MAT2705-04/05 23s Quiz 4 Print Name (Last, First) |

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use EQUAL SIGNS and arrows when appropriate. Always SIMPLIFY expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation). Indicate where technology is used and what type (Maple, GC). **Only use technology to CHECK hand calculations, not substitute for them, unless specifically requested.**

1. Resolving this problem step by step by hand involves relatively simple numbers. You really only need technology to evaluate the numerical value of your formulas.

a) Suppose the number N of people who have heard a certain rumor among a total population of 100,000 people is governed by the differential equation $\frac{dN}{dt} = k\,N\cdot(M-N)$.

[This makes sense since initially the number of people who hear the rumor should scale with the number of people who already know it, but the number has to level off when reaching the whole population, a natural logistic behavior.] Take $M = 100$ using units of 1000 people. If at time $t=0$, half of the population has heard the rumor and the number of those who have heard it is increasing at the rate of 1000 persons per day, how many days will it take for this rumor to spread to 80% of the population? [Hint: Find the value of k by substituting $N(0)$ and $N'(0)$ into the logistic differential equation, then use the solution formula for this equation $N = \frac{M N_0}{N_0 + (M-N_0)\,e^{-Mkt}}$ to answer the question.]

Show step by step first simplifying your expression for $N(t)$ and then finding your the number t_{80} of days by hand using rules of exponents and ln's, giving first an exact expression and then a numerical one to one decimal place.

b) What is the value of the characteristic time for this exponential behavior? How many characteristic times does your answer to part a) represent?

c) How many days does it take to reach 99% of the population? How many characteristic times does this represent?

a) $M = 100$, $N(0) = \frac{1}{2}\cdot 100 = 50$, $N'(0) = 1$

$N'(0) = k\,N(0)\,(M - N(0))$

$1 = k\cdot 50(100-50) = 50^2 k = 2500k$

$k = \frac{1}{2500} = .0004$, $Mk = \frac{100}{2500} = \boxed{\frac{1}{25} = .04 = \tau^{-1}} \rightarrow \boxed{\tau = 25}$ b)

$N = \frac{100(50)}{50 + (100-50)\,e^{-0.04t}} = \frac{50(100)}{50(1+e^{-.04t})} = \boxed{\frac{100}{1+e^{-.04t}} = N}$

$\frac{100}{1+e^{-.04t}} = (.80)\,100 \rightarrow \frac{1}{1+e^{-.04t}} = .8 \rightarrow \underbrace{\frac{1}{.8}}_{1.25} = 1 + e^{-.04t}$

$\underbrace{1.25 - 1}_{.25} = e^{-.04t} \rightarrow \underbrace{\frac{1}{.25}}_{4} = e^{.04t} \rightarrow .04t = \ln 4 \rightarrow \boxed{t = \frac{\ln 4}{.04} = 25\ln 4}$

$\approx \boxed{34.7 \text{ days}}$

b) $\frac{t}{\tau} = \frac{25\ln 4}{25} = \ln 4 = 2\ln 2 \approx \boxed{1.39 \text{ characteristic times}}$

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c) $$\frac{100}{1+e^{-.04t}} = .99(100)$$

$$\frac{1}{1+e^{-.04t}} = .99$$

$$\frac{1}{.99} = 1 + e^{-.04t}$$

$$\underbrace{\frac{1}{.99} - 1}_{\frac{100}{99}-1} = e^{-.04t}$$

$$= \frac{100-99}{99} = \frac{1}{99}$$

$$99 = e^{.04t}$$

$$.04t = \ln 99$$

$$t = \frac{\ln 99}{.04} = \underbrace{25}_{\tau} \ln 99 \approx \boxed{114.9 \text{ days}}$$

$$\frac{t}{\tau} = \ln 99 \approx \boxed{4.60 \text{ characteristic times}}$$