Fermi's Theory of Beta Decay*

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A complete English translation is given of the classic Enrico Fermi paper on beta decay published in *Zeitschrift für Physik* in 1934.

INTRODUCTION

In 1934, Enrico Fermi, then a professor of theoretical physics at the University of Rome, Italy, proposed his clear and simple description of β decay. He assumed the existence of the neutrino which Pauli had suggested to preserve the principle of conservation of energy, and he treated the ejection of electrons and neutrinos from a nucleus by a method similar to the radiation theory of photon emission from atoms. Fermi derived quantitative expressions for the lifetime of β decay, as well as for the shape of the β-ray emission spectrum.

Fermi's theory, aside from bolstering Pauli's proposal of the neutrino, has a special significance in the history of modern physics. One must remember that only the naturally occurring β emitters were known at the time the theory was proposed. Later when positron decay was discovered, the process was easily incorporated within Fermi's original framework. On the basis of his theory, the capture of an orbital electron by a nucleus was predicted and eventually observed. With time much experimental data has accumulated. Although peculiarities have been observed many times in β decay, Fermi's theory always has been equal to the challenge.

The consequences of the Fermi theory are vast. For example, β spectroscopy was established as a powerful tool for the study of nuclear structure. But perhaps the most influential aspect of this work of Fermi is that his particular form of the β interaction established a pattern which has been appropriate for the study of other types of interactions. It was the first successful theory of the creation and annihilation of material particles. Previously, only photons had been known to be created and destroyed.

To appreciate the impact produced by Fermi's theory of β decay on modern physics, one may note that it is rather amazing what varieties of observed phenomena (and what thicknesses of the *Physical Review*) are based on his one paper on the subject. For example, the experiment proposed by Yang and Lee in 1956 to test conservation of parity, involved the properties of β decay of ^{60}Co.

With his paper on β decay, Fermi brought to a close his purely theoretical studies and became an experimentalist. Thus this paper on β decay, which in its main outlines is still considered correct, is of significant importance not only to the development of modern physics, but also to the memory of Enrico Fermi. A complete translation is provided below.

Attempt at a Theory of β Rays[1] I

A quantitative theory of β decay is proposed, in which the existence of the neutrino is assumed. The emission of electrons and neutrinos from a nucleus by β decay will be treated by a method similar to that for the emission of a light quantum from an excited atom in radiation theory. Formulas for the lifetime and for the form of the continuous β-ray emission spectrum are derived and compared with experiment.

I. BASIC ASSUMPTIONS OF THE THEORY

Two well-known difficulties are encountered when one tries to construct a theory of nuclear electrons and of β emission. The first arises from the continuous β-ray spectrum. If the law of conservation of energy is to remain valid, one must assume that a fraction of the energy set free in β decay has, up to now, escaped our means of observation. For example, one can assume in

* E. Fermi, Z. Physik **88**, 161 (1934). Springer-Verlag Berlin, Heidelberg, New York, has given permission to publish this translation.

[1] Compare the preliminary article: La Ricerca Scientifica **2**, No. 12 (1933). (In addition to this reference in the title, a note states that the paper was received on January 16, 1934. The paper lists the author as E. Fermi in Rome.)

conformity with the proposal of Pauli, that not only an electron but also a new particle, the so-called "neutrino" (mass of the order of or smaller than the mass of the electron and no electric charge) is emitted in β decay. We base the proposed theory on the hypothesis of the neutrino.

There is an additional difficulty for the theory of nuclear electrons in that the present relativistic theories of lightweight particles (electrons or neutrinos) are not capable of explaining, in a satisfactory manner, how such particles can be bound in orbits of nuclear dimensions.

It seems appropriate, therefore, to assume with Heisenberg,[2] that a nucleus consists only of heavy particles—the protons and neutrons. Nevertheless, in order to understand that β emission is possible, we want to try to construct a theory of the emission of lightweight particles from the nucleus in analogy with the theory of emission of light quanta from an excited atom by the usual radiation process. In radiation theory, the total number of light quanta is not constant. Light quanta are created when they are emitted from an atom, and are annihilated when they are absorbed. In analogy with this, we wish to base the β-ray theory on the following assumptions:

(a) The total number of electrons, as well as neutrinos, is not necessarily constant. Electrons (or neutrinos) can be created or annihilated. This possibility, however, is not analogous to the creation or annihilation of an electron–positron pair. If one interprets a positron as a Dirac "hole," one is able to understand that this latter process is simply a quantum jump of an electron between a state of negative energy and a state with positive energy, with conservation of the total number (infinitely great) of electrons.

(b) The heavy particles (neutrons, protons) may be treated (as by Heisenberg) as two internal quantum states of the heavy particle. We formulate this by the introduction of an intrinsic coordinate, ρ, of the heavy particle which can assume only two values: $\rho=1$ if the particle is a neutron, $\rho=-1$ if the particle is a proton.[3]

[2] W. Heisenberg, Z. Physik **77,** 1 (1932).

[3] Translator's note: The choice of the sign is clearly arbitrary. However, Fermi's choice is opposite to the one which is more commonly used today. See F. Mandl, *Introduction to Quantum Field Theory* (Interscience Publishers, Inc., New York, 1961), p. 79.

(c) The Hamiltonian function of the system consisting of heavy and lightweight particles must be so chosen that each transition from a neutron to a proton is associated with the creation of an electron and a neutrino. The reverse process (change of a proton into a neutron) must be associated with the annihilation of an electron and a neutrino. Note that by this, conservation of charge is assured.

II. THE OPERATORS WHICH APPEAR IN THE THEORY

In accordance with these three requirements, a mathematical formalism of the theory can be developed most easily with the help of the Dirac–Jordan–Klein[4] method of "second quantization." Therefore, we regard the probability amplitudes of the electrons and the neutrinos, ψ and φ, as well as their complex conjugates ψ^* and φ^*, as operators. On the other hand, for the description of the heavy particles, we use the usual representation in configuration space, where of course ρ must be included as a coordinate in the calculation.

We first introduce two operators, Q and Q^*, which operate like the linear substitutions,

$$Q=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}, \qquad Q^*=\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}, \tag{1}$$

on the functions of the two-valued variable, ρ. It is easily seen that Q represents a transition from a proton to a neutron, and Q^* a transition from a neutron to a proton.

As is well known, the meaning of the probability amplitudes ψ and φ, when interpreted as operators, is the following: let $\psi_1\psi_2\cdots\psi_s\cdots$ be a system of individual quantum states for the electrons. One then can write

$$\psi=\sum_s \psi_s a_s, \qquad \psi^*=\sum_s \psi_s^* a_s^*. \tag{2}$$

The amplitudes a_s and the complex conjugate quantities a_s^* are operators which operate on the functions of the occupation numbers $N_1, N_2, \cdots, N_s, \cdots$, of the individual quantum states. When the Pauli principle is applicable, N_s may take on

[4] For example, compare P. Jordan and O. Klein, Z. Physik **45**, 751 (1927); W. Heisenberg, Ann. Physik **10**, 888 (1931).

only the two values, 0 and 1. The operators a_s and a_s^* are then defined in the following manner[5]:

$$a_s\Psi(N_1N_2\cdots N_s\cdots)=(-1)^{N_1+N_2+\cdots+N_{s-1}}(1-N_s)\Psi(N_1N_2\cdots 1-N_s\cdots),$$
$$a_s^*\Psi(N_1N_2\cdots N_s\cdots)=(-1)^{N_1+N_2+\cdots N_{s-1}}N_s\Psi(N_1N_2\cdots 1-N_s\cdots). \quad (3)$$

The operator a_s^* represents the creation and the operator a_s the annihilation of an electron in a quantum state s.

Corresponding to Eqs. (2), one writes for the neutrinos:

$$\varphi=\sum_\sigma \varphi_\sigma b_\sigma, \qquad \varphi^*=\sum_\sigma \varphi_\sigma^* b_\sigma^*. \quad (4)$$

The complex conjugate quantities, b_σ, b_σ^*, are operators which operate on the functions of the occupation numbers M_1, M_2, $\cdots$, M_σ, $\cdots$ of the individual quantum states φ_1, φ_2, $\cdots$, φ_σ, $\cdots$ of the neutrinos. If one assumes that the Pauli principle is also true for the neutrinos, the numbers M_σ may have only the two values 0, 1. Further,[6]

$$b_\sigma\Phi(M_1M_2\cdots M_\sigma\cdots)=(-1)^{M_1+M_2+\cdots+M_{\sigma-1}}(1-M_\sigma)\Phi(M_1M_2\cdots 1-M_\sigma\cdots),$$
$$b_\sigma^*\Phi(M_1M_2\cdots M_\sigma\cdots)=(-1)^{M_1+M_2+\cdots+M_{\sigma-1}}M_\sigma\Phi(M_1M_2). \quad (5)$$

The operators b_σ and b_σ^* represent respectively the annihilation and the creation of a neutrino in a quantum state σ.

III. CONSTRUCTION OF THE HAMILTONIAN FUNCTION

The energy of the entire system consisting of heavy particles and lightweight particles, is the sum of the energies $H_{\text{h.p.}}$ of the heavy particles plus $H_{\text{l.p.}}$ of the lightweight particles plus the interaction energy H between the heavy particles and the lightweight particles. We write the first term in the form,

$$H_{\text{h.p.}}=\tfrac{1}{2}(1+\rho)N+\tfrac{1}{2}(1-\rho)P, \quad (6)$$

since for the present we consider only a single heavy particle. N and P represent the energy operators of the neutron and of the proton respectively. For $\rho=1$ (neutron), Eq. (6) reduces to N: For $\rho=-1$ (proton), Eq. (6) reduces to P.

The energy $H_{\text{l.p.}}$ of the lightweight particles assumes its simplest form when one takes stationary states for ψ_1, ψ_2, $\cdots$ ψ_l $\cdots$, the quantum states of the electrons, and φ_1, φ_2 $\cdots$, φ_σ $\cdots$, the quantum states of the neutrinos. In doing so, one probably should chose the stationary states of the electrons in the Coulomb field of the nucleus, because of the electron screening. One can simply assume plane deBroglie waves for the neutrinos, since the force acting on the neutrinos probably plays no significant role. Let H_1, H_2, H_3, $\cdots$, $H_s\cdots$ and K_1, K_2, $\cdots$ $K_\sigma\cdots$ be the respective energies of the electrons and the neutrinos. Then we have

$$H_{\text{l.p.}}=\sum_s H_sN_s+\sum_\sigma K_\sigma M_\sigma. \quad (7)$$

The interaction energy must yet be written. First, this consists of the Coulomb energy between the proton and the electrons. For heavy nuclei, however, the attraction by a single proton plays only a subordinate role[7] and, in any case, does not contribute to the process of β decay. For the sake of simplicity, we do not consider this term further. On the other hand, we must add a member to the Hamiltonian function which fulfills the conditions of (c) in Sec. I.

According to Sec. II, the necessary term to couple the change of a proton into a neutron with the annihilation of an electron and a neutrino, has the form,

$$Qa_sb_\sigma. \quad (8)$$

On the other hand, the complex conjugate operator,

$$Q^*a_s^*b_\sigma^*, \quad (8')$$

couples the reverse process (the change of a neutron into a proton, creation of an electron and a neutrino).

[5] Translator's note: a_s and a_s^* are reversed on the left side of Eq. (3). From the right sides of Eq. (3) it is obvious that the upper equation is the equation of creation and the lower equation that of annihilation.

[6] Translator's note: Here, as in Eq. (3), b_σ and b_σ^* are reversed on the left side of Eq. (5).

[7] Naturally, the Coulomb potential of the many other protons must be regarded as a static field.

An interaction term which fulfills requirement (c), therefore, can be written in the following form,

$$H = Q \sum_{s,\sigma} c_{s\sigma} a_s b_\sigma + Q^* \sum_{s,\sigma} c_{s\sigma}^* a_s^* b_\sigma^*, \qquad (9)$$

where $c_{s\sigma}$ and $c_{s\sigma}^*$ represent quantities which may depend on the coordinates, momenta, etc. of the heavy particles.

For a more exact statement of H, one is guided by criteria of simplicity. An essential limitation on the freedom of choice of H is fixed by conservation of momentum, as well as by the requirement that Eq. (9) must remain invariant under a rotation or a translation of the spatial coordinates.

If we first neglect relativistic corrections and spin interaction, the simplest choice for Eq. (9) would probably be the following:

$$H = g[Q\psi(x)\varphi(x) + Q^*\psi^*(x)\varphi^*(x)], \qquad (10)$$

where g represents a constant with dimensions l^5mt^{-2}, and x represents the coordinates of the heavy particles. ψ, φ, ψ^*, φ^* are given by Eqs. (2) and (4), and are to be evaluated at the position (x, y, z) of the heavy particles.

Equation (10) in no way represents the only possible choice for H. Any scalar expression, as for example,

$$L(p)\psi(x)M(p)\varphi(x)N(p) + \text{c.c.},$$

where $L(p)$, $M(p)$, $N(p)$ represent suitable functions of the momentum of the heavy particles, would be just as good. However, since the conclusions from Eq. (10) up to now appear to be in harmony with experience, it probably is better to limit oneself to the simplest choice.

It is essential, however, to generalize Eq. (10) in such a way that one can at least treat the light-weight particles relativistically. Naturally, when making this generalization, a certain arbitrariness cannot be excluded. The simplest solution of this problem might be the following: Relativistically the four Dirac functions ψ_1, ψ_2, ψ_3, ψ_4, and φ_1, φ_2, φ_3, φ_4, take the place of ψ and φ. Now, we consider the 16 independent bilinear combinations of ψ_1, ψ_2, ψ_3, ψ_4, and φ_1, φ_2, φ_3, φ_4. Under a Lorentz transformation of the coordinates, these 16 quantities undergo a linear transformation, which is a representation of order 16 of the Lorentz group. This representation splits into various simpler representations. In particular, the four bilinear combinations:

$$\begin{aligned} A_0 &= -\psi_1\varphi_2 + \psi_2\varphi_1 + \psi_3\varphi_4 - \psi_4\varphi_3, \\ A_1 &= \psi_1\varphi_3 - \psi_2\varphi_4 - \psi_3\varphi_1 + \psi_4\varphi_2, \\ A_2 &= i\psi_1\varphi_3 + i\psi_2\varphi_4 - i\psi_3\varphi_1 - i\psi_4\varphi_2, \\ A_3 &= -\psi_1\varphi_4 - \psi_2\varphi_3 + \psi_3\varphi_2 + \psi_4\varphi_1, \end{aligned} \qquad (11)$$

transform as the components of a polar four-vector; and hence, as the components of the electromagnetic four-potential. The next thing which must be done is to introduce the quantity $g(QA_i + Q^*A_i^*)$ in the Hamiltonian function of the heavy particles in a manner analogous to the placing of the components of the four-potential.

Here, we meet a difficulty originating in the fact that the relativistic wave equation of the heavy particles is unknown. If the velocity of the heavy particles is small relative to c, one can limit oneself to the term analogous to eV, where V is the scalar potential, and write

$$\begin{aligned} H = g[&Q(-\psi_1\varphi_2 + \psi_2\varphi_1 + \psi_3\varphi_4 - \psi_4\varphi_3) \\ &+ Q^*(-\psi_1^*\varphi_2^* + \psi_2^*\varphi_1^* + \psi_3^*\varphi_4^* - \psi_4^*\varphi_3^*)]. \end{aligned} \qquad (12)$$

In addition to this term, other terms of the order v/c should be added. Since in nuclei the velocities of the neutrons and protons usually are small relative to the velocity of light, we, for the present, neglect these terms. (See, however, Sec. IX.)

Equation (12) can be abbreviated symbolically in the following manner:

$$H = g[Q\tilde{\psi}^*\delta\varphi + Q^*\tilde{\psi}\delta\varphi^*], \qquad (13)$$

where ψ and φ are written as vertical column matrices. The $\sim$ sign represents the Hermetian conjugate matrix; and

$$\delta = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}. \qquad (14)$$

With these designations, one obtains by comparison with Eq. (9),

$$c_{s\sigma} = g\tilde{\psi}_s^*\delta\varphi_\sigma,$$

and

$$c_{s\sigma}^* = g\tilde{\psi}_s\delta\varphi_\sigma^*, \qquad (15)$$

where ψ_s and φ_s,[8] represent the normalized four-component eigenfunctions of the states s (of the electron) and σ (of the neutrino). ψ and σ in Eq. (15) are to be evaluated at the positions of the heavy particle, and therefore, are functions of (x, y, z).

IV. THE PERTURBATION MATRIX

With the help of the constructed Hamiltonian function, the theory of β decay can be developed in complete analogy with radiation theory. In the latter case, as is well known, the Hamiltonian consists of the sum of the energy of the atom plus the energy of the pure radiation field plus the interaction energy. This last term is considered as a perturbation to the other two. In our case, as an analogy to this, we treat the sum

$$H_{\text{h.p.}} + H_{\text{l.p.}}, \qquad (16)$$

as the unperturbed Hamiltonian, to which we must add perturbation represented by the interaction term, Eq. (13).

The quantum states of the unperturbed system can be denoted in the following manner:

$$(\rho, n, N_1N_2\cdots N_s\cdots M_1M_2\cdots M_\sigma\cdots), \qquad (17)$$

where the first number, ρ, assumes one of the two values $+1$ or -1, and indicates whether the heavy particle is a neutron or a proton. The second number n, designates the quantum state of the neutron or proton. For $\rho=1$ (neutron), let the suitable eigenfunction be

$$u_n(x), \qquad (18)$$

where x represents the coordinates of the heavy particles with the exception of ρ. For $\rho=-1$ (proton), let the eigenfunction be

$$v_n(x). \qquad (19)$$

The remaining numbers $N_1N_2\cdots N_s\cdots M_1M_2\cdots M_\sigma\cdots$ may have only the two values 0, 1 and indicate whether a particular electron or a neutrino state is filled.

If one considers the general form of the perturbation energy [Eq. (9)], one sees that the elements are different from zero only for such transitions in which either a heavy particle goes from a neutron to a proton state at the same time as an electron and a neutrino are created, or if the reverse process occurs.

With the help of Eqs. (1), (3), (5), (9), (18), and (19), one gets immediately the matrix elements in question

$$H_{-1mN_1N_2\cdots 1_s\cdots M_1M_2\cdots 1_\sigma\cdots}{}^{1nN_1N_2\cdots 0_s\cdots M_1M_2\cdots 0_\sigma\cdots} = \pm\int v_m^* c_{s\sigma}^* u_n d\tau, \qquad (20)$$

where the integration must be carried out over configuration space of the heavy particles (with the exception of the coordinate ρ). The $\pm$ sign means

$$(-1)^{N_1+N_2+\cdots+N_{s-1}+M_1+M_2\cdots+M_{\sigma-1}},$$

and drops out of the following calculation. A complex-conjugate matrix element corresponds to the opposite transition.

If one introduces for c_s^* the value of Eq. (15), one obtains

$$H_{-1m1_s1_\sigma}{}^{1n0_s0_\sigma} = \pm g\int v_m^* u_m \tilde{\psi}_s \delta\varphi_\sigma^* d\tau, \qquad (21)$$

where for brevity all constant indices have been omitted from the first term.

V. THEORY OF β DECAY

A β decay is the process by which a nuclear neutron changes into a proton at the same time as an electron, which is observed as a β ray, and a neutrino are emitted by the described mechanism. In order to calculate the probability of these processes, we assume that at time $t=0$ a neutron exists in a nuclear state with eigenfunction $u_n(x)$ and $N_s=M_\sigma=0$. That is, the electron state s and the neutrino state σ are empty. Then for $t=0$, the probability amplitude for the state $(1, n, 0_s, 0_\sigma)$ is

$$a_{1n0_s0_\sigma} = 1, \qquad (22)$$

and it is zero for the state $(-1, m, 1_s, 1_\sigma)$ where the neutron is changed into a proton with the eigenfunction $v_m(x)$ by emission of an electron and a neutrino.

Using the usual perturbation formula, for a time which is short enough that Eq. (22) remains

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[8] Translator's note: φ_s (sic) should be φ_σ.

approximately true, one gets

$$\dot{a}_{-1m1_s1_\sigma} = -(2\pi i/h) H_{-1m1_s1_\sigma}{}^{1n0_s0_\sigma} \times \exp 2\pi i/h(-W+H_s+K_\sigma)t, \quad (23)$$

where W represents the energy difference of the neutron and the proton states.

From Eq. (23) one obtains, (since for $t=0$, $a_{-1m1_s1_\sigma}=0$)

$$a_{-1m1_s1_\sigma} = H_{-1m1_s1_\sigma}{}^{1n0_s0_\sigma} \times \frac{\exp[2\pi i/h(-W+H_s+K_\sigma)t]-1}{-W+H_s+K_\sigma}. \quad (24)$$

The probability of the transition is therefore, for a time t,

$$|a-1m1_s1_\sigma|^2 = 4|H_{-1m1_s1_\sigma}{}^{1n0_s0_\sigma}|^2 \times \frac{\sin^2(\pi t/h)(-W+H_s+K_\sigma)}{(-W+H_s+K_\sigma)^2}. \quad (25)$$

In order to calculate the lifetime of the neutron state u_n, one must sum the expression (25) over all unoccupied electron and neutrino states. An essential simplification in carrying out this sum is obtained by noticing that the de Broglie wavelength for electrons and neutrinos with energies of a few million volts is significantly greater than the nuclear dimensions. Hence, in first approximation, one can consider the eigenfunctions ψ_s and φ_σ as constant within the nucleus. Equation (21) then becomes

$$H_{-1m1_s1_\sigma}{}^{1n0_s0_\sigma} = \pm g\tilde{\psi}_s\delta\varphi_\sigma{}^* \int v_m{}^* u_n d\tau. \quad (26)$$

Here, and in the following, ψ_s and φ_σ are to be evaluated at the position of the nucleus (see Sec. VIII). From Eq. (26) one obtains

$$|H_{-1m1_s1_\sigma}{}^{1n0_s0_\sigma}|^2 = g^2 \left| \int v_m{}^* u_n d\tau \right|^2 \tilde{\psi}_s\delta\varphi_\sigma{}^*\tilde{\varphi}_\sigma{}^*\tilde{\delta}\psi_s. \quad (27)$$

The neutrino states σ are determined by their momentum p_σ and the spin direction. For normalization purposes we quantize in a volume Ω, whose dimensions subsequently will be allowed to become infinite. Then the normalized neutrino eigenfunctions are plane-Dirac waves with density Ω^{-1}. By simple algebra the average in Eq. (27) can be taken over all directions of p_σ and all spin directions of the neutrino. (Only the positive eigenvalues are to be considered. The negative eigenvalues are removed by an artifice analogous to the Dirac hole theory.) One obtains

$$\langle |H_{-1m1_s1_\sigma}{}^{1n0_s0_\sigma}|^2\rangle_{\rm av} = \frac{g^2}{4\Omega} \left| \int v_m{}^* u_n d\tau \right|^2 \left(\tilde{\psi}_s\psi_s - \frac{\mu c^2}{K_\sigma}\tilde{\psi}_s\beta\psi_s\right), \quad (28)$$

where μ represents the rest mass of the neutrino and β the Dirac matrix

$$\beta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \quad (29)$$

Notice now that the number of neutron states of positive energy with momentum between p_σ and $p_\sigma+dp_\sigma$ is $(8\pi\Omega/h^3)p_\sigma{}^2dp_\sigma$, that $\partial K_\sigma/\partial p_\sigma = v_\sigma$, where v_σ represents the velocity of the neutrino in state σ; and that Eq. (25) has a sharp maximum near the value of p_σ for which the variation of the unperturbed energy disappears, that is

$$-W+H_s+K_\sigma=0. \quad (30)$$

Therefore, Eq. (25) can be summed over σ in a well-known manner,[9] and one obtains

$$t\cdot\frac{8\pi^3g^2}{h^4} \times \left| \int v_m{}^* u_n d\tau \right|^2 \frac{p_\sigma{}^2}{v_\sigma}\left(\tilde{\psi}_s\psi_s - \frac{\mu c^2}{K_\sigma}\tilde{\psi}_s\beta\psi_s\right), \quad (31)$$

where here p_σ means the value of the neutrino momentum for which Eq. (30) is true.

VI. FUNCTIONAL DEPENDENCE OF THE TRANSITION PROBABILITY

The probability that during the time t, a β decay with transition of an electron into a state s takes place is given in expression (31). As it should be, this probability is proportional to the time t (if t is regarded as small with respect to the

[9] For the exact description of the method of carrying out such sums compare any article on radiation theory; e.g., E. Fermi, Rev. Modern Phys. 4, 87 (1932).

lifetime). The coefficient of t is the transition probability for the described process. It is

$$P_s=\frac{8\pi^3 g^2}{h^4}\times\left|\int v_m{}^*u_n d\tau\right|^2\frac{p_\sigma{}^2}{v_\sigma}\left(\tilde{\psi}_s\psi_s-\frac{\mu c^2}{K_\sigma}\tilde{\psi}_s\beta\psi_s\right). \quad (32)$$

One notices the following:

(a) For the free neutrino states, K_σ is always greater than μc^2. Therefore, to satisfy Eq. (30), it is necessary that

$$H_s\leqq W-\mu c^2. \quad (33)$$

The equal sign corresponds to the upper limit of the continuous β-ray spectrum.

(b) Since for the free electron states $H_s>mc^2$, one obtains the following necessary condition for β decay to occur:

$$W\geqq(m+\mu)c^2. \quad (34)$$

Thus, an occupied neutron state n in the nucleus must lie high enough over an unoccupied proton state m so that the β process can go through it.

(c) According to Eq. (32), P_s depends on the eigenfunctions u_n, v_m of the heavy particle in the nucleus through the matrix element,

$$Q_{mn}{}^*=\int v_m{}^*u_n d\tau. \quad (35)$$

In β-ray theory this matrix element plays a role similar to the matrix element of the electric moments of an atom in radiation theory. The matrix element, Eq. (35), normally is of the order of magnitude 1. However, because of special symmetry properties of u_n and v_m it often can occur that $Q_{nm}{}^*$ disappears. In such cases we speak of *forbidden* β transitions. Naturally, one must not expect that a forbidden transition may never occur since Eq. (32) is only an approximate formula. We say something about this type of transition in Sec. IX.

VII. MASS OF THE NEUTRINO

The shape of the continuous β spectrum is determined from the transition probability, Eq. (32). We want to discuss first how this shape depends on the rest mass of the neutrino, μ, in order to determine this constant by comparison with empirical curves. The mass, μ, is contained in the factor $p_\sigma{}^2/v_\sigma$. The dependence of the form of the energy distribution curve on μ is most pronounced near the end point of the distribution curve. If E_0 is the maximum energy of the β rays, then one sees without difficulty that the distribution curve for energy E near E_0, up to a factor independent of E, behaves as

$$p_\sigma{}^2/v_\sigma=c^{-3}(\mu c^2+E_0-E)[(E_0-E)^2+2\mu c^2(E_0-E)]^{1/2}. \quad (36)$$

In Fig. 1, the end of the distribution curve for $\mu=0$ and for large and small values of μ is sketched. The greatest similarity to the empirical curves is given by the theoretical curve for $\mu=0$.

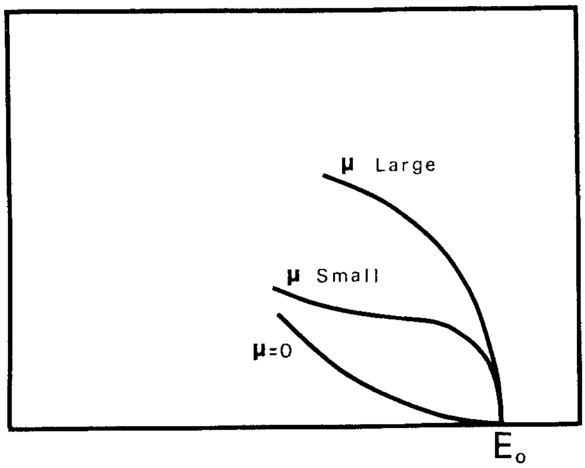

FIG. 1. The end of the distribution curve for $\mu=0$ and for large and small values of μ.

Hence, we conclude that the rest mass of the neutrino is either zero, or, in any case, very small in comparison to the mass of the electron.[10] In the following calculation, we make the simplest hypothesis that $\mu=0$. Then Eq. (30) becomes

$$v_\sigma=c, \qquad K_\sigma=p_\sigma c,$$

and

$$p_\sigma=K_\sigma/c=(W-H_s)/c. \quad (37)$$

The inequalities (33) and (34) become now

$$H_s\leqq W,$$

[10] In a notice appearing recently in Compt. Rend. **197**, 1625 (1933), F. Perrin comes to the same conclusion by qualitative considerations.

and

$$W \geqq mc^2. \qquad (38)$$

The transition probability, Eq. (32), takes on the form,

$$P_s = \frac{8\pi^3 g^2}{c^3 h^4} \left| \int v_m{}^* u_n d\tau \right|^2 \tilde{\psi}_s \psi_s (W - H_s)^2. \qquad (39)$$

VIII. LIFETIME AND SHAPE OF THE DISTRIBUTION CURVE FOR "ALLOWED" TRANSITIONS

From Eq. (39), one can derive a formula which gives how many β transitions take place in a unit of time for which the β particle has a momentum between $mc\eta$[11] and $mc(\eta+d\eta)$. For this purpose, one must derive a formula for the sum of $\tilde{\psi}_s\psi_s$ (evaluated at the position of the nucleus) over all quantum states, of the continuous spectrum, within the interval in question.

It should be noted that the relativistic eigenfunctions in the Coulomb field become infinite for the state with $j=\frac{1}{2}$ ($^2s_{1/2}$ and $^2p_{1/2}$) when $r=0$. However, the nuclear attraction for the electrons obeys the Coulomb law only for r greater than ρ, where ρ here means the nuclear radius. A rough calculation shows that if one makes plausible assumptions of the way the electric field behaves within the nucleus, the value of $\tilde{\psi}_s\psi_s$ at the center lies very near the value that $\tilde{\psi}_s\psi_s$ would take for the case of Coulomb's law at a distance ρ from the center.

By drawing on the well-known formula[12] for the relativistic eigenfunctions of the continuum for hydrogen-like wave functions, one finds after a rather tedious calculation that

$$\sum_{d\eta} \tilde{\psi}_s\psi_s = d\eta \cdot \frac{32\pi m^3 c^3}{h^3[\Gamma(3+2S)]^2} \left(\frac{4\pi m c \rho}{h}\right)^{2S} \eta^{2+2S} \times \exp \pi\gamma \frac{(1+\eta^2)^{1/2}}{\eta} \left| \Gamma\left(1+S+i\gamma \frac{(1+\eta^2)^{1/2}}{\eta}\right) \right|^2, \qquad (40)$$

where

$$\gamma = Z/137; \qquad S = (1-\gamma^2)^{1/2} - 1. \qquad (41)$$

The transition probability into an electron state with a momentum in the interval $mcd\eta$ becomes, according to Eq. (39),

$$P(\eta)d\eta = d\eta \cdot g^2 \frac{256\pi^4}{[\Gamma(3+2S)]^2} \frac{m^5 c^4}{h^7} \left(\frac{4\pi m c\rho}{h}\right)^{2S} \left| \int v_m{}^* u_n d\tau \right|^2 \times \eta^{2+2S} \exp \pi\gamma \frac{(1+\eta^2)^{1/2}}{\eta} \left| \Gamma\left(1+S+i\gamma \frac{(1+\eta^2)^{1/2}}{\eta}\right) \right|^2 [(1+\eta_0{}^2)^{1/2} - (1+\eta^2)^{1/2}]^2, \qquad (42)$$

where η_0 represents the maximum momentum of the emitted β rays, measured in units of mc.

Numerical evaluation of Eq. (42) can be made, e.g., for $\gamma=0.6$, corresponding to $Z=82.2$, since the atomic numbers of the radioactive elements do not lie far from this value. For $\gamma=0.6$, $S=-0.2$, according to Eq. (41). One finds further that for $\eta<10$, the following formula is approximately true:

$$\eta^{1.6} \exp\left[(0.6\pi)\frac{(1+\eta^2)^{1/2}}{\eta}\right] \left| \Gamma\left[0.8+0.6i\frac{(1+\eta^2)^{1/2}}{\eta}\right] \right|^2 \cong 4.5\eta + 1.6\eta^2. \qquad (43)$$

If ρ is set equal to $9(10^{-13})$, Eq. (42) becomes

$$P(\eta)d\eta = 1.75(10^{95}) g^2 \left| \int v_m{}^* u_n d\tau \right|^2 \times (\eta + 0.355\eta^2)[(1+\eta_0{}^2)^{1/2} - (1+\eta^2)^{1/2}]^2. \qquad (44)$$

[11] Translator's note: η is not explicitly defined in the text. It is the ratio $(v/c)/[1-(v/c)^2]^{1/2}$, where v is the electron velocity. This is equivalent to $m_v v/mc$: hence $mc\eta = m_v v$, the relativistic momentum at the velocity v. The rest mass of the electron is m.

The reciprocal lifetime is obtained by integration from $\eta=0$ to $\eta=\eta_0$.

One finds that

$$\tau^{-1} = 1.75(10^{95}) g^2 \left| \int v_m{}^* u_n d\tau \right|^2 F(\eta_0), \qquad (45)$$

[12] R. H. Hulme, Proc. Roy. Soc. (London) **133**, 381 (1931).

TABLE I. The value of $F(\eta_0)$ for large values of η_0.

η_0	$F(\eta_0)$
0	$\eta_0^6/24$
1	0.03
2	1.2
3	7.5
4	29
5	80
6	185
7	380

where

$$F(\eta_0) = \tfrac{2}{3}[(1+\eta_0^2)^{1/2}-1]+\tfrac{1}{12}\eta_0^4-\tfrac{1}{3}\eta_0^2 +0.355\{-\tfrac{1}{4}\eta_0-\tfrac{1}{12}\eta_0^3+\tfrac{1}{30}\eta_0^5+\tfrac{1}{4}[(1+\eta_0^2)^{1/2}] \times \log[\eta_0+(1+\eta_0^2)^{1/2}]\}. \quad (46)$$

For small arguments $F(\eta_0)$ behaves as $\eta_0^6/24$. For larger arguments the values of F are collected in Table I.

IX. FORBIDDEN TRANSITIONS

Before we proceed to comparison with experiment, we will discuss some properties of the forbidden β transitions.

As already pointed out, a transition is forbidden when the corresponding matrix element, (35), disappears. Now, if the description of the nucleus in terms of individual quantum states of the neutrons and protons is a good approximation, Q_{mn}^* always disappears on the basis of symmetry, unless

$$i=i', \quad (47)$$

where i, i' represent the angular momenta of the neutron state u_n and the proton state v_m (in units of $h/2\pi$). If the individual states are not good approximations, selection rule (47) corresponds to the more general condition,

$$I=I', \quad (48)$$

where I and I' are the angular momenta of the nucleus before and after the β decay.

The selection rules (47) and (48) are by no means as rigorous as the selection rules of optics. Principally there are two processes in which a violation of these selection rules is possible:

(a) Equation (26) is obtained by neglecting the variation of ψ_s and φ_σ within the nuclear dimensions. However, if one does not consider ψ_s and φ_σ to be constant within the nucleus, then one also obtains the possibility of β transitions in cases where Q_{mn}^* disappears.

It is easy to see that the intensity of such transitions is in the order of magnitude $(\rho/\lambda)^2$ relative to the intensity of allowed processes, where λ represents the de Broglie wavelength of the lightweight particles. Note that for the same energy, the kinetic energy of the electron at the position of the nucleus is considerably greater than that of the neutrino because of the electrostatic attraction. Therefore, the largest effect arises from the variation of ψ_s. Estimation of the intensity of these forbidden processes shows that they must be of the order of 100 times weaker than the transitions allowed by Eq. (48) if the β particles are emitted with the same energy.

One can see the characteristics of forbidden transitions of this type not only in the relatively longer lifetime, but also in the different shape of the energy distribution curve of the β rays. One finds in particular, that for these transitions the distribution curve for small energies lies much lower than in the normal case.

(b) A second possibility of transitions forbidden according to Eq. (48) follows from the fact pointed out at the end of Sec. III, that if one does not neglect the velocity of the heavy nuclear constituents with respect to the velocity of light, more terms of the order v/c are introduced into the interaction term, Eq. (12). Perhaps, if one also accepts a relativistic wave equation of the Dirac type for the heavy particles, one can add terms to Eq. (12), as for example,

$$gQ(\alpha_x A_1+\alpha_y A_2+\alpha_z A_3)+\text{c.c.}, \quad (49)$$

where $\alpha_x\alpha_y\alpha_z$ are the Dirac matrices for the heavy particles, and A_1, A_2, A_3, are the space components of the four-vector defined in Eq. (11). The term (49) would be related to Eq. (12) in the same way as the term $e(\alpha, U)$ and respectively eV are to the Dirac–Hamiltonian function. (V is equal to the scalar potential, U is the vector potential.)

An interaction term like (49) naturally would also make forbidden transitions possible with a relative intensity of the order of magnitude $(v/c)^2$ with respect to the allowed transitions. There is, therefore, a second possibility for the existence of transitions which are about 100 times weaker than normal.

X. COMPARISON WITH EXPERIMENT

Equation (45) gives a relation between the maximum momentum of the emitted β rays and the lifetime of the β-radiating substance. In this relation there is still, to be sure, an unknown quantity, the integral

$$\int v_m^* u_n d\tau, \tag{50}$$

for whose evaluation a knowledge of the eigenfunctions of the proton and of the neutron in the nucleus would be necessary. In the case of the allowed transitions, however, Eq. (50) is of the order of magnitude 1. One can expect, therefore, that the product

$$\tau F(\eta_0), \tag{51}$$

has the same order of magnitude for all allowed transitions. If, however, the transition in question is forbidden, the lifetime is about 100 times greater than in the normal case and, therefore, the product (51) will be correspondingly larger.

TABLE II. The values of $\tau F(\eta_0)$ for the radioactive elements for which there are sufficient data on the continuous β spectra.

Element	τ(hours)	η_0	$F(\eta_0)$	$\tau F(\eta_0)$
UX_2	0.026	5.4	115	3.0
RaB	0.64	2.04	1.34	0.9
ThB	15.3	1.37	0.176	2.7
ThC″	0.076	4.4	44	3.3
AcC″	0.115	3.6	17.6	2.0
RaC	0.47	7.07	398	190
RaE	173	3.23	10.5	1800
ThC	2.4	5.2	95	230
$MsTh_2$	8.8	6.13	73	640

In Table II, the product (51) is tabulated for the radioactive elements for which one has sufficient data concerning the continuous β spectrum. From Table II the two anticipated groups are immediately recognizable. Indeed, such a classification has already been established empirically by Sargent,[13] from whose work the values of η_0 are taken. (For comparison, one notes that $\eta_0 = (H\rho)_{max}/1700$. The values of η_0 which Sargent lists as not too reliable, do not fit too well into this classification; for UX_1, one has $\tau = 830$, $\eta_0 = 0.76$, $F(\eta_0) = 0.0065$, and $\tau F(\eta_0) = 5.4$. Therefore, this element appears to fit in the first group. For AcB one has the following data: $\tau = 0.87$, $\eta_0 = 1.24$, $F(\eta_0) = 0.102$, $\tau F(\eta_0) = 0.09$, hence a τF-value about 10 times smaller than the first group. For RaD, one has $\tau = 320{,}000$, $\eta_0 = 0.38$ (very uncertain), $F(\eta_0) = 0.00011$, and $\tau F(\eta_0) = 35$. RaD, therefore, lies approximately in the middle between these two groups. I have not found any data concerning the other β-emitting elements such as $MsTh_1$, UY, Ac, AcC, UZ, RaC″.

From the data of Table II, one can obtain, even if very crudely, an evaluation of the constant g. If one assumes, say, that $\tau F(\eta_0) = 1$ (i.e. measured in seconds, $= 3600$) in the cases where the integral (50) equals unity, one obtains from Eq. (45)

$$g = 4(10^{-50}) \text{ cm}^3 \text{ erg}.$$

This value naturally will be only an order of magnitude of g.

To summarize, one can say that this comparison of theory and experiment gives as good an agreement as one could expect. The discrepancies found for the hard-to-pin-down data for elements, RaD and AcB, probably could be explained in part through inaccuracy of the measurements, partly, also, by the abnormally large, although not at all implausible, variations of the matrix elements in Eq. (50). Note further that one can

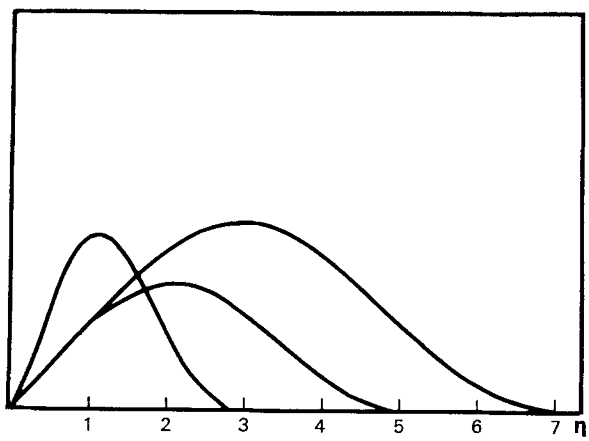

FIG. 2. Velocity distribution curves for different values of η_0.

[13] B. W. Sargent, Proc. Roy. Soc. (London) A139, 659 (1933).

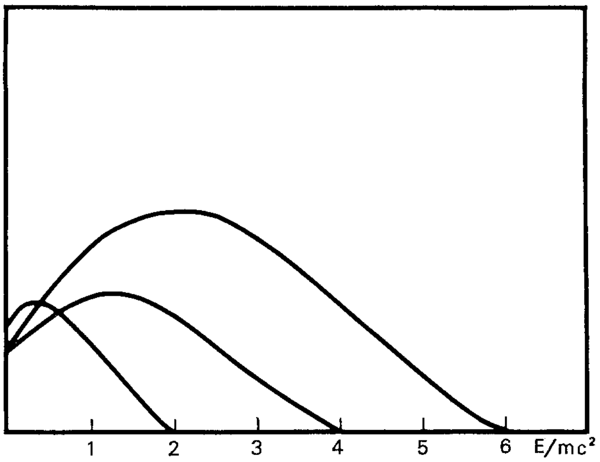

FIG. 3. Velocity distribution curves versus the energy, $E=mc^2/[1-(v/c)^2]^{1/2}-mc^2$.

conclude from the γ radiation accompanying the β decay, that most β decays can lead to different final states of the proton, whereby again variations in the $\tau F(\eta_0)$ value can be explained.

We turn now to the question of the shape of the velocity distribution curve of the emitted β rays. For the case of allowed transitions, the distribution curve of η (or $H\rho$ except for the factor of 1700) is given by Eq. (44). Distribution curves for different values of η_0 are shown in Fig. 2. For convenience, the designations of the units of the ordinates was suitably adjusted for the various cases. These curves show a satisfying similarity to the set of distribution curves given for instance by Sargent.[14] Only in the low-energy portion of the curve do Sargent's curves lie somewhat lower than the theoretical ones. This is clearly seen in Fig. 3, where energy in place of momentum is the abscissa. One must remember, however, that the experimental knowledge of the distribution laws for small energies is particularly precarious.[15] Moreover, one also has to anticipate that for forbidden transitions, the theoretical curves lie lower than those of Figs. 2 and 3 in the region of small energies. This last point is especially pertinent in the comparatively well-known experimental curve for RaE. In Table II, one sees that RaE has a very large $\tau F(\eta_0)$ value. The β decay of RaE is, therefore, certainly forbidden and even if possible, will be allowed only in the second approximation. In a further article, I hope to be able to say something more precise about the shape of the energy distribution curve for forbidden transitions.

In conclusion, one can say that the theory presented in this article is in conformity with the experimental data, which by all means are not always exact. Should one encounter contradictions in a closer comparison of theory and experiment, it might still be possible to alter the theory without disturbing its conceptual fundamentals. Namely, one could retain Eq. (9), but make a different choice of $c_{s\sigma}$. This could lead to a change of the selection rule (48) and yield another form of the energy distribution curve, as well as the dependence of the lifetime on the maximum energy. However, whether such a change is necessary can be shown only through a further development of the theory and possibly through a refinement of the experimental data.

[14] B. W. Sargent, Proc. Cambridge Phil. Soc. **28,** 538 (1932).

[15] For example, compare E. Rutherford, B. Ellis, and J. Chadwick, *Radiations from Radioactive Substances* (Cambridge University Press, London, England, 1932). See especially, p. 407.