MAT2705-04/05 23S Quiz 7 bonus details

a) $4x'' + 4x' + 17x = \cos 2t$

$17\,[\,x_p = c_3 \cos 2t + c_4 \sin 2t$

$+4\,[\,x_p' = -2c_3 \sin 2t + 2c_4 \cos 2t$

$+4\,[\,x_p'' = -4c_3 \cos 2t - 4c_4 \sin 2t$

$$4x_p'' + 4x_p' + 17x_p = [(17-16)c_3 + 8c_4]\cos 2t + [-8c_3 + (17-16)c_4]\sin 2t = \cos 2t$$

$$\begin{bmatrix} 1 & 8 \\ -8 & 1 \end{bmatrix}\begin{bmatrix} c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \to \begin{pmatrix} c_3 \\ c_4 \end{pmatrix} = \frac{1}{65}\begin{bmatrix} 1 & -8 \\ 8 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{65}\begin{bmatrix} 1 \\ 8 \end{bmatrix}$$

$x_p = \frac{1}{65}(\cos 2t + 8 \sin 2t)$

$$x = x_h + x_p = e^{-t/2}(c_1 \cos 2t + c_2 \sin 2t) + \frac{1}{65}(\cos 2t + 8 \sin 2t)$$

$$x' = -\frac{1}{2}e^{-t/2}(c_1 \cos 2t + c_2 \sin 2t) + e^{-t/2}(-2c_1 \sin 2t + 2c_2 \cos 2t) + \frac{1}{65}(-2 \sin 2t + 16 \cos 2t)$$

$x(0) = c_1 + 1/65 = 0$

$x'(0) = -\frac{1}{2}c_1 + 2c_2 + \frac{16}{65} = 0$

$c_1 = -\frac{1}{65},\ c_2 = \frac{1}{2}\left(-\frac{16}{65} + \frac{1}{2}\left(\frac{1}{65}\right)\right) = \frac{-33}{4 \cdot 65}$

$$\boxed{x = \frac{1}{65}\left(16 \cos 2t - \frac{33}{4} \sin 2t\right)e^{-t/2} + \frac{1}{65}(\cos 2t + 8 \sin 2t)}$$

b) $4x'' + 4x' + 17x = \cos \omega t = 4B_0 \cos \omega t$

$17\,[\,x_p = c_3 \cos \omega t + c_4 \sin \omega t$

$+4\,[\,x_p' = -c_3 \omega \sin \omega t + c_4 \omega \cos \omega t$

$+4\,[\,x_p'' = -c_3 \omega^2 \cos \omega t - c_4 \omega^2 \sin \omega t$

$$4x_p'' + 4x_p' + 17x_p = [(17-4\omega^2)c_3 + 4\omega c_4]\cos \omega t + [-4\omega c_3 + (17-4\omega^2)c_4]\sin \omega t = \cos \omega t$$

$17-4\omega^2$ everywhere (careless)

$$\begin{bmatrix} 17-4\omega^2 & 4\omega \\ -4\omega & 17-4\omega^2 \end{bmatrix}\begin{bmatrix} c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} c_3 \\ c_4 \end{bmatrix} = \frac{1}{(17-4\omega^2)^2 + 16\omega^2}\begin{bmatrix} 17-4\omega^2 & -4\omega \\ 4\omega & 17-4\omega^2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{(\dots)}\begin{bmatrix} 17-4\omega^2 \\ 4\omega \end{bmatrix}$$

$$A(\omega) = \sqrt{c_3^2 + c_4^2} = \frac{\sqrt{(17-4\omega^2)^2 + (4\omega)^2}}{(17-4\omega^2)^2 + 16\omega^2} = \frac{1/4}{\sqrt{(\omega^2 - 17/4)^2 + \omega^2}}$$

$$0 = A'(\omega) = \frac{1}{4}\left(-\frac{1}{2}\right)(\dots)^{-3/2}\left[2\left(\omega^2 - \frac{17}{4}\right)(2\omega) + 2\omega\right]$$

$$\underbrace{4\omega\left[\omega^2 - \frac{17}{4} + \frac{1}{2}\right]} \to = \omega^2 - \frac{15}{4} \to \boxed{\omega = \frac{\sqrt{15}}{2}}$$

$$A\left(\frac{\sqrt{15}}{2}\right) = \frac{1/4}{\sqrt{\left(\frac{15}{4} - \frac{17}{4}\right)^2 + \frac{15}{4}}} = \frac{1}{8}$$

c) $4x'' + 17x = \cos 2t$

$17\,[\,x_p = c_3 \cos 2t + c_4 \sin 2t$

$+\ [\,x_p' = -2c_3 \sin 2t + 2c_4 \cos 2t$

$+4\,[\,x_p'' = -4c_3 \cos 2t - 4c_4 \sin 2t$

$$4x_p'' + 17x_p = (17-16)c_3 \cos 2t + (17-16)c_4 \sin 2t = \cos 2t$$

$c_3 = 1,\ c_4 = 0,\ x_p = \cos 2t$

$$x = c_1 \cos \tfrac{\sqrt{17}}{2}t + c_2 \sin \tfrac{\sqrt{17}}{2}t + \cos 2t$$

$$x' = -\tfrac{\sqrt{17}}{2}c_1 \sin \tfrac{\sqrt{17}}{2}t + \tfrac{\sqrt{17}}{2}c_2 \cos \tfrac{\sqrt{17}}{2}t - 2 \sin 2t$$

$x(0) = c_1 + 1 = 0 \to c_1 = -1$

$x'(0) = \frac{\sqrt{17}}{2}c_2 = 0 \qquad c_2 = 0$

$x = \cos 2t - \cos \frac{\sqrt{17}}{2}t$

see textbook or notes

$\omega_- = \frac{\sqrt{17}}{4} - 1,\ \omega_+ = \frac{\sqrt{17}}{4} + 1$

$\approx 0.0308 \ll \approx 2.0308$

$T_- \approx 204.16 \gg T_+ = 3.094$

$T_{beat} = \frac{1}{2}T_- \approx 102.08$

$\frac{T_-}{T_{beat}} \approx 33$ oscillations per beat.