 0, \quad (12)$$

expressing Hamilton's principle.

If one refers to the principal axes of the tensor  $m_{ik}$  instead of arbitrary ones, (10) takes the simple form:

$$X_i = m_{ii} \Gamma_i. \quad (13)$$

§ 4. – This formula holds only if  $V/c$  is negligible. To generalize it to an arbitrary velocity let us denote by  $S \equiv (x_1, x_2, x_3, t)$  the indicated reference frame and by  $S^* \equiv (x, y, z, t)$  a frame fixed with respect to  $S$  with the  $x$ -axis oriented along the velocity of the system at a certain fixed but generic time  $\bar{t}$ , and finally let  $S' \equiv (x', y', z', t')$  be a system with spatial axes parallel to  $xyz$  which moves uniformly with respect to  $S^*$  with velocity equal to that of the moving one at time  $\bar{t}$ , whose magnitude is  $v$ . One will have

$$t' = \beta \left( t - \frac{v}{c^2} x \right); \quad x' = \beta (x - vt); \quad y' = y; \quad z' = z; \quad \beta = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad (14)$$

where, once  $\bar{t}$  is fixed,  $v$  and hence  $\beta$  are constant.

Let us assume that the forces acting on our system are due to an external electromagnetic field  $(\mathbf{E}, \mathbf{H})$ ; since at the instant  $t$  the system has velocity zero with respect to  $S'$ , (10) will hold for it, and so one will therefore have, with an obvious meaning for the symbols:

$$\begin{aligned} e E'_x &= m_{xx} \Gamma'_x + m_{xy} \Gamma'_y + m_{xz} \Gamma'_z \\ e E'_y &= m_{yx} \Gamma'_x + m_{yy} \Gamma'_y + m_{yz} \Gamma'_z \\ e E'_z &= m_{zx} \Gamma'_x + m_{zy} \Gamma'_y + m_{zz} \Gamma'_z. \end{aligned}$$

But one has

$$e E'_x = e E_x, \quad e E'_y = e \beta \left( E_y - \frac{v}{c} H_z \right), \quad e E'_z = e \beta \left( E_z + \frac{v}{c} H_y \right).$$

Therefore setting

$$\mathbf{k} = e \left( \mathbf{E} + \frac{1}{c} \mathbf{V} \times \mathbf{H} \right), \quad (15)$$

one finds

$$e E'_x = e E_x, \quad e E'_y = e \beta k_y, \quad e E'_z = e \beta k_z.$$

On the other hand,

$$\Gamma'_x = \frac{d^2 x' dt' - d^2 t' dx'}{dt'^3},$$

but at time  $\bar{t}$ ,  $\frac{dx'}{dt'} = 0$ , hence  $\Gamma'_x = \frac{d^2x'}{dt'^2}$ . Taking  $t$  as the independent variable, and noting that  $\frac{dx}{dt} = v$ , then  $\Gamma'_x = \beta^3\Gamma_x$ . Analogously,  $\Gamma'_y = \beta^2\Gamma_y$  and  $\Gamma'_z = \beta^2\Gamma_z$ . Substituting

$$\begin{cases} k_x = m_{xx}\beta^3\ddot{x} + m_{xy}\beta^2\ddot{y} + m_{xz}\beta^2\ddot{z} \\ k_y = m_{yx}\beta^3\ddot{x} + m_{yy}\beta^2\ddot{y} + m_{yz}\beta^2\ddot{z} \\ k_z = m_{zx}\beta^3\ddot{x} + m_{zy}\beta^2\ddot{y} + m_{zz}\beta^2\ddot{z} . \end{cases} \quad (16)$$

Denoting by  $\alpha_{xi}$  the cosine of the angle between the  $x$ -axis and the  $x_i$ -axis, one has

$$k_i = \alpha_{xi}k_x + \alpha_{yi}k_y + \alpha_{zi}k_z .$$

On the other hand, since  $m_{io}$  is covariant, one has for example

$$m_{xy} = \sum_r m_{rr}\alpha_{xr}\alpha_{yr} .$$

Analogously

$$\ddot{x} = \sum_j \ddot{x}_j\alpha_{xj} .$$

Multiplying then (16) by  $\alpha_{xi}$ ,  $\alpha_{yi}$ ,  $\alpha_{zi}$  and summing, one finds

$$k_i = \sum_{rj} m_{rr}\ddot{x}_j \begin{bmatrix} \beta^3\alpha_{xr}^2\alpha_{xj}\alpha_{xi} + \beta^2\alpha_{xr}\alpha_{yr}\alpha_{yj}\alpha_{xi} + \beta^2\alpha_{xr}\alpha_{zr}\alpha_{zj}\alpha_{xi} \\ + \beta^2\alpha_{yr}\alpha_{xr}\alpha_{xj}\alpha_{yi} + \beta\alpha_{yr}^2\alpha_{yj}\alpha_{yi} + \beta\alpha_{yr}\alpha_{zr}\alpha_{zj}\alpha_{yi} \\ + \beta^2\alpha_{zr}\alpha_{xr}\alpha_{xj}\alpha_{zi} + \beta\alpha_{zr}\alpha_{yr}\alpha_{yj}\alpha_{zi} + \beta\alpha_{zr}^2\alpha_{zj}\alpha_{zi} \end{bmatrix} .$$

But one has  $\alpha_{xi} = \frac{\dot{x}_i}{v}$ . Taking into account the relations between the  $\alpha$ 's, one finally finds the sought after generalization of (13)

$$k_i = \beta \sum_{rj} \ddot{x}_j m_{rr} \left\{ (\beta - 1)^2 \frac{\dot{x}_i \dot{x}_j \dot{x}_r^2}{v^4} + (\beta - 1) \left[ (jr) \frac{\dot{x}_i \dot{x}_r}{v^2} + (ir) \frac{\dot{x}_j \dot{x}_r}{v^2} \right] + (ir)(jr) \right\} , \quad (17)$$

where

$$(jr) = 1 , \quad \text{if } j = r ; \quad (jr) = 0 , \quad \text{if } j \neq r .$$

In the case of spherical symmetry, setting  $m_{11} = m_{22} = m_{33} = m$ , one can evaluate the sum in (17), finding:

$$k_i = \beta m \ddot{x}_i + m \beta (\beta^2 - 1) \frac{\dot{x}_i}{v^2} \sum_j \dot{x}_j \ddot{x}_j ,$$

from which, recalling that

$$\beta = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}},$$

one recovers the well known formula of electronic dynamics

$$k_i = \frac{d}{dt} \frac{m\dot{x}_i}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

Pisa, January 1921.