MAT2705-04/05 23s Quiz 6 Print Name (Last, First) ____________________|___

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use **proper mathematical notation**, identifying expressions by their proper symbols (introducing them if necessary), and use EQUAL SIGNS and arrows when appropriate. Always SIMPLIFY expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation). Indicate where technology is used and what type (Maple, GC). **Only use technology to CHECK hand calculations, not substitute for them, unless specifically requested.**

1. Consider the initial value problem $y''-3y'+2y=0, y(0)=3, y'(0)=5$.
a) Show that $y_1=e^x$ and $y_2=e^{2x}$ are solutions of the DE, so that the general solution is $y=c_1y_1+c_2y_2$.
b) Write down the matrix form of the two equations needed to impose the initial conditions.
c) Use the inverse matrix to solve the system.
d) Write down the solution of the initial value problem.

2. a) Find the coefficients $\langle y_1, y_2\rangle$ of the new basis vectors $b_1=\langle 2,-1\rangle$, $b_2=\langle -1, 1\rangle$ needed to express the vector $\langle x_1, x_2\rangle=\langle 1, 2\rangle$ as a linear combination of them. First write down the matrix equation required to solve for these coefficients and use the inverse matrix to solve it.

[see next page for more]

① a) $+2\,[y_1=e^x]$
$-3\,[y_1'=e^x]$
$1\,[y_1''=e^x]$

$y_1''-3y_1'+2y_1=(1-3+2)e^x=0$ ✓

$+2\,[y_2=e^{2x}]$
$-3\,[y_2'=2e^{2x}]$
$+1\,[y_2''=4e^{2x}]$

$y_2''-3y_2'+2y_2=(2-6+4)e^{2x}=0$ ✓

b) $y=c_1e^x+c_2e^{2x}$
$y'=c_1e^x+2c_2e^{2x}$

$y(0)=c_1+c_2=3$
$y'(0)=c_1+2c_2=5$

$\boxed{\begin{bmatrix}1&1\\1&2\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}=\begin{bmatrix}3\\5\end{bmatrix}}$

c) $\begin{bmatrix}c_1\\c_2\end{bmatrix}=\begin{bmatrix}1&1\\1&2\end{bmatrix}^{-1}\begin{bmatrix}3\\5\end{bmatrix}=\frac{1}{2-1}\begin{bmatrix}2&-1\\-1&1\end{bmatrix}\begin{bmatrix}3\\5\end{bmatrix}=\begin{bmatrix}2\cdot 3-1(5)\\-1(3)+1(5)\end{bmatrix}=\begin{bmatrix}6-5\\-3+5\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix}$

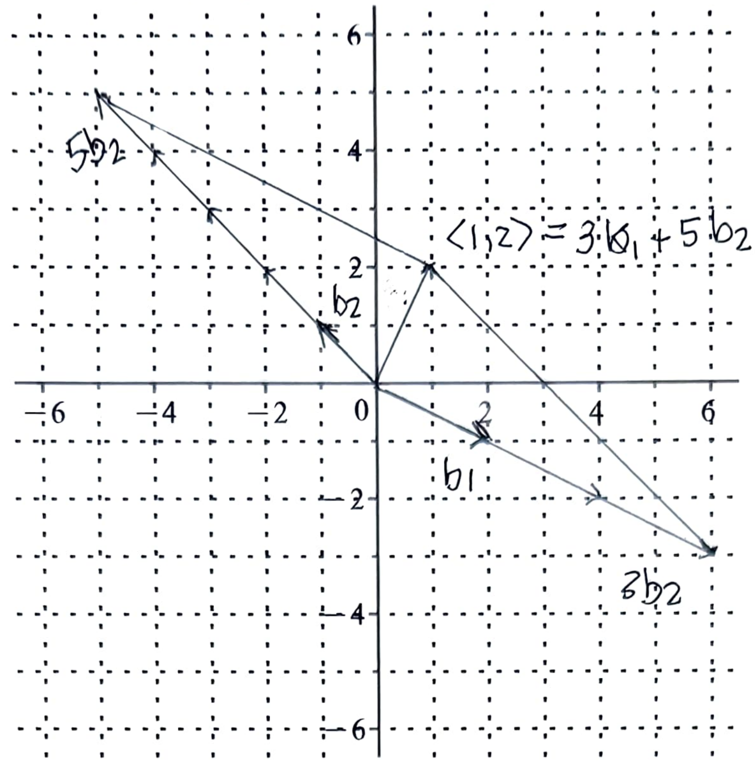
d) $\boxed{y=1e^x+2e^{2x}}$

② a) $y_1b_1+y_2b_2=\langle 1,2\rangle \rightarrow \begin{bmatrix}2&-1\\-1&1\end{bmatrix}\begin{bmatrix}y_1\\y_2\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix}$

$\begin{bmatrix}y_1\\y_2\end{bmatrix}=\begin{bmatrix}2&-1\\-1&1\end{bmatrix}^{-1}\begin{bmatrix}1\\2\end{bmatrix}=\frac{1}{2-1}\begin{bmatrix}1&1\\1&2\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}1\cdot 1+1(2)\\1(1)+2(2)\end{bmatrix}=\begin{bmatrix}3\\5\end{bmatrix}$

so $\boxed{\langle 1,2\rangle=3\langle 2,-1\rangle+5\langle -1,1\rangle}$

2. b) On the grid draw in the 4 vectors $\langle 1, 2\rangle$, b_1, b_2, $y_1 b_1$, $y_2 b_2$ with the values you found for y_1 and y_2, and complete the latter two vectors to a parallelogram by adding each to the tip of the other. The resulting main diagonal vector should be the vector $\langle 1, 2\rangle$ if you did this correctly.



c) If we re-interpret this grid as the c_1-c_2 coordinate plane for the solution space of the initial value problem 1, then the vectors $\langle 2,-1\rangle$, $\langle -1, 1\rangle$ (which are the columns of the inverse Wronskian matrix at $x=0$) correspond to solution functions $Y_1 = 2\,e^x - e^{2x}$, $Y_2 = -e^x + e^{2x}$. Evaluate the Wronskian matrix $\begin{bmatrix} Y_1(0) & Y_2(0) \\ Y_1'(0) & Y_2'(0) \end{bmatrix}$ of this new basis of solutions of the initial value problem and show that it equals the identity matrix. This means that it is a natural basis for initial conditions at $x=0$, so that the initial data are themselves the coefficients needed to find a solution of the initial value problem using these new basis functions. Simplify $3\,Y_1 + 5\,Y_2$ to show that it agrees with your solution to problem 1. [None of this is necessary just to impose initial conditions, but it shows how this procedure corresponds to a change of basis in the solution space.]

c) $Y_1 = 2e^x - e^{2x}$ $\quad Y_1(0) = 2-1=1$

$Y_1' = 2e^x - 2e^{2x}$ $\quad Y_1'(0) = 2-2=0$

$Y_2 = -e^x + e^{2x}$ $\quad Y_2(0) = -1+1=0$

$Y_2' = -e^x + 2e^{2x}$ $\quad Y_2'(0) = -1+2=1$

$$\begin{bmatrix} Y_1(0) & Y_2(0) \\ Y_1'(0) & Y_2'(0) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

$$3Y_1 + 5Y_2 = 3(2e^x - e^{2x}) + 5(-e^x + e^{2x}) = (6-5)e^x + (-3+5)e^{2x}$$

$$= e^x + 2e^{2x} = y \checkmark !$$