MAT2500-01/02 24S Test 2

① a) $f(x,y) = x^3 - 6xy + 8y^3$

$f_x(x,y) = 3x^2 - 6y$

$f_y(x,y) = -6x + 24y^2$

$f(2,1) = 8 - 6\cdot 2 + 8\cdot 1 = 16 - 12 = 4$

$f_x(2,1) = 3\cdot 4 - 6\cdot 1 = 12 - 6 = 6$

$f_y(2,1) = -6\cdot 2 + 24\cdot 1 = 24 - 12 = 12$

$\vec{\nabla} f(2,1) = \langle 6,12\rangle = 6\langle 1,2\rangle$

$\hat{\nabla} f(2,1) = \boxed{\frac{\langle 1,2\rangle}{\sqrt{5}}}$ direction of maximum increase

$|\vec{\nabla} f(2,1)| = 6\cdot\sqrt{1+4} = \boxed{6\sqrt{5}}$ maximum rate of change

b) $\vec{PQ} = \langle 1.96, 1.03\rangle - \langle 2,1\rangle = \langle -0.04, 0.03\rangle = .01\langle -4,3\rangle$

$\hat{u} = \hat{PQ} = \frac{\langle -4,3\rangle}{5}$ $\qquad |\langle -4,3\rangle| = 5$

$D_{\hat{u}} f(2,1) = \hat{u}\cdot\vec{\nabla} f(1,2) = \frac{1}{5}\langle -4,3\rangle\cdot 6\langle 1,2\rangle = \frac{6}{5}(-4+6) = \boxed{\frac{12}{5} = 2.4}$

c) $L(x,y) = \underbrace{f(2,1)}_{4} + \underbrace{f_x(2,1)}_{6}(x-2) + \underbrace{f_y(2,1)}_{12}(y-1)$

$= \boxed{4 + 6(x-2) + 12(y-1)} = 6x + 12y + 4 - 12 - 12 = \boxed{6x + 12y - 20}$

d) $f(1.96, 1.03) \approx L(1.96, 1.03) = 4 + 6(1.96 - 2) + 12(1.03 - 1)$

$= 4 + 6(-0.04) + 12(0.03)$

$= 4 - 0.24 + 0.36 = \boxed{4.12}$

e) $z = L(x,y) = 6x + 12y - 20 \rightarrow \boxed{6x + 12y - z = 20}$

$\vec{n} = \langle 6, 12, -1\rangle$

f) normal line is perpendicular to tangent plane

or $= \langle -6, -12, 1\rangle$

$\vec{r}_0 = \langle 2, 1, f(2,1)\rangle = \langle 2,1,4\rangle$

$\vec{r} = r_0 + t\vec{n} = \langle 2,1,4\rangle + t\langle 6,12,-1\rangle$

$\hookrightarrow \boxed{\langle x,y,z\rangle = \langle 2+6t, 1+12t, 4-t\rangle}$

g) $\vec{\nabla} f(x,y) = \langle 3x^2 - 6y, -6x + 24y^2\rangle$

$\vec{\nabla} f(0,0) = \langle 0,0\rangle$ ✓

$\vec{\nabla} f(1,\frac{1}{2}) = \langle 3\cdot 1 - 6(\frac{1}{2}), -6\cdot 1 + 24(\frac{1}{2})^2\rangle = \langle 3-3, -6+6\rangle = \langle 0,0\rangle$ ✓

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① g) continued

$f_x(x,y) = 3x^2 - 6y$

$f_y(x,y) = -6x + 24y^2$

$f_{xx}(x,y) = \frac{\partial}{\partial x}(3x^2-6y) = 6x$

$f_{yy}(x,y) = \frac{\partial}{\partial y}(-6x+24y^2) = 48y$

$f_{xy}(x,y) = \frac{\partial}{\partial x}(-6x+24y^2) = -6$

$(0,0)$: $f_{xx}(0,0) = 0$ — inconclusive

$f_{yy}(0,0) = 0$ — inconclusive

$f_{xx}(0,0)\,f_{yy}(0,0) - f_{xy}(0)^2 = 0-(-6)^2 = -36 < 0$ saddle pt.

$(1,\frac{1}{2})$: $f_{xx}(1,\frac{1}{2}) = 6 > 0$ local min

$f_{yy}(1,\frac{1}{2}) = 48\cdot\frac{1}{2} = 24 > 0$ local min

$f_{xx}(1,\frac{1}{2})\,f_{yy}(1,\frac{1}{2}) - f_{xy}(1,\frac{1}{2})^2 = 6\cdot 24 - (-6)^2 = 144-36 > 0$

confirms local min in all directions.

② $f(x,y,z) = (x^2+y^2+z^2)^{1/2}$

$f(3,4,12) = (\underbrace{\underbrace{3^2+4^2}_{5^2}+12^2}_{13^2})^{1/2} = 13$

$f_x = \frac{1}{2}(\dots)^{-1/2}(2x) = \frac{x}{(x^2+y^2+z^2)^{1/2}} \to f_y = \frac{y}{(x^2+y^2+z^2)^{1/2}} \to f_z = \frac{z}{(x^2+y^2+z^2)^{1/2}}$

$df(x,y,z) = f_x(xyz)\,dx + f_y(xyz)\,dy + f_z(xyz)\,dz$

$= \frac{1}{(x^2+y^2+z^2)^{1/2}}(x\,dx + y\,dy + z\,dz)$

$df(3,4,12) = \frac{1}{13}(3dx + 4dy + 12dz)$

$|df(3,4,12)| \le \frac{1}{13}(3\underbrace{|dx|}_{\le 0.002} + 4\underbrace{|dy|}_{\le 0.002} + 12\underbrace{|dz|}_{\le 0.002}) \le \frac{19}{13}0.002 \approx 0.00292$

$\approx 0.0029\,m$

$\approx \boxed{0.29\ cm}$