obtained, it has to be considered one of the fundamental contributions to relativistic cosmology, and since that time I have made a special effort to publicize it and assign it as mandatory reading for all my university students.

I then realized that a number of other articles by Fermi were equally insufficiently well known: a possible reason being that they had not yet been translated from Italian into English, especially those by the young Fermi when he was a student at the Scuola Normale Superiore in Pisa dealing mainly with electrodynamics and the special and general theories of relativity. This led to a lengthy process in which with the help of Emanuele Alesci, Donato Bini, Dino Boccaletti, Andrea Geralico, Robert Jantzen, and Simone Mercuri, we translated from Italian to English a selection of Fermi's papers, including the ones of the Pisa period. In the course of our work we also became aware of scientific results published in a series of six papers written by Fermi in 1922–1923 during his Pisa period while still a student and later in a temporary position at the University of Florence, results which he presented in Göttingen in 1924. This work, which has been overlooked in nearly all textbooks, is his solution of the infamous so called "4/3 problem" that plagued the classical theory of the electron introduced by Abraham and Lorentz during the first years of the life of special relativity and which was wrongly interpreted by Poincaré as due to unidentified internal stresses holding the electron together. I discussed this topic with Donato Bini, Andrea Geralico and Robert Jantzen over the period of a few years, resulting in our commentary article Appendix (A.1) and a shortened version (A.2) for the journal *General Relativity and Gravitation*.

While examining Fermi's early papers, we came across two important papers which we also translated. The first is a 1930 lecture delivered in Trento in which he clearly motivated his distrust toward approaching the internal constitutions of stars, an attitude which had negative consequences for the Italian development of astrophysics. The second was greatly rewarding: a crucial lecture that Fermi later delivered in Italian at the University of Rome in October 1949, "Theories on the origins of the elements," recorded by Ettore Pancini, which we have also translated into English. Through this I finally became aware of Fermi's deep knowledge of cosmology and derivation of the key equations, which allowed him to perform the computation in his work with Turkevich. There were also some other later papers related to astrophysics which, although they had been published in English, for a variety of reasons, had not yet reached the attention they deserved from the scientific community at large. We started assembling all of this material. Of course many books and even movies already exist which review the glorious achievements of the Fermi group in Rome on neutron physics, nuclear physics and statistical mechanics, but none of these overlap with our specific interest in the matter of general relativity and astrophysics. I first noticed with curiosity Fermi's apparent lack of interest in general relativity and also in astrophysics during the entire Rome period of this life. This was particularly surprising since many fundamental results were obtained in those years in England and in the United States which had great significance for astrophysics in the following decades. Many of the results were indeed obtained using Fermi's conceptual discoveries.

It became natural to ask why Fermi, one of the first scientists to reach a deep understanding of Einstein's theory of general relativity and to give profound contributions to that theory, already as a student in Pisa, never addressed any issue related to general relativity after transferring to Florence in 1924 and in 1926 to Rome. What could have happened during this Florence transition which inhibited his desire to pursue general relativity further?

While I was mulling over all these issues in the intervening years, I continued my work in relativistic astrophysics and was witnessing on a daily basis the tremendous relevance to the field of relativistic astrophysics of the classic work of three giants: Fermi, Einstein and Heisenberg. The greatest and most fundamental new results have come from the utilization of their ideas not in the isolation that they had created between themselves while alive but in a profound new interaction unhampered by their personal prejudices. From this thinking came the decision to contextualize this material with a companion book [3] dedicated to Einstein, Fermi, Heisenberg and the birth of relativistic astrophysics which took place due to both theoretical and observational advances that came one after the other in the 1960s, seen from my personal perspective as one of the participants in this story from its beginnings to the present time. I purposely avoided there entering into matters already extensively treated elsewhere, including in my own books, and have focused on a historical perspective regarding some particular events in the development of relativistic astrophysics which I have witnessed directly or have reconstructed in Rome, Princeton, Cambridge, Moscow and in locations where relativistic astrophysics after its inception flourished in the following years. I have privileged the indications on some current research which I consider particularly promising.

In ~~the present~~ this volume the introductory Chapter 1 summarizes the contents of the remaining two chapters and appendices. We have reproduced and where necessary translated the fundamental contributions Fermi made which are relevant to astrophysics, starting from his early student days in Pisa (see Fig. 1) and continuing throughout his life. Chapter 2 contains those relevant papers from his Italian period before moving to America, followed by Chapter 3 which includes papers from his American period, including his paper on theories of element formation in the early universe from his 1949 Rome lecture as recorded by E. Pancini, and the Fermi-Turkevich work reproduced by Alpher and Herman. These are discussed in detail in the companion book. Appendix A contains some commentary articles regarding Fermi's early work in Italy, while Appendix B reproduces a selection of papers from the 2001 Meeting on Fermi and Astrophysics published in Nuovo Cimento in 2002.

In addition to remembering ~~in this volume~~ Fermi's contributions to fundamental physics starting from his student days in Pisa, continuing throughout his life, before closing, I recall here the influence Fermi had on science in China. This was commemorated in a special ceremony held in Beijing during the Fourth Galileo-Xu

7

14th

Guangqi Meeting (GX4) in May 2015 (see Fig. 2) just preceding the Fourteenth Marcel Grossmann Meeting in Rome in July 2015. On that occasion both Fermi's former students C.N. Yang and T.D. Lee received Marcel Grossmann Awards (see Figs. 3, 4). Yang (see Fig. 5) then delivered a talk of his personal recollections of Fermi, including an exchange with Eugene Wigner, indicated by "W", as well as their final meeting in the hospital (accompanied oby Murray Gell Mann) in the last minutes of Fermi's life. As the most unique Fermi reminiscence I have ever read and possibly the most touching words expressed by one human being for another, we reproduce them below.

— *Remo Ruffini*

Fig. 1 The young Enrico Fermi.

Yang on Fermi

I remember that it was at the Second Marcel Grossman Meeting in Trieste in 1979, that I formulated the phrase "symmetry dictates interactions", which describes the principle that governs the structure of interactions. I am happy to receive this award from an organization based in Italy, the country I feel closest to, after China and the USA. Enrico Fermi was one of the great sons of Italy in her long history. Prometheus in Greek mythology, Suiren in Chinese mythology, taught mankind how to use chemical energy. Enrico Fermi in reality, taught mankind how to use nuclear energy.

Enrico Fermi was, of all the great physicists of the 20th century, among the most respected and admired. He was respected and admired because of his contributions to both theoretical and experimental physics, because of his leadership in discovering for mankind a powerful new source of energy, and above all, because of his personal character. He was always reliable and trustworthy. He had both of his feet on the ground all the time. He had great strength, but never threw his weight around. He did not play to the gallery. He did not practise one-up-manship. He exemplified, I always believe, the perfect Confucian gentleman.

Fermi from 1950 to 1951 was a member of the General Advisory Committee (GAC) of the Atomic Energy Committee (AEC) chaired by Oppenheimer. He then resigned with a quote:
"You know, I don't always trust my opinions about these political matters".

> Shakespeare's Sonnets No. 94
> *They that have power to hurt and will do none,*
> *That do not do the thing they most do show,*
> *Who, moving others, are themselves as stone,*
> *Unmoved, cold, and to temptation slow;*
> *They rightly do inherit heaven's graces,*
> *And husband nature's riches from expense;*
> *They are the lords and owners of their faces,*
> *Others but stewards of their excellence.*

In my years in Chicago, Fermi was personally very kind to me. I remember in June 1948, I had problems with the US Immigration Office. Fermi and Professor Allison, the Director of Chicago's Institute, went with me to the Immigration Office in Chicago. The Head of the office was so overwhelmed by the presence of Fermi that all my immigration problems were resolved immediately.

Fermi made many first rate contributions to physics. His contemporaries, including himself, considered his beta decay theory the most important. To bring out the great impact that paper had on physicists in the early 1930s, allow to me to tell you a story.

Y: What do you think was Fermi's most important contribution to theoretical physics?

W: Beta decay theory.
Y: How could that be? It is being replaced by more fundamental ideas. Of course it was a very important contribution which had sustained the whole field for some forty years: Fermi had characteristically swept what was unknowable at that time under the rug, and focused on what can be calculated. It was beautiful and agreed with experiment. But it was not permanent. In contrast the Fermi distribution is permanent.
W: No, no, you do not understand the impact it produced at the time. Von Neumann and I had been thinking about beta decay for a long time, as did everybody else. We simply did not know how to create an electron in a nucleus.
Y: Fermi knew how to do that by using a second quantized ψ?
W: Yes.
Y: But it was you and Jordan who had first invented the second quantized ψ?
W: Yes, yes. But we never dreamed that it could be used in real physics.

In the fall of 1954 Fermi was critically ill. Murray Gell-Mann and I went to the Billwigs Hospital to see him for a last time. He was thin, but not sad. He was reading a book full of stories about men who had succeeded, through shear will power, to overcome fantastic obstacles and misfortunes. As we bade goodbye and walked towards the door of his room, he said:
"Now I have to leave physics to your generation."
— Chen-Ning Franklin Yang

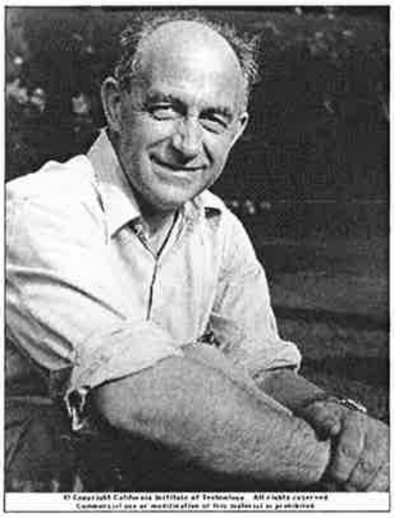

Fig. 6 Enrico Fermi (1901–1954).

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Both volumes are available at the Accademia dei Lincei website:
`http://www.lincei.it/modules.php?name=Content&pa=showpage&pid=125`.

[3] *Einstein, Fermi, Heisenberg and the Birth of Relativistic Astrophysics*, Remo Ruffini, World Scientific, 2017.

[4] Rong-Gen Cai, Remo Ruffini, and Yue-Liang Wu, *Int. J. Mod. Phys. A* **30**, 1502005 (2015),
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[5] The Fourth Marcel Grossmann Meeting Awards booklet
`http://www.icra.it/mg/mg14/mg14_awards.pdf`.

Chapter 1

Introduction

The present volume contains two chapters including translations and reproductions of key papers by Fermi relevant to astrophysics, together with three appendices of some historically relevant papers by other other authors and commentary on some of his articles.

1.1 Fermi's Italian Period

Chapter 2 contains the English translation of the papers originally published in Italian during Fermi's Pisa and Rome periods. The most famous of these introducing Fermi coordinates and Fermi transport (implicitly defining what later became known as Fermi-Walker transport, see Appendix B.2) was indeed a detour from Fermi's initial investigation of electromagnetic mass in special and general relativity that seems to have been largely ignored over the past ninety years. Credit for translation of Fermi's articles from Italian to English goes to: Emanuele Alesci for papers 4c) and 5), Donato Bini and Andrea Geralico for papers (1), (2), and (3), Dino Boccaletti for papers (7), (10), (12), (13), (30), (38) and (80a), and Simone Mercuri for paper (43), using the article labeling system from the two volume set of Fermi's collected works noted in the preface. Robert Jantzen edited these translations for English expression.

This section contains the English translations of a selection of papers from those Fermi published in Italian in the first part of his scientific career. The seminal papers selected are all related to relativity, astronomy and their applications. For a better account of the circumstances under which the papers were written, we also add excerpts of the presentations due to friends and collaborators of Fermi and published in Volume 1 of Fermi's *Note e Memorie*, 1961.

In paper FI 1 *On the Dynamics of a Rigid System of Electric Charges in Translational Motion* (1), Fermi calculates the inertial mass of a spherical distribution of charge with a constant acceleration by considering the reaction of the charge to its own average field. This leads to the formula $mc^2 = (4/3)U$ relating the inertial mass m to the classical electromagnetic energy U of the distribution. This value, in agreement with a calculation of the electromagnetic mass of a spherical homogeneous shell performed by Lorentz, contradicts the formula $mc^2 = U$ that one would expect from the principle of equivalence of mass and energy. Fermi considers the charge distribution at rest in a homogeneous gravitational field equal to the sign-reversed acceleration which appears to be in agreement with the relativistic formula. This topic is further examined in the subsequent article.

In paper FI 2 *On the Electrostatics of a Homogeneous Gravitational Field and on the Weight of Electromagnetic Masses* (2), Fermi reconsiders the calculation of the inertial mass of a spherical distribution of charge using for the first time general relativity, employing a Levi-Civita metric to describe a homogeneous gravitational field in the linear approximation. This approach has been expanded to what we now call today the Rindler metric.[1] His final result leads to the desired relation $mc^2 = U$. Another result derived in this paper is the value of the polarization of an infinitesimal conducting sphere at rest in a static gravitational field. An article by R. Ruffini[2] (see Appendix A.5) discusses some general relativistic developments that have taken place in the intervening years for describing electric charges in strong gravitational fields.

Paper FI 3 *On phenomena occurring close to a world Line* (3) is a classic result obtained by Fermi within the framework of general relativity expressing a system of space-time coordinates particularly suited to follow the behavior in time of phenomena happening in a small spatial region around the world line of a particle. Fermi explores the definition of the related coordinate transport which underlies it, later known as "Fermi transport," expressing the metric in the linear approximation for a general space-time. He also expresses Maxwell's equations in these coordinates, supporting the conclusions reached in the previous article.

The contribution by D. Bini and R. Jantzen (B.2) in Appendix B of this volume gives a summary of what we now call Fermi coordinates and Fermi transport with a historical update including Walker's contribution which led to the terminology of "Fermi-Walker transport." This article also discusses the geometry of the various relativistic contributions to the Fermi-Walker transport of vectors around circular orbits in black hole spacetimes and in their Minkowski limit.

In paper FI 4 *Correction of a Contradiction between Electrodynamic and Relativistic Electromagnetic Mass Theories* (4c), Fermi reconsiders the problem of the electromagnetic contribution to the mass of an elementary particle already discussed in the previous three articles. The discrepancy between the value $(4/3)(U/c^2)$, obtained by Lorentz for the inertial mass of a rigid, spherically symmetrical system of electric charges, and the value U/c^2 predicted by relativity was well known to Fermi from the previous articles. Such a discrepancy had been interpreted by Poincaré as due to the part of the stress-energy tensor contributed by internal non-

[1]See W. Rindler: *Essential relativity; special, general, and cosmological*, Van Nostrand Reinhold Co., 1969.

[2]*R. Ruffini: Charges in gravitational field: From Fermi, via Hanni-Ruffini-Wheeler, to the "electric Meissner effect"*, *Nuovo Cimento* **119B**, 785–807, 2004.

electromagnetic stresses, whose existence was assumed to assure the equilibrium of the charged particles. A vast scientific literature of followers of this Poincaré's conjecture exists. Fermi shows that by assuming the accelerated charge distribution to be spherically symmetric in its rest frame instead of the laboratory frame, he obtains the correct inertial mass expected from the equivalence principle. This essentially reintroduces the crucial lapse factor between coordinate and physical components of the electric field which is responsible for the correction, to first order in the acceleration, of the approximation made in all of his "Fermi coordinate" system calculations. The results obtained by Fermi in this paper went unnoticed and for the most part remain that way today. Some of the crucial Fermi results in this paper and the historical developments of this most unique accident in physics are discussed in Appendix A by D. Bini, A. Geralico, R.T. Jantzen and R. Ruffini (see A.1) and by R.T. Jantzen and R. Ruffini in a brief summary of the key mathematics and their consequences (see A.2), as well as in a historical review by D. Boccaletti (see A.3).[3] Interestingly enough, related considerations were also put forward years later by B. Kwal without mentioning Fermi's work. Appendix A.4 reproduces this 1949 paper.

The paper FI 5 *Masses in the theory of relativity* (5)—a short contribution to a collective volume on the foundations of Einstein's theory of relativity—is evidence of the high reputation enjoyed by the young Fermi (age 22) in the physicists' community. Remarkable appears to be the prophetic premonition of things to come. A very favorable attitude toward Einstein theory by the young Fermi is clear at a time in which the older generation of Italian physicists was skeptical and hostile to relativity as recalled by Emilio Segré in Vol. 1, p. 33 of *Note e Memorie*.[4]

In paper FI 6 *On the mass of radiation in an empty space* (10), written in collaboration with Aldo Pontremoli, Fermi successfully applied the method used in FI 4 (4c) to the calculation of the mass of the radiation contained in a cavity with reflecting walls, for which the standard textbooks had an expression containing the same factor 4/3.

The papers FI 7 *The principle of adiabatics and the systems which do not admit angle coordinates* (12) and FI 8 *Some theorems of analytical mechanics of great importance for quantum theory* (13) are dedicated to the theory of adiabatic invariants. The interest of Fermi in the theory of adiabatic invariants, if we make reference to the published papers, goes from 1923 throughout 1926. As the other theoretical physicists in that period, he was convinced of the fundamental importance of the theory of adiabatic invariants for a rigorous formulation of quantum mechanics. On

[3] Boccaletti's review was written before the publication of the paper "The mass of the particles" by A. Bettini (*Rivista del Nuovo Cimento* **32**, No. 7, 2009, pp. 295–337) where Fermi's priority in first resolving this problem is again noted and continuing ignorance of his result by many outstanding authors is recalled as well (see pp. 302–303).

[4] On this topic see also, e.g., Roberto Maiocchi: *Einstein in Italia—La scienza e la filosofia italiana fra le due guerre—Le Lettere*, Firenze, 1985.

the other hand, Max Born also shared the same opinion[5].

Fermi also devoted a lecture in his university course on theoretical physics[6] to the theory of adiabatic invariants and he gave an elementary exposition of it in his book *Introduzione alla Fisica atomica*.[7] His interest was also awakened in conferences and seminars delivered at the University of Rome and in communications at the *Accademia Nazionale dei Lincei*. It was in those occasions that he sparked the interest of an outstanding listener: Tullio Levi-Civita.[8] The involvement of Levi-Civita was such that he, besides to giving a rigorous mathematical formulation of the subject,[9] also promoted astronomical applications of the theory. Those due to his collaborator Giulio Krall turned out to be particularly interesting. We must add that in those years James Jeans was also concerned with astronomical applications of the theory of adiabatic invariants.[10]

The paper FI 9 *A theorem of calculation of probability and some of its applications* (38b) is the second part of a Fermi's habilitation thesis to the "Scuola Normale Superiore" of Pisa (1922). It concerns the application of a theorem of calculation of probability to the dynamics of comets. The significance and the potentialities of this paper are well elucidated in the paper of C. Sigismondi and F. Maiolino (B.8) in Appendix B.

The paper FI 10 *Formation of images with Röntgen rays* (7) derives from a part of the degree thesis of Fermi at the University of Pisa. The thesis of Fermi was the most complete survey of X-rays physics in his time. He can also be considered a forerunner of techniques which are standard today. As Sigismondi and Mastroianni say in their article (B.9), although Fermi's seminal ideas are not among the sources investigated by Riccardo Giacconi and Bruno Rossi (1960) when they proposed a telescope using X-rays, Fermi's thesis was the most complete study of X-ray physics at his time. Fermi used the technique of 'mandrels' to form optical surfaces. He anticipated the technique used for the mirrors of Exosat, Beppo-SAX, Jet-X and XMM-Newton telescopes, which is now a mainstay of optical manufacturing. The paper by Sigismondi and Mastroianni discusses this noteworthy connection. It is appropriate here also to recall the comments of Franco Rasetti in the introduction of this article in Volume 1 of Fermi's Note e Memorie. Since at that time *"he had already published or at least completed several important theoretical papers, it may be asked why he did not present a theoretical thesis. It must be explained that at*

[5]See Max Born: *Vorlesungen über Atommechanik*, Berlin, 1925, pp. 58–67, 109–114. English translation: *The mechanics of the atom*, London, 1927, pp. 52–59, 95–99.

[6]See A. De Gregorio, S. Esposito: Teaching theoretical physics: The cases of Enrico Fermi and Ettore Majorana, Am. J. Phys. 75 (9), 781–790 (2007).

[7]Enrico Fermi: Introduzione alla Fisica Atomica, Zanichelli, Bologna, 1928, pp. 155–160.

[8]See P. Nastasi, R. Tazzioli: Tullio Levi-Civita, in Lettera Matematica pristem n. 57-58, Springer 2006.

[9]We restrict ourselves to quote the last paper on the subject: T. Levi-Civita, A general survey of the theory of adiabatic invariants, *Journal of Math. and Physics*, Vol. 13, pp. 18–40 (1934).

[10]J.H. Jeans: Cosmogonic problems associated with a secular decrease of mass, MNRAS 85 (1), 2 (1924).
J.H. Jeans: The effect of varying mass on a binary system. MNRAS 85 (9), 912 (1925).

the time in Italy theoretical physics was not recognized as a discipline to be taught in universities, and a dissertation in that field would have been shocking at least to the older members of the faculty. Physicists were essentially experimentalists, and only an experimental dissertation would have passed as physics. The nearest subject to theoretical physics, mechanics, was taught by mathematicians as a field of applied mathematics, with complete disregard for its physical implications. These circumstances explain why such topics as the quantum theory had gained no foothold in Italy: they represented a "no man's land" between physics and mathematics. Fermi was the first in the country to fill the gap." (F. Rasetti, Vol. 1, pp. 55–56).

The paper FI 11 *On the quantization of an ideal monoatomic gas* (30) is the communication (to the Accademia Nazionale dei Lincei) in which Fermi expounds for the first time the statistical theory which will be named after him (together with P.A.M. Dirac). The enormous importance of the Fermi-Dirac statistics in astrophysics is recalled in Section 1.1 of Chapter 1.

In the following we give an excerpt from the presentation of Franco Rasetti *"... the present paper, probably his most famous theoretical contribution, where he formulated the theory of an ideal gas of particles obeying the Pauli exclusion principle, now designated in his honor as "fermion."*

There is conclusive evidence to show that Fermi had been concerned with the problem of the absolute entropy constant at least since January 1924, when he wrote a paper (Fermi 20) on the quantization of systems containing identical particles. He had also been discussing these problems with Rasetti several times in the following year. He told much later to Segré that the division of phase space into finite cells had occupied him very much and that had not Pauli discovered the exclusion principle he might have arrived at it a round-about way from the entropy constant (cfr. No. 20).

As soon as he read Pauli's article on the exclusion principle, he realized that he now possessed all the elements for a theory of the ideal gas which would satisfy the Nerst principle at absolute zero, give the correct Sackur-Tetrode formula for the absolute entropy in the limit of low density and high temperature, and be free of the various arbitrary assumptions that had been necessary to introduce in statistical mechanics in order to derive a correct entropy value. He does not seem to have been greatly influenced by Einstein's theory based on Bose's treatment of the black-body radiation as a photon gas, although he points out the analogy between the two forms of statistics. Apparently it took Fermi but a short time to develop the theory in the detailed and definitive form in which it was published in the German version." (F. Rasetti, Vol. 1, p. 178).

The paper FI 12 *A statistical method for the determination of some properties of the atom* (43), here translated, is the first of the papers Fermi devoted to the theory of what is today called the Thomas-Fermi atom. Fermi was unaware of the results previously reached by Thomas and his work went on independently for two years. Of great importance are the applications of the Thomas-Fermi model in astrophysics. He was, for example, quite familiar with the applications of his statistics (with the required relativistic modifications) to the theory of the structure of white dwarf stars: indeed, T.D. Lee, as a graduate student of Fermi, wrote his Ph.D. thesis on the *Hydrogen Content and Energy-Productive Mechanism of White Dwarf Stars* (*Ap. J.* **111**, 625, 1950). As we showed the general relativistic generalization of the Thomas-Fermi atom has recently led to a new theoretical framework to study both white dwarfs and neutron stars.

The paper FI 13 *An Attempt at a Theory of β Rays* (80a), translated here, can be described as the birth certificate of the theory of β-decay and weak interactions. Its importance is hardly questionable today. At that time (1933) things were not so easy (see Segré's report below).

"Fermi gave the first account of this theory to several of his Roman friends while we were spending the Christmas vacation of 1933 in the Alps. It was in the evening after a full day of skiing; we were all sitting on one bed in a hotel room, and I could hardly keep still in that position, bruised as I was after several falls on icy snow. Fermi was fully aware of the importance of his accomplishment and said that he thought he would be remembered for this paper, his best so far. He sent a letter to Nature advancing his theory but the editor refused it because he thought it contained speculations that were too remote from physical reality; and instead the paper ("tentative theory of beta rays") was published in Italian and in the Zeitschrift für Physik. Fermi never published anything else on this subject, although in 1950 he calculated matrix elements for beta decay as an application of the nuclear shell model." (Emilio Segré: *Enrico Fermi Physicists*, The University Chicago Press, 1970, p. 72).

In this paper there is the first mention to the possible existence of a massive neutrino.

1.2 Fermi's American Period

Chapter 3 reproduces some of Fermi's classic papers from his American period regarding the origin of cosmic rays and the mechanism of their acceleration, the interstellar magnetic field and its importance in astrophysics (in this field, Fermi was a pioneer), and the famous Fermi-Pasta-Ulam paper on nonlinear problems. Paper (240.3) is an article "The origin of the elements" of Fermi in Italian from Fermi's American period recorded by E. Pancini and translated by Dino Boccaletti, the third of nine lectures delivered in an Italian physics conference held in Rome and Milan in 1949, in response to Gamov's attempt to calculate the relative abundances of elements created in the early hot expanding universe. We also include the Fermi-Turkevich article which follows this same argument. A detailed discussion of the story behind these two papers and their relevance for relativistic cosmology can be found in the companion book *Einstein, Fermi, Heisenberg and the Birth of Relativistic Astrophysics* by Remo Ruffini.

We have selected seven of Fermi's papers from his American period to reproduce here, six of which are relevant to astrophysics. We have also added the famous paper "Study of nonlinear problems." All of these have been quoted and commented on numerous times but we think that in order to have a clearer idea of their ideas, it is better to go back to the original sources. As in the preceding chapter, we also include some excerpts of commentary on those papers from Volume 2 of Fermi's *Note e Memorie*.

The first three papers, FA 1 E. Fermi: *On the Origin of the Cosmic Radiation* (237), FA 2 E. Fermi: *An Hypothesis on the Origin of the Cosmic Radiation* (238), FA 3 E. Fermi: *Galactic Magnetic Fields and the Origin of Cosmic Radiation* (265), tackle the problem of the origin of the cosmic rays formulating the hypothesis of a galactic origin and considering the role of the magnetic field. Comments on these papers can also be found in the Ames paper (B.1) in Appendix B.

As recalled by Anderson, *"Paper No. 237 was a direct outcome of heated disputes with Edward Teller on the origin of the cosmic rays. It was written to counter the view that cosmic rays were principally of solar origin and that they could not extend through all galactic space because of the very large amount of energy which would then be required. Taking up the study of the intergalactic magnetic fields, Fermi was able to find not only a way to account for the presence of the cosmic rays, but also a mechanism for accelerating them to the very high energies observed. He presented these same views on the origin of cosmic rays, though less extensively, in a talk at the Como International Congress on the Physics of Cosmic Rays (paper No. 238)."* (H.L. Anderson, Vol. 2, p. 655)

As Chandrasekhar recalls, *"In the fall of 1948, Edward Teller was advancing the view that cosmic rays are of solar origin. Fermi was wont to say—half-jokingly—that this inspired him to take an opposing view and advocate a galactic origin of the cosmic rays."* (S. Chandrasekhar, Vol. 2, p. 924)

It is therefore appropriate to recall here Teller's point of view: *"Fermi mentioned to me his interest in the origin of cosmic rays as early as 1946. Several years before that time he mentioned the subject in some lectures in Chicago. He had the suspicion that magnetic fields could accelerate the cosmic particles. In 1948 Alfvén visited Chicago. He had been interested in electromagnetic phenomena on the cosmic scale for quite some time. At that time I was playing with the idea that cosmic rays might be accelerated in the neighborhood of the sun. I had discussed this question with Alfvén, and he visited us in Chicago in order to carry forward the discussion. During this visit Fermi learned from Alfvén about the probable existence of greatly extended magnetic fields in our galactic system. Since this field would necessarily be dragged along by the moving and ionized interstellar material, Fermi realized that here was an excellent way to obtain the acceleration mechanism for which he was looking. As a result he outlined a method of accelerating cosmic ray particles which serves today as a basis for most discussions on the subject. In his papers published in 1949 (No 237 and 238) he explained most of the observed properties of cosmic rays with one important exception: it follows from his originally proposed mechanism that heavier nuclei will not attain as high velocities as protons do. This is in contradiction with experimental evidence. Fermi returned to this problem in his paper Galactic Magnetic Fields and the Origin of Cosmic Radiation (No. 264). Some details concerning the origin of cosmic rays have not been settled conclusively by Fermi's papers. Another competing theory has been proposed by Stirling Colgate and Montgomery Johnson according to which cosmic rays are produced by shock mechanism in exploding supernovae. The actual origin of cosmic rays continues to remain in doubt."* (E. Teller, Vol. 2, p. 655)

As Anderson recalls *"Fermi's interest in astrophysics was welcomed by the astrophysicists. They asked him to give the Sixth Henry Norris Russell Lecture of the American Astronomical Society. Fermi was quite pleased by this show of regard outside his own field and took the occasion to re-examine his earlier ideas about the origin of the cosmic rays in view of later developments in the knowledge of the strength and behavior of the magnetic fields."* (H.L. Anderson, Vol. 2, p. 970). (See also the introduction to paper No. 237.)

The paper FA 4 *E. Fermi: High energy nuclear events* (241) was published in the issue of the *Progress of Theoretical Physics* dedicated to the 15th anniversary of the Yukawa theory and considers a statistical description for pion production. As mentioned by Anderson in the comments to this paper in the collected work of Fermi,[11] the methods developed by Fermi were relatively simple, and moreover were deliberately simplified and therefore, were rather useful for experimentalists at that initial phase of high energy physics. Since pions are also bosons, at high energies when their rest mass can be neglected, the concept of temperature can be introduced and the energy density will be given by Stefan's law. Obtaining the temperature from the total energy within a given volume, the number densities of the produced

[11] *Fermi: Note e Memorie (Collected Papers)*, Vol. 2, 1965, p. 789.

pions and nucleons can then be estimated. The role played by thermalization in this paper has inspired us, even though the mechanism is different, namely astrophysical applications in the study of the spectra of gamma ray bursts (GRBs).

It is appropriate to recall here the comment of Isador Rabi in reaction to this paper as told by Anderson: *"Rabi's comment after hearing Fermi present this paper at an American Physical Society meeting in Chicago is worth recording here. 'If Fermi is right in saying that he can calculate what will happen at very high energies by purely statistical methods, then we will have nothing new to learn in this field.' Rabi should have had nothing to fear. Fermi's theory was greatly oversimplified as he intended it to be, and while it did not give very well the detailed results which were later found, it did serve as a standard against which one could make a first comparison of the experimental results of multiple production to reveal when something non-statistical was going on. In the later literature this made it appear that this theory was always wrong; a point that Fermi didn't enjoy at all. He had always stressed the purpose and limitations of his calculations and referred ironically to his own authority and to those who took his results beyond what he intended them to be."* (H.L. Anderson, Vol. 2, p. 789)

Fermi's theoretical papers rarely had co-authors. Among his few co-authors was Chandrasekhar, on two papers on magnetohydrodynamics, FA 5 *S. Chandrasekhar, E. Fermi: Magnetic Fields in Spiral Arms* (261) - FA 6 *S. Chandrasekhar, E. Fermi: Problems of Gravitational Stability in the Presence of a Magnetic Field* (262). Chandrasekhar's recollections on their joint work with remarkable details on Fermi's style of work are published in Volume 2 of Note e Memorie.[12] We give below some excerpts from them. D. Boccaletti comments on the two papers in an article (A.3) of Appendix A.

On paper (262) Chandrasekhar recalls: *"As I have already stated, Fermi and I discussed astrophysical problems regularly during 1952–53. The paper Problems of Gravitational Stability in the Presence of a Magnetic Field (No. 262) was an outcome of these discussions. Referring to this largely mathematical paper, several persons have remarked that it is "out of character" with Fermi. For this reason I may state that the problems which are considered in this paper were largely at Fermi's suggestion. The generalization of the virial theorem; the existence of an upper limit to the magnetic energy of a configuration in equilibrium under its own gravitation; the distortion of the spherical shape of a body in gravitational equilibrium by internal magnetic fields; the stabilization of the spiral arms of a galaxy by axial magnetic fields; all these were Fermi's ideas, novel at the time. But they had to be proved; for, as Fermi said: "It is so very easy to make mistakes in magneto-hydrodynamics that one should not believe in a result obtained after a long and complicated mathematical derivation if one cannot understand its physical origin; in the same way, one cannot also believe in a long and complicated piece of physical reasoning if one cannot demonstrate it mathematically." If only this dictum were followed by all!"* (S.

[12] *Fermi: Note e Memorie (Collected Papers)*, Vol. 2, 1965, p. 923–927.

Chandrasekhar, Vol. 2 p. 925)

And again Chandrasekhar: *"Fermi's interest in hydromagnetic turbulence led him to inquire into the physics of ordinary hydrodynamic turbulence. Confessing ignorance of this subject, Fermi asked me (early in 1950) to come to his office and tell him about the ideas of Kolmogoroff and Heisenberg which were then very much in the vogue. However, when I went to tell him, I found that it was not necessary for me to say beyond a few words: such as isotropy, the cascade of energy from large to small eddies etc. With only such words as clues, Fermi promptly went to the blackboard ("to see if I understand these words") and proceeded to derive the Kolmogoroff spectrum for isotropic turbulence (in the inertial range) and the basis of Heisenberg's elementary theory. Fermi's manner of arguing is worth recording for its transparent simplicity.*

Divide the scale of logk (where k denotes the wave number) into equal divisions, say $(\ldots, n, n+1, \ldots)$. *In a stationary state the rate of flow of energy across "n" must be equal to the rate of flow across "n + 1." Therefore:*

$$E_{n,n+1} = \rho \frac{v_n}{k_n}(v_n k_n)^2 - \rho \frac{v_{n+1}}{k_{n+1}}(v_{n+1}k_{n+1})^2 , \tag{1}$$

if one remembers that the characteristic time associated with "eddies" with wave numbers in the interval $(n, n+1)$ *is* $(v_{n+1}k_{n+1})^{-1}$. *From this relation it follows that:*

$$v_n = Constant \times k_n^{-1/3} , \tag{2}$$

and this is equivalent to Kolmogoroff's law. For decaying turbulence, equation (1) should be replaced by:

$$\frac{d}{dt}(\rho v_n)^2 = E_{n,n+1} \tag{3}$$

and this equation expresses the content of Heisenberg's theory." (Chandrasekhar, Vol. 2, pp. 925–926)

The paper FA 7 *E. Fermi, J. Pasta, S. Ulam: Studies of Non linear Problems* (266) (always quoted as F.P.U.) is outstanding for several reasons: (a) It represents the first computer study of a non-linear system; (b) the results contradicted the belief held since Poincaré, that any perturbed Hamiltonian system has to be chaotic. Fermi had considered it 'a little discovery' (as quoted by Ulam), thus immediately evaluating its extraordinary importance; (c) it was one of Fermi's last works, completed after his death in 1954; (d) remained unpublished for a decade; (e) coincides in time with Kolmogorov's theorem (1954), though FPU and Kolmogorov-Arnold-Moser (KAM) theory were linked to each other only in 1966; (f) inspired the discovery of solitons and numerous other studies; (g) its results are not fully understood till now and the FPU model continues its inspiring mission today, after half a century. In his recollections Ulam refers to Fermi's opinion on the importance of the "understanding of non-linear systems" for the future fundamental theories, and the "potentialities of the electronic computing machines" and even mentions

Fermi's learning of the actual coding (programming) during one summer. The FPU paper and its influence on various areas of astrophysics and stochastic dynamics are discussed in Appendix B (see the papers by A. Carati et al. (B.4), S. Ruffo (B.7) and G.M. Zaslavsky (B.11)). Here is the presentation written by S. Ulam.

"After the war, during one of his frequent summer visits to Los Alamos, Fermi became interested in the development and potentialities of the electronic computing machines. He held many discussions with me on the kind of future problems which could be studied through the use of such machines. We decided to try a selection of problems for heuristic work where in absence of closed analytic solutions experimental work on a computing machine would perhaps contribute to the understanding of properties of solutions. This could be particularly fruitful for problems involving the asymptotic-long time or "in the large" behavior of non-linear physical systems. In addition, such experiments on computing machines would have at least the virtue of having the postulates clearly stated. This is not always the case in an actual physical object or model where all the assumptions are not perhaps explicitly recognized.

Fermi expressed often a belief that future fundamental theories in physics may involve non-linear operators and equations, and that it would be useful to attempt practice in the mathematics needed for the understanding of non-linear systems. The plan was then to start with the possibly simplest such physical model and to study the results of the calculation of its long-time behavior. Then one would gradually increase the generality and the complexity of the problem calculated on the machine. The Los Alamos report LA–1940 (paper No. 266) presents the results of the very first such attempt. We had planned the work in the summer of 1952 and performed the calculations the following summer. In the discussions preceding the setting up and running of the problem on the machine we had envisaged as the next problem a two-dimensional version of the first one. Then perhaps problems of pure kinematics, e.g., the motion of a chain of points subject only to constraints but no external forces, moving on a smooth plane convoluting and knotting itself indefinitely. These were to be studied preliminary to setting up ultimate models for motions of system where "mixing" and "turbulence" would be observed. The motivation then was to observe the rates of mixing and "thermalization" with the hope that the calculational results would provide hints for a future theory. One could venture a guess that one motive in the selection of problems could be traced to Fermi's early interest in the ergodic theory. In fact, his early paper (No. 11a) presents an important contribution to this theory.

It should be stated here that during one summer Fermi learned very rapidly how to program problems for the electronic computers and he not only could plan the general outline and construct the so-called flow diagram but would work out himself the actual coding of the whole problem in detail. The results of the calculations (performed on the old MANIAC machine) were interesting and quite surprising to Fermi. He expressed to me the opinion that they really constituted a little discovery in providing intimations that the prevalent beliefs in the universality of "mixing and *thermalization" in non-linear systems may not be always justified.*

A few words about the subsequent history of this non-linear problem. A number of other examples of such physical systems were examined by calculations on the electronic computing machines in 1956 and 1957. I presented the results of the original paper on several occasions at scientific meetings; they seemed to have aroused considerable interest among mathematicians and physicists and there is by now a small literature dealing with this problem. The most recent results are due to N.J. Zabusky. (i) His analytical work shows, by the way, a good agreement of the numerical computations with the continuous solution up to a point where a discontinuity developed in the derivatives and the analytical work had to be modified. One obtains from it another indication that the phenomenon discovered is not due to numerical accidents of the algorithm of the computing machine, but seems to constitute a real property of the dynamical system.

In 1961, on more modern and faster machines, the original problem was considered for still longer periods of time. It was found by J. Tuck and M. Menzel that after one continues the calculations from the first "return" of the system to its original condition the return is not complete. The total energy is concentrated again essentially in the first Fourier mode, but the remaining one or two percent of the total energy is in higher modes. If one continues the calculation, at the end of the next great cycle the error (deviation from the original initial condition) is greater and amounts to perhaps three percent.[13] *Continuing again one finds the deviation increasing—after eight great cycles the deviation amounts to some eight percent; but from that time on an opposite development takes place! After eight more, i.e., sixteen great cycles altogether, the system gets very close better than within one percent to the original state! This supercycle constitutes another surprising property of our non-linear system."* (S.M. Ulam, Vol. 2, pp. 977–978)

Paper FA 8 *E. Fermi: Theories on the Origin of the Elements (240.3)* was a rough calculation of Fermi on the formation of the elements in the early hot big bang universe in response to Gamov's earlier attempt at solving this problem. It is followed by the later publication of the more detailed Fermi–Turkevich work on this problem, namely paper FA 9 *Fermi–Turkevich: An excerpt from "Theory of the Origin and Relative Abundance Distribution of the Elements," by Ralph A. Alpher and Robert C. Herman.* These are discussed in detail in the companion book *Einstein, Fermi, Heisenberg and the Birth of Relativistic Astrophysics.*

[13](i) Exact Solutions for the Vibrations of a non-linear continuous string. A. E. C. Research and Development Report, MATT-102, Plasma Physics Laboratory, Princeton University, October 1961.

1.3 Appendices

Appendix A includes some commentary articles on Fermi's resolution of this "4/3 problem" in the ratio between inertial mass and energy for the classical electron Coulomb field and a shorter journal article summarizing the natural completion of Fermi's original ideas about electromagnetic mass (see A.1–3), followed by a historical context commentary paper. We also reproduce the related article from 1949 by B. Kwal (see A.4) which seems to be the only one to touch upon this topic until the independent work of Rohrlich in 1960, after which Fermi's original contribution was rediscovered.

Appendix B contains a selection of the articles from the proceedings the meeting "Fermi and Astrophysics" organized at the University of Rome "La Sapienza" and at the ICRANet Center in Pescara October 3–6, 2001 and published in *Il Nuovo Cimento B* **117**, Nos. 9–11 (2002). The meeting was focused on the influence of Fermi on astrophysics and general relativity: his activities related to these topics were clustered at the beginning and end of his scientific career. These articles, selected because of their direct commentary on articles by Fermi or related applications of his ideas expressed in those articles, are presented in alphabetical order of their first authors.

Susan Ames discusses the historical background of Fermi's work on cosmic rays, along with current problems and further prospects for the physics of cosmic rays. In particular she points out how the frequently discussed ultra-high cosmic rays cannot be accelerated by the Fermi mechanism. Equipartition between the energy of matter and that of cosmic rays was among the initial points made by Fermi, and in that context Ames mentions also the role of the cosmic microwave background radiation.

Donato Bini and Robert Jantzen give a summary of Fermi's discussion of what we now call Fermi coordinates and Fermi transport with a historical update including Walker's contribution which led to the terminology of "Fermi-Walker transport." This article explicitly estimates the various relativistic contributions to the Fermi-Walker transport for vectors around circular orbits in black hole spacetimes and in their Minkowski limit.

Dino Boccaletti comments on the two papers which resulted from the collaboration of Fermi with Chandrasehkar (see papers 261, 262 of Chapter 4). The first paper is devoted to the study of light dispersion in the polarization plane and using the effect to derive the galactic magnetic field. The second paper contains the generalization of the virial theorem in the presence of a magnetic field. The commentary notes that Fermi was the first scientist to draw attention to the possible existence of a galactic magnetic field.

The review of Andrea Carati, Luigi Galgani, Antonio Ponno and Antonio Giorgilli is devoted to the equipartition problem in the Fermi-Pasta-Ulam paradox both in classical and quantum mechanics. Equipartition is discussed starting from Planck's work and Poincaré's theorem. Numerical results on the dependence of the existence of equipartition and the corresponding time scales on a certain critical energy are mentioned.

Piero Cipriani reviews the work of Fermi in the field of classical analytical mechanics. After a short historical introduction, he emphasizes some aspects of geometrical methods of the description of dynamics and the theory of stochastic differential equations. Interesting recollections on Fermi are quoted.

John G. Kirk reviews the Fermi acceleration mechanism in the context of galactic nuclei and gamma ray bursts, i.e., in processes involving relativistic motion. Diffusive and non-diffusive versions of Fermi's stochastic acceleration are considered, including those predicting a softer spectrum of accelerated particles. The appearance of anisotropy in the accelerated particles with increasing gamma factor is discussed for various astrophysical situations.

Stefano Ruffo reviews evidence for long relaxation time scales in Hamiltonian systems, and shows how complex and diverse is the dynamics of long-range systems. The 'quasi-states' of Fermi-Pasta-Ulam are discussed particularly in the context of two theoretical approaches developed by the author and collaborators, one based on the Vlasov-Poisson equation, and the other based on the averaging of fast oscillations.

Costantino Sigismondi and Francesca Maiolino review an early work by Fermi completed June 20, 1922, the year of his habilitation thesis on statistics at the Scuola Normale Superiore of Pisa, with an application to the case of comets. Fermi studied this case with a coplanar orbit to the one of Jupiter, neglecting the influence of other planets. The probability of ejection of the comet from the solar system (a parabolic or hyperbolic orbit) after interaction with Jupiter is calculated, as well as the probability of an impact with Jupiter. They apply Fermi's results to the case of the Earth in order to recover the time rate of collision of comets with our planet, which reliably produced the extinction of the dinosaurs. In this context the properties of the Oort cloud are discussed as well.

Costantino Sigismondi and Angelo Mastroianni recall that approximately in the same period Fermi studied the formation of X-ray images and presented his first experimental work as a dissertation at the University of Pisa in the spring of 1922. The need for Fermi to make an experimental essay was made mandatory since at that time theoretical physics was not yet considered sufficient to have independent validity. Although his seminal ideas are not among the bibliographical sources investigated by Riccardo Giacconi and Bruno Rossi (1960) when they proposed a telescope using X-rays, Fermi's thesis was the most complete study of X-ray physics in his time. Fermi used the technique of 'mandrels' to form optical surfaces. He anticipated the technique used for the mirrors of the Exosat, Beppo-SAX, Jet-X and XMM-Newton telescopes, a technique which is now a mainstay of optical manufacturing.

Alexei Yu. Smirnov reviews the neutrino flavor transformations in matter, as one of the authors of the original theoretical predictions and related observable effects. In particular, the Sudbury Neutrino Observatory results provide strong evidence of the neutrino flavor conversion. Neutrino conversion is discussed also in the context of supernova neutrinos and the corresponding predictions for the fluxes and energies at the Earth, including the role of the Earth matter effect. The author shows that the data of SN1987 can also be explained by the neutrino oscillations in the matter of Earth as conversions of muon and tau antineutrinos.

George M. Zaslavsky reviews the Fermi-Pasta-Ulam problem with an attempt to find the transition from regular to chaotic dynamics. The Fermi acceleration mechanism is considered as a precursor of the Fermi-Pasta-Ulam problem. The Kepler map introduced by Roald Sagdeev and George Zaslavsky and several other problems are considered, demonstrating the role of the Fermi-Pasta-Ulam work in the discretization methods of differential equations and in the study of chaotic systems when the Lyapunov exponent method is not efficient.

Chapter 2

From Fermi's papers of the Italian period

1) On the dynamics of a rigid system of of electric charges on translational motion

"Sulla dinamica di un sistema rigido di cariche elettriche in moto traslatorio," Nuovo Cimento 22, 199–207 (1921).

§ 1.– When a rigid system of electric charges moves arbitrarily, the electric field it generates is different from that which Coulomb's law would predict. Now, the electric field produced by the entire system exerts some forces on each element of charge of the system. The resultant of these forces, namely the resultant of the internal electric forces, would of course be identically zero if Coulomb's law were valid, but it no longer is, however, at least in general, when the system moves, since in such a case that law is no longer valid.

This resultant gives the electromagnetic inertial reaction, and the aim of the present work is precisely its evaluation in the case of an arbitrary system in translational motion. In the case in which the system is a spherical distribution of surface electricity, as it is assumed in most electronic models, it is known that one finds [1] that such a resultant, at least in the first approximation, is given by

$$-\frac{2e^2}{3\mathrm{R}c^3}\,\mathbf{\Gamma}+\frac{2e^2}{3c^2}\,\dot{\mathbf{\Gamma}}\,, \tag{1}$$

where e, R denote the total charge and the radius of the system, c is the speed of light, $\mathbf{\Gamma}$ and $\dot{\mathbf{\Gamma}}$ are the acceleration and its derivative with respect to time. For quasi-stationary motions the second term of (1) becomes negligible, so that (1) reduces to

$$-m\mathbf{\Gamma}\,, \tag{2}$$

where m is the elecromagnetic mass.

In § 2 one finds the generalization of (1) to the case of any system, referring for example to molecular models, always assuming that the velocity is negligible with respect to the speed of light. If F_i $(i = 1, 2, 3)$ are the components of the resultant in question, one finds

$$\mathrm{F}_i = -\sum_k m_{ik}\Gamma_k + \sum_k \sigma_{ik}\dot{\Gamma}_k\,, \tag{3}$$

where m_{ik}, σ_{ik} are some quantities depending on the properties of the system. Therefore one can no longer refer to a scalar electromagnetic mass, but instead in its place one introduces the tensor m_{ik}.

§ 3 is devoted to the dynamical study of the law for quasi-stationary motions:

$$\mathrm{K}_i = \sum_k m_{ik}\Gamma_k\,, \tag{4}$$

[1] RICHARDSON, *Electron Theory of Matter*, Chapter XIII. The difference between my formulas and those of Richardson is due to the fact that he adopts Heaviside units.

We denote orthogonal Cartesian coordinates by x_1, x_2, x_3, and let (x_i) be the coordinates of M, (x'_i) those of P. The components of **a** are $a_i = \dfrac{x'_i - x_i}{r}$. Writing (6) in scalar form, one thus obtains

$$\mathrm{F}_i = -\sum_k m_{ik}\Gamma_k + \sum_k \sigma_{ik}\dot{\Gamma}_k \; ; \tag{7}$$

noting that, under the assumption of translational motion, Γ_i and $\dot{\Gamma}_i$ are constant when the integration is performed.

Here one has set:

$$\begin{cases} m_{ii} = \dfrac{2\mathrm{U}}{c^2} - \displaystyle\iint \dfrac{\rho\rho'(x'_i - x_i)^2}{c^2 r^3}\, d\tau\, d\tau' \; , \\[2ex] m_{ik} = m_{ki} = -\displaystyle\iint \dfrac{\rho\rho'(x'_i - x_i)(x'_k - x_k)}{c^2 r^3}\, d\tau\, d\tau' \; , \qquad i \neq k \; , \end{cases} \tag{8}$$

$$\begin{cases} \sigma_{ii} = \dfrac{e^2}{c^3} - \displaystyle\iint \dfrac{\rho\rho'(x'_i - x_i)^2}{c^3 r^2}\, d\tau\, d\tau' \; , \\[2ex] \sigma_{ik} = \sigma_{ki} = -\displaystyle\iint \dfrac{\rho\rho'(x'_i - x_i)(x'_k - x_k)}{c^3 r^2}\, d\tau\, d\tau' \; ; \qquad i \neq k \; . \end{cases} \tag{9}$$

In these formulae U represents the electrostatic energy of the system $= \dfrac{1}{2}\displaystyle\iint \dfrac{\rho\rho'}{r}\, d\tau\, d\tau'$, and e the total electric charge $= \displaystyle\int \rho\, d\tau = \int \rho'\, d\tau'$.

From the expressions (8), (9) it immediately follows that if the axes (x_i) are substituted by others (y_i) using the orthogonal substitution

$$y_i = \sum_k \alpha_{ik} x_k \; ,$$

the m_{ik} and σ_{ik} corresponding to the new axes are given by:

$$m'_{ik} = \sum_{rs} \alpha_{ir}\alpha_{ks} m_{ik} \; ,$$

$$\sigma'_{ik} = \sum_{rs} \alpha_{ir}\alpha_{ks} \sigma_{ik} \; .$$

Hence both m_{ik} and σ_{ik} are symmetric covariant tensors. Each of them will have three orthogonal principal directions such that, taking the axes to be parallel to them, one has either $m_{ik} = 0$ or $\sigma_{ik} = 0$ when $i \neq k$.

The principal axes of tensors m, σ, however, will be different in general. If the case that the system has spherical symmetry one can do the integrations (8) and (9), since instead of $\dfrac{(x'_i - x_i)(x'_k - x_k)}{r^2}$ one can put the mean value of this expression over all possible directions MP, since in this case to the two points MP correspond an infinite number of pairs which differ only by orientation. Now, this mean value if $i = k$ is given by $\dfrac{2\pi}{4\pi}\displaystyle\int_0^\pi \cos^2\theta \sin\theta\, d\theta$; if instead $i \neq k$, it is zero.

So one then has

$$m_{11} = m_{22} = m_{33} = \frac{4\mathrm{U}}{3c^2} \; ; \qquad m_{23} = m_{31} = m_{12} = 0 \; ;$$

$$\sigma_{11} = \sigma_{22} = \sigma_{33} = \frac{2}{3}\frac{e^2}{c^3} \; ; \qquad \sigma_{23} = \sigma_{31} = \sigma_{12} = 0 \; .$$

By substituting these values into (7), one obtains well-known formulas if the system consists of a homogeneous spherical layer.

§ 3.– Returning to the general case, we note that for quasi-stationary motions (5) can be replaced by:

$$\mathrm{F}_i = -\sum_k m_{ik}\Gamma_k \; .$$

If one thinks of an external force (X_i) acting on the system, the total force will be ($\mathrm{X}_i + \mathrm{F}_i$). If one now supposes that the system has no material mass one must have $\mathrm{X}_i + \mathrm{F}_i = 0$, and so

$$\mathrm{X}_i = \sum_k m_{ik}\Gamma_k \; . \tag{10}$$

It is easy to show how with the law (10) the principle of the kinetic energy theorem and of Hamilton's principle are preserved. In fact, denoting the velocity by $\mathrm{V} \equiv (\mathrm{V}_1, \mathrm{V}_2, \mathrm{V}_3)$ and multiplying (10) by V_i, then summing with respect to i one obtains

$$\sum_i \mathrm{X}_i \mathrm{V}_i = \sum_{ik} m_{ik} \mathrm{V}_k \frac{d\mathrm{V}_i}{dt} \; .$$

Interchanging i and k in the second sum, and noting that $m_{ik} = m_{ki}$

$$\sum_i \mathrm{X}_i \mathrm{V}_i = \sum_{ik} m_{ik} \mathrm{V}_i \frac{d\mathrm{V}_k}{dt} \; ,$$

and summing

$$2\sum_i \mathrm{X}_i \mathrm{V}_i = \sum_{ik} m_{ik} \left(\mathrm{V}_i \frac{d\mathrm{V}_k}{dt} + \mathrm{V}_k \frac{d\mathrm{V}_i}{dt} \right) = \frac{d}{dt} \sum_{ik} m_{ik} \mathrm{V}_i \mathrm{V}_k \; .$$

The first left-hand side is twice the potential P of the external forces. Thus one has

$$P = \frac{dT}{dt} \; , \qquad \text{where} \qquad T = \frac{1}{2}\sum_{ik} m_{ik} \mathrm{V}_i \mathrm{V}_k \; . \tag{11}$$

Multiplying, instead, the two sides of (10) by δx, and then summing, one similarly gets

$$\sum_i \mathrm{X}_i \delta x_i = \frac{1}{2}\sum_{ik} m_{ik} \left(\frac{d^2 x_k}{dt^2}\,\delta x_i + \frac{d^2 x_i}{dt^2}\,\delta x_k \right)$$

$$= \frac{d}{dt}\left\{ \frac{1}{2}\sum_{ik} m_{ik}(\dot{x}_k \delta x_i + \dot{x}_i \delta x_k) \right\} - \frac{1}{2}\sum_{ik} m_{ik}(\dot{x}_k \delta \dot{x}_i + \dot{x}_i \delta \dot{x}_k)$$

$$= \frac{d}{dt}\left\{ \frac{1}{2}\sum_{ik} m_{ik}(\dot{x}_k \delta x_i + \dot{x}_i \delta x_k) \right\} - \delta T \; .$$

Multiplying by dt and integrating between two limits t', t'' at which the variations δx_i are assumed to be zero, one obtains

$$\int_{t'}^{t''} \left(\delta T + \sum_i \mathrm{X}_i \delta x_i \right) = 0 \ , \tag{12}$$

expressing Hamilton's principle.

If one refers to the principal axes of the tensor m_{ik} instead of arbitrary ones, (10) takes the simple form:

$$\mathrm{X}_i = m_{ii} \Gamma_i \ . \tag{13}$$

§ 4.– This formula holds only if V/c is negligible. To generalize it to an arbitrary velocity let us denote by $\mathrm{S} \equiv (x_1, x_2, x_3, t)$ the indicated reference frame and by $\mathrm{S}^* \equiv (x, y, z, t)$ a frame fixed with respect to S with the x–axis orientated along the velocity of the system at a certain fixed but generic time $\bar{t}$, and finally let $\mathrm{S}' \equiv (x', y', z', t')$ be a system with spatial axes parallel to xyz which moves uniformly with respect to S^* with velocity equal to that of the moving one at time $\bar{t}$, whose magnitude is v. One will have

$$t' = \beta \left(t - \frac{v}{c^2}\, x \right) \ ; \quad x' = \beta\,(x - vt) \ ; \quad y' = y \ ; \quad z' = z \ ; \quad \beta = \frac{1}{\sqrt{1 - \dfrac{v^2}{c^2}}} \ , \tag{14}$$

where, once $\bar{t}$ is fixed, v and hence β are constant.

Let us asumme that the forces acting on our system are due to an external electromagnetic field (**E**, **H**); since at the instant t the system has velocity zero with respect to S', (10) will hold for it, and so one will therefore have, with an obvious meaning for the symbols:

$$\begin{aligned} e\,\mathrm{E}'_x &= m_{xx}\Gamma'_x + m_{xy}\Gamma'_y + m_{xz}\Gamma'_z \\ e\,\mathrm{E}'_y &= m_{yx}\Gamma'_x + m_{yy}\Gamma'_y + m_{yz}\Gamma'_z \\ e\,\mathrm{E}'_z &= m_{zx}\Gamma'_x + m_{zy}\Gamma'_y + m_{zz}\Gamma'_z \ . \end{aligned}$$

But one has

$$e\,\mathrm{E}'_x = e\,\mathrm{E}_x \ , \qquad e\,\mathrm{E}'_y = e\beta \left(\mathrm{E}_y - \frac{v}{c}\,\mathrm{H}_z \right) \ , \qquad e\,\mathrm{E}'_z = e\beta \left(\mathrm{E}_z + \frac{v}{c}\,\mathrm{H}_y \right) \ .$$

So therefore setting

$$\mathbf{k} = e \left(\mathrm{E} + \frac{1}{c}\,\mathrm{V} \times \mathrm{H} \right) \ , \tag{15}$$

one finds

$$e\,\mathrm{E}'_x = e\,\mathrm{E}_x \ , \qquad e\,\mathrm{E}'_y = e\beta k_y \ , \qquad e\,\mathrm{E}'_z = e\beta k_z \ .$$

On the other hand

$$\Gamma'_x = \frac{d^2x'\,dt' - d^2t'\,dx'}{dt'^3} \ ,$$

but at time $\bar{t}$, $\dfrac{dx'}{dt'} = 0$, hence $\Gamma'_x = \dfrac{d^2x'}{dt'^2}$. Taking t as the independent variable, and noting that $\dfrac{dx}{dt} = v$, then $\Gamma'_x = \beta^3\Gamma_x$. Analogously, $\Gamma'_y = \beta^2\Gamma_y$ and $\Gamma'_z = \beta^2\Gamma_z$. Substituting

$$\begin{cases} k_x = m_{xx}\beta^3\ddot{x} + m_{xy}\beta^2\ddot{y} + m_{xz}\beta^2\ddot{z} \\ k_y = m_{yx}\beta^3\ddot{x} + m_{yy}\beta\ddot{y} + m_{yz}\beta\ddot{z} \\ k_z = m_{zx}\beta^3\ddot{x} + m_{zy}\beta\ddot{y} + m_{zz}\beta\ddot{z}\ . \end{cases} \tag{16}$$

Denoting by α_{xi} the cosine of the angle between the x–axis and the x_i–axis, one has

$$k_i = \alpha_{xi}k_x + \alpha_{yi}k_y + \alpha_{zi}k_z\ .$$

On the other hand, being m_{i0} covariant, one has for instance

$$m_{xy} = \sum_r m_{rr}\alpha_{xr}\alpha_{yr}\ .$$

Analogously

$$\ddot{x} = \sum_j \ddot{x}_j\alpha_{xj}\ .$$

Multiplying then (16) by $\alpha_{xi}, \alpha_{yi}, \alpha_{zi}$ and summing, one finds

$$k_i = \sum_{rj} m_{rr}\ddot{x}_j \begin{bmatrix} \beta^3\alpha^2_{xr}\alpha_{xj}\alpha_{xi} + \beta^2\alpha_{xr}\alpha_{yr}\alpha_{yj}\alpha_{xi} + \beta^2\alpha_{xr}\alpha_{zr}\alpha_{zj}\alpha_{xi} \\ +\beta^2\alpha_{yr}\alpha_{xr}\alpha_{xj}\alpha_{yi} + \beta\alpha^2_{yr}\alpha_{yj}\alpha_{yi} + \beta\alpha_{yr}\alpha_{zr}\alpha_{zj}\alpha_{yi} \\ +\beta^2\alpha_{zr}\alpha_{xr}\alpha_{xj}\alpha_{zi} + \beta\alpha_{zr}\alpha_{yr}\alpha_{yj}\alpha_{zi} + \beta\alpha^2_{zr}\alpha_{zj}\alpha_{zi} \end{bmatrix}$$

But one has $\alpha_{xi} = \dfrac{\dot{x}_i}{v}$. Taking into account the relations between the α's, one finally finds the sought after generalization of (13)

$$\begin{aligned} k_i = \beta\sum_{rj}\ddot{x}_j m_{rr}\Big\{ &(\beta-1)^2\frac{\dot{x}_i\dot{x}_j\dot{x}_r^2}{v^4} \\ &+(\beta-1)\left[(jr)\frac{\dot{x}_i\dot{x}_r}{v^2} + (ir)\frac{\dot{x}_j\dot{x}_r}{v^2}\right] + (ir)(jr)\Big\}\ , \end{aligned} \tag{17}$$

where

$$(jr) = 1\ , \quad \text{if} \quad j = r\ ; \qquad (jr) = 0\ , \quad \text{if} \quad j \neq r\ .$$

In the case of spherical symmetry, setting $m_{11} = m_{22} = m_{33} = m$, one can evaluate the sum in (17), finding:

$$k_i = \beta m\ddot{x}_i + m\beta(\beta^2 - 1)\frac{\dot{x}_i}{v^2}\sum_j \dot{x}_j\ddot{x}_j\ ,$$

from which, recalling that

$$\beta = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} ,$$

one recovers the well-known formula of electronic dynamics

$$k_i = \frac{d}{dt} \frac{m\dot{x}_i}{\sqrt{1 - \frac{v^2}{c^2}}} .$$

Pisa, January 1921

2) On the electrostatics of a homogeneous gravitational field and on the weight of electromagnetic masses

"Sull'elettrostatica di un campo gravitazionale uniforme e sul peso delle masse elettromagnetiche,"
Nuovo Cimento 22, 176–188 (1921).

INTRODUCTION

The aim of ~~the present~~ this paper is to investigate in the framework of general relativity how a homogeneous gravitational field modifies the electrostatic phenomena occurring in it. Once the differential equation relating the electrostatic potential to the charge density, which corresponds to the Poisson equation in classical electrostatics, is established, one is able to integrate it at least when the gravitational field is weak enough (and certainly the gravitational field of the Earth amply satisfies this condition), obtaining in this way the corrections to Coulomb's law due to the presence of the gravitational field.

In a first application the distribution of the electric charges on a conducting sphere is studied, showing that the sphere polarizes by means of the gravitational field.

The second application is devoted to studying the weight of an electromagnetic mass, that is the force exerted on a fixed system of electric charges (e.g., sustained by a rigid dielectric), as a consequence of the presence of the gravitational field.

One finds that such a weight is given by the acceleration of gravity times u/c^2, where u denotes the electrostatic energy of the charges of the system, and c is the velocity of light. So the gravitational mass, namely the ratio between the weight and the acceleration of gravity, does not coincide in general with the inertial mass for the system under consideration, since the latter is given by $(4/3)u/c^2$ (with the same notation) if the system is endowed with spherical symmetry for example.

Besides it is known how special relativity leads us to take $\Delta u/c^2$ as the increase of the *inertial* mass of a system getting an energy Δu, and this fact can be easily related to the aforementioned result.

Finally, it is shown how to find a point having the same properties, with respect to the weight of the considered system of charges, as the center of gravity with respect to the weight of an ordinary system of material masses.

PART 1

ELECTROSTATICS IN A GRAVITATIONAL FIELD

§ 1.– Let us consider a portion of the spacetime where a homogeneous gravitational field is present, and assume the electrostatic phenomena that we think are taking place in it to be weak enough to neglect the effect they produce on the metric describing the region under consideration. Under this assumption, the line element of the spacetime manifold can be written as [1]

$$ds^2 = a\,dt^2 - dx^2 - dy^2 - dz^2 \;, \tag{1}$$

where a is a function only of z.

nts

The variables t, x, y, z will also be denoted by x_0, x_1, x_2, x_3, and the coefficients of the quadratic form (1) by g_{ik}. Let φ_i be the vector potential, and F_{ik} the electromagnetic field. Then we have

$$\mathrm{F}_{ik} = \varphi_{i,k} - \varphi_{k,i} \;, \tag{2}$$

referring ourselves to the fundamental form (1).

By limiting our considerations to electrostatic fields, we can set $\varphi_1 = \varphi_2 = \varphi_3 = 0$, and, for the sake of brevity, $\varphi_0 = \varphi$. Thus one has:

$$\mathrm{F}_{ik} = \varphi_{i,k} - \varphi_{k,i} = \frac{\partial\varphi_i}{\partial x_k} - \frac{\partial\varphi_k}{\partial x_i} \;,$$

that is

$$\begin{cases} \mathrm{F}_{01} = \dfrac{\partial\varphi}{\partial x} \;, \qquad \mathrm{F}_{02} = \dfrac{\partial\varphi}{\partial y} \;, \qquad \mathrm{F}_{03} = \dfrac{\partial\varphi}{\partial z} \;, \\ \mathrm{F}_{23} = \mathrm{F}_{31} = \mathrm{F}_{12} = 0 \;, \qquad \mathrm{F}_{ik} = -\mathrm{F}_{ki} \;, \qquad \mathrm{F}_{ij} = 0 \;. \end{cases} \tag{3}$$

In addition one has:

$$\mathrm{F}^{(ik)} = \sum_{hk} g^{(ih)} g^{(jk)} \mathrm{F}_{(hk)} = g^{(ii)} g^{(jj)} \mathrm{F}_{(ij)} \;,$$

from which by noting that:

$$g^{(00)} = \frac{1}{a} \;, \qquad g^{(11)} = g^{(22)} = g^{(33)} = -1 \;,$$

one obtains

$$\begin{cases} \mathrm{F}^{(01)} = -\dfrac{1}{a}\dfrac{\partial\varphi}{\partial x} \;, \qquad \mathrm{F}^{(02)} = -\dfrac{1}{a}\dfrac{\partial\varphi}{\partial y} \;, \qquad \mathrm{F}^{(03)} = -\dfrac{1}{a}\dfrac{\partial\varphi}{\partial z} \;, \\ \mathrm{F}^{(23)} = \mathrm{F}^{(31)} = \mathrm{F}^{(12)} = 0 \;, \qquad \mathrm{F}^{(ik)} = -\mathrm{F}^{(ki)} \;, \qquad \mathrm{F}^{(ii)} = 0 \;. \end{cases} \tag{4}$$

In the case under consideration here, the action can be written in the form

$$W = \int_\omega \sum_{ik} \mathrm{F}_{ik}\mathrm{F}^{(ik)}\, d\omega + \int de \int \varphi\, dx_0 \;, \tag{5}$$

where

$$d\omega = \sqrt{-||g_{ik}||}\, dx_0\, dx_1\, dx_2\, dx_3 = \sqrt{a}\, dx\, dy\, dz\, dt$$

[1] T. Levi-Civita, Note II. "Sui ds^2 einsteiniani". *Rend. Acc. Lincei*, **27**, 1° sem. N° 7.

is the hypervolume element of the manifold, and the integration corresponding to $d\omega$ has to be performed over a specific region of the manifold, while the integrations corresponding to de, dx_0 have to be extended to all the elements of electric charge whose world lines cross the region under consideration and to the portions of those world lines lying in it, respectively.

§ 2.– In the variation of W, φ can be arbitrarily varied, under the single condition that $\delta\varphi = 0$ on the boundary of the integration domain.

The variations δx, δy, δz instead, in addition to the condition $\delta x = \delta y = \delta z = 0$ on the boundary, could also be subjected to further conditions to be determined in each particular case. For example, inside a conducting body they will be quite arbitrary, while in a rigid dielectric they will have to represent the components of a rigid virtual displacement, and so on.

By putting the quantities (3), (4) into (5), one obtains:

$$W = -\frac{1}{2}\iiiint \frac{1}{\sqrt{a}}\left\{\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2 + \left(\frac{\partial\varphi}{\partial z}\right)^2\right\} dx\, dy\, dz\, dt + \int de \int \varphi\, dt\ , \tag{6}$$

from which

$$\begin{aligned}\delta W = &\iiiint \delta\varphi\left[\frac{1}{\sqrt{a}}\Delta_2\,\varphi + \frac{\partial\varphi}{\partial z}\frac{d(1/\sqrt{a})}{dz} + \rho\right] dx\, dy\, dz\, dt \\ &+ \iiiint \rho\left(\frac{\partial\varphi}{\partial x}\,\delta x + \frac{\partial\varphi}{\partial y}\,\delta y + \frac{\partial\varphi}{\partial z}\,\delta z\right) dx\, dy\, dz\, dt\end{aligned} \tag{7}$$

as immediately follows by noting that $dx = dy = dz = 0$ along a given world line, as a consequence of our assumptions, and $\rho\, dx\, dy\, dz = de$, since ρ is the electric density.

Then in order for δW to vanish identically, since $\delta\varphi$ is arbitrary inside the integration domain, one finds that

$$\Delta_2\,\varphi - \frac{d \log\sqrt{a}}{dz}\frac{\partial\varphi}{\partial z} = -\rho\sqrt{a}\ . \tag{8}$$

Moreover, one must also have

$$\iiiint \rho\left(\frac{\partial\varphi}{\partial x}\,\delta x + \frac{\partial\varphi}{\partial y}\,\delta y + \frac{\partial\varphi}{\partial z}\,\delta z\right) dx\, dy\, dz\, dt = 0\ , \tag{9}$$

for every system of values for δx, δy, δz satisfying the assumed constraints.

In equation (8) is contained the generalization of the Poisson's law, to which the (8) reduces if a is constant, that is if the gravitational field is absent.

§ 3.– If we indicate by G the acceleration of gravity of the field under consideration, namely the acceleration with which a free material point begins to move, one has:

$$\mathrm{G} = -\frac{1}{2}\frac{da}{dz}\ . \tag{10}$$

With this (8) becomes:

$$\Delta_2 \varphi + \frac{G}{a}\frac{\partial \varphi}{\partial z} = -\rho\sqrt{a} \,. \tag{11}$$

In order to find the solution of (11), given ρ at each point, we imagine the electric charges to be contained in a small region around the origin of the coordinates. Moreover, we will set $a = c^2$ at the origin (with c the velocity of light near the origin), and we will assume gravity to be so weak that those terms which contain the square of the ratio $l\,G/c^2$ can be neglected, where l represents the maximum length entering into the problem under consideration. Under these assumptions, we can set:

$$\sqrt{a} = c + \frac{1}{2c}\frac{da}{dz}z = c\left(1 - \frac{G}{c^2}z\right) .$$

Therefore (11) can be written as:

$$\Delta_2 \varphi + \frac{G}{c^2}\frac{\partial \varphi}{\partial z} = -c\left(1 - \frac{G}{c^2}z\right)\rho \,. \tag{12}$$

The integral of that equation in this approximation, as can be directly verified, is given by:

$$\begin{aligned}\varphi_P &= \frac{c}{4\pi}\int\left(1 - \frac{G}{c^2}\right) z_M \, d\tau_M \left(\frac{1}{r} - \frac{G}{2c^2}\frac{z_P - z_M}{r}\right)\\ &= \frac{c}{4\pi}\int \rho_M \, d\tau_M \left(\frac{1}{r} - \frac{G}{2c^2}\frac{z_P + z_M}{r}\right) ,\end{aligned} \tag{13}$$

where M is the generic point of the region τ_M containing the electric charges, P is the point at which the potential φ is evaluated, and r is the distance MP.

Given the linearity of equation (12), any integral of the equation:

$$\Delta_2 \varphi + \frac{G}{c^2}\frac{\partial \varphi}{\partial z} = 0 \,, \tag{12}^*$$

obtained by setting $\rho = 0$ in (12), can be added to (13). This integral will represent the field due to causes external to ρ_M. For the application we have in mind it is convenient to consider a particular solution to (12)* given by

$$\varphi = -cE^*_x x - cE^*_y y + \frac{c^2}{G}E^*_z e^{-\frac{G}{c^2}z} \,, \tag{14}$$

with E^*_x, E^*_y, E^*_z constants.

At the origin one has

$$E_x = -\frac{1}{c}F_{01} \,, \qquad E_y = -\frac{1}{c}F_{02} \,, \qquad E_z = -\frac{1}{c}F_{03} \,,$$

since E is the electric force.

From this it follows that the electric force of the external field (14) has components

$$E^*_x \,, \quad E^*_y \,, \quad E^*_z \,.$$

§ 4.– Let us now calculate the electric field due to a charge e concentrated at the origin of the coordinates. From (13) one has:

$$\varphi = \frac{ce}{4\pi}\left(\frac{1}{r} - \frac{\mathrm{G}}{2c^2}\frac{z}{r}\right) \, , \tag{15}$$

and this formula gives the generalization of Coulomb's law, as immediately follows by setting $\mathrm{G} = 0$. Recalling (3) one gets:

$$\begin{cases} \mathrm{F}_{01} = \dfrac{ce}{4\pi}\left(\dfrac{x}{r^3} - \dfrac{\mathrm{G}}{2c^2}\dfrac{zx}{r^3}\right) \, , \\[2ex] \mathrm{F}_{02} = \dfrac{ce}{4\pi}\left(\dfrac{y}{r^3} - \dfrac{\mathrm{G}}{2c^2}\dfrac{zy}{r^3}\right) \, , \\[2ex] \mathrm{F}_{03} = \dfrac{ce}{4\pi}\left(\dfrac{z}{r^3} - \dfrac{\mathrm{G}}{2c^2}\dfrac{z^2}{r^3} + \dfrac{\mathrm{G}}{2c^2}\dfrac{1}{r}\right) \, . \end{cases} \tag{16}$$

We can summarize all three of the preceding formulas in a single vector formula. In fact by indicating by F_0 the vector with components $\mathrm{F}_{01}, \mathrm{F}_{02}, \mathrm{F}_{03}$, with $\vec{a}$ a vector of magnitude 1 and orientation MP, and finally with $\vec{G}$ a vector of magnitude G and orientation z, (16) can be written as:

$$\mathrm{F}_0 = \frac{ce}{4\pi}\left\{\frac{\vec{a}}{r^2} + \frac{\vec{\mathrm{G}} \times \vec{a}}{2c^2 r}\vec{a} - \frac{1}{2c^2 r}\vec{\mathrm{G}}\right\} \, . \tag{17}$$

It is interesting to compare this formula with the one which gives the electric force exerted by an electric charge e which in the absence of gravitational attraction has acceleration $\vec{\Gamma}$, quasi-stationary motion and velocity negligible with respect to the speed of light. Such a force is expressed by

$$\mathrm{E} = \left\{\frac{\vec{a}}{r^2} + \frac{\vec{\Gamma} \times \vec{a}}{c^2 r}\vec{a} - \frac{1}{2c^2 r}\vec{\Gamma}\right\} \, , \tag{18}$$

with the same notation.

From here one sees that, by setting

$$\vec{\Gamma} = -\frac{\vec{\mathrm{G}}}{2} \tag{19}$$

in (18), one obtains

$$\mathrm{F}_0 = c\mathrm{E} \, .$$

This result can be put into words as follows, noting that $c\mathrm{E}$ is the electric part of the electromagnetic field generated by the charge in accelerated motion:

The electric part $(\mathrm{F}_{01}, \mathrm{F}_{02}, \mathrm{F}_{03})$ of the electromagnetic field (F_{ik}) generated by an electric charge at rest in a homogeneous field of strength G is equal to the electric part of the electromagnetic field which the same charge would produce in the absence of gravitational field if it moved under the conditions indicated above with acceleration G/2 in the direction opposite to the gravitational field.

§ 5.– Now, let us study how the distribution of the electricity over a conductor is modified by the gravitational field. To this end, let us note that since δx, δy, δz are arbitrary inside the conductor, from (9) it follows that $\varphi = constant$ inside, and so $\rho = 0$ by (8). Thus the electricity is completely at the surface. Then let us assume that our conductor is a sphere with center O at the origin of the coordinates and of radius R.

Let us try to satisfy the condition $\varphi = constant$ in the interior by assuming the following expression for the surface electric density at a generic point M of the surface:

$$\frac{e}{4\pi R^2} + \frac{e}{r}\, a\cos\theta\ , \tag{20}$$

where θ represents the angle spanned by the radius vector OM from the z–axis, and a is a constant to be determined, which we assume to be of the order of magnitude of G/c^2. From (13), the potential at a point P inside will be given by:

$$\varphi_\mathrm{P} = \frac{c}{4\pi}\int_\sigma \left(\frac{e}{4\pi r^2} + \frac{e}{r}\, a\cos\theta\right)\left(\frac{1}{r} - \frac{\mathrm{G}}{2c^2}\frac{z_\mathrm{P} + z_\mathrm{M}}{r}\right) d\sigma\ ,$$

where the integration must be extended over the whole surface σ of the sphere. By neglecting terms of order greater than G/c^2 one obtains:

$$\begin{aligned}\varphi_\mathrm{P} = {} & \frac{ce}{16\pi^2 r^2}\int\frac{d\sigma}{r} + \frac{cea}{4\pi r}\int\frac{\cos\theta d\sigma}{r} \\ & - \frac{ce\mathrm{G}z_\mathrm{P}}{32\pi^2\mathrm{R}^2c^2}\int\frac{d\sigma}{r} - \frac{ce\mathrm{G}}{22\pi^2 R^2c^2}\int\frac{z_\mathrm{M} d\sigma}{r}\ .\end{aligned} \tag{21}$$

However, since P is inside, one has:

$$\int\frac{d\sigma}{r} = 4\,\pi\,\mathrm{S}\ , \qquad \int\frac{\cos\theta}{r}\,d\sigma = \frac{4}{3}\,\pi\,z_\mathrm{P}\ , \qquad \int\frac{z_\mathrm{M}}{r}\,d\sigma = \frac{4}{3}\,\pi\,\mathrm{R}\,z_\mathrm{P}\ .$$

Thus one finds:

$$\varphi_\mathrm{P} = \frac{ce}{4\pi\mathrm{R}} + \frac{c}{3}\left(\frac{e}{\mathrm{R}}\,a - \frac{e}{2\pi\mathrm{R}c^3}\right) z_\mathrm{P}\ . \tag{22}$$

So if we require φ_P to be constant, we will have to set

$$a = \frac{1}{2\pi}\frac{\mathrm{G}}{c^2}\ .$$

By substituting this value into (20), one finds the following expression for the surface density:

$$\frac{e}{4\pi\mathrm{R}^2} + \left(1 + \frac{2\mathrm{G}}{c^2}\,\mathrm{R}\,\cos\theta\right)\ . \tag{23}$$

Therefore, the fact of being in a gravitational field produces a polarization of the sphere with moment

$$\frac{2}{3}\,\frac{\mathrm{G}}{c^2}\,e\,\mathrm{R}^2\ .$$

PART 2

WEIGHT OF ELECTROMAGNETIC MASSES

§ 6.– Suppose we have a system of charges held by a rigid support in such a way that the δx, δy, δz of § 2 have to be given the form corresponding to the components of a rigid displacement. Leaving the rotational displacements till later, let us consider now the translational ones, that is say assume that δx, δy, δz are arbitrary functions of time, but do not depend on x, y, z.

Then we will try to satisfy (9) by thinking of the potential φ_{P} at a generic point P as the sum of the potential given by (13) and one of the form (14). We will denote these two terms by φ_{P}' and φ_{P}'', and suppose that the ratio between the derivatives of φ_{P}' and φ_{P}'' with respect to any direction whatsoever is of order $l\,\mathrm{G}/c^2$, having decided to neglect the quadratic terms. Hence (9) can be written:

$$\int dt \left\{ \int_{\tau_{\mathrm{P}}} \delta x \left(\frac{\partial \varphi'}{\partial x} + \frac{\partial \varphi''}{\partial x} \right) \rho_{\mathrm{P}}\, d\tau_{\mathrm{P}} + \delta y \left(\frac{\partial \varphi'}{\partial y} + \frac{\partial \varphi''}{\partial y} \right) \rho_{\mathrm{P}}\, d\tau_{\mathrm{P}} \right.$$
$$\left. + \delta z \left(\frac{\partial \varphi'}{\partial z} + \frac{\partial \varphi''}{\partial z} \right) \rho_{\mathrm{P}}\, d\tau_{\mathrm{P}} \right\} = 0 \, .$$

Given that δx, δy, δz are arbitrary functions of time, independent of each other, this equation gives rise to the three equivalent equations:

$$\int_{\tau_{\mathrm{P}}} \left(\frac{\partial \varphi'}{\partial x} + \frac{\partial \varphi''}{\partial x} \right) \rho_{\mathrm{P}}\, d\tau_{\mathrm{P}} = \int_{\tau_{\mathrm{P}}} \left(\frac{\partial \varphi'}{\partial y} + \frac{\partial \varphi''}{\partial y} \right) \rho_{\mathrm{P}}\, d\tau_{\mathrm{P}}$$
$$= \int_{\tau_{\mathrm{P}}} \left(\frac{\partial \varphi'}{\partial z} + \frac{\partial \varphi''}{\partial z} \right) \rho_{\mathrm{P}}\, d\tau_{\mathrm{P}} = 0 \, . \tag{24}$$

Now from the expression (13) for φ_{P}', by noting that

$$\frac{\partial r}{\partial x_{\mathrm{P}}} = \frac{x_{\mathrm{P}} - x_r}{r} \, ,$$

one immediately obtains:

$$\int_{\tau_{\mathrm{P}}} \frac{\partial \varphi'}{\partial x} \rho_{\mathrm{P}}\, d\tau_{\mathrm{P}} = -\frac{c}{4\pi} \int_{\tau_{\mathrm{P}}} \int_{\tau_{\mathrm{M}}} \rho_{\mathrm{P}} \rho_{\mathrm{M}}\, d\tau_{\mathrm{P}} d\tau_{\mathrm{M}} \left\{ \frac{x_{\mathrm{P}} - x_{\mathrm{M}}}{r^3} - \frac{\mathrm{G}}{2c^2} \frac{(x_{\mathrm{P}} - x_{\mathrm{M}})(z_{\mathrm{P}} + z_{\mathrm{M}})}{r^3} \right\} ,$$

where both integrals have to be performed over the region occupied by the charges. By interchanging P and M in the right hand side, which changes nothing, one obtains:

$$\int_{\tau_{\mathrm{P}}} \frac{\partial \varphi'}{\partial x} \rho_{\mathrm{P}}\, d\tau_{\mathrm{P}} = -\frac{c}{4\pi} \int_{\tau_{\mathrm{M}}} \int_{\tau_{\mathrm{P}}} \rho_{\mathrm{M}} \rho_{\mathrm{P}}\, d\tau_{\mathrm{M}} d\tau_{\mathrm{P}} \left\{ \frac{x_{\mathrm{M}} - x_{\mathrm{P}}}{r^3} - \frac{\mathrm{G}}{2c^2} \frac{(x_{\mathrm{M}} - x_{\mathrm{P}})(z_{\mathrm{M}} + z_{\mathrm{P}})}{r^3} \right\} ,$$

from which, by taking half the sum:

$$\int_{\tau_{\mathrm{P}}} \frac{\partial \varphi'}{\partial x} \rho_{\mathrm{P}}\, d\tau_{\mathrm{P}} = 0 \, . \tag{25}$$

In an completely analogous way:

$$\int_{\tau_P} \frac{\partial \varphi'}{\partial y} \rho_P \, d\tau_P = 0 \, . \tag{26}$$

On the other hand, similarly:

$$\int_{\tau_P} \frac{\partial \varphi'}{\partial z} \rho_P \, d\tau_P = -\frac{c}{4\pi} \int_{\tau_P} \int_{\tau_M} \rho_P \rho_M \, d\tau_P d\tau_M \left\{ \frac{z_P - z_M}{r^3} - \frac{G}{2c^2} \frac{(z_P - z_M)(z_P + z_M)}{r^3} + \frac{G}{2c^2} \frac{1}{r} \right\} ,$$

interchanging M and P:

$$\int_{\tau_P} \frac{\partial \varphi'}{\partial z} \rho_P \, d\tau_P = -\frac{c}{4\pi} \int_{\tau_M} \int_{\tau_P} \rho_M \rho_P \, d\tau_M d\tau_P \left\{ \frac{z_M - z_P}{r^3} - \frac{G}{2c^2} \frac{(z_M - z_P)(z_M + z_P)}{r^3} + \frac{G}{2c^2} \frac{1}{r} \right\} ,$$

and by taking half the sum:

$$\int_{\tau_P} \frac{\partial \varphi'}{\partial z} \rho_P \, d\tau_P = -\frac{c}{4\pi} \int_{\tau_P} \int_{\tau_M} \frac{\rho_P \rho_M}{r} \, d\tau_P d\tau_M = -G \frac{U}{c^2} e \, , \tag{27}$$

denoting by U the electrostatic energy of the system (apart from the gravitational correction terms). As a consequence of the assumptions made about the derivatives of φ_P'', we can certainly write, with our approximation:

$$\begin{cases} \displaystyle\int_{\tau} \frac{\partial \varphi''}{\partial x} \rho \, d\tau = -c \, E_x^* \, e \, , \\ \displaystyle\int_{\tau} \frac{\partial \varphi''}{\partial y} \rho \, d\tau = -c \, E_y^* \, e \, , \\ \displaystyle\int_{\tau} \frac{\partial \varphi''}{\partial z} \rho \, d\tau = -c \, E_z^* \, e \, , \end{cases}$$

where $e = \int_{\tau} \rho \, d\tau$ indicates the total charge of the system. By substituting the expression just obtained into (24) one finds:

$$\begin{cases} e \, E_x^* = 0 \, , \\ e \, E_y^* = 0 \, , \\ e \, E_z^* = -G \dfrac{U}{c^2} \, . \end{cases}$$

Our result is contained in these formulas. In fact, they tell us that in order to maintain our system in equilibrium an external field (E^*) is required exerting on the system a force given (in the first approximation) by $e \, E^*$, which must be understood to balance the weight of the system, which is therefore given by $-e \, E^*$, and so has components

$$0, \, 0, \, G \frac{U}{c^2} \, . \tag{28}$$

With this we conclude that *the weight of an electromagnetic mass always has the vertical direction and magnitude equal to the weight of a material mass* u/c^2.

§ 7.– In the preceding section we have taken δx, δy, δz to be the components of a translational displacement. If instead one takes the components of a virtual rotational displacement with the axis passing through the origin of the coordinates, namely setting

$$\delta x = qz - ry \; ; \qquad \delta y = rx - pz \; ; \qquad \delta z = py - qx \; , \tag{29}$$

the integral (9), apart from the contribution due to the external field φ'', becomes:

$$\int d\tau \left\{ p \int_\tau \rho \left(y\frac{\partial \varphi}{\partial z} - z\frac{\partial \varphi}{\partial y} \right) d\tau + q \int_\tau \rho \left(z\frac{\partial \varphi}{\partial x} - x\frac{\partial \varphi}{\partial z} \right) d\tau \right.$$
$$\left. + r \int_\tau \rho \left(x\frac{\partial \varphi}{\partial y} - y\frac{\partial \varphi}{\partial x} \right) d\tau \right\} . \tag{30}$$

The integrals between curly brackets are easily evaluated using (13) through methods similar to that used in the previous section. They have the values:

$$-\frac{\mathrm{G}}{8\pi c} \iint \frac{y_\mathrm{P}}{r} \rho_\mathrm{P} \rho_\mathrm{M} \, d\tau_\mathrm{P} d\tau_\mathrm{M} \; ; \quad +\frac{\mathrm{G}}{8\pi c} \iint \frac{x_\mathrm{P}}{r} \rho_\mathrm{P} \rho_\mathrm{M} \, d\tau_\mathrm{P} d\tau_\mathrm{M} \; ; \quad 0 \; . \tag{31}$$

By taking as the origin the point O′ defined by the point O and the vector

$$\mathrm{O}' - \mathrm{O} = \frac{1}{2\mathrm{U}} \iint \frac{\mathrm{P} - \mathrm{O}}{r} \rho_\mathrm{P} \rho_\mathrm{M} \, d\tau_\mathrm{P} d\tau_\mathrm{M} \; ,$$

one sees immediately that the three integrals vanish for *any* orientation of the system about O′. As a consequence, with respect to the new origin the integral (9) is identically zero, namely the moment of the weight with respect to O′ is zero for any orientation of the system; thus O′ enjoys the properties of the center of gravity.

Pisa, March 1921

3) On Phenomena Occurring Close to a World Line

"Sopra i fenomeni che avvengono in vicinanza di una linea oraria,"
Rend. Lincei, **31** *(I), 21–23, 51–52, 101–103 (1922).*[1]

Note I.

§ 1.– In order to study phenomena which occur close to a world line, i.e., in nonrelativistic language, in a region of space in the spacetime manifold, even varying in time, but always very small compared with the divergences from Euclidean space, it would be convenient to find a particular frame such that close to the line being studied, the spacetime ds^2 will assume a simple form. In order to find such a frame, we must begin with some geometrical considerations.

Let there be given a line L in a Riemannian manifold V_n or in a manifold metrically connected in the sense of Weyl.[2] Let us associate at every point P of L a direction y perpendicular to L, with the rule that the direction $y+dy$, corresponding to the point P+dP, will be derived from that y associated to P in the following way: let η be the direction tangent to L at P; let y and η be parallel transported[3] from P to P+dP and let $y + \delta y$ and $\eta + \delta\eta$ be the directions obtained in this way, which because of the fundamental properties of parallel transport will be still orthogonal. If L is not geodesic $\eta + \delta\eta$ will not coincide with the direction $\eta + d\eta$ of the tangent to L at P+dP, and these two directions at P+dP will define a 2-dimensional subspace. Let us consider at P+dP the element of the S_{n-2} perpendicular to this subspace and let us rigidly rotate around this S_{n-2} all the surrounding particle space until $\eta + \delta\eta$ is superposed on $\eta + d\eta$. Then $y + \delta y$ will be mapped to a position which we will consider to be the direction $y + dy$ relative to the point P+dP. It is clear that, arbitrarily fixing the direction y at a point of L, an integration process will allow it to be obtained at any point of L.

Let us now look for the analytic expressions which translate the indicated operations to a Riemannian manifold, which coincide with those valid for a Weyl metric manifold as long as the "Eichung" is choosen such that the measure of a segment, which moves rigidly around L, will be constant. Let

$$ds^2 = \sum_{ik} g_{ik} dx^i dx^k \qquad i, k = 1, 2, \ldots n \tag{1}$$

and let y_i, $y^{(i)}$; η_i, $\eta^{(i)} = dx_i/ds$ be the co- and contravariant systems of the directions y, η. We will then have

$$\frac{\delta\eta^{(i)}}{ds} = -\sum_{hl} \begin{Bmatrix} h\,l \\ i \end{Bmatrix} \eta^{(h)} \frac{dx_l}{ds} = -\sum_{hl} \begin{Bmatrix} h\,l \\ i \end{Bmatrix} \frac{dx_h}{ds}\frac{dx_l}{ds} \,,$$

[1]Presented by the Correspondent G. Armellini during the session of January 22, 1922.
[2]WEYL, *Space, Time, Matter*, p. 109. Berlin, Springer, 1921.
[3]T. LEVI CIVITA, *Rend. Circ. Palermo*, Vol. XLII, p. 173 (1917).

and moreover $\frac{d\eta^i}{ds} = \frac{d}{ds}\frac{dx_i}{ds} = \frac{d^2x_i}{ds^2}$. Therefore one finds

$$\frac{\delta\eta^{(i)} - d\eta^{(i)}}{ds} = -\left(\frac{d^2x_i}{ds^2} + \sum_{hl}\left\{\begin{matrix} h\,l \\ i \end{matrix}\right\}\frac{dx_h}{ds}\frac{dx_l}{ds}\right) = -C^i \ .$$

The C^i are the contravariant components of the vector **C**, the geodetic curvature, namely of a vector having the same orientation as the geodesic principal normal of L and a magnitude equal to its geodesic curvature.

On the other hand one has

$$\frac{\delta y^{(i)}}{ds} = -\sum_{hk}\left\{\begin{matrix} h\,k \\ i \end{matrix}\right\} y^{(k)}\frac{dx_k}{ds} \ . \qquad (2)$$

Now since y is orthogonal to L, the displacement with which from $y + \delta y$ one gets $y + dy$ will be parallel to the tangent to L and will have magnitude equal to the projection onto the same y of $\delta\eta - d\eta$; that is to say, since y has magnitude 1, equal to the scalar product of $\delta\eta - d\eta$ and y, namely

$$\sum_i(\delta\eta_i - d\eta_i)y^{(i)} = -ds\sum_i C_i y^{(i)} \ .$$

Its contravariant components will be obtained therefore by multiplying its magnitude by the contravariant coordinates of the tangent to L, that is dx_i/ds. These are, in the final analysis, $-dx_i\sum_r C_r y^{(r)}$. From (2) it follows immediately that

$$\frac{dy^{(i)}}{ds} = -\sum_{hk}\left\{\begin{matrix} h\,k \\ i \end{matrix}\right\} y^{(k)}\frac{dx_k}{ds} - \frac{dx_i}{ds}\sum_h C_h y^h \ . \qquad (3)$$

Eq. (3), written for $i = 1, 2, \ldots n$ gives a system of n first order differential equations for the n unknowns $y^{(1)}, y^{(2)}, \ldots, y^{(n)}$, which are therefore determined once the initial data are assigned. It would also be easy to formally verify from (3) that, if the initial values of the $y^{(i)}$ satisfy the condition of perpendicularity to L, such a condition will remain satisfied all along the line.

§ 2.– Let us now assign at a point P_0 of L n mutually orthogonal directions $y_1, y_2, \ldots, y_n$ chosen at will, with the condition that y_n be tangent to L. The directions $y_1, y_2, \ldots, y_{n-1}$ will be perpendicular to L, and we can transport them along L by using the law given in the preceding section, which clearly from its definition preserves their orthogonality. We are then in a position to associate with every point of L n mutually orthogonal directions, the last one of which is tangent to L. Let us now consider our V_n embedded in a Euclidean S_N with a suitable number of dimensions. We can take as coordinates of a point of V_n the orthogonal Cartesian coordinates of its projection onto the S_N tangent to V_n at a generic point P of L, having P as the origin and the directions $y_1, y_2, \ldots, y_n$ relative to the point P as directions. In terms of these coordinates, the line element of V_n at P can be written in the form $ds_P^2 = dy_1^2 + dy_2^2 + \cdots + dy_n^2$; in addition, they are geodesics at P, as one can immediately see. In other words, for the coordinates y it is possible in a neighbourhood of P to set $g_{ii} = 1$, $g_{ik} = 0$ $(i \neq k)$, up to infinitesimals of order greater than the first. Obviously we shall have such a reference frame at every point of L. Let us consider now a point Q_0 of V_n with coordinates $\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_{n-1}, 0$ in the reference frame corresponding to the point P_0 on L. For any other point P of L we can so determine a point Q having in the frame corresponding to P the same coordinates as Q_0 has in the frame corresponding to P_0. The point Q will therefore trace out a line parallel to L. Now we want to find the relation between ds_Q and ds_P, assuming that the point Q is infinitely close to P. In order to do so, we note that the displacement transporting Q to Q + dQ is composed of the displacements denoted in § 1 by δ and $d - \delta$, and that the first one gives $\delta s_Q = ds_P$ up to infinitesimals of greater order since it is a parallel displacement; the second one is a rotation, which gives $(d - \delta)s_Q = ds_P\, \mathbf{C} \cdot (Q - P)$, as is seen from § 1, denoting by $\cdot$ the symbol of the scalar product, and with Q − P the vector with origin at P and endpoint at Q. Moreover, both ds_Q and $(d - \delta)s_Q$ have the direction of the tangent to L. Hence, one has $ds_Q = \delta s_Q + (d - \delta)s_Q$; namely

$$ds_Q = ds_P[1 + \mathbf{C} \cdot (Q - P)] \ . \tag{4}$$

The trajectories of the points Q form a $(n-1)^{ple}$ infinity of lines, so at least with proper limitations through each point M of V_n will pass one of these lines; in this way, we can characterize M through the coordinates of the point Q, $\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_{n-1}$ corresponding to the line passing through M, and the arclength s_P of the line L marked off from an arbitrarily chosen origin to that point P corresponding to the Q one coinciding with M.

If M is infinitely close to L, ds_Q will be perpendicular to the hypersurface $s_P =$ constant. Thus one will have

$$ds_M^2 = ds_Q^2 + d\bar{y}_1^2 + d\bar{y}_2^2 + \cdots + d\bar{y}_{n-1}^2 \ ;$$

and taking into account (4),

$$ds_M^2 = [1 + \mathbf{C} \cdot (M - P)]^2 ds_Q^2 + d\bar{y}_1^2 + d\bar{y}_2^2 + \cdots + d\bar{y}_{n-1}^2 \ . \tag{5}$$

As a result, in the neighborhood of L we have found a very simple expression for ds^2.

Note II.

§ 3.– Before passing to the physical application of the results obtained above, we still want to make some geometrical observations. First of all, it is clear that the previous considerations, and so also the formula (5) representing their conclusion, which for any manifold whatsoever are only valid close to L, are instead completely rigorous for Euclidean spaces. So let us associate to the line L of V_n a line L^* in a Euclidean space S_n, in which we indicate the orthogonal cartesian coordinates by

x_i^*. If we indicate with asterisks the symbols referring to the line L*, we can write for S_n the formula analogous to (5):

$$ds^2_{M^*} = [1 + \mathbf{C}^* \cdot (M^* - P^*)]^2 \, ds^2_{P^*} + d\bar{y}_1^{*2} + d\bar{y}_2^{*2} + \cdots + d\bar{y}_{n-1}^{*\,2} \; ; \qquad (5^*)$$

as in (5) **C** is a function of s_P, so in (5*) **C*** is a function of s_{P^*}.

Let $K^{(1)}$, $K^{(2)}$, $\cdots$, $K^{(n-1)}$ be the contravariant components of **C** with respect to $\bar{y}_1, \bar{y}_2, \cdots, \bar{y}_{n-1}$, and $K^{(1)*}$, $K^{(2)*}$, $\cdots$, $K^{(n-1)*}$ those of **C*** with respect to the $\bar{y}^*$. Let us try to determine L* in such a way that the functions $K^{(r)*}(s_{P^*})$ become equal to the $K^{(r)}(s_P)$. In order so, we shall begin by imposing that $s_P = s_{P^*}$, i.e., by establishing between the points of L and L* a one-to-one correspondence preserving the arclength. We then note that $K^{(r)*}$ is the projection of **C*** on the r^{th} direction y^*. Namely, one has

$$K^{(r)*} = \sum_{i=1}^{i=n} y^*_{i|r} \frac{d^2 x_i^*}{ds_P^2} \qquad (r = 1, 2, \cdots, n-1). \qquad (6)$$

The $K^{(r)}$ are then known functions of s_P. The condition $K^{(r)} = K^{(r*)}$ thus leads to the $(n-1)$ equations

$$K^{(r)}(s_P) = \sum_{i=1}^{i=n} y^*_{i|r} \frac{d^2 x_i^*}{ds_P^2} \qquad (r = 1, 2, \cdots, n-1). \qquad (7)$$

On the other hand, (3) once written for the S_n, gives us another $n(n-1)$ equations. If we add to these equations the following one

$$ds_P^2 = dx_1^{*2} + dx_2^{*2} + \cdots + dx_n^{*\,2} \; , \qquad (8)$$

we obtain a system of $n - 1 + n(n-1) + 1 = n^2$ equations for the n^2 unknowns x_i^*, $y^*_{i|r}$, which can be used to express them in terms of s_P. In this way we can determine the parametric equations $x_i^* = x_i^*(s_P)$ for L*. With that the formula (5*) becomes identical to (5), that is we have represented by applicability the neighborhood of the line L* onto that of L. In addition, since L* is in a Euclidean space, we can say that we have unfolded the neighborhood of L in a Euclidean space, i.e., we have found coordinates which are simultaneously geodesic at each point of L.

Note III.

§ 4. – In order to show the application to the theory of relativity of the results obtained above, we shall assume that V_n is the V_4 spacetime and that L is a world line in whose neighborhood we want to study the phenomena. Setting $ds_M^2 = ds^2$ in (5) for the sake of brevity, one finds in this case:

$$ds^2 = [1 + \mathbf{C} \cdot (M - P)]^2 \, ds_P^2 + d\bar{y}_1^2 + d\bar{y}_2^2 + d\bar{y}_3^2 \; .$$

To avoid the appearance of imaginary terms and to restore the homogeneity, it is convenient to make the following change of variables:

$$s_P = vt \; ; \quad \bar{y}_1 = ix \; ; \quad \bar{y}_2 = iy \; ; \quad \bar{y}_3 = iz \; ,$$

where v is a constant with dimensions of a velocity, so that t has the dimensions of time. Thus one obtains

$$ds^2 = a\,dt^2 - dx^2 - dy^2 - dz^2 \;, \tag{9}$$

where

$$a = v^2[1 + \mathbf{C}\cdot(\mathrm{M}-\mathrm{P})]^2 \;. \tag{10}$$

Hereafter, we refer to the space x, y, z using the ordinary symbols of vector calculus. And it is just in this sense that the scalar product which enters in (10) can be understood, provided that $\mathbf{C}$ is considered as the vector whose components are the covariant components of the geodesic curvature of the world line $x = y = z = 0$, and $\mathrm{M} - \mathrm{P}$ is the vector with components x, y, z. We will call x, y, z spatial coordinates, and t time. Sometimes for uniformity we will write x_0, x_1, x_2, x_3 in place of t, x, y, z, and we will also denote the coefficients of the quadratic form (9) by g_{ik}.

§ 5.– Let[4] F_{ik} be the electromagnetic field and $(\varphi_0, \varphi_1, \varphi_2, \varphi_3)$ the first rank tensor "potential" of F_{ik}, such that $\mathrm{F}_{ik} = \varphi_{i,k} - \varphi_{k,i}$. We set $\varphi_0 = \varphi$ and call $\mathbf{u}$ the vector with components $\varphi_1, \varphi_2, \varphi_3$. First of all, we have:

$$\left.\begin{matrix}\mathrm{F}_{01}\\ \mathrm{F}_{02}\\ \mathrm{F}_{03}\end{matrix}\right\} = \operatorname{grad}\varphi - \frac{\partial \mathbf{u}}{\partial t} \;, \quad \left.\begin{matrix}\mathrm{F}_{23}\\ \mathrm{F}_{31}\\ \mathrm{F}_{12}\end{matrix}\right\} = -\operatorname{curl}\mathbf{u} \;,\; \mathrm{F}_{ii} = 0 \;,\; \mathrm{F}_{ik} = -\mathrm{F}_{ki} \;;$$

analogously

$$\left.\begin{matrix}\mathrm{F}^{(01)}\\ \mathrm{F}^{(02)}\\ \mathrm{F}^{(03)}\end{matrix}\right\} = \frac{1}{a}\left(-\operatorname{grad}\varphi + \frac{\partial \mathbf{u}}{\partial t}\right) \;, \quad \left.\begin{matrix}\mathrm{F}^{(23)}\\ \mathrm{F}^{(31)}\\ \mathrm{F}^{(12)}\end{matrix}\right\} = -\operatorname{curl}\mathbf{u} \;,\; \mathrm{F}^{(ii)} = 0 \;,\; \mathrm{F}^{(ik)} = -\mathrm{F}_{(ki)} \;,$$

so that

$$\frac{1}{4}\sum_{ik}\mathrm{F}_{ik}\mathrm{F}^{(ik)} = \frac{1}{2}\left\{\operatorname{curl}^2\mathbf{u} - \frac{1}{a}\left(-\operatorname{grad}\varphi + \frac{\partial \mathbf{u}}{\partial t}\right)^2\right\} \;.$$

Let $d\omega$ be the hypervolume element of V_4. We will have

$$d\omega = \sqrt{-\|g_{ik}\|}\,dx_0\,dx_1\,dx_2\,dx_3 = \sqrt{a}\,dt\,d\tau \;,$$ (24)

where $d\tau = dx\,dy\,dz$ is the volume element of the space.

One also has:

$$\sum \varphi_i dx_i = \varphi dx + \mathbf{u}\cdot d\mathrm{M} \;, \qquad d\mathrm{M} = (dx, dy, dz) \;.$$

[4]See WEYL, op. cit., pp. 186 and 208 for the notation and the Hamiltonian derivation of the laws of physics.

Apart from the action of the metric field, whose variation is zero since we consider it as given *a priori* by (9), the action will assume the following form:

$$W = \frac{1}{4}\int_\omega \sum_{ik} \mathrm{F}_{ik}\mathrm{F}^{(ik)}\, d\omega + \int_e de \int \varphi_i\, dx_i + \int_m dm \int ds \ ,$$

$$\begin{pmatrix} de \ = \text{element of electric charge} \\ dm = \text{element of mass} \end{pmatrix} .$$

By introducing the indicated notation, one finds

$$\begin{aligned} W = \frac{1}{2}\iint &\left\{\mathrm{curl}^2\,\mathbf{u} - \frac{1}{a}\left(-\mathrm{grad}\,\varphi + \frac{\partial \mathbf{u}}{\partial t}\right)^2\right\}\sqrt{a}\, dt\, d\tau \\ &+ \iint (\varphi + \mathbf{u}\cdot\mathbf{V}_\mathrm{L})\rho\, d\tau\, dt + \iint \sqrt{a - {\mathbf{V}_\mathrm{M}}^2}\, k\, d\tau\, dt \ , \end{aligned} \tag{11}$$

where ρ, k are respectively the density of electricity and of matter, so that $de = \rho\, d\tau$, $dm = k\, d\tau$, $\mathbf{V}_\mathrm{L}$ is the velocity of the electric charges, $\mathbf{V}_\mathrm{M}$ that of the masses.

The integrals on the right-hand side can be extended to an arbitrary region τ between any two times t_1, t_2. Then one has the constraint that on the boundary of the region τ, and for the two times t_1, t_2, all variations are zero.

Apart from these conditions, the variations of φ and of $\mathbf{u}$ are completely arbitrary. Further conditions can be imposed on the variations of x, y, z thought of as coordinates of an element of charge or mass, expressing the constraints of the specific problem under consideration. Then writing that dW vanishes for any variation $\delta\varphi$ of φ whatsoever, one finds

$$0 = -\iint \left(\mathrm{grad}\,\varphi - \frac{\partial \mathbf{u}}{\partial t}\right)\cdot \delta\,\mathrm{grad}\,\varphi \frac{d\tau\, dt}{\sqrt{a}} + \iint \delta\varphi\rho\, dt\, d\tau \ .$$

Transforming the first integral by a suitable application of Gauss's theorem, and taking into account that $\delta\varphi$ vanishes at the boundary, we find

$$0 = \iint \delta\varphi \left\{\rho + \mathrm{div}\left[\frac{1}{\sqrt{a}}\left(\mathrm{grad}\,\varphi - \frac{\partial \mathbf{u}}{\partial t}\right)\right]\right\} dt\, d\tau \ ,$$

and since $\delta\varphi$ is arbitrary, we obtain the equation

$$\rho + \mathrm{div}\left[\frac{1}{\sqrt{a}}\left(\mathrm{grad}\,\varphi - \frac{\partial \mathbf{u}}{\partial t}\right)\right] = 0 \ . \tag{12}$$

Analogously, taking the variation of $\mathbf{u}$, one finds

$$\rho\mathbf{V}_\mathrm{L} + \mathrm{curl}(\sqrt{a}\,\mathrm{curl}\,\mathbf{u}) - \frac{\partial}{\partial t}\left[\frac{1}{\sqrt{a}}\left(\mathrm{grad}\,\varphi - \frac{\partial \mathbf{u}}{\partial t}\right)\right] = 0 \ . \tag{13}$$

These last two equations allow us to determine the electromagnetic field, once the charges and their motion are given.

Another set of equations can be obained by varying the trajectories of the charges and masses in W. Let $\delta\mathrm{P}_\mathrm{M}$ be the variation of the trajectory of the masses, $\delta\mathrm{P}_\mathrm{L}$ that of the charges. Moreover, since $\mathbf{u}$ is a vector function of the position and $\mathbf{V}$ a vector, let us denote by $(\partial\mathbf{u}/\partial\mathrm{P})(\mathbf{V})$ the vector with components $\dfrac{\partial u_x}{\partial x}V_x + \dfrac{\partial u_x}{\partial y}V_y + \dfrac{\partial u_x}{\partial z}V_z$, and so on. Now, writing that the variation of W is zero, one finds through the usual methods:

$$\iint \left(\delta\mathrm{P_M}\cdot\operatorname{grad}\varphi - \delta\mathrm{P_L}\left(\frac{\partial\mathbf{u}}{\partial t} + \frac{\partial\mathbf{u}}{\partial\mathrm{P}}(\mathrm{V_L})\right) + \mathrm{V_L}\cdot\frac{\partial\mathbf{u}}{\partial\mathrm{P}}(\delta\mathrm{P_L})\right)\rho\,dt\,d\tau$$
$$+ \iint \delta\mathrm{P_M}\cdot\left\{\frac{dt}{ds}\frac{\mathbf{grad}\,\boldsymbol{a}}{\mathbf{2}} + \frac{d}{dt}\left(\frac{dt}{ds}\mathbf{V}_\mathrm{M}\right)\right\}k\,dt\,d\tau = 0\,. \qquad (14)$$

If the δP's at a given time do not depend on their values at other times, the coefficient of dt in (14) must be zero. So one finds:

$$\int\left\{\delta\mathrm{P_M}\cdot\operatorname{grad}\varphi - \delta\mathrm{P_L}\left[\frac{\partial\mathbf{u}}{\partial t} + \frac{\partial\mathbf{u}}{\partial\mathrm{P}}(\mathrm{V_L})\right] + \mathrm{V_L}\cdot\frac{\partial\mathbf{u}}{\partial\mathrm{P}}(\delta\mathrm{P_L})\right\}\rho\,d\tau$$
$$+ \iint \delta\mathbf{P}_\mathrm{M}\cdot\left\{\frac{1}{2}\frac{dt}{ds}\operatorname{grad}a + \frac{d}{dt}\left(\frac{dt}{ds}\mathbf{V}_\mathrm{M}\right)\right\}k\,d\tau\,, \qquad (15)$$

which has to be satisfied for all systems of δP satisfying the constraints.

Pisa, March 1921.

4c) Correction of a contradiction between the electrodynamic theory and the relativistic theory of electromagnetic masses ([1])

"Correzione di una contraddizione tra la teoria elettrodinamica e quella relativistica delle masse elettromagnetiche,"
Nuovo Cimento ***25****, 159–170 (1923)*

§ 1.– The theory of electromagnetic masses was studied for the first time by M. Abraham[2] before the discovery of the theory of relativity. Abraham therefore, as was natural, considered in his calculations the mass of a rigid system of charges in the sense of classical mechanics, and he found that, with the hypothesis that such a system had spherical symmetry, its mass varied with the speed and is precisely equal to[3] $\frac{4}{3}\frac{u}{c^2}$ (where u is the electrostatic energy of the system and c is the speed of light) for zero or very small speeds, but for speeds v comparable to c correction terms of order of magnitude v^2/c^2 appear which are a bit complicated. Even before the theory of relativity, FitzGerald introduced the hypothesis that solid bodies underwent a contraction in the direction of motion in the ratio

$$\sqrt{1-\frac{v^2}{c^2}} \; : 1$$

and Lorentz redid Abraham's theory of electromagnetic masses, considering instead of rigid systems of electric charges in the sense of classical mechanics, systems that underwent this contraction. The result was that the rest mass, i.e., the limit of the mass for vanishing speed, was still $\frac{4}{3}\frac{u}{c^2}$, but the correction terms depending on v^2/c^2 changed. The experiences of Kaufmann, Bucherer and others with the mass of the β particles of radioactive bodies, and with high speed cathodic particles, decided in favor of the Lorentz theory, known as the contractile electron, against Abraham's theory of the rigid electron. This fact at the beginning was interpreted as a proof of the exclusively electromagnetic nature of the mass of electrons, because it was thought that otherwise their mass should be constant. Afterwards the discovery of the theory of relativity led to the consequence that all masses, electromagnetic or not, must vary with the speed like the mass of Lorentz's contractile electron; in this way the previous experiences left undecided the electromagnetic nature or not of the electron mass, being only a confirmation of the theory of relativity. On the other hand the special relativity theory first, and after the general theory, led to attribute a mass u/c^2 to a system with energy u and in this way arose a serious discrepancy between the Lorentz electrodynamic theory, which gives to a spherical

[1] On the same argument see my notes in *Rend. Acc. Lincei*, (5), 31, pp. 84 306 (1922).

[2] Abraham, Theory of Electricity; Richardson, Electron Theory of Matter, Chapter XI; Lorentz, The Theory of Electrons, p. 37

[3] The electromagnetic mass of an homogeneous spherical shell of charge e, and radius r is $\frac{2}{3}\frac{e^2}{rc^2}$; but if we observe that the electrostatic energy is $u=\frac{1}{2}\frac{e^2}{r}$, we find the mass $\frac{4}{3}\frac{u}{c^2}$.

distribution of electricity the rest mass $\frac{4}{3}\frac{u}{c^2}$, and special relativity which attributes to this distribution the mass u/c^2. That difference[4] is particularly serious given the great importance of the notion of the electromagnetic mass as a foundation for the electronic theory of matter.

This discrepancy showed up dramatically in two recent articles[5] in one of which, using the ordinary electrodynamic theory I considered the electromagnetic masses of a system with arbitrary symmetry, finding that in general they are represented by tensors instead of scalars, that reduce to $\frac{4}{3}\frac{u}{c^2}$ in the spherical symmetry case; in the other one instead, starting from general relativity, I considered the weight of the same systems which was in every case equal to $\frac{u}{c^2}\, g$, where g is the acceleration of gravity.

In the present work we will demonstrate precisely: that the difference between the two values of the mass obtained in the two ways originates in a concept of a rigid body in contradiction with the principle of relativity, which is applied in the electromagnetic theory (as well as in the contractile electron) and leads to the mass $\frac{4}{3}\frac{u}{c^2}$, while a better justified notion of rigid body conforming to the theory of relativity leads to the value u/c^2.

We note that the relativistic dynamics of the electron was done by M. Born[6] who starting from a point of view not essentially different from the usual one naturally found the rest mass $\frac{4}{3}\frac{u}{c^2}$.

Our considerations will be based on Hamilton's principle as the most suitable one to study a problem subject to very complicated constraints; in fact our system of electric charges must satisfy a constraint of a nature that is different from those considered in ordinary mechanics, since it has to exhibit, depending on its speed, the Lorentz contraction, as a consequence of the principle of relativity. To avoid misunderstandings, we note that while Lorentz contraction is of order v^2/c^2, its influence on the electromagnetic mass is on the principal terms of this one, i.e., on the rest mass and therefore has a rather bigger importance, being appreciable for very small speeds as well.

§ 2.– So we consider a system of electric charges, sustained by a rigid dielectric that, under the action of an electromagnetic field generated partly from the system itself and partly from external sources, moves with a translation motion describing a world tube in the space-time.[7]

[4] The experiences of Kaufmann and others cannot be useful to understand which of the two results is right, because these allow only the measurement of the speed dependent correction terms which are the same in both theories, while the difference is between the rest masses.

[5] E. Fermi, *N. Cim.*, VI, 22, pp. 176, 192 (1921).

[6] Max Born, *Ann. d. Phys.*, 30, p. 1 (1909).

[7] In the following we consider a Euclidean space-time, because we suppose that the considered electromagnetic fields are small enough to not modify the metric structure.

is to be discarded because it is in contradiction with the principle of relativity. Let T be the time tube described by the system. In the figure the space (x, y, z) is represented by only one dimension along the x-axis, and the time t is substituted by ict to have a definite metric.

Variation A: one considers as a variation that satisfies the rigidity constraint an infinitesimal displacement, rigid in the ordinary kinematic sense, parallel to the space (x, y, z), of each section of the tube parallel to the same space. In the figure we will obtain such a variation by shifting each section t=const. of the tube parallel to the x-axis by an arbitrary infinitesimal segment. If we restrict ourselves to consider translational displacement, we will therefore have δx, δy, δz as arbitrary functions only of time, and $\delta t = 0$.

Variation B: one considers as a variation that satisfies the rigidity constraint an infinitesimal displacement perpendicular to the tube of each section normal to the same tube, rigid in the ordinary kinematic sense. In the figure we will obtain this variation by shifting each normal section of the tube parallel to itself by an arbitrary segment.

Of two such variations A *is in obvious contradiction with the principle of relativity* and must be discarded because, not even being Lorentz invariant, it depends on the particular frame (t, x, y, z) we have chosen and can't be the expression of any physical notion, like rigidity. The variation B instead, besides satisfying Lorentz invariance, since it only consists of elements of the tube T completely independent of the position of the frame axes, is the only one presents itself naturally, like that based on a rigid virtual displacement in the frame where at the instant considered the system of charges has zero speed. Now it would be wrong to think that the difference between the consequences of the two methods of variation A and B is significant only for high speeds, i.e., when the tube T has a big slope with respect to the time axis. Instead the calculations we are going to develop will demonstrate immediately that the difference is felt already at zero speed and that precisely A gives $\frac{4}{3}\frac{u}{c^2}$ as the electromagnetic mass the while B gives instead u/c^2.

§ 3.– We indicate the coordinates of time and space by (t, x, y, z) or (x_0, x_1, x_2, x_3) as convenient and let ϕ_i be the four-potential and

$$F_{ik} = \frac{\partial \phi_i}{\partial x_k} - \frac{\partial \phi_k}{\partial x_i}$$

the electromagnetic field, and $\mathbf{E}$ and $\mathbf{H}$ the electric and magnetic forces that derive from it.

Hamilton's principle that summarizes the laws of Maxwell Lorentz and those of mechanics says that:[8] the total action, i.e., the sum of the actions of the electromagnetic field and of the material and electric masses, has zero variation under the effect of an arbitrary variation of the ϕ_i and of the coordinates of the points of the electric charge world lines that respect the constraints and are zero on the

[8] Weyl, *Space, Time, Matter*, pp. 194 196; Berlin, Springer (1921).

boundary of the integration region. In our case there aren't material masses, and the only variable elements are the coordinates of the points on the world lines of the charges; therefore it is enough to consider only the action of the electric charges, i.e.

$$W = \sum_i \int de \int \phi_i \, dx_i$$

where de is the generic element of electric charge and the second integral is calculated on the timeline arc described by de that is contained in the four-dimensional region G of integration. For each system of variations δx_i satisfying the constraints and that *vanishes on the boundary of* G, one must have $\delta W = 0$, i.e.

$$\sum_{ik} \int \int de \; F_{ik} \delta x_i dx_k = 0 \tag{1}$$

Now we must examine separately the results obtained substituting δx_i by the values given by the system of variations A or B.

§ 4.– *Consequences of the system of variations* A. — In this case the region of integration reduces to ABCD. The regions BCG, ADH give no contribution, because in them all the δx_i are zero since they have to vanish on the boundary of G, and therefore along the curves BG, AH and must be constants for $t =$ const, i.e., on the straight lines parallel to the x axis. If we label the times of A and B by t_1 and t_2, the equation (1) can be written, since $\delta t = 0$ and δx, δy, δz are functions of the time only:

$$\sum_{ik} \int_{t_1}^{t_2} dt \; \delta x_i \int de \; F_{ik} \frac{dx_k}{dt} \qquad (i = 1, 2, 3) \quad (k = 0, 1, 2, 3) \, .$$

Since the δx_i are arbitrary functions of t, we obtain the three equations

$$\int de \sum_k F_{ik} \frac{dx_k}{dt} = 0$$

i.e.,

$$\int de \left[E_x + \frac{dy}{dt} H_z - \frac{dz}{dt} H_y \right] = 0 \qquad \text{and the analogous two.}$$

If at the chosen instant the system has zero speed in the frame (t, x, y, z) the three equations can be summarized by a single vector equation:

$$\int \mathbf{E} \, de = 0 \, . \tag{2}$$

We could have obtained this equation without calculations if, as is usually done in the ordinary treatment and as M. Born essentially does in the cited work, we had set to zero from the beginning the total force acting on the system. We wanted deduce it using Hamilton's principle to show the fault of its origin, since it follows from the system of variations A that it is in contradiction with the relativity principle.

From (2) follows immediately the value $\frac{4}{3}\frac{u}{c^2}$ for the electromagnetic mass. Suppose in fact that $\mathbf{E}$ is the sum of a part $\mathbf{E}^{(i)}$ due to the system itself, plus a uniform field $\mathbf{E}^{(e)}$ due to external sources. (2) gives:

$$\int \mathbf{E}^{(i)}\, de + \int \mathbf{E}^{(e)}\, de = 0 \ .$$

Now $\int de = e =$ charge; and then $\mathbf{E}^{(e)} \int de = \mathbf{F} =$ external force. In the spherical symmetry case, both direct calculation, and the well known considerations of the electromagnetic moment[9] show that:

$$\int \mathbf{E}^{(i)}\, de = -\frac{4}{3}\frac{u}{c^2}\mathbf{\Gamma} \ ,$$

where $\mathbf{\Gamma}$ is the acceleration.

The previous equation then becomes:

$$\mathbf{F} = \frac{4}{3}\frac{u}{c^2}\mathbf{\Gamma}$$

that compared to the fundamental law of point dynamics, $\mathbf{F} = m\mathbf{\Gamma}$, gives:

$$m = \frac{4}{3}\frac{u}{c^2} \ .$$

§ 5.– *Consequences of the system of variations* B. — In this case the same considerations of the previous section demonstrate that the region of integration reduces to ABEF, i.e., to the region bounded by two normal sections of the tube T. By the use of infinite normal sections Decomposing it using an infinite number of normal sections into layers of infinitesimal thickness, and in order to calculate the contribution of one of these to the integral (1) we refer to its rest frame, by considering the space (x,y,z) parallel to the layer. For this $\delta t = 0$ will hold, while δx, δy, δz will be arbitrary constants. Moreover $dx = dy = dz = 0$, because the speed of all the points is zero, $dt =$ height of the layer, that will vary for each point, because the layer has for its faces two normal sections which in general are not parallel. If O is a generic point but fixed in the layer, for example the origin of coordinates, in which dt has the value dt_0, and $\mathbf{K}$ is the vector with the orientation of the principal normal to the timeline passing for O and size equal to its curvature, we have manifestly, since dt is the thickness at the generic point P of the layer:

$$dt = dt_0[1 - \mathbf{K}\cdot(P - O)] \ .$$

Since the speed is zero we have

$$\mathbf{K} = -\mathbf{\Gamma}/c^2 \ ,$$

and therefore:

$$dt = dt_0\left(1 + \frac{\mathbf{\Gamma}\cdot(P - O)}{c^2}\right) \ .$$

[9]RICHARDSON loc. cit.

Substituting these values we find that the contribution of our layer to the integral (1) is:

$$-dt_0\Big\{\delta x\int\Big(1+\frac{\mathbf{\Gamma}\cdot(P-O)}{c^2}\Big)E_x de+\delta y\int\Big(1+\frac{\mathbf{\Gamma}\cdot(P-O)}{c^2}\Big)E_y de+ \\ +\delta z\int\Big(1+\frac{\mathbf{\Gamma}\cdot(P-O)}{c^2}\Big)E_z de\Big\}\ .$$

This expression must vanish for all the values of δx, δy, δz and we obtain from it three equations that can be summarized in the single vector equation:

$$\int\Big(1+\frac{\mathbf{\Gamma}\cdot(P-O)}{c^2}\Big)\mathbf{E}\,de=0 \qquad (3)$$

A correct application of Hamilton's principle has then brought us to (3) instead of (2). Now it's easy to examine the consequences. Setting

$$\mathbf{E}=\mathbf{E}^{(i)}+\mathbf{E}^{(e)}$$

we find

$$\int\mathbf{E}^{(i)}de+\int\mathbf{E}^{(i)}\,\frac{\mathbf{\Gamma}\cdot(P-O)}{c^2}de+e\mathbf{E}^{(e)}+\mathbf{E}^{(e)}\int\frac{\mathbf{\Gamma}\cdot(P-O)}{c^2}de=0\ .$$

In the spherical symmetry case we have as before

$$\int\mathbf{E}^{(i)}\,de=-\frac{4}{3}\frac{u}{c^2}\mathbf{\Gamma}\ ;$$

substituting in the previous equation we find that $\mathbf{E}^{(e)}$ is compared only with the terms that contain $\mathbf{\Gamma}$. If we neglect the $\mathbf{\Gamma}^2$ terms[10], we can neglect the last integral, and we obtain:

$$-\frac{4}{3}\frac{u}{c^2}\mathbf{\Gamma}+\int\mathbf{E}^{(i)}\,\frac{\mathbf{\Gamma}\cdot(P-O)}{c^2}de+\mathbf{F}=0\ . \qquad (4)$$

To calculate the integral which appears in (4) we observe that $\mathbf{E}^{(i)}$ is the sum of the Coulomb force

$$=\int\frac{P-P'}{r^3}de'$$

(P' is the point of charge de' and $r=\overline{PP'}$), and of a term containing $\mathbf{\Gamma}$ that can be neglected because it would give a contribution containing Γ^2. Our integral then becomes:

$$\int\int\frac{P-P'}{r^3}\,\frac{\mathbf{\Gamma}\cdot(P-O)}{c^2}de\,de'\ ;$$

or exchanging P with P', which doesn't change matters, and taking the half sum of the two values obtained in this way:

$$\frac{1}{2}\int\int\frac{P-P'}{cr^3}\,[\mathbf{\Gamma}\cdot(P-P')]de\,de'\ .$$

[10]More precisely the number compared to which the quadratic terms are negligible is $\Gamma\ell/c^2$, where ℓ is the largest length which appears in the problem. It is clear that such an approximation is more than justified in common situations.

We observe that, in our approximation $\mathbf{\Gamma}$ is constant for all the points and then can be taken out of the integrals. Therefore the x component of the previous integral is:

$$\frac{1}{2c^2}\left\{\mathbf{\Gamma}_x \int\int \frac{(x-x')^2}{r^3} de\, de' + \mathbf{\Gamma}_y \int\int \frac{(y-y')(x-x')}{r^3} de\, de' + \right.$$
$$\left. + \mathbf{\Gamma}_z \int\int \frac{(z-z')(x-x')}{r^3} de\, de' \right\} .$$

Now, since the system has spherical symmetry, to each segment PP' corresponds an infinite number of other segments differing only in orientation. In the three integrals we can therefore substitute

$$(x-x')^2, (x-x')(y-y'), (x-x')(z-z')$$

by their average values for all the possible orientations of PP', which are; $\frac{1}{3}r^2$, 0, 0.

With that the x component becomes:

$$\frac{\Gamma_x}{3c^2}\frac{1}{2}\int\int \frac{de\, de'}{r}$$

We now observe that the expression

$$\frac{1}{2}\int\int \frac{dede'}{r}$$

is the electrostatic energy u; going back to vector notation we find for the integral appearing in equation (4) the expression: $\frac{u}{3c^2}\mathbf{\Gamma}$. (4) becomes in this way:

$$\frac{u}{c^2}\mathbf{\Gamma} = \mathbf{F} \tag{5}$$

that says the electromagnetic mass is u/c^2.

5) Masses in the theory of relativity

"Le masses nella teoria della relatività,"
from A. Kopff, I fondamenti della relatività Einsteiniana,
Eds. R. Conti and T. Bembo, Hoepli, Milano, 1923, pp. 342–344

The grandiose conceptual importance of the theory of relativity as a contribution to a deeper understanding of the relationships between space and time and the often lively and passionate discussions to which it has as a consequence also given given rise outside of the scientific environment, have perhaps diverted attention away from another of its results that, even though less sensational and let's say, even less paradoxical, nevertheless has consequences for physics no less worthy of note, and whose interest is realistically destined to grow in the near term development of science.

The result to which we refer is the discovery of the relationship that ties the mass of a body to its energy. The mass of a body, says the theory of relativity, is equal to its total energy divided for the square of the speed of light. A superficial examination already shows us how, at least for the physics that is observed in the laboratories, the importance of this relationship between mass and energy is such that it considerably overshadows that of the other consequences, quantitatively much lighter, but to which the mind gets used to with more effort. This merits an example: a body one meter long that moves with the respectable enough speed of 30 km per minute (equal more or less to the speed of the earth through space) would always appear to be one meter long to an observer carried along by its motion, while to a fixed observer it would appear to be one meter long less five millionths of a millimeter; as one sees the result, however strange and paradoxical it can seem, is nevertheless very small, and it is hard to believe that the two observers would start quarreling over so little. The relationship between mass and energy brings us instead to enormous figures. For example if one succeeded in releasing the energy contained in a gram of matter, one would obtain an energy greater than that developed over three years of nonstop work by a motor of a thousand horse power (useless to comment!). One might say with reason that it doesn't appear possible, at least in the near future, to find a way to liberate these incredible quantities of energy, something that moreover one would hope not to be able to do, since the explosion of such an incredible quantity of energy would have as its first result reducing to pieces the physicist who had the misfortune to find a way to produce it.

But even if such a complete explosion of matter doesn't appear possible for now, there are already in progress during the past few years some experiments directed towards transforming the chemical elements into each other. Such a transformation, which happens naturally in radioactive bodies, has been recently done artificially by Rutherford who, bombarding some atoms with some α particles (corpuscles launched with huge speed by radioactive substances), has succeeded in obtaining their decomposition. Now to these transformations of the elements into each other are associated energy exchanges that the relationship between mass and energy allows us to study in a very clear way. To illustrate this it is worth another numerical example. We have reason to think that the nucleus of an atom of helium is composed of four nuclei of the hydrogen atom. Now the atomic weight of helium is 4.002 while that of hydrogen is 1.0077. The difference between four times the mass of hydrogen and the mass of the helium is therefore due to the energy of the bonds that unite the four nuclei of hydrogen to form the nucleus of helium. This difference is 0.029 corresponding, according to the relativistic relationship among mass and energy, to an energy of around six billion calories per gram-atom of helium. These figures show that the energy of the nuclear bonds is some million times greater than those of the most energetic chemical bonds and explains to us how against the problem of transformation of matter, the dream of alchemists, for so many centuries the efforts of the best minds have been useless, and how only now, using the most energetic means to our disposition, one has succeeded in obtaining this transformation; moreover in such a small quantity as to illude the most delicate analyses.

These brief indications are enough to show how the theory of relativity, besides giving us a clear interpretation of the relationships between space and time, will be, perhaps in the near future, destined to be the keystone for the resolution of the problem of the structure of matter, the last and more difficult problem of physics.

10) On the mass of radiation in an empty space

"Sulla massa della radiazione in uno spazio vuoto,"
with A. Pontremoli,
Rend. Lincei 32(1), 162–164 (1923)

Recently, one of us[1] had been able to demonstrate, by introducing a more correct concept of rigidity, that the standard electrodynamics allows us to reach a determination of the electron rest mass not different from that coming from the theory of relativity which, as is known, simply amounts to dividing the energy of the system by the squared speed of light. We have observed that a similar difference, between the value determined following from standard electrodynamics and the one given by the theory of relativity, occurs in the calculation of the mass of the radiation in an empty space.[2] We intend to demonstrate that this discrepancy can be removed by analogous arguments. The procedure followed until now for determining by electrodynamics the mass of the radiation in a cavity consisted first of all in evaluating the electromagnetic momentum $\boldsymbol{G}_0$ for slow and quasi-stationary motions, which, neglecting terms in v^2/c^2, results to be given by[3]

$$\boldsymbol{G}_0 = \frac{4}{3}\frac{W_0}{c^2}\boldsymbol{v}$$

where W_0 is the energy of the radiation for the cavity at rest, $\boldsymbol{v}$ is the actual velocity of the cavity, and c is the speed of light. From this, one deduced that the inertial reaction is given by

$$-\frac{d\boldsymbol{G}_0}{dt} = -\frac{4}{3}\frac{W_0}{c^2}\,\Gamma$$

where Γ is the acceleration; whence an apparent mass of the radiation equals $\frac{4}{3}\frac{W_0}{c^2}$, while, according to the theory of relativity, it should be simply $\frac{W_0}{c^2}$. In this procedure it is implicitly contained the assertion that the external force F is equal to the time derivative of the electromagnetic momentum, i.e., to the resultant of the electromagnetic forces $d\varphi$ acting on every single part of the system; in this way, one then puts:

$$F = \int d\varphi. \tag{1}$$

But this is not correct, because, if one considers the notion of rigidity discussed by one of us in the quoted paper, the external force is given instead by

$$F = \int d\varphi \left[1 + \frac{\Gamma(P-O)}{c^2}\right], \tag{2}$$

[1] E. Fermi, these "Rendiconti", Vol. XXXI, pp. 184 and 306 (1922), "Physikalische Zeit.", Vol. XXIII (1922), p. 340.

[2] F. Hasenöhrl, "Ann. der Physik", Vol. XV, p. 344 (1904) and Vol. XVI, p. 589 (1905); K. von Mosengeil, "Ann. der Physik", Vol. XXII, p. 867 (1927); M. Planck, "Berlin. Sitzber.", p. 542 (1907); M. Abraham, Theorie der Elektrizität, Vol. II, p. 341 (1920).

[3] M. Abraham, loc. cit. p. 345.

$(P-O)$ being the vector from the point P, where the force $d\varphi$ is applied, to a fixed point O, which we can take as the center of coordinates, internal to the system. Now, $d\varphi$ is the resultant of force $d\varphi_1$, exerted by the radiation pressure which would exist if the cavity were at rest, and a force $d\varphi_2$, caused by the perturbations of this pressure due to the motion of the cavity. By applying (1), since evidently $\int d\varphi_1 = 0$, because $d\varphi_1$ is the force exerted by a homogeneous pressure on a closed surface, one finds that the external force is

$$F = \int d\varphi_2. \tag{3}$$

This force is exactly the one calculated as the inertial reaction by the quoted authors, whence

$$\int d\varphi_2 = -\frac{4}{3}\frac{W_o}{c^2}\,\Gamma \tag{4}$$

On the contrary, by applying (2), still taking into account that $\int d\varphi_1 = 0$, one finds

$$F = \int (d\varphi_1 + d\varphi_2)\left[1 + \frac{\Gamma(P-O)}{c^2}\right] = \int d\varphi_1 \frac{\Gamma(P-O)}{c^2} + \int d\varphi_2 + \int d\varphi_2 \frac{\Gamma(P-O)}{c^2}.$$

Neglecting terms in Γ^2 and observing that $d\varphi_2$ is proportional to Γ, one can simply put

$$F = \int d\varphi_1 \frac{\Gamma(P-O)}{c^2} + \int d\varphi_2. \tag{5}$$

In this case the difference between (3) and (5) is not *a priori* negligible, although it contains c^2 at the denominator, since $d\varphi_1/d\varphi_2$ can become considerably large, being the ratio between a force and its perturbation.[4] In fact $d\varphi_2 = pnd\sigma$, where p is the radiation pressure which, as is known, equals $\frac{1}{3}\frac{W_o}{V}$, being V the volume of the cavity, and n a unit vector with the direction of the external normal to element $d\sigma$ of the surface of the cavity with coordinates (x, y, z). The x component of the first integral of (5) is then

$$\left[\int d\varphi_1 \frac{\Gamma(P-O)}{c^2}\right]_x = \frac{W_o}{3c^2V}\int (\Gamma_x dx + \Gamma_y dy + \Gamma_z dz)\cos\widehat{nx}\,d\sigma =$$

$$= \frac{W_o}{3c^2V}\left(\Gamma_x \int dx\,\cos\widehat{nx}\,d\sigma + \Gamma_y \int dy\,\cos\widehat{nx}\,d\sigma + \Gamma_z \int dz\,\cos\widehat{nx}\,d\sigma\right);$$

but an immediate application of Gauss's theorem shows that

$$\int dx\,\cos\widehat{nx}\,d\sigma = V, \qquad \int dy\,\cos\widehat{nx}\,d\sigma = \int dz\,\cos\widehat{nx}\,d\sigma = 0.$$

[4] In the case of electromagnetic masses one has $d\varphi$ equal to the resultant of the Coulomb forces (which are the predominant part) and the forces due to the acceleration. For the former, evidently in this case the relation $\int d\varphi_1 = 0$ also holds; therefore these forces make their presence felt only if we apply (5) instead of (3).

Therefore our component is $(W_o\Gamma_x)/3c^2$ and

$$\int d\varphi_1 \frac{\Gamma(P-O)}{c^2} = \frac{W_o\Gamma_x}{3c^2}$$

Considering this relation and (4), it is easy to see that the ratio between the integrals of the right hand side of (5) is $-1/4$ and thus effectively not negligible. By substituting these values into (5), one finds

$$F = -\frac{W_o}{c^2}\,\Gamma$$

from which the requested rest mass results to be equal to W_o/c^2, in accordance with the principle of relativity.