MAT2500-01/02 24s Quiz 7 Print Name (Last, First) ____________________|__

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, IDENTIFYING expressions by their proper symbols (introducing them if necessary), and use EQUAL SIGNS and arrows when appropriate. Always SIMPLIFY expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation if appropriate). Indicate where technology is used and what type (Maple, GC). Only use technology to CHECK hand calculations, not substitute for them.

1. Consider $f(x, y) = xy - x^2 y - xy^2$.
a) Set up the equations to determine its critical points.
b) Use Maple to find quickly the solutions.
c) Now derive these solutions by hand. [Hint: Factor $f_y(x, y)$ to get two conditions, use each in the other equation to finish the job.]
d) Next evaluate the second derivatives in general, and at the critical points you find. Make a table of values and annotate the entries like in this summary page:
https://www34.homepage.villanova.edu/robert.jantzen/courses/mat2500/handouts/2d2nddertest.pdf
linked to the HW page, and classify these points as local maxima, minima or saddle points, justifying your conclusions as indicated in this table.
e) Confirm that your classification of the 4 critical points looks right by using

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> with(plots) :
> contourplot(f(x, y), x=-1..2, y=-1..2, contours = 500, gridlines = true)
> plot3d(f(x, y), x=-1..2, y=-1..2, view=-1..1 , style=surfacecontour, contours = 50)
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No need to print out the plots (unless you wish to), just make sure your results are in agreement with what you see. Just say that they confirm your results (or do not if they do not).

2. Let $f(x, y) = x^2 y^2 - y^3$, $P(1, 2)$, $Q(-3, 5)$.
a) Find the gradient of f and evaluate it at the point P.
b) Evaluate its magnitude and unit vector direction at this point.
c) Evaluate the directional derivative of f in the direction of Q at this point.
d) Derive the equation of the tangent plane at this point and simplify it to standard form.
e) Derive the vector equation of the normal line to the level surface through this point.

▶ solution

① a) $f(x,y) = xy - x^2 y - xy^2$

$f_x = y - 2xy - y^2 = \boxed{y(1-2x-y) = 0}$

$f_y = x - x^2 - 2xy = \boxed{x(1-x-2y) = 0}$

b) $\boxed{\text{MAPLE SOLN: } (x,y) = (0,0),\ (1,0),\ (0,1),\ (\tfrac{1}{3}, \tfrac{1}{3})}$

c) $\rightarrow y=0$ or $1-2x-y=0$

$\rightarrow x=0$ or $1-x-2y=0$

$x=0 \rightarrow (0,0)$

$1-y=0 \rightarrow y=1 : (0,1)$

$1-x=0 \rightarrow x=1$: $(1,0)$

c) continued: $2x+y=1$

$x+2y=1$

$\ominus$: $x-y=0 \rightarrow y=x$

$\rightarrow x+2x=1 \rightarrow 3x=1 \rightarrow x=\frac{1}{3}=y : (\frac{1}{3}, \frac{1}{3})$

d) $f_{xx} = 0 - 2y - 0 = -2y$

$f_{yy} = 0 - 0 - 2x = -2x$

$f_{xy} = 1 - 2x - 2y$

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d) continued

	$(0,0)$	$(1,0)$	$(0,1)$	$\left(\frac{1}{3},\frac{1}{3}\right)$
$f_{xx}=-2y$	0	0	$-2<0 \Rightarrow$ local max	$-\frac{2}{3}<0 \Rightarrow$ local max?
$f_{yy}=-2x$	0	$-2<0 \Rightarrow$ local max	0	$-\frac{2}{3}<0 \Rightarrow$
$f_{xy}=1-2x-2y$	1	-1	-1	$1-\frac{4}{3}=-\frac{1}{3}$
$f_{xx}f_{yy}-f_{xy}^2$	$-1<0$ saddle	$0-(-1)^2=-1<0$ saddle	$0-(-1)^2=-1<0$ saddle	$\left(-\frac{2}{3}\right)\left(-\frac{2}{3}\right)-\left(-\frac{1}{3}\right)^2 = \frac{4-1}{9}>0 \Rightarrow$ confirms local max

e) The plot3d shows a "monkey saddle" shape with 2 places for the legs to go and one in back for the tail, with only one rising "horn" of the saddle in front, and 2 behind separating the legs from the tail.

The contourplot clearly shows the loops around the first quadrant extremum and the 3 saddle points on the axes, all at the right positions.

These confirm our results.

(2) a) $f(x,y)=x^2y^2-y^3 \quad P(1,2),\ Q(-3,5) \quad f(1,2)=1^2 2^2-2^3=4-8=-4$

$f_x = 2xy^2$

$f_y = 2x^2y-3y^2$

$\boxed{\vec{\nabla} f = \langle 2xy^2,\ 2x^2y-3y^2\rangle}$

$\vec{\nabla} f(1,2) = \langle 2\cdot1\cdot2^2,\ 2\cdot1^2\cdot2-3\cdot2^2\rangle$

$= \langle 8, 4-12\rangle = \boxed{\langle 8,-8\rangle}$

$= 8\langle 1,-1\rangle$

b) $|\vec{\nabla} f(1,2)| = 8\sqrt{1+1} = \boxed{8\sqrt{2}}$

$\hat{\nabla} f(1,2) = \boxed{\frac{\langle 1,-1\rangle}{\sqrt{2}}}$

c) $\vec{PQ} = \langle -3,5\rangle - \langle 1,2\rangle = \langle -3-1, 5-2\rangle$

$= \langle -4,3\rangle \longrightarrow |\vec{PQ}| = \sqrt{4^2+3^2} = \sqrt{25} = 5$

$\hat{u} = \hat{PQ} = \frac{\langle -4,3\rangle}{5}$

$D_{\hat{u}} f(1,2) = \hat{u}\cdot\vec{\nabla} f(1,2) = \frac{1}{5}\langle -4,3\rangle\cdot 8\langle 1,-1\rangle = \frac{8}{5}(-4-3) = -\frac{7\cdot 8}{5} = \boxed{\frac{-56}{5}}$

d) graph:

$F(x,y,z) = z-(x^2y^2-y^3) = 0$

$\vec{\nabla} F(x,y,z) = \langle -2xy^2, -2x^2y+3y^2, 1\rangle$

$\vec{\nabla} F(1,2,-4) = \langle -8, 8, 1\rangle = \vec{n}$

$\vec{r}_0 = \langle 1,2,-4\rangle$

$0 = \vec{n}\cdot(\vec{r}-\vec{r}_0) = \langle -8,8,1\rangle\cdot\langle x-1, y-2, z+4\rangle$

$= -8(x-1)+8(y-2)+z+4$

$= -8x+8y+z+\underbrace{8-16+4}_{-4}$

$\boxed{-8x+8y+z=4}$ or $\boxed{8x-8y-z=-4}$

e) $\vec{r} = \vec{r}_0 + t\vec{n} = \langle 1,2,-4\rangle + t\langle -8,8,1\rangle$

$= \langle 1-8t, 2+8t, -4+t\rangle = \langle x,y,z\rangle$

NO! see next page

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(2) e) This makes no sense. What level surface?
A level curve would make sense, which is how many of you interpreted this. The intention was:
"Derive the vector equation of the normal line to the graph of f at this point."

bob screwed up and wasn't thinking.
He mistakenly intended $F(x,y,z) = z - f(x,y) = 0$
but did not succeed in formulating the question correctly.

The normal line has its orientation along the gradient.

$\vec{n} = \langle 1,-1\rangle$ (simplest)

$\langle x,y\rangle = \langle 1,2\rangle + t\langle 1,-1\rangle = \langle 1+t, 2-t\rangle$

so $\boxed{\langle x,y\rangle = \langle 1+t, 2-t\rangle}$

$\vec{r}(t) = \langle 1-t, 1-2t\rangle$ is not the equation of a parametrized line but the definition of a vector valued function of t.
Setting $\vec{r} = \langle x,y\rangle$ equal to this vector valued function shows how x and y depend on t to trace out a curve in the plane.

re-interpret as tangent line:

2 d) normal line to contour or level curve is proportional to the gradient: $\vec{n} = \langle 1,-1\rangle$

$\vec{r}_0 = \langle 1,2\rangle$

$0 = \vec{n}\cdot(\vec{r}-\vec{r}_0) = \langle 1,-1\rangle \cdot \langle x-1, y-2\rangle = x-1-(y-2)$

$= x - y - 1 + 2 = x - y - 1 \rightarrow \boxed{x - y = 1 \text{ or } y = x-1}$