MAT2705-04/05 23s Quiz 3 Print Name (Last, First) ____________________

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use **proper mathematical notation**, identifying expressions by their proper symbols (introducing them if necessary), and use EQUAL SIGNS and arrows when appropriate. Always SIMPLIFY expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation). Indicate where technology is used and what type (Maple, GC). **Only use technology to CHECK hand calculations, not substitute for them, unless specifically requested.**

1. a) Although this DE is separable, use the linear solution approach to find its general solution
$(x^2+4)\,y' = -3\,x\,y + x,\quad y(0)=1$
including every step by hand.
b) Then find the solution satisfying the initial condition. Does your solution agree with Maple, yes or no?
c) To check your solution, back substitute this initial value problem solution $y=y(x)$ and its derivative into the original DE and simplify both sides without changing their values until both sides are the same.

2. a) A 30 year old woman accepts an engineering position with a starting salary of \$60,000 per year. Her annual salary $S(t)$ increases exponentially with $S(t)=60\,e^{0.05\,t}$ thousand dollars after t years. Meanwhile, 10% of her salary is deposited continously in a retirement account, which accumulates interest at a continuous annual rate of 8%. The amount $A(t)$ in her retirement account is described by the initial value problem:

$$\frac{dA}{dt} = 0.10\cdot 60\,e^{0.05t} + 0.08\,A,\ A(0)=0.$$

Use the linear solution approach to find the general solution, and then the IVP solution, and finally the amount $A(30)$ when she early retires at age 60. Give your final answer to the nearest thousand dollars.
b) What is the characteristic time in years for the exponential growth of her income? What is the doubling time? [One decimal place is sufficient.]

▶ solution

a) $y' = \dfrac{-3x\cancel{y}\cancel{+x}}{x^2+4} = \dfrac{-3x}{x^2+4}\,y + \dfrac{x}{x^2+4}$

$\left[y' + \underbrace{\dfrac{3x}{x^2+4}}\,y = \dfrac{x}{x^2+4}\right](x^2+4)^{3/2} \longrightarrow \dfrac{d}{dx}\left(y\,(x^2+4)^{3/2}\right) = \dfrac{x}{(x^2+4)}(x^2+4)^{3/2} = x\,(x^2+4)^{1/2}$

$\int \dfrac{3x}{x^2+4}\,dx = \dfrac{3}{2}\int \dfrac{2x\,dx}{x^2+4} = \dfrac{3}{2}\int \dfrac{du}{u} = \dfrac{3}{2}\ln|u| = \dfrac{3}{2}\ln(x^2+4)$

$e^{\frac{3}{2}\ln(x^2+4)} = (e^{\ln(x^2+4)})^{3/2} = (x^2+4)^{3/2}$

$\rightarrow y\,(x^2+4)^{3/2} = \int x\,(x^2+4)^{1/2}\,dx = \dfrac{1}{2}\int (x^2+4)^{1/2}\,2x\,dx = \dfrac{1}{2}\int u^{1/2}\,du$

$= \dfrac{1}{2}\dfrac{u^{3/2}}{3/2} + C = \dfrac{1}{3}(x^2+4)^{3/2} + C$

$y = (x^2+4)^{-3/2}\left(\dfrac{1}{3}(x^2+4)^{3/2} + C\right) = \boxed{\dfrac{1}{3} + C\,(x^2+4)^{-3/2} = y}$ gen soln

b) $1 = y(0) = \dfrac{1}{3} + C\,\underbrace{4^{-3/2}}_{=\frac{1}{8}} \rightarrow \dfrac{2}{3} = 1 - \dfrac{1}{3} = \dfrac{C}{8} \rightarrow C = \dfrac{16}{3}$

$\boxed{y = \dfrac{1}{3} + \dfrac{16}{3(x^2+4)^{3/2}}}$

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① c) $y = \frac{1}{3} + \frac{16}{3}(x^2+4)^{-3/2}$

$y' = \frac{16}{3}\left(-\frac{3}{2}\right)(x^2+4)^{-5/2}(2x) = -16x(x^2+4)^{-5/2}$

$(x^2+4)y' = -3xy + x$

$(x^2+4)(-16x)(x^2+4)^{-5/2} = -3x\left[\frac{1}{3} + \frac{16}{3}(x^2+4)^{-3/2}\right] + x$

$-16x(x^2+4)^{-3/2} = \cancel{-x} - 16x(x^2+4)^{-3/2} + \cancel{x} \to 0$

$= -16x(x^2+4)^{-3/2}$ ✓

② a) $\frac{dA}{dt} = \underbrace{0.10\cdot 60}_{6}\, e^{.05t} + 0.08A$

$e^{-.08t}\left[\frac{dA}{dt} - 0.08A = 6e^{.05t}\right] \to \frac{d}{dt}\left(Ae^{-.08t}\right) = 6e^{-.08t}e^{.05t}$

$= 6e^{-.03t}$

$e^{\int -0.08\,dt} = e^{-.08t}$

$Ae^{-.08t} = \int 6e^{-.03t}\,dt = 6\,\frac{e^{-.03t}}{-.03} + C$

$= -200e^{-.03t} + C$

$A = e^{.08t}\left(-200e^{-.03t} + C\right) = \boxed{-200e^{.05t} + e^{.08t} = A}$ gen soln

$0 = A(0) = -200 + C \to C = 200$

$\boxed{A = -200e^{.05t} + 200e^{.08t}}$ IVP soln

$A(30) = 200\left(e^{.08(30)} - e^{.05(30)}\right)$

$= 200\left(e^{2.4} - e^{1.5}\right) \approx 1308.297$

$\approx \boxed{1308}$ K\$ so 1,308 million dollars

b) $S(t) = 60e^{.05t} = 60e^{t/\tau}$ $\quad \tau = \frac{1}{.05} = \boxed{20}$ characteristic time in years

$e^{.05t} = 2 \to .05t = \ln 2 \to t = \frac{1}{.05}\ln 2 = 20\underbrace{\ln 2}_{0.693} \approx \boxed{13.9}$ doubling time in years

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The doubling time or characteristic time ONLY apply to a single exponential function like $Q = Q_0 e^{kt}$, $k > 0$, $Q_0 \neq 0$!

The doubling time is how long Q takes to double (from $t=0$)

so $e^{kt} = 2 \to kt = \ln 2 \quad t = \frac{\ln 2}{k}$

The characteristic time is how long Q takes to increase by a factor of e (from $t=0$):

$$e^{kt} = e^1 \to kt = 1 \to t = \frac{1}{k} \equiv \tau \quad \text{"characteristic time"}$$

In this word problem the salary $S(t)$ is the "income", not the investment amount

$$A(t) = 200\,(\underbrace{e^{.08t} - e^{.05t}}) \to A(0) = 0 \leftarrow \text{cannot double!}$$

two different exponentials, two different characteristic times:

$\tau_1 = \frac{1}{.08} = 12.5$, $\tau_2 = \frac{1}{.05} = 20$

The initial amount zero cannot double or increase by a factor of e.

same exponential

The salary = income is a single exponential: $S = 60e^{.05t}$ $\to \tau_2 = \frac{1}{.05} = 20$ years

doubling time $60 \to 120$: $e^{.05t} = 2$

$.05t = \ln 2$

$t = \ln 2 \left(\frac{1}{.05}\right) = 20 \ln 2 = 13.9$ years

This part will not be graded due to the confusion.