MAT2500-01/02 24S Test 1

a) $\vec{r} = \langle \frac{t^2}{2}, \frac{1-t^3}{3}, \frac{t^3}{3} \rangle = \langle x, y, z \rangle$

Note $y+z = \frac{1-t^3}{3} + \frac{t^3}{3} = \frac{1}{3}$
so curve is confined to this plane
see f)

$\vec{v} = \vec{r}\,' = \boxed{\langle t, -t^2, t^2 \rangle} = t\langle 1, -t, +t \rangle$

$v = |\vec{r}\,'| = t\sqrt{1+t^2+t^2} = \boxed{t\sqrt{1+2t^2}}$ [note $t = |t|$ since $t \geq 0$]

b) $s = \int_0^t |\vec{r}\,'(\tau)|\,d\tau = \int_0^t \tau\sqrt{1+2\tau^2}\,d\tau = \frac{1}{6}(1+2\tau^2)^{3/2}\Big|_0^t$
$= \frac{1}{6}((1+2t^2)^{3/2} - 1)$

$u = 1+2\tau^2$
$du = 4\tau\,d\tau$

$\int \underbrace{\tau\sqrt{1+2\tau^2}\,d\tau}_{= du/4} = \int u^{1/2}\frac{du}{4} = \frac{1}{4}\frac{u^{3/2}}{3/2} + C = \frac{1}{6}u^{3/2} + C$
$= \frac{1}{6}(1+2\tau^2)^{3/2} + C$

$s(2) = \frac{1}{6}(1+2\cdot 4)^{3/2} - 1) = \frac{1}{6}(9^{3/2} - 1) = \frac{1}{6}((3^2)^{3/2} - 1)$
$= \frac{1}{6}(27-1) = \frac{26}{6} = \boxed{\frac{13}{3}} = 4\,\frac{1}{3} \approx \boxed{4.3333}$

c) $\vec{a} = \vec{r}\,'' = \boxed{\langle 1, -2t, 2t \rangle}$

$\vec{a}(1) = \boxed{\langle 1, -2, 2 \rangle}$

$a = |\vec{r}\,''| = \sqrt{1+4t^2+4t^2} = \boxed{\sqrt{1+8t^2}}$

$a(1) = \sqrt{1+8} = \sqrt{9} = \boxed{3}$

d) $\hat{T} = \frac{\vec{r}\,'}{|\vec{r}\,'|} = \frac{\cancel{t}\langle 1, -t, t \rangle}{\cancel{t}\sqrt{1+2t^2}} = \boxed{\frac{\langle 1, -t, t \rangle}{\sqrt{1+2t^2}}}$

$\hat{T}(1) = \boxed{\frac{\langle 1, -1, 1 \rangle}{\sqrt{3}}}$

e) $\vec{r} = \vec{r}(1) + t\,\vec{r}\,'(1) = \langle \frac{1}{2}, 0, \frac{1}{3} \rangle + t\langle 1, -1, 1 \rangle = \boxed{\langle \frac{1}{2}+t, -t, \frac{1}{3}+t \rangle = \langle x, y, z \rangle}$

or $x = \frac{1}{2}+t$, $\underbrace{y = -t, \; z = \frac{1}{3}+t}_{\text{(f) notice that } y+z=\frac{1}{3}}$

f) $\vec{B} = \vec{r}\,' \times \vec{r}\,'' = t\langle 1,-t,t\rangle \times \langle 1,-2t,2t\rangle \underset{\text{Maple}}{=} t\langle 0,-t,-t\rangle = \boxed{t^2\langle 0,1,-1\rangle}$

$|\vec{B}| = t^2\sqrt{1+1} = \boxed{\sqrt{2}\,t^2}$

$\hat{B} = \frac{1}{\sqrt{2}t^2}\, t^2\langle 0,-1,-1\rangle = \boxed{\frac{1}{\sqrt{2}}\langle 0,-1,-1\rangle}$ $\quad \boxed{\hat{B}(1) = \frac{1}{\sqrt{2}}\langle 0,-1,-1\rangle}$

choose $\vec{n} = \langle 0,1,1\rangle \propto \vec{B}$

$0 = \vec{n}\cdot(\vec{r} - r(1)) = \langle 0,1,1\rangle \cdot \langle x-\frac{1}{2}, y-0, z-\frac{1}{3}\rangle = (y-0) + (z-\frac{1}{3})$

$\boxed{y+z=\frac{1}{3}}$ follows from $\vec{r}$ above!

g) $\hat{N} = \hat{B}\times\hat{T} = \frac{1}{\sqrt{2}}\langle 0,-1,-1\rangle \times \frac{\langle 1,-t,t\rangle}{\sqrt{1+2t^2}} = \boxed{\frac{1}{\sqrt{2}\sqrt{1+2t^2}}\langle -2t,-1,1\rangle}$

$\boxed{\hat{N}(1) = \frac{1}{\sqrt{6}}\langle -2,1,1\rangle}$

h) $K = \frac{\sqrt{2}t^2}{(t\sqrt{1+2t^2})^3} = \boxed{\frac{\sqrt{2}}{t(1+2t^2)^{3/2}}}$

i) $a_{\hat{T}} = \hat{T}\cdot\vec{a} = \frac{\langle 1,-t,t\rangle}{\sqrt{1+2t^2}} \cdot \langle 1,-2t,2t\rangle = \frac{(1+2t^2+2t^2)}{\sqrt{1+2t^2}} = \frac{1+4t^2}{\sqrt{1+2t^2}}$

$a_{\hat{N}} = \hat{N}\cdot\vec{a} = \frac{1}{\sqrt{2}\sqrt{1+2t^2}}\langle -2t,-1,1\rangle\cdot\langle 1,-2t,2t\rangle = \frac{1}{\sqrt{2}\sqrt{1+2t^2}}(-2t+2t+2t)$

$= \frac{\sqrt{2}t}{\sqrt{1+2t^2}}$

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optional

Note: $a_{\hat{T}}^2 + a_{\hat{N}}^2 = \frac{(1+4t^2)^2}{1+2t^2} + \frac{2t^2}{1+2t^2} = \frac{1+8t^2+16t^4+2t^2}{1+2t^2}$

$= \frac{1+10t^2+16t^4}{1+2t^2} = \frac{(1+2t^2)(1+8t^2)}{(1+2t^2)} = 1+8t^2 = a^2$

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j) $K(t)\,v(t)^2 = \left(\frac{\sqrt{2}}{t(1+2t^2)^{3/2}}\right)(t\sqrt{1+2t^2})^2 = \frac{\sqrt{2}t^2}{t}\frac{(1+2t^2)}{(1+2t^2)^{3/2}}$

$= \frac{\sqrt{2}t}{\sqrt{1+2t^2}} = a_{\hat{N}}$ from above! yes!