10) On the mass of the radiation in an empty space

"Sulla massa della radiazione in uno spazio vuoto,"
Rend. Lincei 32(1), 162-164 (1923)

Recently, one of us[1] had been able to demonstrate, by introducing a more correct concept of rigidity, that the standard electrodynamics allows us to reach a determination of the electron rest mass not different from that coming from the theory of relativity which, as is known, simply amounts to dividing the energy of the system by the squared speed of light. We have observed that a similar difference, between the value determined following from standard electrodynamics and the one given by the theory of relativity, occurs in the calculation of the mass of the radiation in an empty space[2]. We intend to demonstrate that this discrepancy can be removed by analogous arguments. The procedure followed until now for determining by electrodynamics the mass of the radiation in a cavity consisted first of all in evaluating the electromagnetic momentum $\boldsymbol{G}_0$ for slow and quasi-stationary motions, which, neglecting terms in v^2/c^2, results to be given by[3]

$$\boldsymbol{G}_0 = \frac{4}{3}\frac{W_0}{c^2}\boldsymbol{v}$$

where W_0 is the energy of the radiation for the cavity at rest, $\boldsymbol{v}$ is the actual velocity of the cavity, and c is the speed of light. From this, one deduced that the inertial reaction is given by

$$-\frac{d\boldsymbol{G}_0}{dt} = -\frac{4}{3}\frac{W_0}{c^2}\,\Gamma$$

where Γ is the acceleration; whence an apparent mass of the radiation equals $\frac{4}{3}\frac{W_0}{c^2}$, while, according to the theory of relativity, it should be simply $\frac{W_0}{c^2}$. In this procedure it is implicitly contained the assertion that the external force F is equal to the time derivative of the electromagnetic momentum, i.e., to the resultant of the electromagnetic forces $d\varphi$ acting on every single part of the system; in this way, one then puts:

$$F = \int d\varphi. \qquad (1)$$

But this is not correct, because, if one considers the notion of rigidity discussed by one of us in the quoted paper, the external force is given instead by

$$F = \int d\varphi \left[1 + \frac{\Gamma(P-O)}{c^2}\right], \qquad (2)$$

[1]E. Fermi, these "Rendiconti", Vol. XXXI pp. 184 and 306 (1922), "Physikalische Zeit.", Vol. XXIII (1922), p. 340.

[2]F. Hasenöhrl, "Ann. der Physik", Vol. XV, p. 344 (1904) and Vol. XVI, p. 589 (1905); K. von Mosengeil, "Ann. der Physik", Vol XXII, p. 867 (1927); M. Planck, "Berlin. Sitzber.", p. 542 (1907); M. Abraham, Theorie der Elektrizität, Vol. II, p. 341 (1920).

[3]M. Abraham, loc. cit. p. 345.

(P - O) being the vector from the point P, where the force $d\varphi$ is applied, to a fixed point O, which we can take as the center of coordinates, internal to the system. Now, $d\varphi$ is the resultant of force $d\varphi_1$, exerted by the radiation pressure which would exist if the cavity were at rest, and a force $d\varphi_2$, caused by the perturbations of this pressure due to the motion of the cavity. By applying (1), since evidently $\int d\varphi_1 = 0$, because $d\varphi_1$ is the force exerted by a homogeneous pressure on a closed surface, one finds that the external force is

$$F = \int d\varphi_2. \tag{3}$$

This force is exactly the one calculated as the inertial reaction by the quoted authors, whence

$$\int d\varphi_2 = -\frac{4}{3}\frac{W_0}{c^2}\ \Gamma \tag{4}$$

On the contrary, by applying (2), still taking into account that $\int d\varphi_1 = 0$, one finds

$$F = \int (d\varphi_1 + d\varphi_2)\left[1 + \frac{\Gamma(P-O)}{c^2}\right] = \int d\varphi_1 \frac{\Gamma(P-O)}{c^2} + \int d\varphi_2 + \int d\varphi_2 \frac{\Gamma(P-O)}{c^2}.$$

Neglecting terms in Γ^2 and observing that $d\varphi_2$ is proportional to Γ, one can simply put

$$F = \int d\varphi_1 \frac{\Gamma(P-O)}{c^2} + \int d\varphi_2. \tag{5}$$

In this case the difference between (3) and (5) is not *a priori* negligible, although it contains c^2 at the denominator, since $d\varphi_1/d\varphi_2$ can become considerably large, being the ratio between a force and its perturbation[4]. In fact $d\varphi_2 = p\boldsymbol{n}d\sigma$, where p is the radiation pressure which, as it is known, equals $\frac{1}{3}\frac{W_0}{V}$, being V the volume of the cavity, and n a unit vector with the direction of the external normal to element $d\sigma$ of the surface of the cavity with coordinates *(x, y, z)*. The x component of the first integral of ((5) is then

$$\left[\int d\varphi_1 \frac{\Gamma(P-O)}{c^2}\right]_x = \frac{W_0}{3c^2V}\int (\Gamma_x dx + \Gamma_y dy + \Gamma_z dz) cos\widehat{nx}\, d\sigma =$$
$$= \frac{W_0}{3c^2V}\left(\Gamma_x \int dx\, cos\widehat{nx}\, d\sigma + \Gamma_y \int dy\, cos\widehat{nx}\, d\sigma + \Gamma_z \int dz\, cos\widehat{nx}\, d\sigma\right);$$

but an immediate application of the Gauss theorem shows that

$$\int dx\, cos\widehat{nx}\, d\sigma = V, \qquad \int dy\, cos\widehat{nx}\, d\sigma = \int dz\, cos\widehat{nx}\, d\sigma = 0.$$

[4]In the case of electromagnetic masses one has $d\varphi$ equal to the resultant of the Coulomb forces (which are the predominant part) and the forces due to the acceleration. For the Coulomb forces, evidently also in this case the relation $\int d\varphi_1 = 0$ holds; therefore they have effect only if we apply (5) instead of (3).

Therefore our component is $(W_0\Gamma_x)/3c^2$ and

$$\int d\varphi_1 \frac{\Gamma(P-O)}{c^2} = \frac{W_0\Gamma_x}{3c^2}$$

Considering this relation and (4), it is easy to see that the ratio between the integrals of the right hand side of (5) is $-1/4$ and thus effectively not negligible. By substituting these values into (5), one finds

$$F = -\frac{W_0}{c^2}\ \Gamma$$

from which the requested rest mass results to be equal to W_0/c^2, in accordance with the principle of relativity.