one can perform the sum of (25) with respect to $\sigma$ in the usual way[§] and one finds

$$t\frac{8\pi^3 g^2}{h^4}\left|\int v_m^* u_n d\tau\right|^2 \frac{p_\sigma^2}{v_\sigma}\left(\tilde{\psi}_s\psi_s - \frac{\mu c^2}{K_\sigma}\tilde{\psi}_s\beta\psi_s\right) , \tag{31}$$

where $p_\sigma$ is the value of the momentum of the neutrino for which (30) holds.

## 6. Determining elements of the transition probability

~~§ 6.~~ (31) expresses the probability that in a time $t$ a $\beta$ decay takes place in which the electron is emitted in the state $s$. As must be the case, this probability turns out to be proportional to the time ($t$ has been considered small with respect to the lifetime); the coefficient of $t$ gives the transition probability for the process we consider; it turns out to be

$$P_s = \frac{8\pi^3 g^2}{h^4}\left|\int v_m^* u_n d\tau\right|^2 \frac{p_\sigma^2}{v_\sigma}\left(\tilde{\psi}_s\psi_s - \frac{\mu c^2}{K_\sigma}\tilde{\psi}_s\beta\psi_s\right) . \tag{32}$$

Note that:

(a) For the free states of the neutrinos one always has $K_\sigma \geq \mu c^2$. Then it is necessary, in order that (30) can be satisfied, that

$$H_s \leq W - \mu c^2 \tag{33}$$

The upper limit of the $\beta$ ray spectrum corresponds to the = sign.

(b) Secondly, since for the unoccupied electron state one has $H_s \geq mc^2$, we obtain, in order that the decay be possible, the following condition:

$$W \geq (m + \mu)c^2 \tag{34}$$

Then, in order that the $\beta$ decay be possible, one must have a rather high occupied neutron state over a free proton state.

(c) According to (32), $P_s$ depends on the eigenfunctions $u_n$ and $v_m$ of the heavy particle in the nucleus, through the matrix element

$$Q_{mn}^* = \int v_m^* u_n d\tau \tag{35}$$

This matrix element plays a role, in the ~~in the~~ theory of $\beta$ rays, which is analogous to that of the matrix element of the electric moment in the theory of radiation. The matrix element (35) has normally the order of magnitude 1; nevertheless it often happens that, due to particular symmetries of the eigenfunctions $u_n$ and $v_m$, $Q_{mn}^*$ exactly vanishes. In that case we shall speak of "forbidden $\beta$ transitions". On the other hand, one should not expect that the forbidden transitions are really impossible, since (32) is only an approximate formula. We shall come back to this matter in § 9.

[§]For a description of the methods used for performing such sums, cf. any expository article on the theory of radiation. For instance, E. Fermi *Rev. Mod. Phys.* **4**, 87, (1932).

## 7. The mass of the neutrino

~~§ 7. —~~ The transition probability (32) determines among other things the shape of the continuous spectrum of $\beta$ rays. We will discuss here how the shape of this spectrum depends on the rest mass of the neutrino, in order to be able to determine this mass through a comparison with the experimental shape of the spectrum itself. The mass $\mu$ also enters into (32) through the factor $p_\sigma^2/v_\sigma$. The dependence of the shape of the curve of the energy distribution on $\mu$ is particularly pronounced in the proximity of the maximum energy $E_0$ of the $\beta$ rays. It is easy to recognize that the distribution curve for energies $E$ close to the maximum value $E_0$, behaves, apart from a factor independent of $E$, as

$$\frac{p_\sigma^2}{v_\sigma} = \frac{1}{c^3}\left(\mu c^2 + E_0 - E\right)\sqrt{\left(E_0 - E\right)^2 + 2\mu c^2\left(E_0 - E\right)}\,. \tag{36}$$

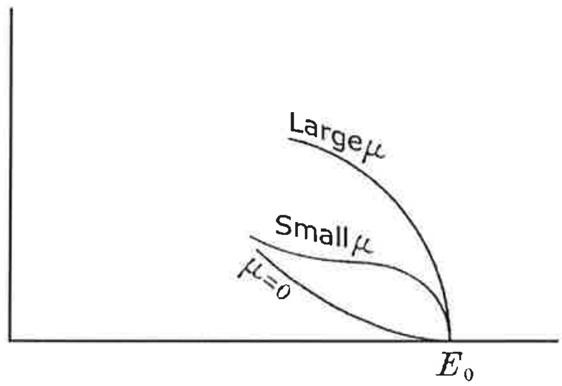


Fig. 1

In Fig~~ure~~. 1 the end of the distribution curve is represented for $\mu = 0$, and for a small value and a large value of $\mu$. The closest similarity of the theoretical curve to the experimental curves corresponds to $\mu = 0$. Thus we arrive at concluding that the mass of the neutrino is equal to zero or, in any case, much smaller than the mass of the electron¶. In the calculations below, for the sake of simplicity, we always set $\mu = 0$.

Then we have, also taking (32) into account

$$v_\sigma = c\,; \qquad K_\sigma = cp_\sigma\,; \qquad p_\sigma = \frac{K_\sigma}{c} = \frac{W - H_s}{c} \tag{37}$$

and the inequalities (33) and (34) become

$$H_s \leq W\,; \qquad W \geq mc^2\,. \tag{38}$$

¶In a recent note F. Perrin, *C.R.*, **197**, 1625 (1933), by means of quantitative arguments arrives at a similar conclusion.

Finally the transition probability takes the form

$$P_s = \frac{8\pi^3 g^2}{c^3 h^4} \left| \int v_m^* u_n \, d\tau \right|^2 \tilde{\psi}_s \psi_s \left(W - H_s\right)^2 \ . \tag{39}$$

## 8. Lifetime and shape of the energy distribution curve for allowed transitions

next line

~~§ 8.1~~ From (39) one can derive a formula which expresses how many $\beta$ transitions in which a $\beta$ particle gets a momentum ranging from $mc\eta$ to $mc(\eta + d\eta)$ take place in unit time. For this it is necessary to calculate the sum of the values of $\tilde{\psi}_s \psi_s$ in the nucleus, extended to all the states (of the continuum) which belong to the indicated range of momentum. In this regard we point out that the relativistic eigenfunctions in the Coulomb field for the states with $j{=}1/2$ ($^2s_{1/2}$ ~~e~~ $^2p_{1/2}$) become infinite in the center. On the other hand the Coulomb law does not hold up to the center of the nucleus, but only up to a distance from it larger than $R$, where $R$ is the nuclear radius. At this point, a tentative calculation shows that, if we make plausible assumptions on the behavior of the electric potential inside the nucleus, the value of $\tilde{\psi}_s \psi_s$ in the center of the nucleus turns out to be very close to the value which $\tilde{\psi}_s \psi_s$ should assume if the Coulomb law were valid at a distance $R$ from the center. Applying the known formulas‖ for the relativistic eigenfunctions of the continuum spectrum in a Coulomb field, after a rather long but easy calculation, one finds

$$\sum_{d\eta} \tilde{\psi}_s \psi_s = d\eta \cdot \frac{32\pi m^3 c^3}{h^3 \left[\Gamma(3+2S)\right]^2} \left(\frac{4\pi mcR}{h}\right)^{2S} \eta^{2+2S} e^{\pi\gamma \frac{\sqrt{1+\eta^2}}{\eta}}$$
$$\times \left| \Gamma\left(1 + S + i\gamma \frac{\sqrt{1+\eta^2}}{\eta}\right) \right|^2 , \tag{40}$$

where we have set

$$\gamma = Z/137 \ ; \qquad S = \sqrt{1-\gamma^2} - 1 \ . \tag{41}$$

The transition probability in an electric state in which the momentum has a value in the interval $mc\, d\eta$ then becomes (see (39))

$$P(\eta) d\eta = d\eta \cdot g^2 \frac{256\pi^4}{\left[\Gamma(3+2S)\right]^2} \frac{m^5 c^4}{h^7} \left(\frac{4\pi mcR}{h}\right)^{2S} \left| \int v_m^* u_n \, d\tau \right|^2 \eta^{2+2S}$$
$$\times e^{\pi\gamma \frac{\sqrt{1+\eta^2}}{\eta}} \left| \Gamma\left(1 + S + i\gamma \frac{\sqrt{1+\eta^2}}{\eta}\right) \right|^2 \left(\sqrt{1+\eta_0^2} - \sqrt{1+\eta^2}\right)^2 , \tag{42}$$

where $\eta_0$ is the maximum momentum of the emitted $\beta$ rays, as measured in units of $mc$.

‖R.H. Hulme, *Proc. Roy. Soc.* **133**, 381 (1931).

For a numerical evaluation of (42) we refer to the particular value $\gamma = 0.6$, which corresponds to $Z = 82.2$ since the atomic numbers of the radioactive substances are not far from this value. For $\gamma = 0.6$, we have from (41) $S = -0.2$. Moreover one finds that, for $\eta < 10$ it is possible to set, with a sufficient approximation

$$\eta^{1.6} e^{0.6\pi \frac{\sqrt{1+\eta^2}}{\eta}} \left| \Gamma\left(0.8 + 0.6i \frac{\sqrt{1+\eta^2}}{\eta}\right) \right|^2 \cong 4.5\eta + 1.6\eta^2 \, . \tag{43}$$

With this, (42) becomes, setting $R = 9 \cdot 10^{-13}$ in it,

$$P(\eta)d\eta = 1.75 \cdot 10^{95} g^2 \left| \int v_m^* u_n d\tau \right|^2 (\eta + 0.355\eta^2) \left( \sqrt{1+\eta_0^2} - \sqrt{1+\eta^2} \right)^2 \, . \tag{44}$$

The inverse of the lifetime is obtained by integrating (44) from $\eta = 0$ to $\eta = \eta_0$; one finds

$$\frac{1}{\tau} = 1.75 \cdot 10^{95} g^2 \left| \int v_m^* u_n d\tau \right|^2 F(\eta_0) \, , \tag{45}$$

where we have set

$$F(\eta_0) = \frac{2}{3}\sqrt{1+\eta_0^2} - \frac{2}{3} + \frac{\eta_0^4}{12} - \frac{\eta_0^2}{3} + 0.355 \left[ -\frac{\eta_0}{4} - \frac{\eta_0^3}{12} + \frac{\eta_0^5}{30} + \frac{\sqrt{1+\eta_0^2}}{4} \log\left(\eta_0 + \sqrt{1+\eta_0^2}\right) \right] \, . \tag{46}$$

For small values of the argument, $F(\eta_0)$ behaves like $\eta_0^6/24$; for larger values of the argument, the values of $F$ are gathered together in the following table.

Table 1

| $\eta_0$ | $F(\eta_0)$ | $\eta_0$ | $F(\eta_0)$ | $\eta_0$ | $F(\eta_0)$ | $\eta_0$ | $F(\eta_0)$ |
|---|---|---|---|---|---|---|---|
| 0 | $\eta_0^6/24$ | 2 | 1.2 | 4 | 29 | 6 | 185 |
| 1 | 0.03 | 3 | 7.5 | 5 | 80 | 7 | 380 |

## 9. The forbidden transitions

~~§ 9.~~ Before moving on to a comparison of the theory with experience, we still want to illustrate some properties of the forbidden transitions.

As we have already ~~said~~, a transition is forbidden when the corresponding matrix element (35) vanishes. If the representation of the nucleus by means of individual quantum states of the protons and neutrons turns out to be a good approximation, the matrix element $Q_{mn}^*$ vanishes, due to symmetry, when mentioned

$$i = i' \tag{47}$$

does not hold, where $i$ and $i'$ are the angular momentum, in units $h/2\pi$, of the neutron state $u_n$ and the proton state $v_m$, respectively. When the individual quantum states do not turn out to be a good approximation, to the selection rule (47) corresponds the other one

please rephrase

$$I = I' \; , \tag{48}$$

where $I$ and $I'$ represent the angular momentum of the nucleus before and after the $\beta$ decay.

The selection rules (47) and (48) are much less rigorous than the selection rules of optics. It is possible to find exceptions to them, particularly with the two following processes:

(a) Formula (26) has been obtained by neglecting the variations of $\psi_s$ and $\varphi_s$ inside the region of the nucleus. If on the contrary these variations are taken into account, one has the possibility of obtaining $\beta$ transitions even when $Q^*_{mn}$ vanishes. It is easy to recognize that the intensity of these transitions has a ratio, as an order of magnitude, with the intensity of the allowed processes given by $(R/\lambda)^2$, where $\lambda$ is the De Broglie wave length of the light particles. It must be noted that, if the electron and the neutrino have the same energy, when the former is near the nucleus it has a higher kinetic energy, due to the electrostatic attraction and so the most important effect comes from the variations of $\psi_s$. An evaluation of the order of magnitude of the intensity of these forbidden processes show that, at the same energy of the emitted electrons, they must have an intensity of one hundredth of the intensity of the normal processes. Besides the relatively small intensity, a characteristics of the forbidden transitions of this type can be found in the different shape of the curve of the energy distribution of $\beta$ rays, which, for the forbidden transitions, must give a number of particles with small energy lower than in the normal case. one word

(b) A second possibility to have $\beta$ transitions forbidden by the rule (48) depends on the fact, already pointed out at the end of § 3, that when the velocity of neutrons and protons is not negligible in comparison with the velocity of light we must add to the interaction term (12) other terms of order $v/c$. If e.g. one would assume a relativistic wave equation of the Dirac type also for the heavy particles, one could add to (12) terms like ,

$$gQ\,(\alpha_x A_1 + \alpha_y A_2 + \alpha_z A_3) \; + \; \text{complex conjugate} \; , \tag{49}$$

where $\alpha_x \alpha_y \alpha_z$ are the usual Dirac matrices for the heavy particle and $A_1 A_2 A_3$ the spatial components of the four vector defined by (12). A term of the type (49) allows also $\beta$ transitions which do not satisfy the selection rule (48), and their intensity is, with respect to that of normal processes, of the order of magnitude $(v/c)^2$, that is about 1/100. Thus we find a second possibility for forbidden transitions nearly 100 times less intense than the normal ones. s

## 10. Comparison with experience

~~§ 10.)~~ (45) establishes a relation between the maximum momentum $\eta_0$ of the $\beta$ rays emitted by a substance and its lifetime. In this relation, really, also an unknown element enters, the integral

$$\int v_m^* u_n d\tau \tag{50}$$

whose evaluation requires knowledge of the nuclear eigenfunctions $u_n$ and $v_m$ of the neutron and the proton. However, in the case of the allowed transitions, (50) is of the order of magnitude of unity. Then we expect the product

$$\tau F(\eta_0) \tag{51}$$

to have the same order of magnitude in all the allowed transitions. Instead, for the forbidden transitions, the lifetime will be, as an order of magnitude, one hundred times larger, and correspondingly also the product (51) will be larger. In the following table we collect the products $\tau F(\eta_0)$ for all the substances which disintegrate by emitting $\beta$ rays and for which we have sufficiently exact data.

Table 2

| Element | $\tau$(hours) | $\eta_0$ | $F(\eta_0)$ | $\tau F(\eta_0)$ |
|---|---|---|---|---|
| $UX_2$ | 0.026 | 5.4 | 115 | 3.0 |
| $RaB$ | 0.64 | 2.04 | 1.34 | 0.9 |
| $ThB$ | 15.3 | 1.37 | 0.176 | 2.7 |
| $ThC''$ | 0.076 | 4.4 | 44 | 3.3 |
| $AcC''$ | 0.115 | 3.6 | 17.6 | 2.0 |
| $RaC$ | 0.47 | 7.07 | 398 | 190 |
| $RaE$ | 173 | 3.23 | 10.5 | 1800 |
| $ThC$ | 2.4 | 5.2 | 95 | 230 |
| $MsTh_2$ | 8.8 | 6.13 | 73 | 640 |

In this table the two groups we have expected are certainly recognizable; moreover such a division of the elements which emit primary $\beta$ rays into two groups had been already observed experimentally by Sargent.** The values of $\eta_0$ have been taken from the quoted paper of Sargent (for a comparison, note that: $\eta_0 = (H\rho)_{max/1700}$). Besides the data in this table, Sargent gives the data for three other elements, warning that they are not as reliable as the other ones. They are $UX_1$ for which $\tau = 830$; $\eta_0 = 0.76$; $F(\eta_0) = 0.0065$; $\tau F(\eta_0) = 5.4$; then this element appears to be attributable to the first group. For $AcB$ one has: $\tau = 0.87$; $\eta_0 = 1.24$; $F(\eta_0) = 0.102$; $\tau F(\eta_0) = 0.09$; then one finds a value of $\tau F(\eta_0)$ about ten

trs

**B.W. Sargent, *Proc. Roy. Soc.* **139**, 659, (1933).

times smaller than those of the first group. Finally for $RaD$ one has: $\tau = 320000$; $\eta_0 = 0.38$ (largely uncertain); $F(\eta_0) = 0.00011$; $\tau F(\eta_0) = 35$. Then this element can be put roughly half-way between the two groups. I have not succeeded in finding data for the other elements which emit primary $\beta$ rays, that is $Ms$, $Th_1$, $UY$, $Ac$, $AcC$, $UZ$, $RaC''$.

On the whole one can conclude from this comparison between theory and experience that the agreement is certainly as good as one would have expected. The discrepancies observed for the elements with uncertain experimental data, $RaD$ and $AcB$, can be explained well partly by the lack of precision of the measures, partly also by oscillations, quite plausible, in the value of the matrix element (50). Moreover one must notice ~~that~~ the fact that the majority of $\beta$ decays are accompanied by emission of $\gamma$ rays indicates that the larger part of the $\beta$ processes can leave the proton in different excitation states and this gives a further mechanism which can determine oscillations in the value of $\tau F(\eta_0)$.

From the data of Table 2 one can infer an evaluation, even if rough, of constant $g$. If we admit, for instance, that when the matrix element (50) has the value 1, one has $\tau F(\eta_0) = 1\ hour = 3600\ s$; one finds from (45)

$$g = 4 \cdot 10^{-50}\ cm^3 \cdot erg$$

which gives nothing more than the order of magnitude.

Let us move on to discuss the shape of the curve of the velocity distribution of $\beta$ rays. In the case of allowed processes, the distribution curve, as a function of $\eta$ (that is, apart from a factor 1700, of $H\rho$) is represented in Fig. 2, for values of the maximum momentum $\eta_0$.

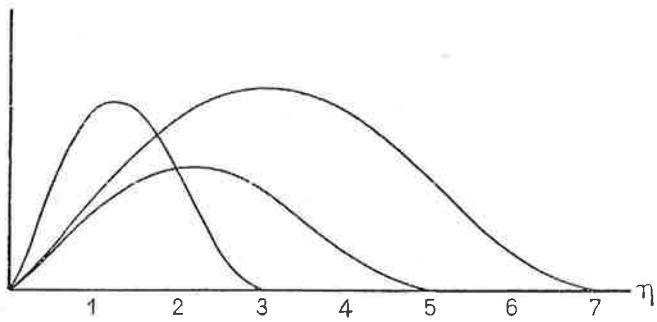


Fig. 2

The curves are satisfactorily similar to experimental ones collected by Sargent.[††] Only in the range of small energy Sargent's curves are a little lower than the theoretical ones, and this is more easily evident in the curves of Fig. 3 where the abscissas are the energies instead of the momenta. But we must remark that the part of

[††] B.W. Sargent, *Proc. Camb. Phil. Soc.* **28**, 538 (1932).

the curves of small energy is not perfectly known experimentally.[‡‡] Moreover, for the forbidden transitions, also theoretically, in the range of small energies the curve must be lower than the curves of the allowed transitions, represented in Figs. 2 and 3.

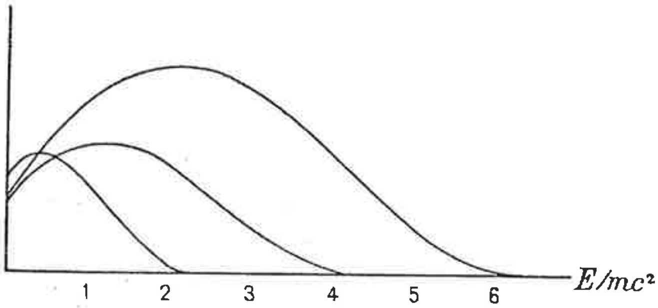


Fig. 3

Of this fact one must particularly take into account for the case of $RaE$, which is the best known from an experimental point of view. The emission of $\beta$ rays from this element, as results from the abnormally large value of $\tau F(\eta_0)$ (Cf. Table 2), is certainly forbidden, or better it is possible that it is allowed only in the second approximation. I hope, in a future article, to be able to better specify the behavior of distribution curves for the forbidden transitions. cf.

To summarize, it seems justified to assert that the theory in the form described here does agree with the experimental data, which in any case are not always sufficiently accurate. On the other hand, even if in a further comparison of the theory with experience, one should arrive at some discrepancy, it would be always possible to modify the theory without changing its conceptual foundations in an essential way. It would be possible precisely to keep equation (9) but choose the $c_{s\sigma}$ in a different way. This will carry us, in particular, to a different form of the selection rule (48) and to a different form of the curve of the energy distribution.

Only a further development of the theory, as also an increase in the precision of the experimental data, will be able to indicate if such a change will be necessary.

‡‡Cf. e.g., Rutherford, Ellis and Chadwick, *Radiation from Radio-active Substances*, Cambridge, 1930. See, in particular p. 407.

### A.1 D. Bini, A. Geralico, R.T. Jantzen and R. Ruffini: On Fermi's resolution of the "4/3 problem" in the classical theory of the electron and its logical conclusion: hidden in plain sight

**Abstract.** We discuss the solution proposed by Fermi to the so-called "4/3 problem" in the classical theory of the electron, a problem which puzzled the physics community for many decades before and after his contribution to the discussion. Unfortunately his early resolution of the problem in 1922–1923 published in three versions in Italian and German journals (after three preliminary articles on the topic) went largely unnoticed, and even recent texts devoted to classical electron theory still do not present his argument or acknowledge the actual content of those articles. The calculations initiated by Fermi at the time are finally brought to their logical physical conclusion here.

#### *Introduction*

The simplest classical model of the nonrotating electron in special relativity consists of a static spherically symmetric distribution of total electric charge $e$ over the surface of a rigid sphere of radius $r_0$, as measured by an observer at rest with respect to the sphere. This model was first developed and studied during the first decade of the 1900s by Abraham [1], Lorentz [2] and Poincaré [3], based entirely on Maxwell's theory of electromagnetism. For an unaccelerated electron, the rest frame integral of the local energy density of the Coulomb field over the exterior of the electron sphere representing the total energy $W = e^2/(2r_0)$ stored in that field, is equal to the self-energy of the charge distribution. For any static isolated configuration of charge, this self-energy is equal to the work needed to assemble it by slowly bringing the charge elements in from spatial infinity.

The factor of $1/2$ in the energy formula is a geometric factor which is replaced by $3/5$ if the model of the electron is a constant charge density solid sphere rather than a constant density spherical surface charge distribution and one also considers the additional contribution to the electromagnetic field energy inside the sphere (zero in the surface distribution case by spherical symmetry): $1/2 + 1/10 = 3/5$. Dropping these factors and converting the Coulomb energy to the entire observed mass $m_e$ of the electron by Einstein's famous mass-energy relation $E = mc^2$ defines a corresponding radius $r_e = e^2/(m_e c^2)$ that pure dimensional analysis would lead to, called the classical radius of the electron.

With the birth of special relativity occurring during the same ~~time~~ years as the Abraham-Lorentz model development, there was the expectation that apart from any additional "bare mass" $m_0$ that the electron might have, the electromagnetic energy $W$ should contribute to the inertial mass of the electron an amount $m_{\rm em} = W/c^2$ satisfying Einstein's mass-energy relation, leading to a total mass $m_e = m_0 + m_{\rm em}$. Instead they had found $m_{\rm em} = \frac{4}{3}W/c^2$ in the limit of nonrelasystem adapted to the central world line is Born rigid.

In order to fix the 4/3 problem Poincaré [3] (followed up by von Laue [13]) seriously confused the issue by mixing it with the question of explaining the rigid configuration of charge through internal stresses. Long after Fermi's resolution of the 4/3 problem, even in the commentary by his friend Persico on Fermi's paper in the collected work of Fermi, it was thought that Poincaré stresses were necessary to explain this discrepancy. In fact the stability of the electron is an entirely different matter from the correct relation of the inertial mass to the electromagnetic energy as explained by Fermi.

Although Wilson [14] discussed the problem of the proper definition of the 4-momentum of the electromagnetic field in 1936 with no citations, he did not succeed in clarifying matters. In 1949 Kwal [9] showed that a slight modification of Abraham's original integral definitions for the unaccelerated electron leads to an electromagnetic 4-momentum endowed with the correct Lorentz transformation properties. Even later Rohrlich [10] in 1960 came to the same conclusion without being aware of previous work. They both explained that the correct result can only be obtained from the usual special relativistic integrals over a hypersurface of constant inertial time if that hypersurface represents a time slice in the rest frame of the electron, although Kwal only discussed changing the element of hypersurface volume without relating the region of integration to that rest frame. The classical electron model has continued to intrigue people ever since, see for example, Feynman [15], Teitelboim [16–17], Boyer [18], Rohrlich [19], Nodvik [20], Schwinger [21], Campos and Jimenéz [22], Cohen and Mustafa [23], Comay [24], Moylan [25], Kolbenstvedt [26], Rohrlich [27], Appel and Kiessling [28], de Leon [29], Harte [30], Pinto [31], Bettini [32], Galley et al [33], and more. At least three entire books are devoted to the topic of the classical theory of the charge distributions, those by Rohrlich [34], Yaghjian [35], and Spohn [36], and the model is described in detail by Jackson [37], the universally accepted reference textbook on classical electrodynamics (see also Chapter 8 of Anderson [38]). Some interesting historical details may be found in the recent article of Janssen and Mecklenburg [39]. This whole problem is not without explicit controversy, as detailed by Parrott in his archived exchange with *Physical Review* which would not publish his criticism of Rohrlich's recent work [40]. et al.

Except for Aharoni [11] and much later Kolbenstvedt [26] in 1997, and for Nodvik [20] and Appel and Kiessling [28] who consider a spinning generalization of the relativistically rigidly rotating electron model reviewed by Spohn [36], none of these references seem to take into account Fermi's actual argument nor connect it to that of Kwal and Rohrlich even though most of them cite Fermi's original article. Kolbenstvedt [26] called attention to Fermi's argument with a slightly different but equivalent explanation of his own, and not in an obscure physics journal, and yet the latest edition of the books of Jackson, Rohrlich, and Yaghjian, all published after that year still do not reflect this news. Jackson does explain that his non-relativistic treatment can be relativistically corrected, referring to Fermi, and to or be fair, the stated purpose of Yaghjian was to update the Abraham-Lorentz model which he did, apparently unaware of the content of Fermi's articles. Misner, Thorne and Wheeler's tome *Gravitation* [41], affectionately known as MTW, really raised the level of mathematical discussion of special and general relativity after 1973, and allowed Spohn to more cleanly and covariantly discuss the relativistically rigid electron model to include spin, but without discussing the observer-dependent 4-momentum integral for the electromagnetic self-field.

An important element of this discussion is the conserved nature of the integrals of the local densities of energy and momentum associated with the divergence-free stress-energy tensor of the sourcefree electromagnetic field when integrated over an entire spacelike hyperplane of Minkowski spacetime due to Gauss's law. Such a conservation law fails to exist when the divergence is instead nonzero in the presence of sources or if a world tube containing sources is excluded from the integral, leading either to a spacetime volume divergence integral or (equivalently) to an internal boundary integral that must be taken into account in Gauss's law. This is an important discussion since none of the textbooks on special or general relativity describe this more general situation, while textbooks on classical electrodynamics typically only use local such integrals within bounded regions of space.

Since this discussion is crucial in understanding the present problem, it is included in the section following this introduction where the preliminary details about the electromagnetic field needed to consider the spherical model of an unaccelerated electron are introduced together with the definitions of the 4-momentum in the field as observed by any inertial observer, and the role played by Gauss's law in conservation laws is then explained, leaving the details of more exotic regions of spacetime integration to the appendix. The calculation of the 4-momentum integrals for the Abraham-Lorentz model of the unaccelerated electron is then reproduced in the subsequent section to explain the role played by Kwal and Rohrlich in this matter. Next we present Fermi's re-analysis of the Abraham-Lorentz calculation of the inertial mass for their model of the accelerated electron taking into account Born's rigidity condition. Finally the Kwal-Rohrlich definition of 4-momentum is related directly back to this correction using Gauss's law.

One finds that the Kwal-Rohrlich restriction of the observer-dependent electromagnetic field 4-momentum integrals to the electron rest frame time hyperplanes associates a unique 4-momentum with the unaccelerated electron which is the one special relativity assigns, which has long been known. However, for a single static electron configuration in the absence of interaction, the 4-momentum is not so interesting since there is no way even of revealing its inertial mass from at most uniform translational motion in flat spacetime. To get information about the inertial mass and 4-momentum, the electron must be accelerated and if we limit our attention to electromagnetic interactions, it will be accelerated by an external electromagnetic field through the Lorentz force law. We expect that the total momentum of the electron and the electromagnetic field (for a closed system) should be conserved.

We will show here for the first time that indeed the logical conclusion of Fermi's calculation of the lowest order contributions to the equations of motion of the electron is that the total 4-momentum as observed in the time slices in the sequence of instantaneous rest frames along its path is conserved, i.e., is independent of time, and is the usual one we associate with the system. The key idea of Fermi of the importance of this sequence of hyperplanes orthogonal to the path of a given world line in spacetime was imbedded in his Fermi coordinate system adapted to that world line and which outlived the purpose for which he introduced it in those initial days of the theory of general relativity.

### *Electrodynamic preliminaries*

Although Fermi does not specify the density profile of the spherically symmetric charge distribution that he analyzes in his re-examination of the earliest classical electron theory proposed by Abraham [1] and improved by Lorentz [2], he refers specifically to their spherical shell model of the electron in his introduction. Without acceleration of the electron this model cannot help identify the inertial mass which arises as the proportionality constant between the applied force and the resulting acceleration. However, it was the interest in their unaccelerated model which helped push towards the understanding of the 4-momentum hypersurface integrals for the electromagnetic field so it is useful to review this case first. We re-examine their work in light of modern notation and perspective.

The model for the electron first proposed by Abraham [1] and improved by Lorentz [2] consisted of a uniform spherically symmetric distribution of total electric charge $e$ over the surface of a rigid sphere of radius $r_0$ in its rest frame. This was called the contractile electron since it would then undergo Lorentz contraction with respect to an inertial frame in relative motion, while Abraham had assumed that the electron remained a rigid sphere with respect to all inertial observers. Einstein's understanding of special relativity only came after this model had been developed, and Lorentz had interpreted the Lorentz contraction as a dynamical effect rather than as a universal property of spacetime itself. They attempted to explain the mass-energy of the electron as due wholy to the electromagnetic field of the electron, equating the electron's energy and momentum to the energy and momentum of its electromagnetic field, which can be evaluated by suitably integrating the normal components of the stress-energy tensor of the electromagnetic field over a spacelike hyperplane representing a moment of time in an inertial reference frame. This is a useful example to keep in mind.

In an inertial system of Cartesian coordinates $(x^\mu) = (t = x^0, x^1, x^2, x^3)$ associated with an inertial reference frame in Minkowski spacetime with signature $(-+++)$ following the conventions of Misner, Thorne and Wheeler [41] with $c = 1$, Maxwell's equations for the electromagnetic field tensor $F_{\alpha\beta}$ due to the 4-current density $J^\alpha$ are

$$F^{\alpha\beta}{}_{,\beta} = 4\pi J^\alpha\,, \quad F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} = 0\,, \tag{1}$$

where Greek indices assume the values $0, 1, 2, 3$, and Latin indices only ~~instead~~ $1, 2, 3$. only
Indices may be raised and lowered with the flat Minkowski spacetime metric whose inertial coordinate components are $(g_{\alpha\beta}) = \mathrm{diag}(-1, 1, 1, 1) = (g^{\alpha\beta})$.

Of course when these equations are expressed in noninertial coordinate systems the comma here signifying partial coordinate derivatives $f_{,\alpha} = \partial_\alpha f = \partial/\partial x^\alpha$ must be replaced by the semicolon indicating the components of the covariant derivative. We will have need later for an arbitrary covariant constant covector field $Q_\alpha$ of vanishing covariant derivative $Q_{\alpha;\beta} = 0$, the components of which reduce to $Q_{\alpha,\beta} = 0$ in an inertial coordinate system where the components $Q_\alpha$ (and $Q^\alpha$) are actual constants. In fact such covariant constant vector fields $Q^\alpha$ correspond to the translational Killing vector fields of Minkowski spacetime, which are special solutions of the general Killing equations that the symmetrized covariant derivative $Q_{(\alpha;\beta)} = 0$ vanish. The noncovariant constant Killing vectors generate the rotations and boost symmetries of Minkowski spacetime.

The stress-energy tensor of the electromagnetic field

$$T^{\mu\nu}_{\mathrm{em}} = \frac{1}{4\pi}\left(F^{\mu\alpha}F^{\nu}{}_{\alpha} - \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right) \tag{2}$$

has the following explicit inertial coordinate components

$$\begin{aligned}
T^{00}_{\mathrm{em}} &= \frac{1}{8\pi}(E^2 + B^2) = U_{\mathrm{em}}\,,\\
T^{0i}_{\mathrm{em}} &= \frac{1}{4\pi}(E \times B)^i = S^i\,,\\
T^{ij}_{\mathrm{em}} &= \frac{1}{4\pi}[-E^iE^j - B^iB^j + \frac{1}{2}g^{ij}(E^2 + B^2)]\,,
\end{aligned} \tag{3}$$

where $U_{\mathrm{em}}$ and $S$ are the electromagnetic energy density and the Poynting vector respectively, and of course $E$ and $B$ are the usual electric and magnetic fields observed in the associated reference frame in index-free notation, with nontrivial inertial coordinate components $E^i = F^{0i} = F_{i0}$ and $B^1 = F_{23}$ etc. In general if $u^\alpha$ is the 4-velocity of an observer at a point of spacetime, the electric field as seen by that observer there is $E(u)^\alpha = F^\alpha{}_\beta u^\beta$. In a system of inertial coordinates adapted to that observer, then $u^\alpha = \delta^\alpha{}_0$, so that one has $E(u)^\alpha = F^\alpha{}_\beta\delta^\beta{}_0 = F^\alpha{}_0 = \delta^\alpha{}_i F^i{}_0$ since due to the change of sign under index raising and the antisymmetry of the field tensor $F^0{}_0 = -F^{00} = 0$. Note that in inertial coordinates associated with a second inertial observer in relative motion to a given 4-velocity $u^\alpha$, its components are given by $(u^\alpha) = (\gamma, \gamma v^i)$, where $v^i$ are the components of the relative velocity of the first observer and $\gamma = (1 - v^i v_i)^{-1/2}$ is the associated gamma factor.

The divergence of this stress-energy tensor in inertial coordinates is easily calculated using Maxwell's equations

$$T^{\mu\nu}_{\mathrm{em}\,,\nu} = -F^\mu{}_\nu J^\nu\,, \tag{4}$$

as shown by Exercise 3.18 of Misner, Thorne and Wheeler [41], for example. Thus in source-free regions where the 4-current $J^\mu = 0$ vanishes, this divergence is zero, which is the condition needed to obtain a conserved 4-momentum for the free electromagnetic field in textbook discussions using Gauss's law. When the ~~the~~ 4-current density $J^\alpha = \rho U^\alpha$ is due to the motion of a distribution of charge moving with 4-velocity field $U^\alpha$ and rest frame charge density $\rho$, then this divergence has the value

$$T_{\text{em};\nu}^{\mu\nu} = -\rho F^\mu{}_\nu U^\nu = -\rho E(U)^\mu \,, \tag{5}$$

which apart from the sign is the 4-force density exerted by the electromagnetic field on the charge distribution, expressable as the product of the charge density and the electric field in the rest frame of the moving charge. This divergence plays a crucial role in the Lagrangian equations of motion of the electron and in the conservation or not of the 4-momentum of the electromagnetic field. Unlike the 4-momentum of a particle which is locally defined and independent of the observer (but whose components depend on the choice of coordinates of course), the 4-momentum of the electromagnetic field is nonlocal and can only be defined at a momentum of time with respect to some inertial observer through an integral over an entire hyperplane $\Sigma$ of spacetime corresponding to the extension of the local rest space of that observer at that moment. In the presence of sources $J^\alpha \neq 0$, this 4-momentum not only generally depends on the time for nonstationary sources, but also on the choice of observer, since there is no a priori reason to expect integrals over different regions of spacetime to agree. When instead $J^\alpha = 0$ as is the case for a free electromagnetic field, a conservation law applies due to the vanishing divergence and if those integrals are finite, they in fact all define the same 4-momentum vector on Minkowski spacetime.

The components of the 4-momentum of the electromagnetic field as seen by an inertial observer with 4-velocity $u^\alpha$ at a moment of time $t$ in the observer rest frame represented by a time coordinate hyperplane $\Sigma$ (for which $u^\alpha$ is in fact the future-pointing unit normal vector field) is given by the integral formula

$$P(\Sigma)^\alpha = \int_\Sigma T_{\text{em}}^{\alpha\beta} d\Sigma_\beta \,, \tag{6}$$

where one can integrate over an object with a free index only if that index is expressed in some inertial coordinate system where it makes sense to compare 4-vectors at different spacetime points in the flat spacetime due to the path independence of parallel transport. The contracted pair of indices can be evaluated in any coordinates. For any spacelike hyperplane $\Sigma$ with future-pointing timelike unit normal $u^\alpha$, the hyperplane volume element

$$d\Sigma_\alpha = -u_\alpha dV_\Sigma \tag{7}$$

and induced volume element $dV_\Sigma$ are most easily evaluated in inertial coordinates $(t, x^i)$ adapted to the observer with 4-velocity $u^\alpha$, where $u^\alpha = \delta^\alpha{}_0$ while $u_\alpha =$

region of integration, so there is no common agreement among inertial observers about the 4-momentum in the field, nor is the result independent of time for a single inertial observer. This is the source of the complication for defining the 4-momentum of the electromagnetic field in the classical model of the electron.

A covariant constant vector field is a Killing vector generating translational symmetries of Minkowski spacetime from which the conservation of linear momentum follows for translation invariant Lagrangians according to Noether's theorem. The arbitrary translational Killing vector field $Q^\alpha$ allows us to pick out the components of linear momentum. A general Killing vector field satisfies the condition that its symmetrized covariant derivative vanish $Q_{(\alpha;\beta)} = 0$. If instead we use a nontranslational Killing vector field in the above argument, then since the stress-energy tensor is symmetric and only the symmetric part contributes to its contraction with the covariant derivative of $Q_\alpha$, we get the same divergence formula as before

$$\mathcal{J}^\beta{}_{;\beta} = Q_\alpha T^{\alpha\beta}{}_{;\beta} + Q_{(\alpha;\beta)} T^{\alpha\beta} = Q_\alpha T^{\alpha\beta}{}_{;\beta} \,. \tag{16}$$

For the nontranslational Killing vector fields which generate rotations, for example, this process leads to picking out the components of the conserved angular momentum in the case of vanishing divergence. See Misner, Thorne and Wheeler [41], for example. However, we will not consider angular momentum here.

For a static electric field due to a static charge distribution $\rho$ in its rest frame, when expressed in terms of inertial coordinates in that rest frame for a time slice $\Sigma_{\text{rest}}$ in that frame, the quantity

$$P(\Sigma)^0 = \frac{1}{8\pi} \int_{\Sigma_{\text{rest}}} E^2(\mathbf{x}) d^3\mathbf{x} = W \tag{17}$$

is just the self-energy of the charge configuration defined alternatively by

$$W = \frac{1}{2} \int \int d^3\mathbf{x} d^3\mathbf{x}' \, \frac{\rho(t,\mathbf{x})\rho(t,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \,, \tag{18}$$

+ using the vector notation $\mathbf{x} = (x^i)$, $d^3\mathbf{x} = dx^1 dx^2 dx^3 = dV$. Jackson (see p. 41 of the Third edition [37]) shows how the latter formula for the self-energy of such a static charge configuration is equivalent to the energy in its associated electric field using the integral formula for the potential

$$\phi(\mathbf{x}) = \int d^3\mathbf{x}' \, \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \tag{19}$$

and the Poisson equation $\nabla^2\phi = -4\pi\rho$. Then replacing the primed factors in the double integral for $W$ by this expression for the potential, and with a crucial integration by parts identity, we get

$$\begin{aligned} W &= \frac{1}{2} \int \int d^3\mathbf{x} d^3\mathbf{x}' \, \frac{\rho(\mathbf{x})\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} = \frac{1}{2} \int d^3\mathbf{x} \, \rho(\mathbf{x})\phi(\mathbf{x}) \\ &= -\frac{1}{8\pi} \int d^3\mathbf{x} \, \phi(\mathbf{x})\nabla^2\phi(\mathbf{x}) = \frac{1}{8\pi} \int d^3\mathbf{x} \, \left[\nabla_i\phi(\mathbf{x})\nabla^i\phi(\mathbf{x}) - \nabla_i(\phi(\mathbf{x})\nabla^i\phi(\mathbf{x}))\right] \\ &= \frac{1}{8\pi} \int d^3\mathbf{x} \, \left[E(\mathbf{x})^2 - \nabla_i(\phi(\mathbf{x})\nabla^i\phi(\mathbf{x}))\right] \,. \end{aligned} \tag{20}$$

one word

This integral is only over the charge distribution but one can extend it to over all space since the extra contribution is zero where the charge density is zero, but as an integral over all space, the divergence term by Gauss's law is equivalent to a surface integral at spatial infinity, where the integrand goes to zero fast enough in this static case so that the surface integral evaluates to zero in the limit. The result is just the first term representing the total energy in the electric field.

$$W = \frac{1}{8\pi}\int E(\mathbf{x})^2\, d^3\mathbf{x} = \int T^{00} d^3\mathbf{x}\,. \tag{21}$$

This self-energy plays a key role in the lowest order approximation to the equations of motion of the charge distribution.

Returning now to the divergence $-\rho E(U)^i$ in inertial coordinates of the rest frame of a static distribution of charge, its spatial integral reversed in sign is just the total electric force on the charge distribution which of course must be zero for a static configuration of charge, assuming that the charge elements are held in place by forces that are not addressed yet in this model. Otherwise the situation would not remain static. However, if the charge distribution is accelerated, there is no a priori reason to expect that the total electric force in its instantaneous rest frame be zero, and this was the error made in the Abraham and Lorentz model. Fermi showed that by requiring that the rigidity of the model respect Born's special relativistic rigidity condition, this total force integral is modified by a simple factor that his Fermi coordinate system provided, and resolves the 4/3 problem. Gauss's law is then the key to picking out the correct conserved 4-momentum of the total system which remains ambiguous in the static unaccelerated case, as we will show in the final section.

### *The static electron model*

Fermi considers an arbitrary spherically symmetric static distribution of total charge $e$ with density $\rho$ in the rest frame of the electron, while referring specifically to the Abraham-Lorentz model of a uniform surface distribution of charge on a sphere of radius $r_0$ as the motivation for his analysis. The latter is an instructive example to keep in mind. Let the spherically symmetric charge distribution remain at rest at the spatial origin of a system of inertial coordinates $(t, x^i)$ associated with the inertial frame $K$ in which it is at rest for all time. The inertial observer 4-velocity is $u = \partial_t$ (in index-free notation). Let $(t, r, \theta, \phi)$ be a corresponding system of spherical coordinates in terms of which the sphere containing the charge has the equation $r = r_0$. The metric is

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\,. \tag{22}$$

Then whatever the internal distribution of charge, the exterior field outside its outer surface at $r = r_0$ in index-free notation is

$$F = -\frac{e}{r^2} dt \wedge dr\,, \qquad E = \frac{e}{r^2}\partial_r\,, \qquad B = 0 \quad (r \geq r_0)\,. \tag{23}$$

so that the Poynting vector is also zero. Introducing orthonormal components with respect to the normalized spherical coordinate frame via

$$X^{\hat{0}} = X^0\,,\ X^{\hat{r}} = X^r\,,\ X^{\hat{\theta}} = r^{-1}X^\theta\,,\ X^{\hat{\phi}} = (r\sin\theta)^{-1}X^\phi\,, \tag{24}$$

the nonvanishing such components of the stress-energy tensor of the exterior field (for $r \geq r_0$) are

$$T^{00}_{\rm em} = -T^{rr}_{\rm em} = T^{\hat{\theta}\hat{\theta}}_{\rm em} = T^{\hat{\phi}\hat{\phi}}_{\rm em} = \frac{1}{8\pi}E^2 = \frac{e^2}{8\pi r^4} = U_{\rm em}\,. \tag{25}$$

Its divergence is zero in the exterior of the electron sphere. For the shell model, the interior electromagnetic field is zero by spherical symmetry, but if instead one assumes a constant density model inside a ball of radius $r_0$, the interior electric field has magnitude $er/r_0^3$ (for $r \leq r_0$). This interior field then contributes to the total energy of the field.

The inertial coordinate components of the 4-momentum (9) in this rest frame $K$ for any rest frame time slice $\Sigma_{\rm rest}$ are

$$P(\Sigma_{\rm rest})^0 = \int_{\Sigma_{\rm rest}} U_{\rm em}dV = W\,, \quad P(\Sigma_{\rm rest})^k = \int_{\Sigma_{\rm rest}} S^k dV = 0\,, \tag{26}$$

where $W$ is the self-energy of the static charge distribution due to its own electric field. These are time-independent because of the time-independence of the electric field in this frame, leading to the constant 4-momentum

$$P(\Sigma_{\rm rest})^\alpha = WU^\alpha\,. \tag{27}$$

For the Coulomb field of the spherical shell electron, evaluating this quantity in spherical coordinates gives the energy of the Coulomb field

$$W = \frac{e^2}{2r_0}\,. \tag{28}$$

For the constant density model of the electron, this integral over the internal field produces an additional contribution of $e^2/(10r_0)$ leading to the total energy $3e^2/(5r_0)$. If one assumes that this electromagnetic energy makes a contribution $m_{\rm em}$ to the inertial mass of the electron via Einstein's mass-energy relation $E = mc^2$, then $m_{\rm em} = W$ (in units where $c = 1$). However, the inertial mass can only be ascertained from the equation of motion of an accelerated electron, so this must be confirmed by the evaluation of the equation of motion.

Note that the tracefree condition $T^{00}_{\rm em} = T^{11}_{\rm em} + T^{22}_{\rm em} + T^{33}_{\rm em}$ in the Cartesian inertial coordinates when integrated over the same region yields the condition

$$\int_{\Sigma_{\rm rest}} T^{00}_{\rm em}dV = \int_{\Sigma_{\rm rest}} (T^{11}_{\rm em} + T^{22}_{\rm em} + T^{33}_{\rm em})\,dV\,, \tag{29}$$

but by the spherical symmetry of the electric field in the rest frame each of the terms on the right-hand side has the same value

$$\int_{\Sigma_{\rm rest}} T^{11}_{\rm em}dV = \int_{\Sigma_{\rm rest}} T^{22}_{\rm em}dV = \int_{\Sigma_{\rm rest}} T^{33}_{\rm em}dV = \frac{1}{3}\int_{\Sigma_{\rm rest}} T^{00}_{\rm em}dV\,. \tag{30}$$

### *Fermi's contribution*

Fermi's first paper in 1921 (Fermi 1: "On the dynamics of a rigid system of electric charges in translational motion," [4] studied a special relativistic system of electrons in rigid motion as then understood by Abraham and Lorentz and found the 4/3 factor in its inertial mass formula, while this factor was not present in the mass corresponding to the "weight" he calculated using general relativity in his second paper (Fermi 2: "On the electrostatics of a homogeneous gravitational field and on the weight of electromagnetic masses," [5]), referring to Levi-Civita's uniformly accelerated metric for the calculations [42]. This contradicted the assumed equivalence of these two masses in general relativity. These papers were both written within five years of the birth of Einstein's theory of general relativity in 1916, during which Fermi was first a high school student and then a university student writing his first two scientific papers. ~~During the next year~~ in 1922 in preparation for his revisit to the problem, Fermi published his third paper on his famous Fermi comoving coordinate system adapted to the local rest spaces along the world line of a particle in motion (Fermi 3: "On phenomena occurring close to a world line, [6]), and calculated the variation of the action for a system of charges and masses interacting with an electromagnetic field in such a coordinate system. He then used this approach to resolve the 4/3 puzzle in his fourth paper (two versions: Fermi 4a and 4c published in Italian and one Fermi 4b in German, the most complete of which is Fermi 4c: "Correction of a contradiction between electrodynamic theory and the relativistic theory of electromagnetic masses") without explicitly referring to the third paper. These were published in 1922–1923. Still in 1923 collaborating with A. Pontremoli (Fermi 10, [43]), Fermi applied his same argument to correct the calculation of the inertial mass of the radiation in a cavity with reflecting walls, where the same 4/3 factor had appeared when the cavity is in rigid motion not respecting the Born criterion; Boughn and Rothman provide a detailed alternative analysis which confirms Fermi's result in that case [44].

In

His approach was to use a variational principle in a region of spacetime containing the world tube of an accelerated electron charge distribution within which one has to make certain assumptions on how the relative motion of the individual charge elements in the distribution behaves. Following the Born notion of rigidity compatible with special relativity, the only way an electron can move rigidly so that its shape in its rest frame does not change is if the individual world lines of the charge distribution all cut the local rest frame time slices orthogonally, a Lorentz invariant geometrical condition which is equivalent to stating that their relative velocities are all zero at that moment. This condition must hold in a sequence of different inertial observers with respect to which the charge distribution is at rest. If instead one takes the family of time slices associated with a single inertial observer and require that the shape not change, i.e., that the relative velocities are all zero at each such time, this corresponds to the nonrelativistic notion of rigidity, and the world lines may be varied by arbitrary time-dependent translations, so that their origin of these spatial coordinates at $t = 0$ when $v^i = 0$, its time hypersurface $T = 0$ can be chosen to coincide with $t = 0$, but after a small interval $dt$ of laboratory time along the central world line, equal to the increment $dT$ in the proper time along that world line to first order, the Fermi time slice is instead tilted slightly to remain orthogonal to that world line as shown in Fig. 1. The metric in the Fermi coordinate system is 2?

$$ds^2 = -N^2 dT^2 + \delta_{ij} dX^i \, dX^j \,, \quad N = c(1 + \Gamma_i X^i/c^2) \,, \tag{43}$$

where $\Gamma_i = \dot{v}^i = dv^i/dT$ are the Cartesian components of the proper acceleration of the central world line (functions of $T$), and the speed of light $c$ is not taken to be unity in this paragraph only in order to appreciate how factors of $c$ enter the discussion. The proper time along the central Fermi coordinate time line is initially approximately $dT = dt$ at $t = 0 = T$, but away from the spatial origin at that world line there is a linear correction factor due to the lapse function $N$ in the Fermi coordinate system. The proper time interval along the normal to the initial hypersurface (measured by the increment in $t$ or $T$ to first order) to a nearby Fermi time slice is the increment $c^{-1} N \, dT = (1 + \Gamma_i x^i/c^2) dT$, namely the proper time along the time lines in the Fermi coordinate system. Misner, Thorne and Wheeler discuss the Fermi coordinate system in detail [41]. Of course because the proper time of each charge element world line varies by the Fermi lapse function factor compared to the central world line, the accelerations of the actual charge elements away from the central world line differ slightly from that of the central world line.

If we imagine doing a variation of the action integral over a spacetime region in inertial coordinates between two slices of inertial time (his variation A), then if we use the same coordinate symbols $(t, x^i)$ for the corresponding variation in Fermi coordinates between two slices of Fermi coordinate time (his variation B), the only formal difference in the action integrand is the additional Fermi lapse factor which enters through the spacetime volume element. This lapse correction factor is the entire basis for Fermi's correction, and multiplies the coordinate volume element to provide the covariant spacetime volume element in Minkowski spacetime: $d^4V$ which is $d^4x = dt \, dV$ in inertial coordinates but $N dt \, dV$ in Fermi coordinates, where $N dt = d\tau$ is the proper time along the time world lines orthogonal to the flat time slices and $dV = dx^1 dx^2 dx^3$ is the spatial volume element in both cases. Fermi does not mention his mathematical article on these coordinates, but just presents a short derivation of the correction factor based on the curvature of the world line. The extra acceleration term in the integral with coefficient $\Gamma_i x^i$ (with $c = 1$ again) provides exactly the necessary correction to produce the desired result in the inertial mass coefficient in the equations of motion for any smooth spherically symmetric model of the electron.

However, to justify this variation of the action yielding the Lagrange equations, the variations must vanish on the bounding time slices and be arbitrary functions of time for the intermediate times. For the variation A, Fermi explicitly states that the variations of the spatial coordinates are arbitrary functions of $t$ which vanish at the end slices, but for the variation B he only examines an infinitesimal contribution of an interval of Fermi time to the whole 4-dimensional integral and he emphasizes that for that interval of time, the variations in the spatial coordinates of the world lines should be arbitrary constants to represent an overall translation of those world lines. However, in order to claim his resulting Lagrangian equation is valid, it has to be understood that as in the first case, the variations in the spatial coordinates must be arbitrary functions of the time coordinate which vanish at the end times. This implies that the Lagrangian variation extremizes the action among all those world lines which break the rigid Born symmetry assumed in the solution about which the variation takes place. It does not allow for a variation among the family of Born rigid motions of the electron nearby the given solution. None of this is made explicit in Fermi's article.

If the spatial variations were arbitrary constants in the Fermi coordinate system in order to preserve the rigidity in the variation, and if they were to vanish on the end time slices, they would vanish everywhere, so could one not conclude that at every time along the world tube of the electron that the spatial integral coefficients of the variation must vanish. On the other hand if they did not vanish at the end times, one could not ignore the boundary terms which result from the integration by parts along the time lines. Furthermore, without being independent variations at each time, one cannot conclude that their coefficients must vanish. This is a very tricky point since in general one cannot impose symmetries on a Lagrangian and be guaranteed to get the same equations of motion for the restricted variational principle as those that result from imposing the symmetries on the Lagrangian equations of motion derived from the general variational principle as discussed by Maccallum and Taub for the complementary problem of spatial rather than temporal symmetry imposed on a Lagrangian [45]. It is the boundary terms which play the key role in this discussion. By not requiring that the variations about a symmetric solution conform to the symmetry, Fermi appears to have avoided these difficulties.

Note that the model of the charge distribution as some kind of rigid body is necessary in order to assign some common acceleration to the system at each moment of time (that of the central world line) so that its coefficient in the equations of motion can be interpreted as the inertial mass. Consider therefore as Fermi does such an accelerated system of electric charge in special relativity held at rest relative to each other by some external forces (i.e., in conventional or relativistic rigid motion). The corresponding action is given in inertial coordinates by the usual Lagrangian integral in inertial coordinates with the additional term in the mechanical mass added back into the discussion representing a rest mass distribution with differential mass $dm$ assumed to have the same rigidity properties as the charge distribution with differential charge $de$, i.e., they mass and charge elements share the same world lines 7

$$S = S(A_\mu, x^\alpha) = \int \left( -\frac{1}{16\pi} F^{\alpha\beta} F_{\alpha\beta} + A_\mu J^\mu \right) d^4x - \int d\tau \, dm \,. \qquad (44)$$

The region of integration is an arbitrary region of spacetime, and the 4-current equivalent to assuming the noncovariant rigidity condition, which Fermi concludes must obviously invalidate that model.

The only difference for his variation B in the Fermi coordinate system is the additional factor of the Fermi lapse in the differential of proper time needed to define the electric field in that coordinate system

$$0 = \int F_{\sigma\mu} \frac{dx^\mu}{d\tau} N de = \int F_{\sigma\mu} U^\mu N de = \int \rho E(U)_\mu N dV \,, \qquad (52)$$

an expression which only has nonzero components $E(U)_\mu = \delta^i{}_\mu E(U)_i$ in either Fermi coordinates or in inertial coordinates in which the electron is momentarily at rest, where $E(U)_i = E_i$ then agree. Clearly when the acceleration is identically zero $\Gamma_i = 0$ and $N = 1$, the final conditions are the same for both cases A and B, so one must have nonzero acceleration to see a difference in these two cases. Of course without acceleration one cannot measure the inertial mass.

To finish the story we must analyze these conditions in terms of the internal forces exerted on the charge elements by other charge elements and the forces exerted by the external electromagnetic field responsible for the acceleration of the electron. It is the separation of the self-field and the external field that allows one to extract the Lorentz force law relation to the acceleration of the central world line (corrected by radiation reaction terms if one expands ~~it~~ far enough in the acceleration) and thus identify the inertial mass coefficient where the 4/3 problem is apparent, and Fermi's correction restores this factor to 1. The uncorrected Abraham-Lorentz condition is discussed in detail in Jackson [37] (although the ~~T~~hird ~~E~~dition omits the final explicit evaluation of the famous 4/3 term), so we only summarize it here. We then follow Fermi in explicitly evaluating the correction term to see its effect in removing the unwanted 4/3 factor. Finally we will consider the additional mechanical mass term in the Lagrangian to follow Fermi's original Lagrangian discussion in his third paper. For the moment we set this term to zero as in Fermi's fourth paper.

t e

- Field separation for variations of type A

Consider first the system of variations A.

$$0 = \int E_a \, de \,. \qquad (53)$$

Let $E = E_{\rm self} + E_{\rm ext}$, where $E_{\rm self}$ and $E_{\rm ext}$ the contributions to the total field due to the self-interaction of the system and to the external electric field respectively, the latter of which is assumed to be sufficiently uniform over the small dimensions of the system that it can be pulled out of the integral, which results in the total charge multiplying the external electric field evaluated at the central world line. Eq. (53) thus becomes

Equation (to spell in full if it is the first word of the sentence)

$$F^a_{\rm ext} \equiv \int E^a_{\rm ext} \, de = E^a_{\rm ext} \int de = -\int E^a_{\rm self} \, de \equiv -F^a_{\rm self} \,. \qquad (54)$$

of the preceding discussion when the electron is momentarily at rest. Thus the vanishing integral $n = 0$ term, namely Eq. (60), of the original expansion now becomes

$$\begin{aligned} &\int d^3x \int d^3x' \, \rho(t,\mathbf{x})[1+\dot{\mathbf{v}}(t)\cdot\mathbf{x}/c^2]\rho(t,\mathbf{x}')\nabla|\mathbf{x}-\mathbf{x}'|^{-1} \\ &= \int d^3x \int d^3x' \, \rho(t,\mathbf{x})[\dot{\mathbf{v}}(t)\cdot\mathbf{x}/c^2]\rho(t,\mathbf{x}')\nabla|\mathbf{x}-\mathbf{x}'|^{-1} . \end{aligned} \tag{67}$$

Fermi noted that this double spatial integral will give the same value if the two dummy vector integration variables are switched, and hence can also be replaced by the average of these two ways of writing the same integral. Letting $\nabla|\mathbf{x}-\mathbf{x}'|^{-1} = -(\mathbf{x}-\mathbf{x}')/|\mathbf{x}-\mathbf{x}'|^3$

$$\begin{aligned} &c^{-2}\int d^3x \int d^3x' \, \rho(t,\mathbf{x})\rho(t,\mathbf{x}')[\dot{\mathbf{v}}(t)\cdot\mathbf{x}](\mathbf{x}'-\mathbf{x})/|\mathbf{x}-\mathbf{x}'|^3 \\ &= c^{-2}\int d^3x \int d^3x' \, \rho(t,\mathbf{x}')\rho(t,\mathbf{x})[\dot{\mathbf{v}}(t)\cdot\mathbf{x}'](\mathbf{x}-\mathbf{x}')/|\mathbf{x}-\mathbf{x}'|^3 \\ &= -c^{-2}\frac{1}{2}\int d^3x \int d^3x' \, \rho(t,\mathbf{x})\rho(t,\mathbf{x}')[\dot{\mathbf{v}}(t)\cdot(\mathbf{x}'-\mathbf{x})](\mathbf{x}'-\mathbf{x})/|\mathbf{x}-\mathbf{x}'|^3 . \end{aligned} \tag{68}$$

Now imposing spherical symmetry about the origin, the components of this vector integral are nonzero only along the acceleration vector, with a coefficient which can be replaced by the average value of the vector component integral

$$-[\dot{\mathbf{v}}(t)\cdot(\mathbf{x}'-\mathbf{x})](\mathbf{x}'-\mathbf{x}) \to -\dot{\mathbf{v}}(t)\frac{1}{3}(\mathbf{x}'-\mathbf{x})\cdot(\mathbf{x}'-\mathbf{x}) = -\dot{\mathbf{v}}(t)\frac{1}{3}|\mathbf{x}'-\mathbf{x}|^2 \tag{69}$$

so it reduces to

$$-\frac{1}{3}\frac{\dot{\mathbf{v}}(t)}{c^2}\left[\frac{1}{2}\int d^3x \int d^3x' \, \rho(t,\mathbf{x})\rho(t,\mathbf{x}')/|\mathbf{x}-\mathbf{x}'|\right] = -\frac{1}{3}\frac{W}{c^2}\dot{\mathbf{v}}(t) , \tag{70}$$

since the expression in square brackets is the self-energy of the charge distribution at the time $t$. This is the only additional term linear in the acceleration which contributes to the lowest terms of the previous calculation (so that the lowest order radiation reaction term is unchanged, although not shown here)

$$F^{\mathrm{NR}}_{\mathrm{ext}} = \frac{4}{3}\frac{W}{c^2}\dot{\mathbf{v}} - \frac{1}{3}\frac{W}{c^2}\dot{\mathbf{v}} = \frac{W}{c^2}\dot{\mathbf{v}} , \tag{71}$$

which leads to the desired result

$$F^{\mathrm{NR}}_{\mathrm{ext}} = m_{\mathrm{em}}\dot{\mathbf{v}} , \qquad m_{\mathrm{em}} = \frac{W}{c^2} \tag{72}$$

in the nonrelativistic limit, according to Newton's law with the electromagnetic mass $m_{\mathrm{em}} = W/c^2$.

Finally to consider the contribution to the Lagrangian from a mechanical mass distribution, we must vary the final term in the Lagrangian which has been ignored until now. In the Fermi coordinate system this is trivial. The Lagrangian term is simply

$$-\int d\tau \, dm = -\int \left(\int N \, dt\right) dm = -\int \left(\int 1 + \Gamma_i x^i \, dt\right) dm , \tag{73}$$

and its variation is

$$-\delta\int\left(\int 1+\Gamma_i x^i\,dt\right)dm = -\int\left(\int \Gamma_i\delta x^i\,dt\right)dm$$

$$= -\left(\int dm\right)\int \Gamma_i\delta x^i dt = -\int (m_0\Gamma_i)\delta x^i dt\,, \tag{74}$$

where $m_0$ is the total mechanical mass. The contribution to the above Fermi condition are the coefficients of the arbitrary variations $\delta x^i = \delta x^i(t)$, namely just the term $-\int(m_0\Gamma_i) = -m_0\dot{v}^i$. The complete equation of motion is then first

is

$$\int \rho E(U)_i(1+\Gamma_j x^j)\,dV - m_0\Gamma_i = 0\,, \tag{75}$$

and then after splitting off the self-force and passing to the lowest order approximation

$$(m_0 + m_{\rm em})\dot{\mathbf{v}} = F^{\rm NR}_{\rm ext}\,. \tag{76}$$

Thus mechanical mass and the electromagnetic mass contribute in the same way to the total inertial rest mass of the spherical distribution of charged matter.

***Relating Kwal-Rohrich back to Fermi through Gauss***

Given the Kwal-Rohrlich 4-momentum evaluated for an unaccelerated electron and the inertial mass contribution from the electromagnetic field found by Fermi for the accelerated electron, it is natural to look for a relation between them. In the unaccelerated case, one has an entire family of distinct 4-momenta which depend on the inertial observer, but the one we usually associate with the electron of a certain rest energy is the one defined by the rest frame observer. Although Fermi stopped his analysis once he achieved his limited goal, in light of the 4-momentum integral situation in which interest later arose, it is natural to continue his line of thought to its logical conclusion. We do this here and find that Fermi's corrected condition which generates the correct equations of motion guarantees the conservation of the total 4-momentum as seen in the instantaneous rest frame of the accelerated electron at each point of its world line.

to

All we need ~~do~~ do is specialize the Gauss law discussion begun in Section 2 to the electromagnetic stress-energy tensor over the spacetime region $R$ between two successive time hyperplanes $\Sigma_t$ and $\Sigma_{t+\Delta t}$ associated with a Fermi coordinate system adapted to the central world line of the accelerated electron, as in Fig. ~~A.1~~ 5 of the Appendix where the case of 1-dimensional motion is illustated. Let $\Delta t > 0$ so $t + \Delta t$ is to the future of $t$ along the central world line where $t$ measures the elapsed proper time. Fig. A.1 shows the tilting of the Fermi time slices to remain orthogonal to the central world line of the electron and to the common local rest space of the elements of charge which make up the electron sphere. Then Eqs. (13) and (5) lead to the fundamental relation

to / ?

Figure 5

$$-\int_R Q_\alpha \rho E(U)^\alpha d^4V = Q_\alpha\left[P(\Sigma_{t+\Delta t})^\alpha - P(\Sigma_t)^\alpha\right]\,, \tag{77}$$

where on the right hand side the components have to be expressed in inertial coordinates or the components $Q_\alpha$ are not constant and cannot be factored out of the integral. On the left hand side, if evaluated in the Fermi coordinate system, these components are functions of time to compensate for the time-dependent change of direction of the 4-velocity of the central world line, and so can only be pulled out of the spatial integral. Recall that $E(U)^\alpha$ is the electric field seen in the electron rest frame and $\rho$ is the rest frame charge density.

Let $R_-$ be the half-region for which the hyperplane $\Sigma_{t+\Delta t}$ is in the future of $\Sigma_t$, while $R_+$ has the reverse relationship, as in Fig. A.1, so that the world tube of the electron cuts through the region $R_-$ as shown there. Splitting the integral into the spatial integral and then the temporal integral, using the spacetime volume element $d^4V = (1+\Gamma_i x^i)dV\,dt$, one then has

$$-\int_t^{t+\Delta t} Q_\alpha \int_{\Sigma_\tau \cap R_-} \left[\rho E(U)^\alpha (1+\Gamma_i x^i) dV\right] d\tau = Q_\alpha \left[P(\Sigma_{t+\Delta t})^\alpha - P(\Sigma_t)^\alpha\right] . \tag{78}$$

For the Born rigid distribution of charge according to the Fermi condition (75), the spatial integral in parentheses on the left hand side of Eq. (78) at each Fermi time (which the proper time parameter along the central world line) equals the mechanical mass times the proper time covariant derivative $D/d\tau$ of the 4-velocity of the central world line

$$m_0 \delta^\alpha{}_i \Gamma^i = m_0 \frac{Du^\alpha}{d\tau} = \frac{D}{d\tau}(m_0 u^\alpha) = \frac{D}{d\tau} p_0^\alpha \tag{79}$$

where $p_0^\alpha = DU^\alpha/d\tau$ is the mechanical momentum. Here we use the notation $D/d\tau$ to remind us that in noninertial coordinates like those of Fermi, the covariant derivative along the parametrized curve does not coincide with the action of the ordinary such derivative, but when we evaluate the expression in components with respect to a fixed inertial coordinate system, it does. The final integral with respect to the Fermi time coordinate, if performed with the components taken in an inertial coordinate system, is then just the difference of the mechanical momentum between the two Fermi times

$$\int_t^{t+\Delta t} Q_\alpha \left(\frac{dp_0^\alpha}{dt}\right) dt = Q_\alpha \left[p_0(t+\Delta t)^\alpha - p_0(t)^\alpha\right] , \tag{80}$$

so that

$$-Q_\alpha \left[p_0^\alpha(t+\Delta t) - p_0^\alpha(t)\right] = Q_\alpha \left[P(t+\Delta t)^\alpha - P(t)^\alpha\right] \tag{81}$$

Expressing this in inertial coordinates, since $Q_\alpha$ are arbitrary constants, we find

$$p_0(t+\Delta t)^\alpha + P(t+\Delta t)^\alpha = p_0(t)^\alpha + P(t)^\alpha , \tag{82}$$

namely ~~that~~ the sum of the mechanical 4-momentum and the 4-momentum of the external electromagnetic field $p_0^\alpha + P^\alpha$ must be the same on the two Fermi time slices and hence on every Fermi time slice. In other words the Fermi condition is equivalent to the conservation of the Kwal-Rohrlich 4-momentum for the total system, a fact which no one seems to have realized until now. Thus Fermi also pointed the way towards selecting the only observer-defined total 4-momentum which is conserved and which corresponds to what we associate with this system. The proper time derivative of this relation gives its rate of change version

$$\frac{D}{dt}(p_0(t)^\alpha + P(t)^\alpha) = 0 \,. \tag{83}$$

Thus the calculations initiated by Fermi nearly a century ago have finally reached their natural conclusion.

Apart from Kolbenstvedt [26] much later in 1997, only Aharoni [11] seems to have seen and understood Fermi's argument, explaining exactly what Fermi did in detail in his 1965 textbook revised because of the then recent Rohrlich work on this topic and re-interpreting it in his own way, explaining in detail how the 4-momentum integrals first explained by Kwal and later Rohrlich are connected to Fermi's approach to the problem. Anaroni's equations (6.5), (6.18) and (6.19) for the total self-force due to the electron charge distribution involve through his (6.18) the proper time rate of change of an integral over the spacetime region between two successive proper time hypersurfaces of the electron (his own reformulation of the self-force in view of the Kwal-Rohrlich integral definition as noted in a footnote). Aharoni considers the following equivalent reformulation of the previous equations valid for the total electromagnetic field, but restricted only to the self-field in order to define the self-force due only to the self-field of the charge distribution

$$\frac{dP^\mu}{d\tau} = -\frac{d}{d\tau}\int_{\tau_0}^{\tau}\int F_{\text{self}}{}^{\mu}{}_{\nu} J^\nu \, d\tau d\Sigma = -\delta^\mu{}_i \int (1+\Gamma_j x^j) E^i_{\text{self}} \rho \, d^3x \,. \tag{84}$$

However, Aharoni failed to relate his "postulated" self-force expression to Gauss's law to show that it actually is related to the proper time rate of change of the Kwal-Rohrlich 4-momentum integral restricted to the self-field. Spohn and Yaghjian both have long bibliographies in their textbooks, but neither mentions Aharoni, while Rohrlich has an author index indicating Aharoni's name on page 283 where no reference to anyone can be found. Only the much later work of Kolbenstvedt acknowledges Fermi's approach, rederiving it in a slightly different but equivalent form, also ignored by Rohrlich, Spohn and Yaghjian in their later editions.

### *Concluding Remarks*

It is unfortunate that the first four papers by one of the leading physicists of the twentieth century were never translated from their original Italian. The fourth paper which concludes this series and which appeared in preliminary versions in both Italian and German, was the culmination of Fermi's early work in relativity only a few years after the birth of general relativity and written while he was a university student. Its actual contents seem to have remained a mystery to nearly all those who have cited it in discussions of the classical theory of the electron which still interests people even today, while the leading textbook on classical electrodynamics still

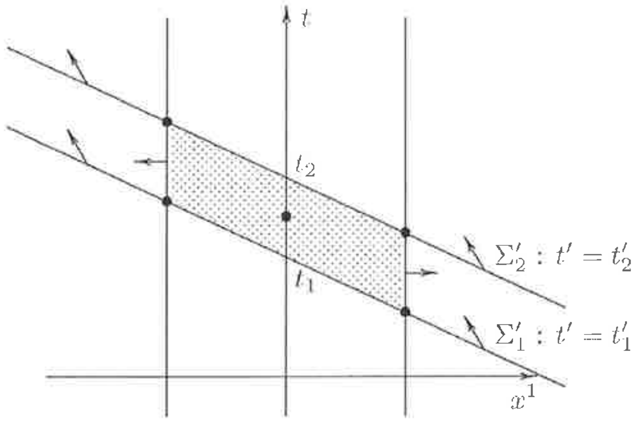


Fig. 3 Figure 5.3.b from Misner, Thorne and Wheeler redrawn with an inner cylindrical boundary which is the world tube of the electron sphere boundary. The arrows show the chosen unit normal direction for the orientation of each hypersurface, but in the single Gauss law relation for the region of spacetime between $\Sigma_1$ and $\Sigma_2$ excluding the shaded region inside the cylinder, the sum of the outward normally oriented integral contributions is zero for a divergence-free vector field. Here the boundary term due to the portion $\sigma$ of the cylinder between the two parallel hyperplanes vanishes by spherical symmetry.

repeats the Abraham-Lorentz derivation of the equations of motion without Fermi's correction, although admitting that it can be relativistically corrected following Fermi. Ironically Fermi's third paper (see [46] for a historical discussion), which he considered only a tool for obtaining his result in that fourth paper, and which Fermi never even explicitly cited there, did make an indelible mark on relativity with the terms Fermi coordinates and Fermi-Walker transport, although even the much later paper by Walker that coupled together their names forever also ignores Fermi's original paper in Italian. Surprisingly even the text by Rohrlich updated only recently four decades after its original publication fails to connect his own adjustment of the definition of the 4-momentum of the electromagnetic field of the classical electron to Fermi's argument about the equations of motion, while the more recent books by Yaghjian and Spohn devoted to this area also show no sign that they have ever seen Fermi's argument. We hope the present work restores Fermi's message to its rightful place and perhaps provoke some thought about its meaning. A shorter version of this discussion has been published elsewhere [47] and reproduced in Appendix B.

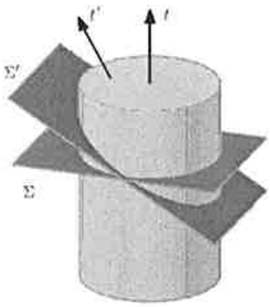


Fig. 4 The world tube of the electron sphere is a cylinder in spacetime about the $t$ axis, shown here with one spatial dimension suppressed. The time slices $t = 0$ $(\Sigma)$ and $t' = 0$ $(\Sigma')$ cut this cylinder, intersecting in the spacelike 2-plane $x^1 = 0, t = 0$, which separates the spacetime region between these time slices into two disjoint subregions $x^1 > 0$ and $x^1 < 0$. Gauss's law applies separately to each of these two simply connected regions outside the electron sphere cylinder, but the signs of the outward normals of the time slices switch between these two regions, while remaining the same for the cylindrical portion of their boundaries.

## *Appendix. Gauss's theorem and "conservation laws"*

For a divergence-free stress-energy tensor in all of Minkowski spacetime which falls off sufficiently fast at spatial infinity, its integral over any two parallel inertial time hyperplanes would be the same by Gauss's law, as explained in most standard textbooks in relativity, see Chapter 5 of Misner, Thorne and Wheeler [41], for example, or Appendix A1–5 of Rohrlich's Third Edition [34], or Anderson [38]. This gives the usual 4-momentum conservation law that the 4-momentum has the same value for different time slices for a given inertial observer. However, for two time slices associated with a pair of inertial observers in relative motion, the time slices necessarily intersect so one has to be more careful in applying Gauss's law to this more general situation, though again one finds that the 4-momentum is independent of the observer as well as the time slice. However, in the present case the nonzero divergence due to the source inside the timelike world tube of the electron sphere surface, or equivalently the boundary term on that world tube if one excludes the sources from Gauss's law, interferes with this more familiar picture, forcing the 4-momentum of the electromagnetic field to depend explicitly on the inertial observer. We consider these complications in detail in this appendix since they do not seem to be discussed in standard textbooks. The spherical shell model of the electron discussed in the first section is used to illustrate the evaluation of the Gauss law integrals.

Figure 3

Fig. 1 generalizes Fig. 5.3.b of Misner, Thorne and Wheeler: it represents a constant $x^2, x^3$ slice of the unaccelerated electron world tube centered at the origin of the unprimed spatial coordinates in spacetime. As in section 2, the unprimed coordinates are associated with the rest frame $K$ of the electron, while the primed coordinates are associated with a frame $K'$ in relative motion with respect to the unprimed frame is in the $x^1$ direction with velocity $-v < 0$ as shown in the figure. Consider the spacetime region devoid of electromagnetic sources between two spacelike hyperplanes $\Sigma'_1$ and $\Sigma'_2$ of constant inertial times $t'_1$ and $t'_2 > t'_1$ and outside of an internal lateral boundary $\sigma$ between them which is a subset of the cylindrical timelike surface representing the world tube of the electron spherical surface ($r = r_0$ in its rest frame). Let $\overline{\Sigma}'_1$ and $\overline{\Sigma}'_2$ be the portions of those planes exterior to this cylinder. Suppose $\Sigma'_1$ and $\Sigma'_2$ are oriented by their future-pointing unit normal vector fields and $\sigma$ by its inward unit normal $\partial/\partial r$ relative to the region of spacetime in question. Let $Q$ be any covariant constant 4-vector so that $q^\mu = Q_\nu T^{\nu\mu}_{\rm em}$ is a divergence-free vector field in the spacetime region bounded by the three hypersurfaces $\overline{\Sigma}$, $\overline{\Sigma}'$ and $\sigma$, as well as by the lateral boundary at spacelike infinity, a region to which Gauss's law with zero volume integral and outward pointing normals applies. Taking the orientations into account relative to the outward normal on each boundary hypersurface, one then has

$$\int_{\overline{\Sigma}'_2} Q_\mu T^{\mu\nu}_{\rm em} d\Sigma_\nu - \int_{\overline{\Sigma}'_1} Q_\mu T^{\mu\nu}_{\rm em} d\Sigma_\nu = \int_\sigma Q_\mu T^{\mu\nu}_{\rm em} d\sigma_\nu \,. \tag{85}$$

If the lateral boundary term vanishes, then the integral is the same over each of the two time hypersurfaces outside the world tube of the electron sphere. Indeed for time slices in the rest frame of the electron, or in the moving frame, these integrals are time-independent, which corresponds exactly to the vanishing of the integral over the electron surface tube between the two slices. This follows for all possible projections $Q_\alpha$ in the explicit evaluation of the lateral integral from the vanishing of $T^{0r}_{\rm em}$ itself and of the surface integral of the spatial stress components

$$T^{x^i r}_{\rm em} = T^{r x^i}_{\rm em} = T^{rr}_{\rm em} \frac{\partial x^i}{\partial r} = -T^{00}_{\rm em} \frac{x^i}{r} \tag{86}$$

over the 2-sphere $r = r_0$, which follows from the spherical symmetry and the fact that the integral along the time direction on the cylinder is the constant rest frame time difference $t_2 - t_1 = \gamma(t'_2 - t'_1)$. However, even though for each such inertial coordinate system, the integral at constant time is time-independent, we must do a second calculation to relate the results of the integration with respect to inertial coordinate systems in relative motion.

In the usual textbook situation of a free electromagnetic field with no sources, one does not exclude any world tube from the Gauss law application so the internal boundary integral is not present and the divergence integral is zero. As a result the difference of the integrals over the two time parallel hyperplanes is zero. The same remarks will apply to the Gauss law application to two intersecting time hyperplanes, extending the equality of the 4-momentum integral to all inertial time slices.

The situation between the time hyperplanes of two different inertial frames is more complicated since the hyperplanes necessarily intersect, as shown in Fig. 2 4
with one spatial dimension suppressed, assuming that the relative velocity $v$ along the direction $x^1$ of the electron rest frame relative to the moving primed frame is

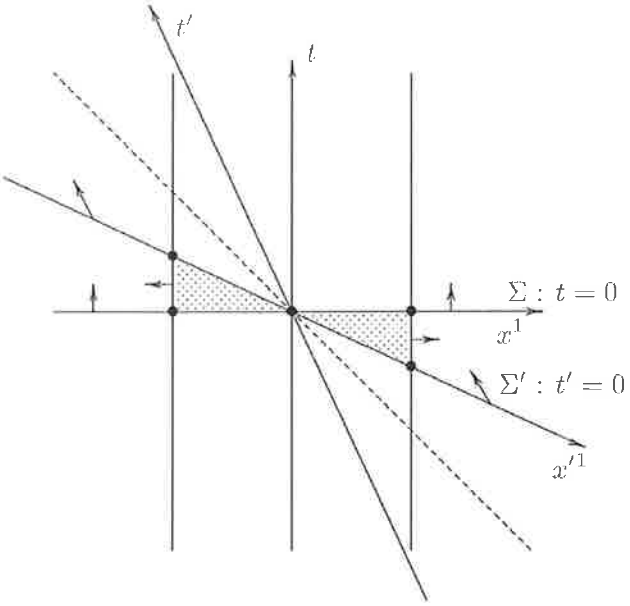


Fig. 5 Fig. 5.3.c from Misner, Thorne and Wheeler (or Fig. A1–3 from Rohrlich's Third Edition) redrawn with an inner cylindrical boundary which is the world tube of the electron sphere boundary, showing a constant $x^2, x^3$ slice of the previous figure. The arrows show the chosen unit normal direction for the orientation of each hypersurface, which changes sign relative to the unit outward normal of the exterior region outside the cylinder going from $x^1 > 0$ to $x^1 < 0$. Here the boundary term due to the portion $\sigma$ of the cylinder between the two parallel hyperplanes is now nonvanishing. The two halves $\sigma_+$ ($x^1 > 0$) and $\sigma_-$ ($x^1 < 0$) contribute terms with opposite signs to the two separate Gaussian integral relations because of the change in sign of the outward normals on $\Sigma$ and $\Sigma'$, and hence in the difference relation needed to reassemble the two halves of those time hypersurface integrals, they contribute a nonzero correction term.

positive, as in the previous figure. Figure Fig. ~~3~~ 5 shows a constant $x^2, x^3$ slice of Fig. 2 4 ? generalizing Fig. 5.3.c of Misner, Thorne and Wheeler [41] (or Fig. A1–3 from Rohrlich's Third Edition), but with an additional internal lateral boundary, here the portion $\sigma$ of the cylinder representing the electron sphere centered around the $t$ axis and extending between the two time slices. Consider the region of spacetime exterior to the electron sphere bounded by the time hypersurfaces $t = 0$ and $t' = 0$, with unit future-pointing normals $U = \partial/\partial t$ and $U' = \partial/\partial t'$. Let $\sigma = \sigma_- \cup \sigma_+$ be the portion of the cylindrical world tube of the electron sphere between these two time hyperplanes, divided into two disjoint parts $\sigma_+$ for $x^1 > 0$ and $\sigma_-$ for $x^1 < 0$, each with the orientation induced by the outward radial normal $\partial/\partial r$ relative to the sphere. For each point on the electron sphere, $\sigma$ consists of the region between $t = 0$ and $t = -vx^1$, so the integral on $\sigma$ along $t$ leads to a factor $\Delta t = 0 - (-vx^1) = vx^1 > 0$ for $x^1 > 0$ and a factor $\Delta t = -vx^1 - 0 = -vx^1 > 0$ for

$x^1 < 0$ since the integrand is independent of $t$ along the cylinder.

Similarly let $\Sigma = \Sigma_- \cup \Sigma_+$ and $\Sigma' = \Sigma'_- \cup \Sigma'_+$, each with the future-pointing normal orientation, and let $\overline{\Sigma} = \overline{\Sigma}_- \cup \overline{\Sigma}_+$ and $\overline{\Sigma}' = \overline{\Sigma}'_- \cup \overline{\Sigma}'_+$ be the portions of those regions outside the world tube of the electron sphere. One can separately apply Gauss's law to the two disjoint regions with these boundaries and reassemble the pieces to get a relation between the integrals over $\overline{\Sigma}$, $\overline{\Sigma}'$ and $\sigma$. Since the outer normal directions switch directions for $\overline{\Sigma}$ and $\overline{\Sigma}'$ but not $\sigma$ going from $x^1 > 0$ to $x^1 < 0$, one must take the difference of the two separate Gauss law relations to reassemble the total integrals over $\overline{\Sigma}$ and $\overline{\Sigma}'$, which leads to a net nonvanishing contribution from $\sigma$ in spite of the spherical symmetry. One has

$$\begin{aligned}
\int_{\overline{\Sigma}_+} Q_\mu T^{\mu\nu}_{\rm em} d\Sigma_\nu - \int_{\overline{\Sigma}'_+} Q_\mu T^{\mu\nu}_{\rm em} d\Sigma_\nu &= \int_{\sigma_+} Q_\mu T^{\mu\nu}_{\rm em} d\sigma_\nu \,, \\
\int_{\overline{\Sigma}'_-} Q_\mu T^{\mu\nu}_{\rm em} d\Sigma_\nu - \int_{\overline{\Sigma}_-} Q_\mu T^{\mu\nu}_{\rm em} d\Sigma_\nu &= \int_{\sigma_-} Q_\mu T^{\mu\nu}_{\rm em} d\sigma_\nu \,,
\end{aligned} \tag{87}$$

and therefore taking the difference

$$\begin{aligned}
&\int_{\overline{\Sigma}'} Q_\mu T^{\mu\nu}_{\rm em} d\Sigma_\nu - \int_{\overline{\Sigma}} Q_\mu T^{\mu\nu}_{\rm em} d\Sigma_\nu \\
&= \int_{\overline{\Sigma}'_+} Q_\mu T^{\mu\nu}_{\rm em} d\Sigma_\nu - \int_{\overline{\Sigma}_+} Q_\mu T^{\mu\nu}_{\rm em} d\Sigma_\nu \\
&\qquad - \left( \int_{\overline{\Sigma}_-} Q_\mu T^{\mu\nu}_{\rm em} d\Sigma_\nu - \int_{\overline{\Sigma}'_-} Q_\mu T^{\mu\nu}_{\rm em} d\sigma_\nu \right) \\
&= - \int_{\sigma_+} Q_\mu T^{\mu\nu}_{\rm em} d\sigma_\nu + \int_{\sigma_-} Q_\mu T^{\mu\nu}_{\rm em} d\sigma_\nu \,.
\end{aligned} \tag{88}$$

Consider applying the above relation in this setting for $Q = -U'$, so that $q^\alpha = -U'_{\nu'} T^{\nu'\alpha}_{\rm em} = T^{t'\alpha}_{\rm em} = \gamma(T^{t\alpha}_{\rm em} + v T^{x^1\alpha}_{\rm em})$. Then

$$\int_{\overline{\Sigma}'} q^\alpha d\Sigma_\alpha = \int_{\overline{\Sigma}'} T^{t't'}_{\rm em}\, dV' = W' \tag{89}$$

while

$$\int_{\overline{\Sigma}} q^\alpha d\Sigma_\alpha = \int_{\overline{\Sigma}} T^{t't}_{\rm em}\, dV = \gamma W \,. \tag{90}$$

Using the exterior field in the source free region outside the electron spherical shell model as an example, one finds that the cylindrical world tube integrals, since the integrand is independent of $t$, are explicitly

$$\begin{aligned}
\int_{\sigma_+} q^\alpha d\sigma_\alpha &= \int_{\sigma_+} T^{t'r}_{\rm em}\, dt\, dS = \int_{\sigma_+} \gamma(T^{tr}_{\rm em} + v T^{x^1 r}_{\rm em})\, dt\, dS \\
&= \int_{-\pi/2}^{\pi/2} \int_0^\pi (0 - (-vx^1))\gamma v \left( \frac{x^1}{r_0} T^{rr}_{\rm em} \right) r_0^2 \sin\theta\, d\theta d\phi \\
&= \gamma v^2 r_0^3 T^{rr}_{\rm em} \int_{-\pi/2}^{\pi/2} \int_0^\pi (\sin\theta\cos\phi)^2 \sin\theta\, d\theta d\phi \\
&= \frac{1}{6}\gamma v^2 (4\pi r_0^3 T^{rr}_{\rm em}) = -\frac{1}{6}\gamma v^2 W
\end{aligned} \tag{91}$$

9

and

$$\begin{aligned}
\int_{\sigma_-} q^\alpha d\sigma_\alpha &= \int_{\pi/2}^{3\pi/2} \int_0^\pi ((-vx^1) - 0)\gamma v \left(\frac{x^1}{r_0} T^{rr}_{\text{em}}\right) r_0^2 \sin\theta \, d\theta d\phi \\
&= -\gamma v^2 r_0^3 T^{rr}_{\text{em}} \int_{\pi/2}^{3\pi/2} \int_0^\pi (\sin\theta\cos\phi)^2 \sin\theta \, d\theta d\phi \\
&= -\gamma v^2 r_0^3 (4\pi T^{rr}_{\text{em}}) \frac{1}{6} = \frac{1}{6}\gamma v^2 W \,. \qquad (92)
\end{aligned}$$

Since the outward normals on $\Sigma$ and $\Sigma'$ reverse direction on the second set of integrals, but the outward normal on $\sigma$ does not, the separate Gauss's law relations are

$$\begin{aligned}
\int_{\Sigma'_+} q^\alpha d\Sigma'_\alpha - \int_{\Sigma_+} q^\alpha d\Sigma_\alpha &= -\int_{\sigma+} q^\alpha d\sigma_\alpha \\
\int_{\Sigma'_-} q^\alpha d\Sigma'_\alpha - \int_{\Sigma_-} q^\alpha d\Sigma_\alpha &= \int_{\sigma_-} q^\alpha d\sigma_\alpha \qquad (93)
\end{aligned}$$

and their sum is

$$\begin{aligned}
W' - \gamma W &= \int_{\Sigma'} q^\alpha d\Sigma'_\alpha - \int_{\Sigma} q^\alpha d\Sigma_\alpha \\
&= -\int_{\sigma+} q^\alpha d\sigma_\alpha + \int_{\sigma_-} q^\alpha d\sigma_\alpha = \frac{1}{3} v^2 \gamma W \,. \qquad (94)
\end{aligned}$$

Thus the unwanted correction factor is exactly the integral over the cylindrical boundary over the electron sphere of the moving frame 4-velocity component of the stress-energy tensor, with the factor of 1/3 equal to

$$\begin{aligned}
&\frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi (\sin\theta\cos\phi)^2 \sin\theta \, d\theta d\phi \\
&= \frac{1}{4\pi} \int_{S_2} \frac{(x^1)^2}{r_0^2} d\Omega = \frac{1}{3} \frac{1}{4\pi} \int_{S_2} \frac{r_0^2}{r_0^2} d\Omega = \frac{1}{3} \,, \qquad (95)
\end{aligned}$$

whose value follows from spherical symmetry as expressed in Eq. (30). This term which causes the result to differ from the 4-momentum as seen in the rest system is exactly due to the unbalanced outward radial stress on the charge distribution at the surface of the electron sphere.

One can repeat this calculation for $Q = \partial/\partial x'^1$ in order to express the momentum correction factor as an integral over this boundary, with one less factor of $v$ in the correction term since

$$T^{x^{1'} r}_{\text{em}} = \gamma(T^{x^1 r}_{\text{em}} + vT^{tr}_{\text{em}}) = \gamma T^{x^1 r}_{\text{em}} \qquad (96)$$

compared to the previous calculation where

$$T^{t' r}_{\text{em}} = \gamma(T^{tr}_{\text{em}} + vT^{x^1 r}_{\text{em}}) = \gamma v T^{x^1 r}_{\text{em}} \,. \qquad (97)$$

With this corresponding correction term the integral relationship now becomes

$$p^{1'} - \gamma v W = \frac{1}{3}\gamma v W \to p^{1'} = \frac{4}{3}\gamma v W \qquad (98)$$

explaining the famous factor of 4/3.

On the other hand for the model with a uniform distribution of charge within the electron sphere, one must extend the hypersurface integrals over the interior region to evaluate the total 4-momentum in the electromagnetic field since the field is no longer zero there. Only by doing this does the self-energy integral of the static charge configuration agree with the energy in the electric field it generates. This forces one instead to consider the spacetime volume divergence integral over that region in applying Gauss's law, rather than the spherical boundary hypersurface integral. One could do the same for the spherical shell model, where the divergence integral would yield the same result as the spherical boundary integral evaluated above when excluding the region containing the charge.

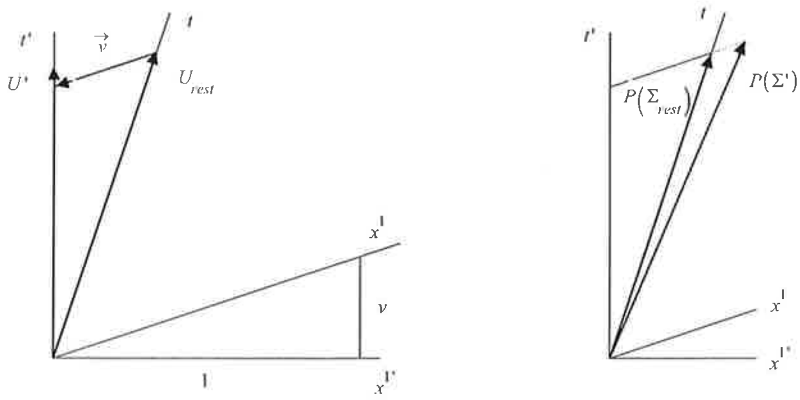


Fig. 6 Left: The plane of the two inertial observer 4-velocities for motion along the $x^1$-axis. The rest frame axis $x^1$ has slope $\nu$. A unit vector along this axis has primed $0'$ and $1'$ components $\langle \gamma\nu, \gamma \rangle$. The relative velocity of $U'$ as seen by the rest frame observer with 4-velocity $U_{\text{rest}}$ is $\vec{\nu}(U', U_{\text{rest}})$, which extends from the tip of $U_{\text{rest}}$ to the vertical axis along $U'$, and whose $0'$ and $1'$ components are $-\nu\langle \gamma\nu, \gamma \rangle$. Right: The rest frame 4-momentum and the moving frame 4-momentum.

We can easily re-express the above component relationships (94) and (98) in 4-vector form. The subtracted terms on the left hand side are exactly the moving frame inertial coordinate components of the 4-momentum as seen by the rest frame

$$\langle P(\overline{\Sigma}_{\text{rest}})^{0'}, P(\overline{\Sigma}_{\text{rest}})^{1'} \rangle = \langle \gamma W, \gamma v W \rangle \,. \tag{99}$$

The right hand sides instead have corresponding primed components $\frac{1}{3}\gamma v \langle v, 1 \rangle$, which can be re-expressed as follows. The 4-vector with its first two primed components equal to $v\gamma\langle v, 1 \rangle$ is just the sign-reversed relative velocity of the moving frame compared to the rest frame as seen in the rest frame, call its components $-\nu(U', U_{\text{rest}})^{\alpha'}$. See Fig. 5. ~~The rest energy is just $W = P(\overline{\Sigma}_{\text{rest}})^0 =$~~ 6? $-P(\overline{\Sigma}_{\text{rest}})_\beta U^\beta_{\text{rest}}$, where for emphasis we include the subscript notation for the rest

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[35] A.D. Yaghjian, *Relativistic Dynamics of a Charged Sphere: Updating the Lorentz-Abraham Model*, (Berlin: Springer, Berlin, 1992; Second Edition 2006); Phys. Rev. **E78**, 046606 (2008).

[36] H. Spohn, *Dynamics of Charged Particles and their Radiation Field* (Cambridge University Press, Cambridge, 2004).

[37] J. Jackson, *Classical Electrodynamics* (Wiley, New York, Second Edition: 1975, Third Edition: 1999); see respectively Chapters 17, 16.

[38] J.L. Anderson, *Principles of Relativity Physics*, (Academic Press, New York, 1967).

[39] M. Janssen and M. Mecklenburg, pp. 65-134 in V.F. Hendricks, K.F. Jørgensen, J. Lützen, and S.A. Pedersen (Eds.), *Interactions: Mathematics, Physics and Philosophy, 1860-1930* (Springer, Dordrecht 2007); available at http://www.tc.umn.edu/~janss011/; see also M. Janssen, PhD thesis "A comparison between Lorentz's ether theory and special relativity in the light of the experiments of Trouton and Noble," see Chapter 2, http://www.mpiwg-berlin.mpg.de/litserv/diss/janssen_diss/ and M. Janssen, Studies in History and

## A.3 D. Boccaletti: When a problem is solved too early. Enrico Fermi and the infamous 4/3 problem

### *Introduction*

It has often happened, particularly in the past few centuries, that some scientific results had been reobtained more than once, each time ignoring the authors of the preceding discoveries. In the case of mechanics this happened many times, as recalled by A. Wintner in the preface to his famous book:[1] "... even the classical literature of the great century of celestial mechanics appears to be saturated with rediscoveries (sometimes *bona fide* and sometimes not assuredly so) ...". In times closer to us, this has happened again for "the infamous 4/3 problem."[2] It took ~~thirty~~ 30 years for the result obtained by Fermi to have its "consecration" in an authoritative book (see below) and ten more to begin circulating among the community of experts. In the next pages we shall first try to historically contextualize Fermi's paper in an extremely concise way and then to bring into question the procedures through which the paper itself has been interpreted. At the end we shall advance a conjecture which, as with all conjectures, is based on circumstantial but not incontrovertible evidence.

### *The story in short*

In the early ~~twenties~~ '20s, when Fermi was concluding his studies at the University of Pisa, in Italy the problems related to the rising quantum mechanics had not yet filtered into academic circles. Instead the electromagnetic theory and the theory of relativity (special and general) were well-known and studied (even if in restricted circles) through the works of Abraham, Lorentz, Poincaré, Richardson ... for electromagnetic theory and the papers of Einstein, Levi-Civita, and the book of Weyl for the theory of relativity. Fermi, still a student, had a deep knowledge of these theories and of classical analytical mechanics. Besides being testified to in Fermi's biography written by Emilio Segrè,[3] this appears clearly in the first papers he published.[4] Paper 1) is substantially a generalization of a result which, at that time,

[1] A. Wintner: *The Analytical Foundations of Celestial Mechanics*, Princeton University Press, 1947, p. IX.

[2] The expression is due to J. D. Jackson in his textbook *Classical Electrodynamics*, Third Edition, Wiley 1998, p. 755.

[3] E. Segrè: *Enrico Fermi Physicist*, The University of Chicago Press, 1970

[4] For the first few Fermi's papers also see, in Italian,
C. Tarsitani: *I lavori di Fermi sulla relatività nei commenti di Persico e Segrè*, Atti del IV congresso nazionale di storia della fisica, Como, 1983,
F. Cordella, F. Sebastiani: *Il debutto di Enrico Fermi come fisico teorico: I primi lavori sulla relatività (1921–1922–23)*, Quaderno di Storia della Fisica N. 5, 1999 and
F. Cordella, A. De Gregorio, F. Sebastiani: *Enrico Fermi: Gli anni italiani*, Editori Riuniti, 2001.
To avoid possible misunderstandings, we follow the convention of the present volume and refer to Fermi's papers making use of the numbered classification scheme given in *Enrico Fermi: Note e Memorie (Collected Papers)*, Accademia Nazionale dei Lincei and University of Chicago Press,

was quoted in the circulating textbooks on electrodynamics.

Besides the various editions of the Abraham's *Theorie der Elektrizität* (which originated as a second part of the treatise of Föppl published for the first time in 1894) and Lorentz's *The theory of electrons* (1909, second edition 1915), the textbook having a larger circulation was Richardson's *The Electron theory of matter* (1914). Fermi refers to this latter textbook. At that time (1921–23) it was generally accepted that, for a charged particle moving with variable velocity, the electromagnetic mass was 4/3 times the inertial mass.[5] The whole theoretical work done in the last two decades, mainly by Abraham and Lorentz, had led to considering the electron (discovered by J. J. Thomson in 1897) to be a rigid sphere with a uniform charge distribution on its surface. In particular, Abraham was convinced that the electron's entire mass was of electromagnetic origin and in 1902 announced the realization of an "electromagnetic mechanics." He also called "longitudinal mass" the mass associated only with a force oriented along the electron's trajectory and called "transverse mass" that associated with a force oriented perpendicular to the electron's trajectory.[6] (These terms had a long life since were used in various papers on the special theory of relativity, including the fundamental Einstein paper of 1905). Since $E_0^e = \frac{e^2}{2R}$ (R radius of the sphere) is the electrostatic energy, the current theory drove to evaluate the electromagnetic contribution to the electron's mass as $m_e = \frac{2}{3}\frac{e^2}{c^2R}$. As a consequence this made the electromagnetic mass equal to 4/3 times the mass entering into Einstein's equation $E = mc^2$.

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Fermi demonstrates that, in the context of the then current theory, one obtains the same result for any system of moving charges, i.e., the factor 4/3. Therefore inertial mass and electromagnetic mass do not match. He also announces that in a forthcoming paper he will consider electromagnetic masses as masses endowed with weight from the point of view of the general theory of relativity. In point of fact, in paper 2), Fermi obtains the result that the electromagnetic mass and the passive gravitational mass (the weight of the charged particle) do match. This is a blatant contradiction: either this result disproves the equivalence principle (largely accepted by that time) or a new problem arises on the possible electromagnetic nature of mass (remember Abraham's ideas on "electromagnetic mechanics"!). In paper 4c), Fermi first solves the problem. He is well aware of the importance of the result obtained. In fact he writes and publishes three equivalent versions of his work (in *Il nuovo Cimento*, in the *Rendiconti dell'Accademia Nazionale dei Lincei*

Vol. 1, 1961, Vol. 2, 1965. The relevant papers 1), 2), 3), 4c) of which we will be concerned in the following are given in English translation in Chapter 2 of this volume.

Chp.

[5]See, for instance O. W. Richardson: *The Electron theory of matter*, Cambridge University Press, 1914, Chapters XI, XII.

[6]For a historical analysis of the problem of the electromagnetic mass see
A. I. Miller: *Albert Einstein's Special Theory of Relativity, Emergence (1905) and Early Interpretation (1905–1911)*, Springer, 1998 and
E. T. Whittaker: *A History of the Theories of the Aether and Electricity*, Thomas Nelson & Sons, London 1951 and 1953.

and in *Physikalische Zeitschrift*).[7] He also confides to his friend Enrico Persico that there will be some troubles to obtain an agreement with his ideas: "... *I am trying with great effort to launch the business of the 4/3. The main difficulty derives from the fact that they have a hard time understanding—in part because the thing is not easy to understand, in part because I express myself too concisely—but little by little they begin to understand what it is all about* ...".[8]

But, as the saying goes, no man is a prophet in his own country and the three versions (even the German one) went unnoticed. Thus, as Rohrlich said,[9] the result was bound to be rediscovered. It did not find its way into the standard references or textbooks until 1953 when E. T. Whittaker, in the second volume of his *History* (on p. 51, see footnote 6) quoted Fermi's Lincei communications saying "It was shown long afterwards by E. Fermi that the transport of the stress system set up in the material of the sphere should be taken into account, and that when this is done, Thomson's result becomes

$$\textit{Additional mass} = \frac{1}{c^2}\ \textit{Energy of the field}\text{"}$$

In the meantime two papers had appeared. W. Wilson obtained the same result of Fermi in a different way[10] and analogously B. Kwal 13 years later in a short note arrived at the same conclusions exploiting the relativistic transformation of the electromagnetic energy-momentum tensor.[11] Finally, the result was discovered for a fourth time by F. Rohrlich,[12] again (apparently) without the knowledge of any of the previous papers. Fundamentally, Fermi showed that factor 4/3 was produced by an incorrect application of (or more precisely by failing to apply) the theory of relativity. The circumstance which, at first sight, might appear rather strange is that Fermi, in his teaching activity of those years continued teaching the old result. Only in his textbook *Introduzione alla Fisica Atomica*[13] he introduced a short sentence mentioning relativistic corrections (without demonstration). In this connection, W. Joffrain[14] put forward the hypothesis of a sort of deontological scruple: not to teach, in an institutional course, results which are not yet universally accepted. Subsequently, in collaboration with A. Pontremoli,[15] Fermi applied successfully

[7] Besides the *Nuovo Cimento* version 4c), Fermi published the two Lincei communications XXI, 1922, pp. 184–187 and 306–309 (4a) and the paper *Über eine Widerspruch zwischen der elektrodynamischen und relativistischen theorie der elektromagnetischen Masse* in Physikalische Zeitschrift XXIII, 340-344, 1922 (4b).

[8] E. Segrè, op. cit., p. 197.

[9] F. Rohrlich: *Charged Classical Particles*, Addison-Wesley, 1965, p. 17.

[10] W. Wilson: The mass of a convected field and Einstein's mass-energy law, Proc. Phys. Soc. (London) **48**, 736–740 (1936). This paper is also mentioned in Whittaker's book.

[11] B. Kwal: Les expressions de l'énergie et de l'impulsion du champ électromagntique propre de l'électron en mouvement, J. Phys. Radium **10**, 103–104 (1949).

[12] F. Rohrlich: Self-energy and stability of the classical electron, Am. J. Phys. **28**, 639–643 (1960).

[13] E. Fermi: *Introduzione alla Fisica Atomica*, Zanichelli, **1928**, p. 66.

[14] W. Joffrain: *Un inedito di Enrico Fermi — Elettrodinamica*, Atti del XVIII Congresso di Storia della Fisica e dell'Astronomia, Como (Italy), May 15–16, 1998.

[15] E. Fermi, A. Pontremoli: *Sulla massa della radiazione in uno spazio vuoto*, Rend. Lincei, **32** (1), 162–164 (1923).

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the same method to the calculation of the mass of the radiation contained in a cavity with reflecting walls, for which the standard textbooks of the time had an expression containing the same factor 4/3. Anyway, the problem of the nature of the electromagnetic mass ~~was~~ has been dragging on for various decades through the contributions, after that of Fermi, of Rohlrich, Dirac, etc.? However, almost always, the successive results—at least apparently—went unnoticed.

### *The resistible path of Fermi's paper*

At this point, in retrospect, if we look at the whole story some circumstances appear at the very least to be strange. Let us start from the beginning of the sequence. Fermi obtained the result published in 4c) in January 1922.[16] It is clear that he feels proud of the conclusions obtained. This turns out clearly in the letter to Persico in which he already announces his intent of publishing the paper also on a German review (which will result ~~to be~~ in *Physikalische Zeitschrift*), to make it known outside of Italy.

At this point we can notice that, completely immersed in the academic context of those times, Fermi thought that the paper concerning the factor 4/3 was much more important, since it was solving a problem already several decades old, than paper 3), only published in Italian in *Rendiconti dell'Accademia Nazionale dei Lincei* presented by G. Armellini in January 1922. As we know, paper 3), after the generalization due to Walker (1932), spread far and wide and still is considered of lasting importance. The German version of 4c), i.e., 4b), sent to *Physikalische Zeitschrift*, was received by the journal ~~the ninth of~~ on 9 May 1922. The paper was immediately published and also reviewed in *Physikalische Berichte* by Erich Kretschmann in the issue of December 15.[17] We point out that Erich Kretschmann, who was a habitual reviewer of the journal for at least three sections regarding the foundations of physics (in German: *Allgemeines, Allgemeine Grundlagen der Physik, Mechanik*, respectively), was not an obscure physicist, but a quite well-known expert in the theory of relativity. In fact a paper published by him in 1917 on the physical meaning of the postulates of the theory of relativity[18] had caused a lot of talk and even aroused a reply by Einstein himself.

Then Fermi's paper, which clarified how one can correctly apply the principles of the (special) theory of relativity to solve the problem of the factor 4/3, outwardly

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[16]This date can be fixed at a sufficiently good approximation by comparing the Fermi's letter to Enrico Persico (see note 8), which is of January ~~twenty-five~~ 25, 1922, with what Persico writes in *Note e Memorie* Vol. 1, p. 24, introducing the paper. Persico also reports a discussion of Fermi with Luigi Puccianti and Giovanni Polvani regarding the factor 4/3 which seems to coincide with what Fermi writes in the letter (where, however, Fermi does not name the names). The strange thing is that Persico does not quote the letter here. Moreover Fermi dates "January 1922" the German version of the paper.

[17]*Physikalische Berichte, Dritter Jahrgang* 1922, N. 24, p. 1293.

[18]E. Kretschmann: Über den physikalischen Sinn der Relativitätstheorie, Annalen der Physik **53**, 575 614 (1917).

appears to have fallen into the hands of the right person. Unfortunately, this was not the case. In fact instead of investigating the method used by Fermi to applying correctly the relativistic concepts, Kretschmann limited himself to repeating Fermi's words describing the two possible ways of performing the variation for applying Hamilton principle and then to conclude that the solution of the problem of the factor 4/3 given by von Laue in his book was *"much more transparent"*.[19] We know from Fermi's biography written by Segrè and from the reminiscences published by Persico that Fermi had studied Weyl's textbook[20] thoroughly, which moreover is quoted in the paper itself when Fermi follows Weyl in applying Hamilton's principle. It is enough to make a comparison of Fermi's paper with the page where Weyl says *"This theory does not, of course, explain the existence of the electron, since cohesive forces are lacking in it"*[21] for understanding that Fermi, following Weyl, only means to deal with a charged sphere (with a surface distribution of charge) without tackling the problem of its internal structure and stability. Then the comparison that Kretschmann makes with von Laue's solution, which involves the introduction of the so called "Poincaré stresses" which turn out to be necessary for ensuring the stability of the electron, is completely misleading. Fermi, as those who will find the solution of the factor 4/3 after him, considers this problem as having nothing to do with the problem of stability. It is curious that even Enrico Persico, who in January 1922 received the letter in which Fermi mentioned the subject, in 1961 writes *"It is now well known that the factor 4/3 can be interpreted as due to the part of the energetic tensor contributed by the internal non-electromagnetic stresses, whose existence must be assumed to assure the equilibrium of the charges. However, in the books known to Fermi, this discrepancy was not explained (he had evidently overlooked the explanation contained in M. von Laue, Die Relativittstheorie, 1, third edition, 1929, p. 218) and so he found for it an explanation of his own, essentially equivalent to the former but obtained through Weyl's variational method"*.[22] At that date Rohrlich's paper had already been published, but perhaps Persico had not had enough time to see it. However, a good eight years before (1953) the second volume of Whittaker's book[23] had been published in which Fermi's paper (the Lincei version) was mentioned with the explanation reported above. We point out that Whittaker's book did not go unnoticed, both for the reputation of the author and for the *vexata quaestio* of the authorship of the special theory of relativity. As is known, Whittaker ascribed to Poincaré the authorship of the special theory of relativity and was also charged with ahistoricisms concerning the theory of relativ-

[19]See 17 Kretschmann quotes the 1919 third edition of von Laue's book, but the author continued to maintain the same conclusions in the subsequent fourth edition (see Die Relativitätstheorie on Dr. M. von Laue, Braunschweig, 1929, pp. 224–227 and also its French translation).

[20]Fermi always quoted the fourth 1921 edition of H. Weyl: Raum. Zeit. Materie, Springer, Berlin.

[21]This excerpt is from the English translation of the 1921 German edition republished by Dover in 1952 with the title *"Space-Time-Matter"*.

[22]See *Note e Memorie* Vol. 1, p. 24. This strange and uncorrected (for what regards "it is now well known...") sentence has been also remarked by Tarsitani, loc. cit. in 4.

[23]See 6.

ity.[24] Then it is strange that even Rohrlich did not know about the quotation of Fermi by Whittaker, particularly if we bear in mind that the subject of Whittaker's book was the origin and the development of the e. m. theory (Abraham, Poincaré, Lorentz, ...). In the 1960 paper (see 12), which is the first Rohrlich dedicated to the problem of the electromagnetic mass of the electron and related questions, Fermi's paper is not mentioned and the same for Wilson's and Kwal's papers.

Apparently Rohrlich solved independently the "4/3 problem" without knowing the contributions of his predecessors. Two years later, in a lecture given before the Joseph Henry Society[25] he said "*... For a finite electron this was first pointed out by Fermi in 1922. It is closely related to the definition of rigidity in special relativity where the difference in the simultaneity of relatively moving observers plays an essential role. Unfortunately, Fermi's paper was either never understood or soon forgotten*". Rohrlich, at this point, quotes as a reference the German version (see 7) of Fermi's paper but there is no mention of the papers of Wilson and Kwal. In his 1965 book (see 9), he mentions all three authors (Fermi, Wilson, Kwal) who had preceded him. As a matter of fact this is the last time Rohrlich mentions Fermi's contribution. On this subject he has published papers for about forty years 40
but, as one can check considering the most important journals, Fermi's name is no longer mentioned. The odd thing is that, even in the last paper known to us[26] which contains an appendix with the title "*The history and eventual solution of the stability problem (the 4/3 problem)*", Fermi's name does not appear. A prospective reader could only find the reference to Fermi in the bibliographies of the books and the papers quoted.

In the same year (1965) in which Rohrlich's book appeared, the second revised edition of *The Special Theory of Relativity* by J. Aharoni also came out.[27] In the preface the author says that Rohrlich's 1960 paper "*...initiated new interest in the problem and it turned out that actually a similar solution had already been proposed by B. Kwal in 1949 and the same result obtained as for back as 1922 by E. Fermi who used a different method. It can now be stated that the abolition of the 4/3 factor is also implicit in Dirac's paper on the classical theory of the electron (1938). It is difficult to explain why all the earlier papers passed unnoticed. Possibly this was due to Poincaré's idea to link the 4/3 factor with the instability of an electric charge on purely electrostatic forces*". It should be noted that Aharoni cannot have had far ?

[24] See G. Holton: On the Origins of the Special Theory of Relativity (1960) in G. Holton: *Thematic Origins of Scientific Thought. Kepler to Einstein*, Harvard University Press, 1977 and A.J. Miller: A study of Henri Poincaré's "Sur la Dynamique de l'Electron." Arch. Hist. Exact. Scis. **10**, 207–328 (1973).

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[25] The theory of the electron, ~~Thirty-first~~ Joseph Henry Lecture (read before the Society May 11, 1962).

[26] F. Rohrlich: The dynamics of a charged sphere and the electron, Am. J. Phys. **65**, 1051-1056, (1997). n dash

[27] J. Aharoni: The special Theory of Relativity, Second revised edition, Oxford University Press, 1965 (reprinted by Dover, 1985).

knowledge from Rohrlich's paper, which he quotes, of the name of Fermi, Wilson and Kwal since in that paper they are not mentioned. Before the two 1965 books, Kwal's name does not appear, while Wilson's name only appears in Whittaker's book together with Fermi's. Evidently, there is a missing link in the chain!

Let us turn again to Aharoni's book which, from our point of view, assumes a particular importance. In fact, Aharoni is the only one, among all who spoke about Fermi's paper, who devoted himself to effectively understand the method Fermi used for applying his relativistic concept of rigidity. He spends about two pages (170–171) to explain and explicitly reconstruct Fermi's calculations omitted by the author who sums up "...*we have manifestly*...". Therefore, on the part of Aharoni there is a true appreciation for the work done by the man who first solved the problem. Retrospectively, it comes to mind that to the early readers those calculations might not be so transparent (see the above letter to Persico where Fermi admits to expressing himself "*too concisely*") and also that the course of differential geometry given by Luigi Bianchi about which Fermi speaks in a letter to Persico, came in very useful to him so to consider obvious the calculations and then to omit them.[28] All the known biographies of Fermi report that he went to Göttingen with a fellowship from the Italian Ministry of Public Instruction in the winter 1922–23 to study with the group headed by Max Born and he remained there seven months. "...when Fermi arrived at Göttingen, he found several brilliant contemporaries there, among them Werner Heisenberg and Pascual Jordan, two of the brightest luminaries of theoretical physics. Indeed the two had already been recognized for their exceptional abilities, and Born was writing papers in collaboration with them at about the time of Fermi's residence in Göttingen. Unfortunately it seems that Fermi did not become a member of that extraordinary group or interact with them. I do not know the reason for this ...".[29] "...Born himself was kind and hospitable. But he did not guess that the young man from Rome, for all his apparent self-reliance, was at the very moment going through that stage of life which most young people cannot avoid. Fermi was groping with uncertainty and seeking reassurance. He was hoping for a pat on the back from Professor Max Born ...".[30] Both the biographers (Emilio Segrè and Fermi's wife) agree in maintaining that Fermi came back to Rome not satisfied with the German experience, somehow disappointed. It is known from other sources that Born and his collaborators thought the best of Fermi and this is born out from the fact that subsequently he was on friendly terms with them. Moreover, the biographers confirm that Fermi's German was certainly good enough to allow easy communication and then not to be excluded. Why then, as far as we know, he did not join the Born's group and went back to Rome disappointed?

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[28]In the letter Fermi writes "I will pass the examination in higher analysis (differential geometry) which is a terrific bore, in which the problem studied are chosen by the sole criterion that they should lack all interest," see Segrè, op. cit. p. 201–202.

[29]See Segrè, op. cit. p. 33.

[30]Laura Fermi: *Atoms in the Family. My life with Enrico Fermi*, The University of Chicago Press, 1954, p. 31

Both Emilio Segrè and Laura Fermi put forward the hypothesis that the ideas of Born's group at that time appeared to be very concrete, even philosophical, and then not able to catch the interest of Fermi or Fermi himself was not mature enough to get himself to be appreciated in that environment.

Our conjecture is that Fermi had effectively taken his paper to Göttingen to be appreciated, but he did not achieve his aim. When Fermi arrived in Göttingen, the paper on the "4/3 problem" had already been published in German and so readable by Born and the others. It is unthinkable that Fermi, who was so proud of his result, had not exhibited it and asked Born for his opinion. We recall that, what's more, in his paper Fermi quotes a relativistic definition of rigidity due to Born (in a paper of 1909).[31] The most obvious thing to do for a brilliant young physicist, as Fermi was, would have been to display the paper he was proud of to the authoritative professor. To the best of our knowledge, no proof exists even if it is reasonable to suppose that this had happened. The only thing we can say for certain is that Born's book on Relativity theory,[32] which in its second edition of 1921 held the "traditional" point of view of the "4/3 problem", continued to give the same version till to the last edition.[33] The same thing happened for Pauli's famous lectures[34] as if Fermi's paper had never existed. Born and Pauli were not alone in ignoring Fermi's paper and related conclusions; to the list we can add even Feynman.[35] Coming back to Born, from his authobiography[36] it turns out that over the years he continued to think about the problem of the electromagnetic mass of the electron, but there is no connection with Fermi's conclusions which are never mentioned. Our conjecture, for all its worth, is that the disappointment for having not received appreciation embittered Fermi and also deterred him from the subject. Moreover, the problems raised by the new quantum mechanics and statistical theories definitively averted his interest from classical electrodynamics.

[31] Max Born: *Die Theorie des starren Elektrons in der Kinematik des Relativitätsprinzips*, Annalen der Physik IV, **11**, 1–56 (1909).

[32] Max Born: *Die Relativitätstheorie Einsteins und ihre physikalischen Grundlagen*, Springer, Berlin, 1921, p. 157.

[33] Max Born: *Einstein's Theory of Relativity*, revised edition prepared with the collaboration of Günther Leibfried and Walter Biem, Dover, 1962, pp. 207–214 and 278–289.

[34] W. Pauli: *Pauli Lectures on Physics, Vol. 1. Electrodynamics*, MIT Press, 1972 (reprinted by Dover, 2000), p. 151

[35] *The Feynman Lectures on Physics. The Electromagnetic Field*, Addison-Wesley, 1964, Sect. 28–3 and ff.

[36] Max Born: *My Life. Recollection of a Nobel Laureate*, Taylor & Francis Ltd, 1978, Part 2, IV, pp. 254–255

# Appendix B

# Selected papers reprinted from *Il Nuovo Cimento*, Vol. 117B, Nos. 9–11, ~~1992~~ 2002

This Appendix contains a selection of the articles from the proceedings of the meeting "Fermi and Astrophysics" organized at the University of Rome "La Sapienza" and at the ICRANet Center in Pescara October 3–6, 2001 and published in *Il Nuovo Cimento B* **117**, Nos. 9–11. The meeting was focused on the influence of Fermi on astrophysics and general relativity: his activities related to these topics were clustered at the beginning and end of his scientific career. These articles, selected because of their direct commentary on articles by Fermi or related applications of his ideas expressed in those articles, are presented in alphabetical order of their first authors.

Susan Ames discusses the historical background of Fermi's work on cosmic rays, along with current problems and further prospects for the physics of cosmic rays. In particular she points out how the frequently discussed ultra-high cosmic rays cannot be accelerated by the Fermi mechanism. Equipartition between the energy of matter and that of cosmic rays was among the initial points made by Fermi, and in that context Ames mentions also the role of the cosmic microwave background radiation.

Donato Bini and Robert Jantzen give a summary of Fermi's discussion of what we now call Fermi coordinates and Fermi transport with a historical update including Walker's contribution which led to the terminology of "Fermi-Walker transport." This article explicitly estimates the various relativistic contributions to the Fermi-Walker transport for vectors around circular orbits in black hole spacetimes and in their Minkowski limit.

Dino Boccaletti comments on the two papers which resulted from the collaboration of Fermi with Chandrasehkar (see papers 261, 262 of Chapter ~~4~~). The first 3 paper is devoted to the study of light dispersion in the polarization plane and using the effect to derive the galactic magnetic field. The second paper contains the generalization of the virial theorem in the presence of a magnetic field. The commentary notes that Fermi was the first scientist to draw attention to the possible existence of a galactic magnetic field.

The review of Andrea Carati, Luigi Galgani, Antonio Ponno and Antonio Giorgilli is devoted to the equipartition problem in the Fermi-Pasta-Ulam para-

Alexei Yu. Smirnov reviews the neutrino flavor transformations in matter, as one of the authors of the original theoretical predictions and related observable effects. In particular, the Sudbury Neutrino Observatory results provide strong evidence of the neutrino flavor conversion. Neutrino conversion is discussed also in the context of supernova neutrinos and the corresponding predictions for the fluxes and energies at the Earth, including the role of the Earth matter effect. The author shows that the data of SN1987 can also be explained by the neutrino oscillations in the matter of Earth as conversions of muon and tau antineutrinos.

George M. Zaslavsky reviews the Fermi-Pasta-Ulam problem with an attempt to find the transition from regular to chaotic dynamics. The Fermi acceleration mechanism is considered as a precursor of the Fermi-Pasta-Ulam problem. The Kepler map introduced by Roald Sagdeev and George Zaslavsky and several other problems are considered, demonstrating the role of the Fermi-Pasta-Ulam work in the discretization methods of differential equations and in the study of chaotic systems when the Lyapunov exponent method is not efficient.